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Mathematical theory of defects. Part II. Dynamics

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THE BASIC formulations of the dynamic theory of defects in the infinite, homogeneous, linearly — elastic continuum are discussed. The relations between the theory of surface defects in the displacement description and the theory of the medium with defects represented by the incompatible deformation and velocity fields are demonstrated. The dynamic incompatibility problem is examined.

Omówione są podstawy sformułowania dynamicznej teorii defektów w nieskończonym, jednorodnym kontinuum liniowo sprężystym. Pokazane jest przejście od teorii defektów powierzchniowych w opisie przemieszczeniowym do teorii ośrodka z defektami, opisywanego przez niezgodne pola deformacji i prędkości. Pokazane jest rozwiązanie dynamicznego problemu niezgodności.

Обсуждены основные формулировки динамической теории дефектов в бесконечном, однородном линейно упругом континуум. Показан переход от теории поверхностных дефектов в описании в перемещениях до теории среды с дефектами описанной несовместимыми полями деформации и скорости. Указано решение динамической задачи несовместности.

1. Introduction

IN THE PRESENT paper the dynamic theory of defects in the linearly-elastic, infinite, homogeneous medium will be presented. In the same way as in [1], where the static problems were considered, the relations between different formulations of the theory will be demonstrated.

The theory of defects in motion is not studied to such an extent as the static theory, in particular the dynamic incompatibility problem, and the disclinations dynamics has been worked out only recently [7, 14]. In the present paper some results of [1] will be made use of. Some questions, which are similar in the dynamic and static case, and were considered in details in [1], are handled in short here. Special attention will be paid to questions, which are essential in dynamics. In every formulation of the theory, the geometric and kinematic constraints equations, together with the solutions for the elastic fields will be examined. The problems of forces acting on defects, defects self-energy and energy radiation will be omitted.

In the dynamic case, the state of the medium is represented, in addition to the deformation field, also by the velocity field. Together with this new physical field, the new constraints equations appear in the theory. Dynamic defects are characterised not only by the position vector, but also by the velocity vector; thus besides the defects densities, the defects currents appear.

In the displacement description we consider an ideal medium with the forced, time-dependent, singular deformations: along certain moving surfaces the discontinuities of the displacement field $\mathbf{u}$ were produced. Such a discontinuity surface we call a surface defect. In the classical linear theory of elasticity, to defects correspond the dynamic elastic potentials of a double layer (see [2, 3]), characterised by the discontinuity vector $\mathbf{U}$ of the field $\mathbf{u}$.

When the motion of a discontinuity surface S is arbitrary, not only the distortion field $u_{i,k}$ but also the velocity field $\dot{\mathbf{u}}$, possess on S the singularities of the type of a delta-function. We represent thus, as in the static case, the distortion field $u_{i,k}$ in the form of the regular part $\boldsymbol{\beta}$ and singular $\mathring{\boldsymbol{\beta}}$, and the velocity field $\dot{\mathbf{u}}$ in the form of the sum of the regular part $\mathbf{v}$ and singular $\mathring{\mathbf{v}}$. The fields $\boldsymbol{\beta}$ and $\mathbf{v}$ we call elastic, $\mathring{\boldsymbol{\beta}}$ and $\mathring{\mathbf{v}}$ the plastic (or initial) distortion and velocity.

The strain and velocity fields corresponding to dynamic potentials of a double layer satisfy the dynamic equilibrium equation in the form $\varrho\dot{v}_i - c_{iklm}\, e_{lm,k} = 0$, which is the basic equation for the theory of initial deformations in the linearly-elastic medium. When we pass from the ideal medium to the incompatible medium with defects, we assume that the elastic fields satisfy this very equation and the constraints equations.

In the theory of dislocations, the medium is to be described by the elastic fields $\boldsymbol{\beta}$ and $\mathbf{v}$. The two constraints equations describe the influence of the dislocation density tensor α and the dislocation current $\mathbf{J}$ on the fields $\boldsymbol{\beta}$ and $\mathbf{v}$. The medium with disclinations is to be described by four elastic fields: the strain $\mathbf{e}$, the bend-twist $\boldsymbol{\varkappa}$, linear velocity $\mathbf{v}$ and the rotational velocity $\mathbf{w}$. To them correspond the four constraints equations and the four source functions: dislocations and disclinations density tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$, and current tensors $\mathbf{J}$ and $\mathbf{I}$.

In the fourth chapter the general formulation of the so-called dynamic incompatibility problem will be presented. In the dynamic case, these are $\mathbf{e}$ and $\mathbf{v}$, which have always the good physical interpretation. In addition to geometric incompatibilities, represented by the incompatibility tensor $\boldsymbol{\eta}$, we deal with kinematic incompatibilities, represented by the incompatibility current $\mathbf{F}$. The solutions for $\mathbf{e}$ and $\mathbf{v}$ in terms of $\boldsymbol{\eta}$ and $\mathbf{F}$ are found.

2. The displacement description

2.1. Geometry and kinematics

In the dynamic case, we consider defects as the moving surfaces of discontinuity of the displacement field $\mathbf{u}$. As is known, dislocations have the comparatively great freedom to move in the medium. But also the complex defects as cracks can move through the medium. We shall assume, that the surface of a defect can vary in time in an arbitrary way. This general model is important when considering point defects (see [6]); in the theory of linear defects, as dislocations and disclinations, it is sufficient to consider surfaces which vary in time only through the motion of their boundaries; the defects surface can have then the interpretation of a real slip surface (at least from a certain instant of time). However, because the physical quantities do not depend on the surface — they depend on the line only, moreover the choice of the surface and its motion are arbitrary — it will be convenient to consider arbitrary moving surfaces. When a defect is a crack, its surface is a real one; we consider only the motion of the type of cracking.

The discontinuity $\mathbf{U}$ of the displacement field $\mathbf{u}$ at the point $\boldsymbol{\zeta}(t)$ of the moving surface can depend additionally on time:

$$|[\mathbf{u}(\boldsymbol{\zeta}(t)), t]| = \mathbf{U}(\boldsymbol{\zeta}(t), t); \quad \boldsymbol{\zeta}(t) \in S. \tag{2.1}$$

We assume, as in [1], that $\mathbf{U}$ can be represented as a function of $\boldsymbol{\zeta}$.

In the dynamic case, to the geometric compatibility condition comes the kinematic compatibility condition. Namely, the increment of $\mathbf{U}$ at the unit of time (represented by the total time derivative of $\mathbf{U}$) must be equal to the difference of the increments of $\mathbf{u}$ at both sides of the moving surface; the latter quantity depends on $\dot{\mathbf{u}}$ and the surface motion:

$$\frac{d}{dt} U_i = |[\dot{u}_i + \zeta_k \nabla_k u_i]|. \tag{2.2}$$

Making use of $(2.4)_2$[1] we obtain $\left(\dot{\mathbf{U}} = \frac{\partial}{\partial t}\mathbf{U}\right)$:

$$\dot{U}_i + U_{i,k}\dot{\zeta}_k = |[\dot{u}_i]| + U_{i,k}\dot{\zeta}_k - n_k n_s U_{i,s}\dot{\zeta}_k + n_k n_s |[u_{i,s}]| \dot{\zeta}_k. \tag{2.3}$$

Hence the kinematic compatibility condition:

$$\dot{U}_i + (\dot{\boldsymbol{\zeta}}\mathbf{n}) n_s U_{i,s} = |[\dot{u}_i]| + (\dot{\boldsymbol{\zeta}}\mathbf{n}) n_s |[u_{i,s}]|. \tag{2.4}$$

2.2. Theory of elasticity

In the elastodynamics we have at our disposal the dynamic potentials of a double layer, which will serve us to describe defects. By the dynamic potential of a double layer we shall mean the expression (see [2, 3]):

$$u_i(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt' \int_{S(t')} ds_b U_n \left[c_{nbrs} \nabla_s + \delta_{nr} \varrho \dot{\zeta}_b \frac{\partial}{\partial t} \right] G_{ir}(\mathbf{x} - \boldsymbol{\zeta}(t'), t - t'). \tag{2.5}$$

$\mathbf{G}$ is the dynamic retarded Green tensor of the Lamé equation:

$$\check{L}_{il} G_{jl}(\mathbf{x}, t) = \delta_{ij} \delta(t) \delta_3(\mathbf{x}), \tag{2.6$_1$}$$

$$\check{L}_{il} = \delta_{il} \varrho \frac{\partial^2}{\partial t^2} - c_{iklm} \nabla_k \nabla_m; \tag{2.6$_2$}$$

$\check{\mathbf{L}}$ is the Lamé operator.

For the isotropic medium (see [8, 7]):

$(2.7)_1$

$$\overset{\text{ret}}{G}_{ik} = \frac{1}{4\pi\varrho} \left\{ \frac{\delta_{ik}}{c_2^2} \frac{\delta\left(t - \frac{r}{c_2}\right)}{r} + \nabla_i \nabla_k \left[\left(\frac{t}{r} - \frac{1}{c_1}\right) \theta\left(t - \frac{r}{c_1}\right) - \left(\frac{t}{r} - \frac{1}{c_2}\right) \theta\left(t - \frac{r}{c_2}\right) \right] \right\},$$

$$c_1^2 = \frac{\lambda + 2\mu}{\varrho}, \quad c_2^2 = \frac{\mu}{\varrho}, \tag{2.7$_2$}$$

where θ is the Heaviside function, c_1 and c_2 are the longitudinal and transversal wave velocities.

The expression (2.5) has the following properties:

$$(2.8)_1 \qquad |[u_i]| = U_i,$$

$$(2.8)_2 \qquad |[\check{t}_{nr} u_r]| = 0,$$

where

$$(2.9)_1 \qquad \check{t}_{nr} = \delta_{nr}\varrho n_l \dot{\zeta}_l \frac{\partial}{\partial t} + n_b c_{nbrs} \nabla_s,$$

$$(2.9)_2 \qquad |[\check{t}_{nr} u_r]| = \varrho n_b \dot{\zeta}_b |[\dot{u}_n]| + |[n_b \sigma_{nb}]|.$$

We use the retarded Green tensor having in mind the physical interpretation of the field $\mathbf{u}$. (2.5) represents the total field produced by a defect in a period of time $t \geqslant -\infty$; the influence of the initial conditions is neglected. The integration with respect to time represents the summation of all the impacts of the defect history to the present state of the medium; the complicated process of propagation of the field due to a defect describes the tensor $\mathbf{G}$.

The condition $(2.8)_2$ is the condition of dynamic equilibrium of a defect. The expression $\varrho(\dot{\boldsymbol{\zeta}}\mathbf{n})|[\dot{u}_n]|$ represents the momentum influx to the defect surface, $|[n_b \sigma_{nb}]|$ describes, as in the static case, the action of elastic forces on the defect surface.

The expression (2.5) can formally be written as follows:

$$(2.10) \qquad u_i = G_{ir} * \left[-c_{nbrs} \nabla_s \int_S ds_b U_n \delta_3(\mathbf{x}' - \boldsymbol{\zeta}) - \varrho \frac{d}{dt} \int_S ds_b \dot{\boldsymbol{\zeta}}_b U_r \delta_3(\mathbf{x}' - \boldsymbol{\zeta}) \right],$$

the star denotes now the convolution with respect to the three spatial variables and the time. The expression in bracket has an interpretation of the force distribution producing the dynamic defect.

From the condition $(2.8)_2$ we can calculate the discontinuity of the normal derivative of the $\mathbf{u}$ field on the defect surface, for the prescribed discontinuity $\mathbf{U}$. The detailed calculations for the isotropic case are presented in [3]. For the general anisotropy from $(2.4)_2$ [1], (2.4) we obtain:

$$(2.11) \qquad 0 = \varrho(\dot{\boldsymbol{\zeta}}\mathbf{n})\{\dot{U}_i + (\dot{\boldsymbol{\zeta}}\mathbf{n}) n_k U_{i,k} - (\dot{\boldsymbol{\zeta}}\mathbf{n}) n_k |[u_{i,k}]|\} + n_b c_{ibrs}\{U_{r,s} - n_s n_k U_{r,k} + n_s n_k [|u_{r,k}]|\}.$$

Hence

$$(2.12) \qquad [n_b n_s c_{ibrs} - \varrho(\dot{\boldsymbol{\zeta}}\mathbf{n})^2 \delta_{ir}][n_k |[u_{r,k}]| - n_k U_{r,k}] = -n_b c_{ibrs} U_{\langle r,s \rangle} - \varrho(\dot{\boldsymbol{\zeta}}\mathbf{u}) \dot{U}_i.$$

From the above system of linear equations the vector $n_k(|[u_{r,k}]| - U_{r,k})$, which enters into the expression for $|[u_{i,k}]|$ and $|[\dot{u}_i]|$, can be calculated in terms of $U_{\langle i,k \rangle}$ and $\dot{U}_i$. We are not going to analyse in detail the system (2.12), we notice only that when $U_{\langle i,k \rangle}$ and $\dot{U}_i$ are equal to zero at the same time, (2.12) has only the trivial solution equal to zero [we mean the solution for arbitrary $(\dot{\boldsymbol{\zeta}}\mathbf{n})$, we do not consider special cases of degenerate system of equations]. From $(2.4)_2$ [1], (2.4) follows thus that the strain $u_{\langle i,k \rangle}$ and velocity $\dot{u}_i$ fields are then continuous through the defect surface. The very conditions are satisfied for the case of dislocation and disclination.

For dislocation:

$$(2.13) \qquad U_i = -b_i, \quad U_{i,k} = 0, \quad \dot{U}_i = 0,$$
$$|[u_{i,k}]| = 0; \quad |[\dot{u}_i]| = 0.$$

The dynamic potential of a double layer with constant discontinuity $\mathbf{U} = -\mathbf{b}$ can thus serve as a model of a dislocation, which is de facto a linear defect; the corresponding to it "physical fields" $\dot{u}_{i,k}$ and u_i should not have discontinuities on S.

A dynamic disclination is represented by the $\mathbf{U}$ vector constructed in the same way as for the static case [see(2.18), [1]]:

$$\begin{aligned} U_i &= -\varepsilon_{ipq}\Omega_p(\zeta_q-\mathring{\zeta}_q), \\ U_{i,k} &= U_{[i,k]} = \varepsilon_{ikp}\Omega_p, \\ \dot{U}_i &= 0 \quad \text{for} \quad \dot{\mathring{\boldsymbol{\zeta}}} = 0, \quad \boldsymbol{\zeta} \in S. \end{aligned} \tag{2.14}$$

One can have doubts, whether to describe a moving disclination loop as a defect with a fixed rotation axis, or let the axis to move together with the loop. The analysis of the formulae (2.12), $(2.4)_2$[1], (2.4) indicates, that the assumption $\dot{\mathring{\boldsymbol{\zeta}}} \neq 0$ and consequently $\dot{\mathbf{U}} \neq 0$, causes $|[u_{\langle i,k\rangle}]| \neq 0$ and $|[\dot{u}_i]| \neq 0$; such a defect in any case will not be a linear defect; the fields describing a state of the medium will have discontinuities on S, and so the surface S will be visibly marked in space.

For completeness we present here the formula for the displacement field of a moving dislocation expressed by the Green potential $\mathbf{K}$. The dynamic Green potential, which will be important when solving the dynamic incompatibility problem (see [7]), satisfies the Poisson equation to:

$$\Delta K_{ir} = -G_{ir}, \quad K_{ir} = -\Delta^{-1}G_{ir}, \tag{2.15}$$

and at the same time the Lamé equation:

$$\check{L}_{jr}K_{ir} = \delta_{ij}\,\delta(t)\frac{1}{4\pi r}, \quad r = |\mathbf{x}|. \tag{2.16}$$

For the isotropic medium, $\mathbf{K}$ satisfying Eq. (2.15) with the retarded Green tensor at the right-hand side is equal to:

$$\overset{\text{ret}}{K}_{ik} = -\frac{1}{4\pi\varrho}\left\{\frac{\delta_{ik}}{c_2}\frac{1}{r}(r-c_2t)\theta(r-c_2t) + \frac{1}{6}\nabla_i\nabla_k\frac{1}{r}\left[\frac{1}{c_1}(r-c_1t)^3 \times\theta(r-c_1t) - \frac{1}{c_2}(r-c_2t)^3\theta(r-c_2t)\right]\right\}, \quad t \geqslant 0. \tag{2.17}$$

By the appropriate transformations, for $\mathbf{U} = -\mathbf{b}$, we can bring (2.5) to the form (see [9]):

$$u_i = b_u\int_{-\infty}^{\infty}dt' \oint_{L(t')} d\zeta_k\varepsilon_{bak}\left[c_{ubrs}\nabla_s + \varrho\dot{\zeta}_b\frac{\partial}{\partial t}\delta_{ur}\right]\nabla_a K_{ir} + \frac{b_i}{4\pi}\int_S ds_a\nabla_a\cdot\frac{1}{r}, \quad r = |\mathbf{x}-\boldsymbol{\zeta}|. \tag{2.18}$$

The above expression reminds the static one very much [see (2.24) [1]], however in the dynamic case the term depending on the dislocation line L has the much more complicated structure than in the static case. Nevertheless, it is significant that the second term in (2.18), being responsible for the discontinuity of the $\mathbf{u}$ field, has also a simple form of the harmonic potential of a double layer. It depends on the dislocation position at the instant t only the part of the field corresponding to it is thus so to say "dragged" by the moving dislocation. It is essential also that only the velocity of the dislocation line appears in (2.18), the surface velocity being absent at the same time. The assumptions about the surface motion are thus of no importance for the case of a dislocation.

3. Velocities and currents

3.1. Geometry and kinematics

To the **u** field, having the discontinuity **U** on the moving surface S, corresponds, besides the singularity $\mathring{\boldsymbol{\beta}}$ of the spatial derivative, also the singularity $\mathring{\mathbf{v}}$ of the time derivative, having the character of a delta function. We can cross the surface not only making a step in the space, but also when the moving surface passes by our point of observation. We represent the derivatives of the **u** field in the form:

$$u_{i,k} = \beta_{ik}+\mathring{\beta}_{ik}, \tag{3.1$_1$}$$

$$\dot{u}_i = v_i+\mathring{v}_i, \tag{3.1$_2$}$$

where $\mathring{\beta}$ and $\mathring{v}$ are equal (see[10]):

$$\mathring{\beta}_{ik} = \int_S ds_k U_i \delta_3(\mathbf{x}-\boldsymbol{\zeta}), \tag{3.2$_1$}$$

$$\mathring{v}_i = -\int_S ds_b \dot{\zeta}_b U_i \delta_3(\mathbf{x}-\boldsymbol{\zeta}), \quad \boldsymbol{\zeta}\in S. \tag{3.2$_2$}$$

Notice, that the quantity $\mathring{\mathbf{v}}$ depends only on the normal velocity of the surface S, for $(\dot{\zeta}\cdot\mathbf{n}) = 0$ it is equal to zero. In the same way as $\boldsymbol{\beta}$ and $\mathring{\boldsymbol{\beta}}$, the fields **v** and $\mathring{\mathbf{v}}$ are called appropriately the elastic and plastic (or initial) velocity. For a point defect, $\mathring{\mathbf{v}}$ is concentrated at the point (see[6]):

$$\mathring{v}_i = -p\dot{\zeta}_i \delta_3(\mathbf{x}-\boldsymbol{\zeta}), \tag{3.3}$$

where p is the intensity and $\boldsymbol{\zeta}$ the position vector of a point defect.

For linear defects, the plastic velocity $\mathring{\mathbf{v}}$, like the plastic distortion $\mathring{\boldsymbol{\beta}}$ is not the uniquely defined quantity, determined by the position and the motion of a defect. It follows from the free choice of the surface S.

For a dislocation:

$$\mathring{v}_i = b_i \int_S ds_b \dot{\zeta}_b \delta_3(\mathbf{x}-\boldsymbol{\zeta}), \quad \boldsymbol{\zeta}\in S. \tag{3.4}$$

The field $\mathring{\mathbf{v}}$ of an arbitrary surface defect can be understood as due to a superposition of infinitesimal dislocations loops with Burgers vectors — $\mathbf{U}^{(n)}$:

$$\mathring{v} = -\sum_n U_i^{(n)}(\dot{\boldsymbol{\zeta}}\mathbf{n})^{(n)}\delta(\Delta S^{(n)}). \tag{3.5}$$

In the dynamic case we represent the state of the medium with the help of the fields representing a deformation (distortions, strains), to which correspond stresses, and the velocity field. When we are to describe the incompatible medium with defects, $\mathbf{v}$ is identified with the real velocity of material points. In addition to the constraints equations of the static theory, having the geometrical character, the new kinematical constraints equations appear, being the relations between deformations and velocities. Into these kinematical constraints enter the currents, being the new source functions, depending on defects velocities.

Let us consider dislocations, for which the distortion field has a good physical interpretation. The constraints equation (3,7), [1] and the definition (3.8)[1] of the dislocation density tensor $\boldsymbol{\alpha}$ are also valid in the dynamic case. We obtain the additional constraints equation differentiating with respect to time the Eq. $(3.1)_1$:

$$\dot{u}_{i,k} = \dot{\beta}_{ik} + \dot{\mathring{\beta}}_{ik} \tag{3.6}$$

and subtracting the Eq. $(3.1)_2$ differentiated with respect to x_k:

$$\dot{u}_{i,k} = v_{i,k} + \mathring{v}_{i,k}. \tag{3.7}$$

We obtain the following relation between $\boldsymbol{\beta}$, $\mathring{\boldsymbol{\beta}}$, $\mathbf{v}$, $\mathring{\mathbf{v}}$:

$$\dot{\beta}_{ik} - v_{i,k} = -[\dot{\mathring{\beta}}_{ik} - \mathring{v}_{i,k}]. \tag{3.8}$$

We introduce the dislocation current tensor $\mathbf{J}$:

$$J_{ik} = -\dot{\mathring{\beta}}_{ik} + \mathring{v}_{i,k}. \tag{3.9}$$

From the definitions (3.8) [1], (3.9) results the following compatibility equation for the functions $\boldsymbol{\alpha}$ and $\mathbf{J}$:

$$\dot{\alpha}_{ik} - \varepsilon_{klm} J_{im,l} = 0. \tag{3.10}$$

For a single dislocation line:

$$J_{ik} = b_i \oint_L d\zeta_a \varepsilon_{kba} \dot{\zeta}_b \delta_3(\mathbf{x} - \boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in L. \tag{3.11}$$

The dislocation current $\mathbf{J}$ is concentrated on the dislocation line L. It depends in a linear way on the Burgers vector, the tangent vector and the velocity of a dislocation. It is thus,

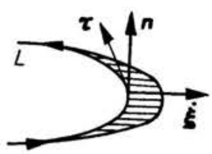

FIG. 1.

beside $\boldsymbol{\alpha}$, the good source function of the dynamic theory of dislocations. The functions $\mathring{\boldsymbol{\beta}}$ and $\mathring{\mathbf{v}}$ have to be constructed in such a way, that (3.9) be satisfied. The tensor $\mathbf{J}$ can be generalised to the case of the continuous flow of dislocations. The expression $\dot{\boldsymbol{\zeta}} \times d\boldsymbol{\zeta}$ under the integral sign in (3.11) has an interpretation of the vector element of the surface, outlined in the unit time by the dislocation line L; see Fig. 1.

In the considerations carried out till now we have assumed that the dislocation motion can be arbitrary. However, the essential restrictions are imposed by the condition of the mass balance (see [12, 13]).

In the ideal medium, described by the displacement $\mathbf{u}$, the mass balance is automatically satisfied. In the case of existence of moving defects in the medium, that is formally, when the deformation and velocity fields describing the medium do not satisfy compatibility conditions, the mass balance is not always satisfied.

Let us denote by m the mass of a unit volume of the deformed medium; the density of the medium is denoted by ϱ. The equation of balance for the quantity m:

$$\frac{dm}{dt}+mv_{k,k} = -j_{k,k}. \tag{3.12}$$

In the above equation $\mathbf{v}$ is the velocity of the points of the medium, $\mathbf{j}$ represents the mass flow. In the linear theory of elasticity $\mathbf{v} = \dot{\mathbf{u}}$, while

$$m = \varrho(1-u_{k,k}), \tag{3.13}$$

and the balance equation in the linearised form is automatically satisfied:

$$\frac{dm}{dt}+\varrho v_{k,k} = 0. \tag{3.14}$$

In the medium with defects the real strain is represented by the elastic field $\mathbf{e}$, the mass of a unit volume is thus equal to:

$$m = \varrho(1-e_{kk}), \tag{3.15}$$

the field $\mathbf{v}$ we shall identify with the field of elastic velocity. Taking into account the constraints equation (3.8) and the definition of the current (3.9), we obtain the balance equation in the form:

$$-\varrho\dot{e}_{kk}+\varrho v_{k,k} = -\varrho J_{kk}. \tag{3.16}$$

The trace of the dislocation current tensor $\mathbf{J}$ determines thus the divergence of the mass flow $\mathbf{j}$. The dislocation motion proceeds thus without the mass flow if $J_{kk} = 0$, that is $\dot{\boldsymbol{\zeta}}\perp(\boldsymbol{\tau}\times\mathbf{b})$, what means that the line moves along the so-called slip plane, determined by the Burgers vector $\mathbf{b}$ and the tangent vector $\boldsymbol{\tau}$. The condition $J_{kk} = 0$ is significant only for a dislocation being not of the screw type; $\boldsymbol{\tau}\times\mathbf{b} \neq 0$. The mass flow in a crystal can be realised by the vacancies absorption or creation. The dislocations motions with $J_{kk} \neq 0$, the so-called climbing motions of edge dislocations, can be realised thus with the help of vacancies influx (positive or negative) to the dislocation line; it makes the significant restriction for the motion of this kind. A dislocation motion along the slip plane is called conservative. The corresponding to it plastic deformation is a "pure plastic" deformation, without the change of the volume. Notice that J_{kk} determines only the divergence of the mass flow $\mathbf{j}$; the mass flow which permits a dislocation to climb can be realised in many ways.

In the case of disclination, to the discontinuity of the displacement $\mathbf{u}$, also corresponds the singularity of the time derivative $^\circ$; the role of the Burgers vector plays now the vector $\Omega\times(\boldsymbol{\zeta}-\mathring{\boldsymbol{\zeta}})$. The representation $(3.1)_{1,2}$ also takes place in this case. However, in addition to $\mathring{\mathbf{v}}$, there appears the singular plastic rotational velocity, corresponding to the discontinuity of the rotation field $\boldsymbol{\omega}$. The time derivative of the distortion field is equal [see $(3.12)_1$ [1]]:

$$\dot{u}_{i,k} = \dot{e}_{ik} - \varepsilon_{ika}\dot{\omega}_a + \dot{\overset{\circ}{e}}_{ik} - \varepsilon_{ika}\dot{\overset{\circ}{\omega}}_a. \tag{3.17}$$

The time derivative of the discontinuous rotation field ω we represent as a sum of the regular part $\mathbf{w}$ and singular part $\overset{\circ}{\boldsymbol{\psi}}$:

$$\dot{\omega}_a = w_a + \overset{\circ}{\psi}_a. \tag{3.18}$$

By the plastic rotational velocity we shall mean the quantity:

$$\overset{\circ}{w}_a = \overset{\circ}{\psi}_a + \dot{\overset{\circ}{w}}_a. \tag{3.19}$$

Hence

$$\dot{u}_{i,k} = \dot{e}_{ik} - \varepsilon_{ika} w_a + \dot{\overset{\circ}{e}}_{ik} - \varepsilon_{ika}\overset{\circ}{w}_a. \tag{3.20}$$

For a single disclination:

$$\dot{v}_i = \int_S ds_b \dot{\zeta}_b \varepsilon_{ipq}\Omega_p(\zeta_q - \overset{\circ}{\zeta}_q)\delta_3(\mathbf{x} - \boldsymbol{\zeta}), \tag{3.21_1}$$

$$\overset{\circ}{\psi}_i = \int_S ds_b \dot{\zeta}_b \Omega_i \delta_3(\mathbf{x} - \boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in S. \tag{3.21_2}$$

The comparison of the formulae (3.7), (3.20) leads us to the following constraints equation for the fields $\dot{\mathbf{e}}$, $\mathbf{w}$ and $\mathbf{v}$:

$$\dot{e}_{ik} - \varepsilon_{ika} w_a - v_{i,k} = -[\dot{\overset{\circ}{e}}_{ik} - \varepsilon_{ika}\overset{\circ}{w}_a - \overset{\circ}{v}_{i,k}]. \tag{3.22}$$

We define the dislocation density current $\mathbf{J}$(1):

$$\begin{aligned} J_{ik} &= -[\dot{\overset{\circ}{e}}_{ik} - \varepsilon_{ika}\overset{\circ}{w}_a - \overset{\circ}{v}_{i,k}] \\ &= -[\dot{\overset{\circ}{\beta}}_{ik} - \overset{\circ}{v}_{i,k} - \varepsilon_{ika}\overset{\circ}{\psi}_a]. \end{aligned} \tag{3.23}$$

We find the second constraints equation by equating to each other the gradient of the expression (3.20) with the time derivative of $(3.15)_1$ [1]:

$$\dot{u}_{i,km} = \dot{e}_{ik,m} - \varepsilon_{ika} w_{a,m} + \dot{\overset{\circ}{e}}_{ik,m} - \varepsilon_{ika}\overset{\circ}{w}_{a,m}, \tag{3.24}$$

and on the other side:

$$\dot{u}_{i,km} = \dot{e}_{ik,m} - \varepsilon_{ika}\dot{\varkappa}_{am} + \dot{\overset{\circ}{e}}_{ik,m} - \varepsilon_{ika}\dot{\overset{\circ}{\varkappa}}_{am}. \tag{3.25}$$

Hence

$$\dot{\varkappa}_{am} - w_{a,m} = -[\dot{\overset{\circ}{\varkappa}}_{am} - \overset{\circ}{w}_{a,m}]. \tag{3.26}$$

We define the disclination current $\mathbf{I}$(2):

$$\begin{aligned} I_{am} &= -[\dot{\overset{\circ}{\varkappa}}_{am} - \overset{\circ}{w}_{a,m}] \\ &= -[\dot{\overset{\circ}{\varphi}}_{am} - \overset{\circ}{\psi}_{a,m}]. \end{aligned} \tag{3.27}$$

(1,2) The transposes of $\mathbf{J}$ and I are used in [14].

Notice that disclinations being absent ($\mathring{\boldsymbol{\varphi}} = 0$, $\mathring{\boldsymbol{\psi}} = 0$), the expression (3.23) for the dislocation current is identical with (3.9); moreover, the disclination current is equal to zero. From the definitions $(3.20)_{1,2}$ [1] and (3.23), (3.27) result the following compatibility equations for the tensors $\boldsymbol{\alpha}$, $\boldsymbol{\theta}$, $\mathbf{J}$ and $\mathbf{I}$:

$$(3.28)_1 \qquad \dot{\alpha}_{ik} - \varepsilon_{klm} J_{im,l} + \varepsilon_{klm}\varepsilon_{ima} I_{al} = 0,$$

$$(3.28)_2 \qquad \dot{\theta}_{an} - \varepsilon_{nlm} I_{am,l} = 0.$$

For a single disclination:

$$(3.29)_1 \qquad J_{ik} = \oint_L d\zeta_a \varepsilon_{kma} \dot{\zeta}_m \varepsilon_{ipq} \Omega_p (\zeta_q - \mathring{\zeta}_q) \delta_3(\mathbf{x} - \boldsymbol{\zeta}),$$

$$(3.29)_2 \qquad I_{am} = \oint_L d\zeta_k \varepsilon_{mlk} \dot{\zeta}_l \Omega_a \delta_3(\mathbf{x} - \boldsymbol{\zeta}), \quad \boldsymbol{\zeta} \in L.$$

In the same way as densities $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$, the currents $\mathbf{J}$ and $\mathbf{I}$ are concentrated on the disclination line only. $\mathbf{J}$ in the same way as $\boldsymbol{\alpha}$ is equal to zero when the disclination line coin-

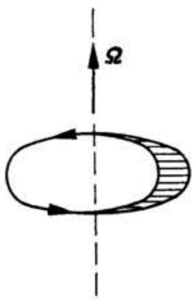

FIG. 2.

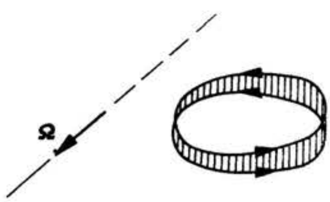

FIG. 3.

cides with the rotation axis; $\boldsymbol{\Omega} \times (\boldsymbol{\zeta} - \mathring{\boldsymbol{\zeta}}) = 0$. The motion of a disclination with respect to the rotation axis is equivalent to the dislocations current with the Burgers vector $\boldsymbol{\Omega} \times (\boldsymbol{\zeta} - \mathring{\boldsymbol{\zeta}})$.

For a disclination, the condition for the motion to be conservative, that is to proceed without the vacancies creation or absorption, imposes also restrictions on the disclination line velocity $\dot{\boldsymbol{\zeta}}$. The condition $J_{kk} = 0$ takes the form:

$$(3.30) \qquad 0 = \dot{\zeta}_m \varepsilon_{mak} \tau_a \varepsilon_{kpq} \Omega_p (\zeta_q - \mathring{\zeta}_q) = \tau_q (\zeta_q - \mathring{\zeta}_q) \dot{\zeta}_p \Omega_p - \tau_a \Omega_a \dot{\zeta}_q (\zeta_q - \mathring{\zeta}_q),$$

what means that the conservative motion of a line element occurs in the plane determined by the vector $\boldsymbol{\tau} \times [\boldsymbol{\Omega} \times (\boldsymbol{\zeta} - \mathring{\boldsymbol{\zeta}})]$. We can speak of the glide surface of a disclination. This problem is examined in details in [15]. The two conservative motions of a disclination loop of the twist type ($\boldsymbol{\tau} \perp \boldsymbol{\Omega}$) and the wedge type ($\boldsymbol{\Omega}$ is contained in the plane of the loop) are demonstrated by Figs. 2, 3.

3.2. Theory of elasticity

If we compare the Eqs. (2.10) and $(3.2)_{12}$, it becomes evident that the dynamic elastic potential of a double layer can formally be represented as follows:

$$(3.31)\qquad u_i = -G_{ir} * \left[c_{ubrs}\nabla_s\mathring{\beta}_{ub} - \varrho\frac{\partial}{\partial t}\mathring{v}_r\right] = -G_{ir} * \left[c_{ubrs}\nabla_s\mathring{e}_{ub} - \varrho\frac{\partial}{\partial t}\mathring{v}_r\right],$$

the convolution in (3.31) being with respect to four variables.

The gradient of $\mathbf{u}$ can be represented as follows:

$$(3.32)\qquad \begin{aligned} u_{i,k} &= -G_{ir} * \left[c_{ubrs}\nabla_s\nabla_k\mathring{\beta}_{ub} - \varrho\frac{\partial}{\partial t}\nabla_k\mathring{v}_r\right] \pm G_{ir} * \left[c_{ubrs}\nabla_b\nabla_s\mathring{\beta}_{uk} - \varrho\frac{\partial^2}{\partial t^2}\mathring{\beta}_{rk}\right] \\ &= -G_{ir} * \left\{c_{nbrs}\nabla_s[\mathring{\beta}_{nb,k} - \mathring{\beta}_{nk,b}] + \varrho\frac{\partial}{\partial t}[\dot{\mathring{\beta}}_{rk} - \mathring{v}_{r,k}]\right\} + \beta_{ik}, \end{aligned}$$

or with the help of $\mathbf{e}$:

$$(3.33)\qquad u_{i,k} = -G_{ir} * \left\{c_{nbrs}\nabla_s[\mathring{e}_{nb,k} - \mathring{e}_{nk,b}] - \varrho\frac{\partial}{\partial t}[\dot{\mathring{e}}_{rk} - \mathring{v}_{r,k}]\right\} + \mathring{e}_{ik}.$$

The time derivative of $\mathbf{u}$:

$$(3.34)\qquad \dot{u}_i = -G_{ir} * \left\{c_{nbrs}\nabla_s\frac{\partial}{\partial t}\mathring{e}_{nb} - \varrho\frac{\partial^2}{\partial t^2}\mathring{v}_r \pm c_{nbrs}\nabla_b\nabla_s\mathring{v}_n\right\} = -G_{ir}c_{nbrs}\nabla_s[\dot{\mathring{e}}_{nb} - \mathring{v}_{n,b}] + \mathring{v}_i.$$

To be in agreement with $(3.1)_{1,2}$, $(3.2)_{1,2}$ we identify:

$$(3.35)_1\qquad \beta_{ik} = -G_{ir} * \left\{c_{nbrs}\nabla_s[\mathring{\beta}_{nb,k} - \mathring{\beta}_{nk,b}] + \varrho\frac{\partial}{\partial t}[\dot{\mathring{\beta}}_{rk} - \mathring{v}_{r,k}]\right\},$$

$$(3.35)_2\qquad e_{ik} = -G_{ir} * \left\{c_{nbrs}\nabla_s[\mathring{e}_{nb,k} - \mathring{e}_{nk,b}] + \varrho\frac{\partial}{\partial t}[\dot{\mathring{e}}_{rk} - \mathring{v}_{r,k}]\right\}_{\langle ik\rangle},$$

$$(3.35)_3\qquad v_i = -G_{ir} * c_{nbrs}\nabla_s[\dot{\mathring{v}}_{nb} - \mathring{e}_{n,b}].$$

From $(3.35)_{1,3}$ we immediately obtain the expressions for the elastic distortion and velocity fields of a dislocation in terms of dislocation density $\boldsymbol{\alpha}$ and current $\mathbf{J}$:

$$(3.36)_1\qquad \beta_{ik} = G_{ir} * \left[c_{nbrs}\nabla_s\varepsilon_{kbp}\alpha_{np} + \varrho\frac{\partial}{\partial t}J_{rk}\right],$$

$$(3.36)_2\qquad v_i = G_{ir} * c_{nbrs}\nabla_s J_{nb}.$$

Notice moreover, that the $\mathbf{u}$ field given by the Eq. (3.31) is a solution of the following equation:

$$(3.37)\qquad \varrho\frac{\partial^2}{\partial t^2}u_i - c_{iklm}\nabla_k\nabla_m u_l = \varrho\frac{\partial}{\partial t}\mathring{v}_i - c_{isnb}\nabla_s\mathring{e}_{nb}.$$

From the above follows that the fields $\mathbf{e}$ and $\mathbf{v}$ satisfy the following equation:

$$(3.38)\qquad \varrho\frac{\partial}{\partial t}v_i - c_{iklm}\nabla_k e_{lm} = 0,$$

what we write in the form:

$$\varrho \frac{\partial}{\partial t} v_i - \sigma_{ik,k} = 0. \tag{3.39}$$

The stress $\boldsymbol{\sigma}$ is understood here as due to the elastic strain $\mathbf{e}$. The elastic parts $\mathbf{e}$ and $\mathbf{v}$ of the strain and velocity corresponding to the potential of a double layer, being the model of a defect, thus satisfy the equilibrium equation (3.38). In the linear model of the medium with defects, we identify $\mathbf{e}$ and $\mathbf{v}$ with the real strain and velocity fields. Further on, when formulating the theory in terms of elastic fields, we can assume the Eq. (3.38) together with the constraints equations as the basic set of equations of the theory.

4. Dynamic incompatibility problem

4.1. Kinematics

In the dynamic case the state of the elastic body is represented by the two physical fields: the elastic strain $\mathbf{e}$ and velocity $\mathbf{v}$. The general formulation of the incompatibility problem has to take into account also the kinematical incompatibilities.

The time derivative of the total strain has the form:

$$\dot{u}_{\langle i,k \rangle} = \dot{e}_{ik} + \dot{\mathring{e}}_{ik}. \tag{4.1}$$

On the other side:

$$\dot{u}_{\langle i,k \rangle} = v_{\langle i,k \rangle} + \mathring{v}_{\langle i,k \rangle}. \tag{4.2}$$

Subtracting (4.2) from (4.1) we obtain the following constraints equation for the fields $\mathbf{e}$ and $\mathbf{v}$:

$$\dot{e}_{ik} - v_{\langle i,k \rangle} = -[\dot{\mathring{e}}_{ik} - \mathring{v}_{\langle i,k \rangle}]. \tag{4.3}$$

The quantity on the left-hand side of (4.3) we call the incompatibility current $\mathbf{F}$:

$$F_{ik} = -[\dot{\mathring{e}}_{ik} - \mathring{v}_{\langle i,k \rangle}], \tag{4.4}$$

hence the constraints equation takes the form:

$$\dot{e}_{ik} - v_{\langle ik \rangle} = F_{ik}. \tag{4.5}$$

From (3.9), (3.23) is evident, that $\mathbf{F}$ coincides with the symmetric part of the dislocation current:

$$F_{ik} = J_{\langle ik \rangle}. \tag{4.6}$$

The disclination current $\mathbf{I}$ does not contribute to $\mathbf{F}$.

The trace of $\mathbf{F}$ describes the mass flow in the medium with incompatibilities [see (3.16)].

Let us recall the definition of the incompatibility tensor $\boldsymbol{\eta}$:

$$\eta_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\mathring{e}_{ln,km}, \tag{4.7}$$

to which corresponds the following representation of $\mathring{\mathbf{e}}$ [see (4.13)[1]:

$$(4.8)_1 \qquad \mathring{e}_{ik} = \Phi_{\langle i,k\rangle} - \varepsilon_{iab}\varepsilon_{kcd}\nabla_a\nabla_c\eta_{bd} * \frac{r}{8\pi},$$

$$(4.8)_2 \qquad \Phi_i = \frac{r}{8\pi} * [\mathring{e}_{ab,abi} - 2\mathring{e}_{ia,abb}].$$

Differentiating twice and contracting with the product $\varepsilon_{ali}\,\varepsilon_{bmk}$ the Eq. (4.4), we obtain the following compatibility condition for the tensors $\boldsymbol{\eta}$ and $\mathbf{F}$:

$$(4.9) \qquad -\dot{\eta}_{ab} = \varepsilon_{ali}\,\varepsilon_{bmk}F_{ik,lm}.$$

Notice that for $\mathring{\mathbf{v}} = 0$, the incompatibility current is equal to minus time derivative of the plastic strain.

As is evident from (4.8), $\dot{\mathring{\mathbf{e}}}$ depends on $\dot{\boldsymbol{\Phi}}$ and $\dot{\boldsymbol{\eta}}$; we could accept as the incompatibility current the quantity $\dot{\boldsymbol{\Phi}} - \mathring{v}$, independent of $\dot{\boldsymbol{\eta}}$; for $\mathring{\mathbf{v}} = 0$ — just $\dot{\boldsymbol{\Phi}}$. For the reasons of physical interpretation, it is more convenient to deal with the quantity $\mathbf{F}$, being close to the dislocation current $\mathbf{J}$.

4.2. Elastic strain and velocity fields

The dynamic incompatibility problem for the elastic medium is thus formulated as follows. The medium with incompatibilities satisfies the following set of equations:

$$(4.10)_1 \qquad \varrho\frac{\partial}{\partial t}v_i - \sigma_{ij,j} = 0,$$

$$(4.10)_2 \qquad -\varepsilon_{ikl}\varepsilon_{jmn}e_{ln,km} = \eta_{ij},$$

$$(4.10) \qquad \dot{e}_{ik} - v_{\langle i,k\rangle} = F_{ik}.$$

Equation $(4.10)_1$ is the dynamic equilibrium equation, $(4.10)_{2,3}$ are the constraints equations. As we mentioned in [1], the method of solution of the above set of equations was presented in [7]. Here, like in the static theory, we demonstrate how to calculate $\mathbf{e}$ and $\mathbf{v}$ fields in terms of $\boldsymbol{\eta}$ and $\mathbf{F}$ from the expressions of the chapter 3. The elastic strain equals [see $(3.35)_2$]:

$$(4.11) \qquad e_{is} = -G_{ij} * \left\{c_{jklm}\nabla_k[\mathring{e}_{lm,s} - \mathring{e}_{ls,m}] + \varrho\frac{\partial}{\partial t}[\dot{\mathring{e}}_{js} - \mathring{v}_{j,s}]\right\}_{\langle is\rangle}.$$

We make use of (4.20)[1]:

$$(4.12) \qquad \mathring{e}_{ls,m} - \mathring{e}_{lm,s} = \Phi_{[s,m]l} + \varepsilon_{smp}\varepsilon_{alb}\nabla_a\eta_{bp} * \frac{1}{4\pi r}.$$

From the above follows:

$$(4.13)_1 \qquad \Phi_{[s,m]ll} = \mathring{e}_{ls,ml} - \mathring{e}_{lm,sl},$$

$$(4.13)_2 \qquad \Phi_{[s,m]} = [\mathring{e}_{ma,sa} - \mathring{e}_{sa,ma}] * \frac{1}{4\pi r}.$$

The specific dynamic term, occuring in (4.11), we transform as follows:

$$(4.14) \qquad \dot{\mathring{e}}_{js} - \mathring{v}_{j,s} = -[\dot{\mathring{e}}_{js,a} - \mathring{v}_{j,sa}] * \nabla_a\frac{1}{4\pi r} = \dot{\Phi}_{[j,s]} - [\dot{\mathring{e}}_{js,a} + \dot{\mathring{e}}_{ja,s} - \dot{\mathring{e}}_{sa,j} - \mathring{v}_{j,sa}] * \nabla_a\frac{1}{4\pi r}.$$

The expression in bracket can afterwards be represented in terms of the incompatibility current **F**:

$$(4.15)\qquad [\mathring{\dot{e}}_{js,a}+\mathring{\dot{e}}_{ja,s}-\mathring{\dot{e}}_{sa,j}-\mathring{v}_{j,sa}] = \mathring{\dot{e}}_{js,a}+\mathring{\dot{e}}_{ja,s}-\mathring{\dot{e}}_{sa,j}-\mathring{v}_{\langle j,s\rangle a}-\mathring{v}_{\langle j,a\rangle s} +\mathring{v}_{\langle s,a\rangle j} = F_{js,a}+F_{ja,s}-F_{sa,j}.$$

Because

$$(4.16)\qquad G_{ij} * \check{L}_{jm}\Phi_{[m,s]_{\langle i,s\rangle}} = \Phi_{[i,s]_{\langle i,s\rangle}} = 0,$$

we obtain the following expression for the field **e**, written in terms of the dynamic Green potential **K**:

$$(4.17)\qquad e_{is} = K_{ij,a} * \left\{c_{jklm}\nabla_k\varepsilon_{smp}\varepsilon_{alb}\eta_{bp}-\varrho\frac{\partial}{\partial t}[F_{js,a}+F_{ja,s}-F_{sa,j}]\right\}_{\langle is\rangle}.$$

The expression for **v** we obtain immediately from $(3.35)_3$:

$$(4.18)\qquad v_i = G_{ij} * c_{jklm}\nabla_k F_{lm}.$$

From (4.17) we can obtain the strain field of a dynamic disclination given in terms of the source functions **α**, **θ**, **J**, **I**; it will be published in [14].

References

1. E. KOSSECKA, *Mathematical theory of defects*. Part I. Statics, Arch. Mech. Stos., **26**, 6, 995–1010, 1974.
2. V. D. KUPRADSE, *The methods of potential in the theory of elasticity* [in Russian], Gos. Izdat. Mat. Lit., Moscow 1963.
3. H. ZORSKI, *Theory of discrete defects*, Arch. Mech. Stos., **18**, 3, 301, 1966.
4. E. KOSSECKA, *Theory of dislocation lines in a continuous medium*, Arch. Mech. Stos., **21**, 2, 167–190, 1969.
5. E. KOSSECKA, H. ZORSKI, *Linear equations of motion of a concentrated defect*, Int. J. Solids Struc., **3**, 881, 1967.
6. E. KOSSECKA, *Point defects in motion*, Bull. Acad. Polon. Sci., Série Sci. Techn., Vol. XXII, No 6, 297[467], 1974.
7. E. KOSSECKA, *Dynamic incompatibility problem*, Arch. Mech. Stos., **26**, 4, 665–674, 1974.
8. W. NOWACKI, *Theory of elasticity* [in Polish], PWN, Warszawa 1970.
9. E. KOSSECKA, *The dislocation displacement field in terms of the dynamic Green potential*, Bull. Acad. Polon. Sci., Serié Sci. Tech., Vol. XXII, No 6, 303 [473], 1974.
10. E. KOSSECKA, *Defects as initial deformations*, Arch. Mech. Stos., **25**, 1, 3–11, 1973.
11. E. KOSSECKA, H. G. SCHÖPF, *On the uniqueness problem of a dislocation field*, Arch. Mech. Stos., **24**, 1, 27–34, 1972.
12. A. M. KOSEVITCH, *Dynamic theory of dislocations* [in Russian], Usp. Phys. Nauk, **84**, 4, 579, 1964.
13. L. D. LANDAU, E. M. LIFSCHITZ, *Theory of elasticity* [in Russian], Izdat. Nauka, Gl. Red. Phys. Mat. Lit., Moscow 1965.
14. E. KOSSECKA, R. deWIT, *Disclination dynamics*, 1974.
15. E. S. P. DAS, M. J. MARCINKOWSKI, R. W. ARMSTRONG, R. deWIT, *The movement of Volterra disclinations and the associated mechanical forces*, Phil. Mag., **27**, 2, 369, 1973.

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