Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 1, pp. 103–114, Warszawa 1975

# Unsteady laminar combined convection near the lower stagnation point of an isothermal circular cylinder

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THE EFECTS of buoyancy forces on the unsteady incompressible flow near the lower stagnation point of an infinite circular cylinder are described. The external stream is assumed to be set in an impulsive motion from rest towards the cylinder and the temperature of the cylinder is suddenly increased from that of the surrounding fluid. Series at small time are used for velocity, temperature, skin friction coefficient and Nusselt number. By applying Shanks's method the values of the skin friction and heat transfer coefficients at small time are used to estimate their values for the steady problem.

Omówiono wpływ sił wyporu na nieustalony przepływ nieściśliwy w pobliżu niższego punktu stagnacji nieskończonego walca kołowego. Założono, że strumień zewnętrzny zostaje gwałtownie wprawiony w ruch w kierunku walca, a temperatura walca wzrasta raptownie w stosunku do temperatury otaczającego ośrodka. Do wyrażenia prędkości, temperatury, tarcia powierzchniowego oraz liczby Nusselta zastosowano przybliżenie szeregami dla małych wartości czasu. Stosując metodę Shanksa wykorzystano otrzymane wartości tarcia powierzchniowego i współczynników przewodzenia ciepła do oceny ich odpowiednich wartości w przypadku zagadnienia ustalonego.

Обсуждено влияние сил подъема на неустановившееся несжимаемое течение вблизи низшей критической точки бесконечного кругового цилиндра. Предположено, что внешний поток внезапно вводится в движение в направлении цилиндра, а температура цилиндра возрастает внезапно по отношению к температуре окружающей среды. Для выражения скорости, температуры, поверхностного трения и числа Нуссельта применено приближение рядами для малых значений времени. Применяя метод Шанкса использованы полученные значения поверхностного трения и коэффициентов теплопроводности для оценки их соответствующих значений в случае установившейся задачи.

## Notations

$C_f$ coefficient of skin friction, $\tau_w/8\varrho x U_\infty^2$,
$D$ cylinder diameter,
$f, h$ functions, Eqs. (3.3),
$f_i, h_i$ functions, Eqs. (3.6),
$F_D$ buoyancy parameter, $U_\infty/\sqrt{\beta g(T_w - T_\infty)D}$,
$g$ acceleration due to gravity,
$G_D$ Grashof number, $\beta g(T_w - T_\infty)D^3/\nu^2$,
$k$ thermal conductivity,
$N_D$ local Nusselt number, $q_w D/(T_w - T_\infty)k$,
$q$ heat transfer, $-k\left(\frac{\partial T}{\partial y}\right)_{y=0}$,
$R_D$ Reynolds number, $U_\infty D/\nu$,
$t$ time,
$T$ temperature,
$u, v$ velocity components,
$x, y$ coordinates,

$\alpha$ thermal diffusivity,
$\beta$ coefficient of thermal expansion,
$\eta$ similarity variable, $y/2\sqrt{t}$,
$\mu$ dynamic viscosity,
$\nu$ kinematic viscosity,
$\varrho$ density,
$\sigma$ Prandtl number, $\nu/\alpha$,
$\tau$ skin friction, $\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$,
$\Psi$ stream function,
$\Phi$ angular position, $2x/D$.

Superscripts

$+$ dimensionless variables,
$'$ derivatives with respect to $\eta$.

Subscripts

$w$ wall conditions,
$i, j$ $0, 1, 2, \ldots$
$\infty$ ambient condition.

## 1. Introduction

Although much work, both theoretical and experimental, has been published on the theory of unsteady boundary layer, there are only a few works which contribute to the problem of the effects of buoyancy forces on the unsteady flow. Ašković (1967, 1972) and Pop (1971) have dealt with the analytical study of the unsteady three-dimensional combined flow. Quite recently, Soundalgekar (1973) has directed his study at the unsteady combined convection over a vertical infinite flat plate.

Problems involving the buoyancy effects are important in technology, meteorology, oceanography, etc. A few more special examples are associated with the mechanics of cloud formation and cloud top oscillation, buoyancy driven ocean circulations, and the thermal circulation in lakes resulting perhaps from water discharges.

The aim of this paper is to present an information on the effects of buoyancy forces on the unsteady incompressible flow in the region of the lower stagnation point of an infinite cylinder which is immersed in the external stream. Therefore, we consider the situation when forced and free convection act simultaneously in establishing the flow and temperature fields adjacent to the stagnation point of a heated or cooled cylinder. The external stream is assumed to be set in an impulsive motion from rest towards the cylinder at time $t = 0$ and kept steady thereafter. The temperature of the cylinder is suddenly increased from that of the surrounding fluid at time $t = 0$. Initially the fluid particles in contact with the cylinder have the same temperature as the cylinder, thus producing a discontinuity in the temperature field. The diffusion of heat from the cylinder which dominates over convection for small time creates variations in the density field and this produces bouyancy forces and hence there arises an additionally fluid motion around the cylinder.

As in Oosthuizen (1970) we will designate as assisting flows those flows for which the buoyancy forces have a positive component in the direction of free stream velocity.

Those flows for which the buoyancy forces have a component opposite to the free stream velocity will be designated asopposing flows. For convenience the calculations have been restricted to Prandtl number $\sigma$ unity since they are very tedious and are liable to cause errors. But the general ideas hold for other values. The analysis is carried out for the case of uniform surface temperature $T_w$. Thus the governing boundary-layer equations are reduced to a set of ordinary differential equations by an expansion method into power series of small time. The first three approximations to the velocity and temperature distributions are analytically evaluated using a method analogous with that given by POP (1969). Numerical calculations are performed for various values of the parameter $F_D$ which characterizes the present problem and graphs for the velocity, temperature, skin friction and heat transfer coefficients are presented. In considering these results it should be mentioned that $F_D \to \infty$ corresponds to purely forced convection.

When the problems of the boundary layer are seen to have steady solutions with positive skin friction, separation would not occur and unsteady flows would approach the steady flow as a limit of time $t \to \infty$. An unsteady flow at the forward stagnation point is cited as an example of such problems. So, the values of the skin friction and heat transfer coefficients for small times we have extrapolated to infinite time. It should be pointed out that a theoretical study of the effect of a magnetic field on the transient phenomena to the steady flow near the forward stagnation point of an infinite plane wall was performed by KATAGIRI (1969) by a direct numerical integration of the unsteady boundary-layer equations.

## 2. Basic equation

The curvilinear orthogonal coordinate system and notations adopted for this analysis are shown in Fig. 1. The arc length $x$ is measured along the surface of the cylinder and

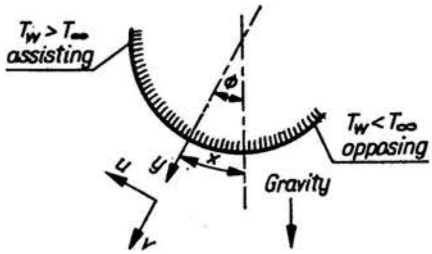


FIG. 1. Coordinate system.

has its initial value of zero at the lower stagnation point, and $y$ is the normal distance from the surface.

If the effects of the component of the buoyancy forces normal to the cylinder are neglected and if the fluid properties are assumed constant and viscous dissipation is disregarded, the governing equations for unsteady and incompressible flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\text{(2.1) [cont.]}\qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}\pm\beta g(T-T_\infty)\sin\Phi,$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}.$$

The plus-minus signs in the buoyancy term correspond to assisting and opposing flows, respectively.

The specification of the initial and the boundary conditions is necessary to complete the statement of the problem. It is as follows. For $t<0$, the flow is assumed to remain at rest. At $t=0$, the flow starts to move impulsively with the velocity $U(t,x)$. In addition, the temperature of the cylinder is suddenly increased from that of the surrounding fluid at time $t=0$. Formally, these conditions may be stated as

$$\text{(2.2)}\qquad \begin{array}{llll} t<0: & u(t,x,y)=0, & T(t,x,y)=T_\infty & \multirow{2}{*}{\text{everywhere;}}\\ t=0: & u=U(t,x), & T=T_\infty & \\ t>0: & u=v=0, & T=T_w=\text{const} & \text{at}\quad y=0,\\ & u\to U, & T\to T_\infty & \text{as}\quad y\to\infty. \end{array}$$

Having thus completed the statement of the problem, attention may next be directed toward finding a solution. The following dimensionless variables are introduced

$$\text{(2.3)}\qquad \begin{gathered} U^+=U/U_\infty,\quad u^+=u/U_\infty,\quad v^+=v\sqrt{R_D}/U_\infty,\quad T^+=(T-T_\infty)/(T_w-T_\infty),\\ t^+=tU_\infty/D,\quad x^+=x/D,\quad y^+=y\sqrt{R_D}/D. \end{gathered}$$

In terms of these variables Eqs. (2.1) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\text{(2.4)}\qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{\partial^2 u}{\partial y^2}\pm\frac{T\sin\Phi}{F_D^2},$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2},$$

where, for the sake of simplicity, the cross in Eqs. (2.4) has been omitted and

$$\text{(2.5)}\qquad F_D=U_\infty/[\beta g(T_w-T_\infty)D]^{1/2}=R_D/\sqrt{G_D}$$

is the buoyancy parameter.

The transformations (2.3) give the boundary conditions as

$$\text{(2.6)}\qquad \begin{array}{llll} t<0: & u(t,x,y)=0, & T(t,x,y)=0 & \multirow{2}{*}{\text{everywhere;}}\\ t=0: & u=U(t,x), & T=0 & \\ t>0: & u=v=0, & T=1 & \text{at}\quad y=0,\\ & u\to U, & T\to 0 & \text{as}\quad y\to\infty. \end{array}$$

### 3. Method of solution

We shall restrict our study to the case when the free stream is independent of time. Near the stagnation point, where $x$ is small, the following approximations can be used

$$\sin\Phi \approx \Phi = 2x \quad \text{and} \quad U \approx 4x. \tag{3.1}$$

If we now define the stream function $\Psi$ by

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}, \tag{3.2}$$

and we write

$$\Psi(t, x, y) = 8x\sqrt{t}f(t, \eta), \qquad T(t, y) = h(t, \eta), \tag{3.3}$$

the Eqs. (2.4) may be written

$$\begin{gathered} \frac{\partial^3 f}{\partial \eta^3} + 2\eta \frac{\partial^2 f}{\partial \eta^2} - 4t \frac{\partial^2 f}{\partial t\, \partial \eta} = -16t\left\{1 - \left(\frac{\partial f}{\partial \eta}\right)^2 + f \frac{\partial^2 f}{\partial \eta^2} \pm h/8F_D^2\right\}, \\ \frac{1}{\sigma}\frac{\partial^2 h}{\partial \eta^2} + 2\eta \frac{\partial h}{\partial \eta} - 4t \frac{\partial h}{\partial t} = -16t \frac{\partial h}{\partial \eta}, \end{gathered} \tag{3.4}$$

with the boundary conditions

$$\begin{gathered} f = \frac{\partial f}{\partial \eta} = 0, \ h = 1 \quad \text{at} \quad \eta = 0, \\ \frac{\partial f}{\partial \eta} \to 1, \qquad h \to 0 \quad \text{as} \quad \eta \to \infty. \end{gathered} \tag{3.5}$$

We look for a solution of the Eqs. (3.4) by an expansion of $f(t, \eta)$ and $h(t, \eta)$ in power of $t$

$$f(t, \eta) = \sum_{i=0}^{\infty} (4t)^i f_i(\eta), \qquad h(t, \eta) = \sum_{i=0}^{\infty} (4t)^i h_i(\eta). \tag{3.6}$$

Expressions (3.6) give the following differential equations and boundary conditions:

$$\begin{gathered} f_0''' + 2\eta f_0'' = 0, \quad \frac{1}{\sigma} h_0'' + 2\eta h_0' = 0, \\ f_0 = f_0' = 0, \quad h_0 = 1 \quad \text{at} \quad \eta = 0; \quad f_0' \to 1, \quad h_0 \to 0 \quad \text{as} \quad \eta \to \infty; \\ f_1''' + 2\eta f_1'' - 4f_1' = -4(1 - f_0'^2 + f_0 f_0'' \pm h_0/8F_D^2), \\ \frac{1}{\sigma} h_1'' + 2\eta h_1' - 4h_1 = -4f_0 h_0', \\ f_1 = f_1' = h_1 = 0 \quad \text{at} \quad \eta = 0, \quad f_1' \to 0, \quad h_1 \to 0 \quad \text{as} \quad \eta \to \infty; \\ f_i''' + 2\eta f_i'' - 4if_i' = -4\sum_{j=0}^{i-1} (f_{i-j-1} f_j'' - f_{i-j-1}' f_j') \mp \frac{1}{2F_D^2} h_{i-1}, \\ \frac{1}{\sigma} h_i'' + 2\eta h_i' - 4ih_i = -4\sum_{j=0}^{i-1} f_{i-j-1} h_i', \\ f_i = f_i' = h_i = 0 \quad \text{at} \quad \eta = 0, \quad f_i' \to 0, \quad h_i \to 0 \quad \text{as} \quad \eta \to \infty, \end{gathered} \tag{3.7}$$

for $i \geqslant 2$.

The system of linear differential equations (3.7) may be solved successively in a manner of analytical procedures proposed by BLASIUS (1908) or by GOLDSTEIN and ROSENHEAD (1936). However, the solutions of higher order equations may be supposed to have complicated forms and it is much troublesome and laborious work to obtain solutions in an analytical form. In this paper, we attempt to obtain analytical solutions up to the third order of approximation. In order to simplify the mathematical description, throughout the remainder of the analysis $\sigma$ is taken as unity although the same ideas hold for different values. Thus the solution of (3.7) up to the third order of approximation, without going into details, is

$$f_0'(\eta) = \operatorname{erf}\eta,\; f_0(\eta) = \eta\operatorname{erf}\eta + \frac{1}{\sqrt{\pi}}(e^{-\eta^2}-1), \quad h_0(\eta) = 1-\operatorname{erf}\eta,$$

$$f_1'(\eta) = -\left(1+\frac{2}{3\pi}\right)(1+2\eta^2)+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\left[(1+2\eta^2)\operatorname{erf}\eta+\frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right]$$
$$+\left(\left(\eta^2-\frac{1}{2}\right)\operatorname{erf}^2\eta+\frac{3}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta+\frac{2}{\pi}e^{-2\eta^2}-\frac{4}{3\pi}e^{-\eta^2}\right.$$
$$\left.+1\pm\frac{1}{4F_D^2}\left(\eta^2\operatorname{erf}\eta+\frac{1}{\sqrt{\pi}}\eta e^{-\eta^2}-\eta^2\right)\right),$$

$$f_1(\eta) = -\left(1+\frac{2}{3\pi}\right)\left(\eta+\frac{2}{3}\eta^3\right)+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\left[\left(\eta+\frac{2}{3}\eta^3\right)\operatorname{erf}\eta+\frac{2}{3\sqrt{\pi}}(1+\eta^2)e^{-\eta^2}\right]$$
$$+\left(\frac{1}{3}\eta^3-\frac{1}{2}\eta\right)\operatorname{erf}^2\eta+\frac{1}{3\sqrt{\pi}}\left[\left(2\eta^2-\frac{11}{2}\right)e^{-\eta^2}-2\right]\operatorname{erf}\eta+\frac{4\sqrt{2}}{3\sqrt{\pi}}\operatorname{erf}(\sqrt{2}\eta)$$
$$+\frac{1}{3\pi}\eta e^{-2\eta^2}+\eta\pm\frac{1}{F_D^2}\left[\frac{1}{12}\eta^3\operatorname{erf}\eta+\frac{1}{12\sqrt{\pi}}\left(\eta^2-\frac{1}{2}\right)e^{-\eta^2}-\frac{1}{12}\eta^3\right.$$
$$\left.+\frac{1}{24\sqrt{\pi}}\right]-\frac{1}{3\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right),$$

$$h_1(\eta) = \left(\eta^2+\frac{1}{2}\right)\operatorname{erf}^2\eta+\frac{1}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta-\left(\frac{1}{2}-\frac{4}{3\pi}\right)\left[(1+2\eta^2)\operatorname{erf}\eta+\frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right]$$
$$+\frac{4}{3\pi}e^{-\eta^2}-\frac{4}{3\pi}(1+2\eta^2),$$

(3.8)
$$f_2'(\eta) = \left(\frac{1}{4}-\eta^2-\frac{1}{3}\eta^4\right)\operatorname{erf}^3\eta+\left[-\frac{4}{\sqrt{\pi}}\left(\eta+\frac{1}{3}\eta^3\right)e^{-\eta^2}+\frac{1}{3}\left(\frac{1}{2}+\frac{2}{3\pi}\right)\eta^4\right.$$
$$\left.+\left(\frac{1}{2}+\frac{2}{3\pi}\right)\eta^2-\frac{4}{3\sqrt{\pi}}\eta-\frac{3}{8}-\frac{1}{2\pi}\pm\frac{1}{F_D^2}\left(-\frac{1}{32}-\frac{1}{4}\eta^2-\frac{1}{8}\eta^4\right)\right]\operatorname{erf}^2\eta$$
$$+\left\{-\frac{1}{3\pi}(14+5\eta^2)e^{-2\eta^2}+\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}+\frac{2}{3\pi}\right)\left(\eta^3+\frac{23}{6}\eta\right)e^{-\eta^2}-\frac{2}{3\pi}(1-2\eta^2)e^{-\eta^2}\right.$$
$$-\frac{4}{3\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right)\eta+\frac{2}{3\pi}\pm\left[-\frac{5}{8\sqrt{\pi}}\left(\frac{1}{2}\eta+\frac{1}{3}\eta^3\right)e^{-\eta^2}-\frac{1}{8}\left(1-\frac{8}{3\pi}\right)\eta^2\right.$$

$$-\frac{1}{3\pi}\eta-\frac{1}{16}\left(1+\frac{8}{3\pi}\right)\Bigg]\Bigg\}\operatorname{erf}\eta+\frac{9\sqrt{3}}{20\pi}(3+12\eta^2+4\eta^4)\operatorname{erf}(\sqrt{3}\eta)$$

$$+\frac{16\sqrt{2}}{15\pi}e^{-\eta^2}\operatorname{erf}(\sqrt{2}\eta)+\frac{1}{30\pi^{3/2}}(133\eta+54\eta^3)e^{-3\eta^2}+\frac{1}{3\pi}\Bigg[\left(1+\frac{4}{3\pi}\right)\eta^2$$

$$+\frac{4}{\sqrt{\pi}}\eta+4\left(1+\frac{4}{3\pi}\right)\Bigg]e^{-2\eta^2}-\frac{1}{\sqrt{\pi}}\Bigg[\left(\frac{1}{3}+\frac{2}{9\pi}\right)\eta^3+\left(\frac{3}{2}+\frac{7}{3\pi}\right)\eta$$

$$+\frac{8}{5\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right)\Bigg]e^{-\eta^2}+\frac{8}{3\sqrt{\pi}}\left(1+\frac{2}{3\pi}\right)\eta\pm\frac{1}{F_D^2}\Bigg[-\frac{1}{12\pi}(1+\eta^2)e^{-2\eta^2}$$

$$-\frac{1}{4\sqrt{\pi}}\Bigg[\left(\frac{5}{4}-\frac{4}{3\pi}\right)\eta+\frac{1}{6}\eta^3\Bigg]e^{-\eta^2}-\frac{2}{15\pi}e^{-\eta^2}-\frac{1}{3\pi}\eta^2+\frac{1}{3\sqrt{\pi}}\eta-\frac{1}{6\pi}\Bigg]$$

$$+\left(\frac{4}{45\pi}+\frac{16}{135\pi}\pm\frac{23}{180\pi}\frac{1}{F_D^2}\right)(3+12\eta^2+4\eta^4)+\Bigg[\frac{1}{24}-\frac{26+81\sqrt{3}}{180\pi}-\frac{16}{135\pi}$$

$$\pm\left(\frac{1}{32}-\frac{23}{180\pi}\right)\frac{1}{F_D^2}\Bigg]\Bigg[(3+12\eta^2+4\eta^4)\operatorname{erf}\eta+\frac{2}{\sqrt{\pi}}(5\eta+2\eta^3)e^{-\eta^2}\Bigg],$$

$$h_2(\eta)=-\frac{1}{3}\left(\frac{7}{4}+7\eta^2+\frac{13}{3}\eta^4\right)\operatorname{erf}^3\eta+\Bigg[-\frac{2}{3\sqrt{\pi}}(7\eta+6\eta^3)e^{-\eta^2}+\frac{1}{6}\left(7-\frac{44}{3\pi}\right)\eta^4$$

$$+\frac{1}{2}\left(3-\frac{4}{\pi}\right)\eta^2-\frac{4}{3\sqrt{\pi}}\eta+\frac{1}{8}\left(3-\frac{4}{\pi}\right)\pm\frac{1}{F_D^2}\left(-\frac{1}{32}-\frac{1}{8}\eta^2-\frac{1}{24}\eta^4\right)\Bigg]\operatorname{erf}^2\eta$$

$$+\Bigg\{-\frac{1}{3\pi}(8+11\eta^2)e^{-2\eta^2}+\frac{1}{3\sqrt{\pi}}\Bigg[\left(\frac{17}{4}-\frac{19}{3\pi}\right)\eta+\left(\frac{13}{2}-\frac{46}{3\pi}\right)\eta^3\Bigg]e^{-\eta^2}$$

$$-\frac{2}{\pi}\left(1+\frac{2}{3}\eta^2\right)e^{-\eta^2}-\frac{16}{3\pi}\eta^2+\frac{4}{3\sqrt{\pi}}\left(1-\frac{8}{3\pi}\right)\eta-\frac{4}{3\pi}\pm\frac{1}{\sqrt{\pi}F_D^2}\left(-\frac{1}{8}\eta^3-\frac{11}{48}\right)e^{-\eta^2}\Bigg\}$$

$$\times\operatorname{erf}\eta+\frac{21\sqrt{3}}{20\pi}(3+12\eta^2+4\eta^4)\operatorname{erf}(\eta\sqrt{3})-\frac{16\sqrt{2}}{15\pi}e^{-\eta^2}\operatorname{erf}(\eta\sqrt{2})$$

$$+\frac{1}{90\pi^{3/2}}(971\eta+378\eta^3)e^{-3\eta^2}+\frac{1}{\pi}\Bigg[-\frac{4}{3\sqrt{\pi}}\eta+\left(1-\frac{8}{3\pi}\right)\eta^2\Bigg]e^{-2\eta^2}$$

$$+\frac{1}{\sqrt{\pi}}\Bigg[\left(\frac{1}{3}+\frac{2}{9\pi}\right)\eta^3+\left(\frac{1}{6}-\frac{35}{9\pi}\right)\eta+\frac{8}{5\sqrt{\pi}}\left(1-\frac{2}{\pi}\right)\Bigg]e^{-\eta^2}+\frac{32}{9\pi^{3/2}}\eta$$

$$\pm\frac{1}{F_D^2}\Bigg\{-\frac{1}{12\pi}(1+\eta^2)e^{-2\eta^2}+\frac{1}{24\sqrt{\pi}}\left(\eta^3+\frac{1}{2}\eta\right)e^{-\eta^2}-\frac{1}{30\pi}e^{-\eta^2}\Bigg\}\Bigg]$$

$$+\left(-\frac{8}{15\pi}+\frac{16}{15\pi^2}\pm\frac{7}{180\pi}\frac{1}{F_D^2}\right)(3+12\eta^2+4\eta^4)+\Bigg[\frac{5}{72}+\frac{206-189\sqrt{3}}{180\pi}$$

$$-\frac{16}{15\pi^2}\pm\left(\frac{1}{96}-\frac{7}{180\pi}\right)\frac{1}{F_D^2}\Bigg]\Bigg[3+12\eta^2+4\eta^4)\operatorname{erf}\eta+\frac{2}{\sqrt{\pi}}(5\eta+2\eta^3)e^{-\eta^2}\Bigg],$$

where $\operatorname{erf}\eta$ is the error function defined by

$$\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-s^2}ds. \tag{3.9}$$

From (3.8) we have

$$\begin{aligned}
f_0''(0) &= \frac{2}{\sqrt{\pi}} = 1\cdot 128379, \quad h_0'(0) = -\frac{2}{\sqrt{\pi}} = -1\cdot 128379,\\
f_1''(0) &= \frac{2}{\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right) \pm \frac{1}{4\sqrt{\pi}}\,\frac{1}{F_D^2} = 1\cdot 607278 \pm 0\cdot 141047\,\frac{1}{F_D^2},\\
h_1'(0) &= -\frac{2}{\sqrt{\pi}}\left(1-\frac{8}{3\pi}\right) = -0\cdot 170581,\\
f_2''(0) &= \frac{1}{\sqrt{\pi}}\left(\frac{11}{6}+\frac{89-108\sqrt{3}}{15\pi}-\frac{256}{135\pi^2}\right) \pm \frac{1}{\sqrt{\pi}}\left(\frac{19}{48}-\frac{62}{45\pi}\right)\frac{1}{F_D^2}\\
&= -0\cdot 248091 \pm (-0\cdot 024147)\,\frac{1}{F_D^2},\\
h_2'(0) &= \frac{1}{\sqrt{\pi}}\left(\frac{23}{18}+\frac{451-252\sqrt{3}}{15\pi}-\frac{256}{15\pi^2}\right) \pm \frac{1}{\sqrt{\pi}}\left(\frac{3}{16}-\frac{28}{45\pi}\right)\frac{1}{F_D^2}\\
&= -0\cdot 080817 \pm (-0\cdot 006771)\,\frac{1}{F_D^2}.
\end{aligned} \tag{3.10}$$

## 4. Discussions

To clarify the influence of the buoyancy forces on the flow, the velocity and temperature profiles versus $y$ are presented in Figs. 2–5 for $F_D = 0.3$ and $\infty$, respectively. It will be noted from these graphs that the effects of buoyancy forces are comparatively small. They decrease the velocity profiles in assisting flow and increase them in opposing flow, while the temperature profiles are increasing in assisting flow and decreasing in opposing flow.

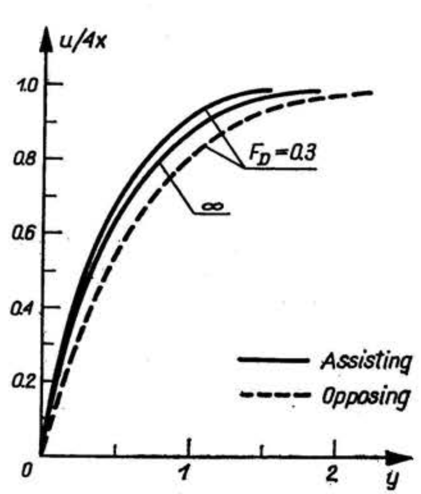


FIG. 2. Velocity profiles for $4t = 0.25$.

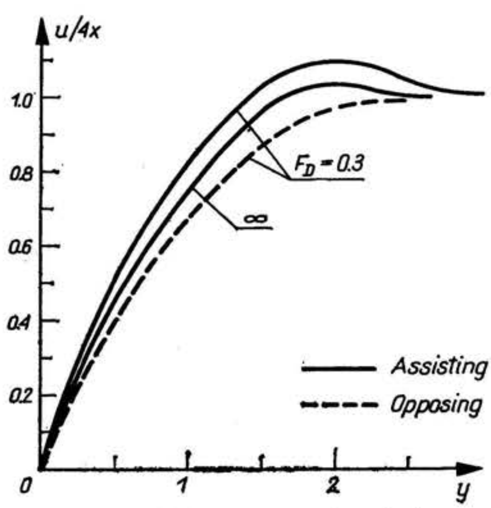


FIG. 3. Velocity profiles for $4t = 0.64$.

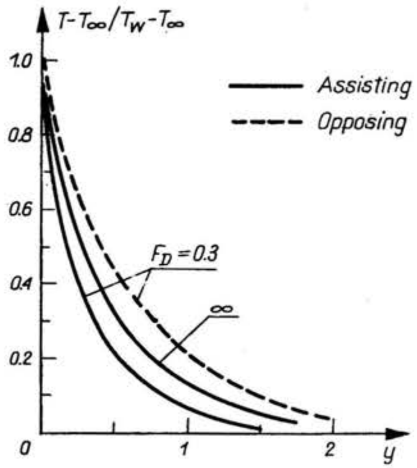


FIG. 4. Temperature profiles for $4t = 0.25$.

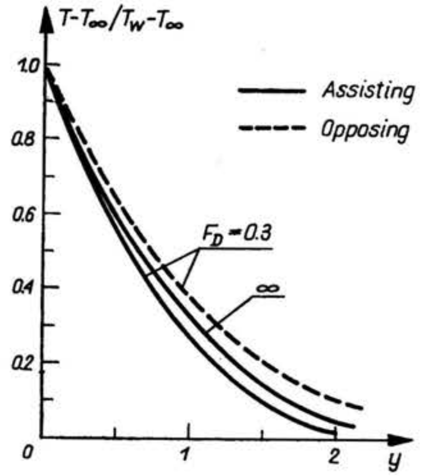


FIG. 5. Temperature profiles for $4t = 0.64$.

Once the distributions of $u$ and $T$ are known, any other required property of the flow can be determined. Thus the coefficients of skin friction and heat transfer at the wall are given by

$$C_f\sqrt{R_D} = \frac{1}{2\sqrt{4t}}\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} = \frac{1}{2\sqrt{4t}}\{f_0''(0)+(4t)f_1''(0)+(4t)^2 f_2''(0)+\ \dots\}$$

$$= 0\cdot 564189(4t)^{-1/2}+(0\cdot 803639\pm 0\cdot 070523\frac{1}{F_D^2}(4t)^{1/2}$$

$$-\left(0\cdot 124045\pm 0\cdot 012073\frac{1}{F_D^2}\right)(4t)^{3/2}+\ \dots, \tag{4.1}$$

$$N_D/\sqrt{R_D} = -\frac{1}{\sqrt{4t}}\left(\frac{\partial h}{\partial \eta}\right)_{\eta=0} = -\frac{1}{\sqrt{4t}}\{h_0'(0)+(4t)h_1'(0)+(4t)^2 h_2'(0)+\ \dots\}$$

$$= 1\cdot 128379(4t)^{-12/}+0\cdot 170581(4t)^{1/2}+\left(0\cdot 080817\pm 0\cdot 006771\frac{1}{F_D^2}\right)(4t)^{3/2}+\ \dots.$$

The coefficients of skin friction and heat transfer are plotted versus time in Figs. 6 and 7 for different values of $F_D$, the buoyancy parameter. They are singular at $t = 0$ because the flow was started impulsively and the temperature of the cylinder is suddenly increased. It is observed that the solutions for the skin friction and heat transfer are valid only for small values of $t$. Further, Figs. 6 and 7 show that the buoyancy forces increase the skin friction and heat transfer in assisting flow and decrease them in opposing flow. Nevertheless, it is worth noting that the buoyancy forces have a completely negligible effect on the flow when $F_D$ is greater than 5 (that is, $G_D/R_D^2$ less than 0.04).

Although series (4.1) hold only for a small time, they may be used to estimate values for the steady problem ($t \to \infty$). The procedure for extrapolating to infinite time was devised by SHANKS (1955) and has successfully been applied to several problems in fluid

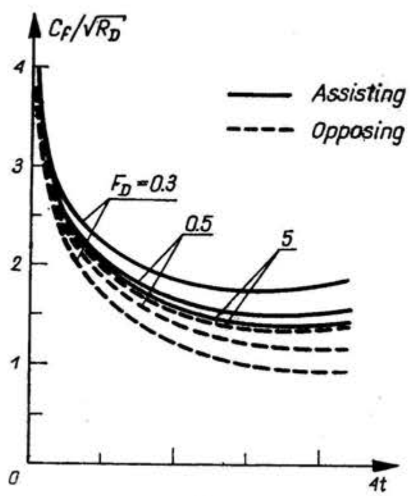


FIG. 6. Results for the skin friction coefficient.

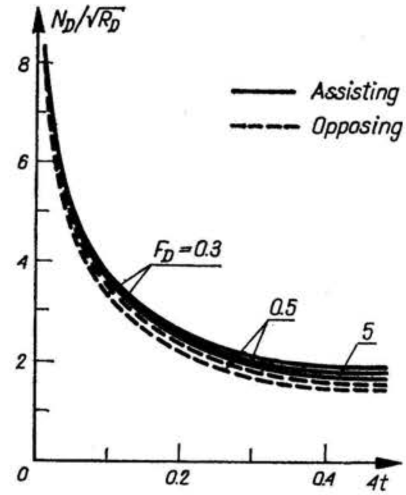


FIG. 7. Results for the heat transfer coefficient.

dynamics by VAN DYKE (1964) and recently for free convection by ELLIOTT (1970). In order to use this technique, the series (4.1) are first expressed in the form

$$\begin{aligned}\left(\frac{4t}{1+4t}\right)^{1/2} C_f\sqrt{R_D} &= \frac{1}{2}f_0''(0)\left\{1+\left(\frac{f_1''(0)}{f_0''(0)}-\frac{1}{2}\right)(4t)\right.\\ &\left.+\left(\frac{f_2''(0)}{f_0''(0)}-\frac{f_1''(0)}{2f_0''(0)}+\frac{3}{8}\right)(4t)^2+\left(\frac{3f_1''(0)}{8f_0''(0)}-\frac{f_2''(0)}{2f_0''(0)}-\frac{5}{16}\right)(4t)^3+\ldots\right\},\\ \left(\frac{4t}{1+4t}\right)^{1/2} N_D/\sqrt{R_D} &= -h_0'(0)\left\{1+\left(\frac{h_1'(0)}{h_0'(0)}-\frac{1}{2}\right)(4t)\right.\\ &\left.+\left(\frac{h_2'(0)}{h_0'(0)}-\frac{h_1'(0)}{2h_0'(0)}+\frac{3}{8}\right)(4t)^2+\left(\frac{3h_1'(0)}{8h_0'(0)}-\frac{h_2'(0)}{2h_0'(0)}-\frac{5}{16}\right)(4t)^3+\ldots\right\}.\end{aligned} \tag{4.2}$$

Applying Shanks's non-linear transformation to the power series on the right-hand side of expressions (4.2) yields single rational fractions. These rational fractions remain bounded at infinite time and so give an estimate of the steady state values. (As is known, Shanks's transformation requires not necessarily the convergence of a series). The values for the skin friction and heat transfer as $t \to \infty$ are given in Table 1 for various values

Table 1. Values of $C_f\sqrt{R_D}$ and $N_D\sqrt{R_D}$ for infinite time

| | $C_f\sqrt{R_D}$ | | $N_D\sqrt{R_D}$ | |
|---|---|---|---|---|
| $F_D$ | assisting | opposing | assisting | opposing |
| 0.3 | 2.604116 | 0.248243 | 0.807913 | 0.683792 |
| 0.5 | 1.848012 | 1.041985 | 0.778989 | 0.726670 |
| 1.0 | 1.546899 | 1.284005 | 0.763906 | 0.749802 |
| 5.0 | 1.447496 | 1.438582 | 0.756234 | 0.755669 |
| ∞ | 1.443196 | 1.443196 | 0,756014 | 0.756014 |

of $F_D$. By an examination on this Table, it is found that for $F_D \to \infty$ (purely forced convection), the results for the skin friction and heat transfer are generally in agreement with those of SIBULKIN (1962), and HAYDAY and BOWLUS (1967) for the steady flow near a two-dimensional stagnation point. Presumably better agreement exists for higher terms in (4.2).

**Acknowledgements**

I wish to express my sincere gratitude in more than a formal way to the Alexander von Humboldt Foundation for a research fellowship at the Technical University of Hannover during the year 1973. The author also wishes to express his grateful thanks to Prof. I. TEIPEL for providing ideal conditions of research.

**References**

1. R. AŠKOVIĆ, *Étude de la couche limite laminaire tridimensionnelle en régime instationnaire*, RAPPORT A-9, Univ. Laval, Québec (Canada), 1967.
2. R. AŠKOVIĆ, *Traitement de la couche limite thermique laminaire tridimensionnelle en régime instationnaire compte-tenu de la poussée d'Archimede*, Int. J. Heat Mass Transfer, **15**, 91–98, 1972.
3. I. POP, *On unsteady three-dimensional laminar flow*, Proc. IUTAM Symp. on Unsteady Boundary Layers (Univ. Laval, Québec, Canada), 1, 302–322, 1971 (ed. E.A. EICHELBRENNER).
4. V. M. SOUNDALGEKAR, *Free convection effects on the oscillatory flow past an infinite, vertical, porous flat plate with constant suction.* I, II. Proc. R. Soc. London, A **333**, 25–36; 37–50, 1973.
5. P. H. OOSTHUIZEN, *Laminar combined convection from an isothermal circular cylinder to air*, Trans. Inst. Chem. Engrs, **48**, T227–T231, 1970.
6. I. POP, *Unsteady free convection on a vertical flat plate* [in Polish], Rozpr. Inżyn., **17**, 173–184, 1969.
7. M. KATAGIRI, *Unsteady magnetohydrodynamic flow at the forward stagnation point*, J. Phys. Soc. Japan, **27**, 1662–1668, 1969.
8. H. BLASIUS, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. Phys., **56**, 1–37, 1908.
9. S. GOLDSTEIN and L. ROSENHEAD, *Boundary layer growth*, Proc. Camb. Phil. Soc., **32**, 392–401, 1936.
10. D. SHANKS, *Non-linear transformations of divergent and slowly convergent sequences*, J. Math. Phys., **34**, 1–42, 1955.
11. M. VAN DYKE, *Perturbation methods in fluid mechanics*, New York 1964.
12. L. ELLIOTT, *Free convection on a two-dimensional or axisymmetric body*, Quart. J. Mech. Appl. Math., **23**, 153–161, 1970.
13. M. SIBULKIN, *Heat transfer near the forward stagnation point of a body of revolution*, J. Aeronaut. Sci., **19**, 570–571, 1952.
14. A. A. HAYDAY and D. A. BOWLUS, *Integration of coupled non-linear equations in boundary-layer theory with specific reference to heat transfer near the stagnation point in three-dimensional flow*, Int. J. Heat Mass Transfer, **10**, 415–426, 1967.

**Additionally references**

1. A. ACRIVOS, *Combined laminar free and forced convection heat transfer in external flows*, AIChE Journal, **4**, 285, 1958.
2. E. M. SPARROW, *R.* EICHHORN and J.L. GREEG, *Combined forced and free convection in a boundary-layer flow*, Phys. Fluids, **2**, 319, 1959.
3. E. M. SPARROW and J. L. GREEG, *Buoyancy and heat transfer*, J. Appl. Mech., Trans. ASME, Ser. E, **81**, 133, 1959.

4. J. R. KLIEGEL, *Laminar free and forced convection heat transfer from a vertical flat plate*, Ph. D. thesis, University of California, Berkeley, Calif., 1959.
5. E. M. SPARROW and W. J. MINKOWYCZ, *Buoyancy effects on horizontal boundary-layer flow and heat transfer*, Int. J. Heat Mass Transfer, **5**, 505, 1962.
6. T. CHIANG and J. KAYE, *On laminar free convection from a horizontal cylinder*, Proc. 4th U.S. Nat. Congress Appl. Mech., **2**, 1213, 1962.
7. A. A. SZEWCZYK, *Combined forced and free-convection laminar flow*, J. Heat Transfer, Trans. ASME, Ser. C, **86**, 501, 1964.
8. S. ESHGHY, *Forced-flow effects on free-convection flow and heat transfer*, J. Heat Transfer, Trans. ASME, Ser. C, **86**, 290, 1964.
9. A. ACRIVOS, *On the combined effect of forced and free convection heat transfer on laminar boundary-layer flows*, Chem. Engng. Sci., **21**, 343, 1966.
10. J. H. MERKIN, *The effects of buoyancy forces on the boundary-layer flow over a semi-infinite vertical flat plate in a uniform free stream*, J. Fluid Mech., **35**, 439, 1969.
11. I. POP, *De la convection naturelle instationnaire sur une plaque verticale infinie*, ZAMP, **20**, 560, 1969.
12. P. H. OOSTHUIZEN and S. MADAN, *Combined convective heat transfer from horizontal cylinders in air*, J. Heat Transfer, Trans. ASME, Ser. C, **92**, 194, 1970.
13. I. POP, *Oscillatory free convection from a vertical plate*, Arch. Mech. Stos., **23**, 303, 1971.
14. I. POP, *On unsteady compressible free convection near an infinite vertical flat plate*, ZAMM, **51**, 489, 1971.
15. N. D. JOSHI and S. P. SUKHATME, *An analysis of combined free and forced convection heat transfer from a horizontal circular cylinder to a transverse flow*, J. Heat Transfer, Trans. ASME, Ser. C, **93**, 441, 1971.
16. P. SINGH and M. C. CHATURVEDI, *Fluctuating flow and heat transfer from a vertical surface*, Canadian J. Phys., **51**, 1800, 1973.
17. V. M. SOUNDALGEKAR and I. POP, *Viscous dissipation effects on unsteady free convective flow past an infinite vertical porous plate with variable suction*, Int. J. Heat Mass Transfer, **17**, 85, 1974.
18. I. POP and V. M. SOUNDALGEKAR, *On unsteady free convection flow of a compressible fluid past a vertical plate*. To be published.

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*Received May* 28, 1973.