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Stability criterion of a dynamic system described by equations with a deviated argument

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A CERTAIN type of a set of integro-differential equations is analyzed. By transforming the system into an operator equation determined in a BANACH space [5], the conditions are given under which the solution is either bounded in the interval ⟨0, ∞), or tends to zero for $t \to \infty$. Using the results derived, the criterion of asymptotic global stability in the sense of Lapunov are formulated with respect to a set of equations with a deviated argument.

Przeprowadzono analizę pewnego typu układu równań różniczkowo-całkowych. Przekształcając rozpatrywany układ w równanie operatorowe określone w przestrzeni Banacha [5], podano warunki, przy spełnieniu których jego rozwiązanie jest ograniczone w przedziale ⟨0, ∞), bądź też zbieżne do zera dla $t \to \infty$. W oparciu o otrzymane rezultaty sformułowano kryterium stateczności asymptotycznej globalnej w sensie Lapunowa dla układu równań z odchylonym argumentem.

Проведен анализ некоторого типа системы интегро-дифференциальных уравнений. Преобразуя рассматриваемую систему в операторное уравнение, определенное в банаховом пространстве [5], даются условия, при удовлетворении которым его решение ограничено в интервале ⟨0, ∞), или же стремится к нулю для $t \to \infty$. Опираясь на полученные результаты сформулирован критерий глобальной асимптотической устойчивости в смысле Ляпунова для системы уравнений с отклоняющимся аргументом.

1. Introduction

LET the dynamic system be described by the following system of differential equations

$$\frac{dx_i}{dt} - \sum_{m=1}^{M} p_{im} x_m(t) - \sum_{m=1}^{M} q_{im} x_m(t-t_{im}) - \sum_{p=2}^{\infty} \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} a_{pim_1\ldots m_p}(t) x_{m_1}(t) \ldots x_{m_p}(t) = 0 \quad (i = 1, \ldots, M). \tag{1.1}$$

Analysis of stability of such a system of equations (in the sense of LAPUNOV [1]) is connected with substantial difficulties. The method most frequently applied consists in seeking the Lapunov functional, although this method yields practical results in very simple cases only. It may be shown, however, that in the case of such systems (or even those of a much more general class), another method may successfully be used [2, 3] which consists in transforming the system considered into the form of an operator equation determined in a certain Banach space.

2. Analysis of the set of integro-differential equations

The system (1.1) constitutes a particular case of a system having the following vectorial form

$$\frac{dx}{dt} - Px(t) - \int_0^T dQ(\tau)x(t-\tau) = y(t) + \sum_{p=2}^{\infty} [G_p x^p(t)]. \tag{2.1}$$

Here

$$x(t) = \{x_i(t)\}, \quad y(t) = \{y_i(t)\}, \quad P = \{p_{im}\}, \quad Q(t) = \{q_{im}(t)\}$$

with $i, m = 1, \ldots, M$, and the operators $G_p = \{G_{pi}\}_{(i=1,\ldots,M)}$ are defined by one of the following formulae:

$$[G_{pi}x^p](t) = \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} a_{\mathrm{pim}_1 \ldots m_p}(t) x_{m_1}(t) \ldots x_{m_p}(t), \tag{2.2}$$

$$[G_{pi}x^p](t) = \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} \int_0^t \ldots \int_0^t k_{\mathrm{pim}_1 \ldots m_p}(t, \tau_1, \ldots, \tau_p) \times \\ \times x_{m_1}(\tau_1) \ldots x_{m_p}(\tau_p)\, d\tau_1 \ldots d\tau_p, \tag{2.3}$$

or by their combinations; it should be remembered that

$$\int_0^T dQ(\tau)x(t-\tau) \overset{\mathrm{df}}{=} \left\{ \sum_{m=1}^{M} \int_0^{t_{im}} dq_{im}(\tau) x_m(t-\tau) \right\}_{(i=1, \ldots M)}. \tag{2.4}$$

Each of the functions $q_{im}(t)$ has a bounded variation in the interval $\langle 0, i_{im} \rangle$: $\operatorname*{Var}_{\langle 0, t_{im} \rangle} q_{im} < \infty$.

Let the Eq. (2.1) describe the action of a certain dynamic system for $t > 0$, $x(t)$ being equal to a known continuous function $f(t)$ for $t \in \langle -T_0, 0 \rangle$; here $T_0 = \max\limits_{i,m} t_{im}$. Let moreover $y(t)$ be a continuous and bounded function for $t \in \langle 0, \infty)$.

Let us consider the linear part of Eq. (2.1), that is

$$\frac{dx}{dt} - Px(t) - \int_0^T dQ(\tau)x(t-\tau) = y(t) \quad \text{for} \quad t > 0. \tag{2.5}$$

On defining the functions $q'_{in}(t)$ in the following manner:

$$q'_{im}(t) = \begin{cases} q_{im}(t) & \text{for} \quad t \in \langle 0, t_{im} \rangle, \\ q_{im}(t_{im}) & \text{for} \quad t > t_{im}, \end{cases} \tag{2.6}$$

we may easily observe that

$$\int_0^{t_{im}} dq_{im}(\tau) x_m(t-\tau) = \int_0^t dq'_{im}(\tau) x_m(t-\tau) - u_{im}(t), \tag{2.7}$$

where

$$u_{im}(t) = \begin{cases} -\int_t^{t_{im}} dq_{im}(\tau) f_m(t-\tau) & \text{for} \quad t \in \langle 0, t_{im} \rangle, \\ 0 & \text{for} \quad t > t_{im}. \end{cases} \tag{2.8}$$

The equation is now true

$$\operatorname*{Var}_{\langle 0,\infty)} q'_{im} = \operatorname*{Var}_{\langle 0, t_{im}\rangle} q_{im}. \tag{2.9}$$

Substituting now

$$Q_1(t) = \{q'_{im}(t)\}_{(i,\, m=1,\, \dots,\, M)},$$

$$u(t) = \Big\{\sum_{m=1}^{M} u_{im}(t)\Big\}_{(i=1,\, \dots,\, M)},$$

into the Eq. (2.5) we have

$$\frac{dx}{dt} - Px(t) - \int_0^t dQ_1(\tau)x(t-\tau) = y(t)+u(t) \quad \text{for} \quad t>0. \tag{2.10}$$

Here

$$\sup_{t\geq 0} \|u(t)\|_2 \leq \|\{\operatorname*{Var}_{\langle 0,t_{im}\rangle} q_{im}\}_{(i,m)}\|2\cdot\Big\{\sum_{i=1}^{M}[\sup_{t\in\langle -T_0,0\rangle} |f(t)|]^2\Big\}^{1/2}.$$

$$\lim_{t\to\infty} \|u(t)\|_2 = 0.$$

Under the assumption that $f(t)$ is continuous and bounded for $t \in (-\infty, 0\rangle$, the considerations presented above remain valid also in the case $T_0 = \infty$.

In what follows, the notion of Laplace-Stieltjes transforms will be used; the transform is defined as follows: Let $h(t)$ be a function defined for $t \in (-\infty, +\infty)$, equal to zero for $t \leq 0$ and having a bounded variation for $t \geq 0$, ($\operatorname*{Var}_{\langle 0,\infty)} h < \infty$). The L–S transform of that function is the function $H(s)$ given by the formula

$$H(s) = \int_0^\infty e^{-st}dh(t) \quad \text{for} \quad \operatorname{Re} s \geq 0.$$

By means of this definition, the following lemma may be proved:

LEMMA. *If $G(t) = \{g_{im}(t)\}_{(i,m=1,\cdots,M)}$ is the matrix of a functions of bounded variation and without singularities, and if $\inf_{\operatorname{Re} s\geq 0} |\det \bar{G}(s)| > 0$, where $\bar{G}(s) = \int_0^\infty e^{-st}dG(t) \overset{\text{df}}{=} \Big\{\int_0^\infty e^{-st}dg_{im}(t)\Big\}_{(i,m)}$, then there exists exactly one matrix $H(t) = \{h_{im}(t)\}_{(i,m)}$ of a functions of boundedvariation and without the singular part, which satisfies the condition*

$$\bar{G}(s)\cdot\bar{H}(s) = I \quad \text{for} \quad \operatorname{Re} s \geq 0.$$

Here $\bar{H}(s) = \int_0^\infty e^{-st}dH(t)$, and I — unit matrix.

The proof follows directly from the following theorem [7]:

THEOREM 1. *If $g(t)$ is a function of finite variation and non-singular, and if $\inf_{\operatorname{Re} s\geq 0} |\int_0^\infty e^{-st}dg(t)| > 0$, then there exists exactly one function $h(t)$, also of finite variation and non-singular, such that $\int_0^\infty e^{-st}dg(t).\int_0^\infty e^{-st}dh(t) = 1$ for $\operatorname{Re} s \geq 0$.*

Taking into account that, according to the assumptions of the lemma, all minors $W_{lm}(s)$ of the matrix $\bar{G}(s)$ and its determinant are the L–S transforms of functions with bounded variations, and without the singular parts, it may easily be observed (Theorem 1) that the functions $H_{lm}(s) = (-1)^{l+m} W_{lm}(s)[\det \bar{G}(s)]^{-1}$ have the same property. If we assume $\bar{H}(s) = \{H_{lm}(s)\}_{(l,m)}$, then it is seen that $\bar{H}(s) = [\bar{G}(s)]^{-1}$ what concludes the proof of the lemma.

Let us now consider the Eq. (2.10) under the assumption that $u(t) = 0$, that is with the condition $x(t) = 0$ for $t \leqslant 0$. In connection with that equation, the following theorem may be formulated and proved.

THEOREM 2. *If*

$$|\det_{\mathrm{Re}\, s \geqslant 0} [sI-P-\bar{Q}_1(s)]| > 0, \tag{2.11}$$

where $\bar{Q}_1(s)$ is the L–S transform of the matrix $Q_1(t)$, then the solution of the Eq. (2.10) *with the condition $x(t) = 0$ for $t \leqslant 0$ may be represented in the form*

$$x(t) = \int_0^t K(t-\tau)y(\tau)d\tau, \tag{2.12}$$

$K(t)$ being the matrix of functions bounded for $t \geqslant 0$ and convergent to zero with $t \to \infty$, and $\int_0^\infty ||K(t)||_2 dt < \infty$.

P r o o f. Applying formally the Laplace transform to the Eq. (2.10) under the condition proposed, we obtain

$$[sI-P-\bar{Q}_1(s)]X(s) = Y(s).$$

The latter equation is rewritten in the form

$$\frac{1}{s+\gamma}[sI-P-\bar{Q}_1(s)]X(s) = \frac{1}{s+\gamma}Y(s).$$

It is easily seen that

(1) Matrix $G(s) = \dfrac{1}{s+\gamma}[sI-P-\bar{Q}_1(s)]$ is the L–S transform of a matrix of the function with bounded variation and without the singular part;

(2) For each s such that $\mathrm{Re}\, s \geqslant 0$, the inequality holds true

$$\left|\det\left\{\frac{1}{s+\gamma}[sI-P-\bar{Q}_1(s)]\right\}\right| > 0;$$

(3) $\lim\limits_{s\to\infty(\mathrm{Re}\, s\geqslant 0)} \det\left\{\dfrac{1}{s+\gamma}[sI-P-\bar{Q}_1(s)]\right\} = 1.$

Thus if $\bar{H}(s) = \left\{\dfrac{1}{s+\gamma}[sI-P-\bar{Q}_1(s)]\right\}^{-1}$, then there exists such a matrix $H(t) = \{h_{lm}(t)\}_{(l,m)}$ that $\bar{H}(s) = \int_0^\infty e^{-st}dH(t)$, with $\operatorname{Var}_{\langle 0,\infty)} h_{lm} < \infty$ for every $i, m = 1, \ldots, M$.

If we now assume

$$K(t) = \int_0^t e^{-\gamma(t-\tau)}dH(\tau), \tag{2.13}$$

then, by means of [4], it is observed that $K(t)$ possesses the properties stated in the theorem, what concludes the proof.

From the theorem it follows that the solution of the Eq. (2.10) under the condition $x(t) = f(t)$ for $t \leqslant 0$ assumes the form

$$x(t) = \int_0^t K(t-\tau)y(\tau)d\tau + w(t). \tag{2.14}$$

Here

$$w(t) = K(t)f(0) + \int_0^t K(t-\tau)u(\tau)d\tau, \tag{2.15}$$

and the following estimate holds true

$$\sup_{t\geqslant 0} ||x(t)||_2 \leqslant \sup_{t\geqslant 0}||K(t)||_2 \cdot ||f(0)||_2 + \int_0^\infty ||K(t)||_2 dt [\sup_{t\geqslant 0}||y(t)||_2 + \sup_{t\geqslant 0}||u(t)||_2,$$

$x(t)$ being at the same time a continuous function [6], and hence $x \in C^{M2}$. Here C^{M2} denotes the Banach space of vector functions which are continuous and bounded for $t \geqslant 0$ with the norm

$$||x||_{2,C} = \sup_{t\geqslant 0}\Big\{\sum_{i=1}^{M} |x_i(t)|^2\Big\}^{1/2}.$$

It was shown in [5] that if U is an operator given by

$$[Ux](t) = \int_0^t K(t-\tau)x(\tau)d\tau, \tag{2.16}$$

then $U \in (C^{M2} \to C^{M2})$ and $U \in (K^{M2} \to K^{M2})$, K^{M2} being the quotient space with elements of the class of elements of the C^{M2} space, differing by a function converging to zero, with the norm

$$||x||_{2,K} = \lim_{T\to\infty} \sup_{t\geqslant T}\Big\{\sum_{i=1}^{M} |x(t)|^2\Big\}^{1/2}.$$

Returning now to the Eq. (2.1) we may, by using the previous considerations, transform that equation to the form

$$x(t) = z(t) + \int_0^t K(t-\tau) \sum_{p=2}^{\infty} [G_p x^p](\tau)d\tau, \tag{2.17}$$

where

$$z(t) = \int_0^t K(t-\tau)y(\tau)d\tau + w(t). \tag{2.18}$$

Thus if G_p (for $p = 2, 3, \ldots$) are the multilinear operators transforming the space C^{M2} (or K^{M2}) into itself, then for each of those spaces the Eq. (2.17) may be written in the operator form

$$x = z + U \sum_{p=2}^{\infty} G_p x^p. \tag{2.19}$$

Making use of the properties of the corresponding solution (which was analysed in [5]), it is easily demonstrated that the following theorems hold true.

THEOREM 3. *If the following conditions are fulfilled*:

(a) $\operatorname*{Var}_{\langle 0, t_{im}\rangle} q_{im} < \infty$ *for* $i, m = 1, \ldots, M$;

(b) $\inf_{\mathrm{Re}\, s \geqslant 0} |\det[sI - P - G_1(s)]| > 0$;

(c) *functions* $a_{pim_1 \cdots m_p}(t)$ *are continuous and bounded for* $t \in \langle 0, \infty)$;

(d) *functions* $k_{pim_1 \cdots m_p}(t, \tau_1, \ldots, \tau_p)$ *are continuous in the region* $\{\langle 0, \infty); \ldots; \langle 0, \infty)\}$;

$$\sup_{t \geqslant 0} \int_0^t \ldots \int_0^t |k_{pim_1 \cdots m_p}(t, \tau_1, \ldots, \tau_p)|\, d\tau_1 \ldots d\tau_p < \infty;$$

(e) *the radius of convergence of the series is positive*:

with reference to operators (2.2)

$$\sum_{p=1}^{\infty} \left\{ \sup_{t \geqslant 0} \sum_{i=1}^{M} \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} |a_{pim_1 \cdots m_p}(t)|^2 \right\}^{1/2} \eta^p,$$

with reference to operators (2.3):

$$\sum_{p=2}^{\infty} \left\{ \sup_{t \geqslant 0} \sum_{i=1}^{M} \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} \left[\int_0^t \ldots \int_0^t |k_{pim_1 \cdots m_p}(t, \tau_1, \ldots, \tau_p)|\, d\tau_1 \ldots d\tau_p \right]^2 \right\}^{1/2} \eta^p;$$

then there exists such a number $\alpha_1 > 0$ *and a function* $\varphi_1(\xi)$ *continuous, non-decreasing in* $\xi \in (0, \alpha_1\rangle$ *and such that* $\varphi_1(0) = 0$, *that for each* $z \in C^{M2}$ *satisfying the inequality* $\|z\|_{2,C} \leqslant \alpha_1$, *the Eq.* (2.17) *has exactly one solution* $x^* \in C^{M2}$ *the norm of which satisfies the estimate*

$$\|x^*\|_{2,C} \leqslant \varphi_1(\|z\|_{2,C}). \tag{2.20}$$

THEOREM 4. *If the assumptions of Theorem* 3 *are fulfilled and for every* $t_0 > 0$ *satisfying*

$$\lim_{t \to \infty} \int_0^{t_0} \ldots \int_0^{t_0} |k_{pim_1 \cdots m_p}(t, \tau_1, \ldots, \tau_p)|\, d\tau_1 \ldots d\tau_p = 0,$$

there exists such a number $\alpha_2 > 0$ *and a function* φ_2 *(defined analogously to* φ_1*), that if* $\|z\|_{2,K} \leqslant \alpha_2$, *then the Eq.* (2.17) *possesses in the space* K^{M2} *exactly one solution* x^* *satisfying the estimate*

$$\|x^*\|_{2,K} \leqslant \varphi_2(\|z\|_{2,K}). \tag{2.21}$$

From the latter theorem it follows in particular that if $y(t)$ converges to zero at $t \to \infty$, then the solution $x^*(t)$ has the same property.

The results derived above may easily be generalized to the case when $P = P(t) = P_0+P_1(t)$. To that end it is sufficient to replace P with P_0 in the Eq. (2.11) and to assume that $P_1(t)$ is a matrix of functions continuous and bounded within the interval $\langle 0, \infty)$, and that the inequality holds true

$$\sup_{t \geq 0} \int_0^t \|K(t-\tau)P_1(\tau)\|_2 d\tau < 1. \tag{2.22}$$

3. Criterion of stability for the equation with a deviated argument

On the basis of the analysis presented in Sec. 2, the criterion of stability in the Lapunov sense [1] may be formulated with respect to the equation

$$\frac{dx}{dt} - P_0 x(t) - P_1(t)x(t) - \int_0^t dQ(\tau)x(t-\tau) - \sum_{p=2}^{\infty} [G_p x^p](t) = 0 \tag{3.1}$$

in the case of operators G_{pl} being defined by Eq. (2.2).

It is easily seen that if the Eq. (3.1) is determined for $t > t_0$, then by means of the substitutions $\bar{x}(t) = x(t+t_0)$, $\bar{P}_1(t) = P_1(t+t_0)$, $\bar{Q}(t) = Q(t)$ and $\bar{a}_{plm_1\cdots m_p}(t) = a_{plm_1\cdots m_p}(t+t_0)$, we obtain again the equation in the original form with respect to $\bar{x}(t)$, determined for positive $t > 0$. Making use of the estimate

$$\|z\|_{2,C} \leq \int_0^\infty \|K(t)\|_2 dt \cdot \Big[\|y\|_{2,C} + \|\{\operatorname*{Var}_{\langle 0,t_{lm}\rangle} q_{lm}\}_{(l,m)}\|_2 \times \times \Big\{\sum_{l=1}^{M} \sup_{t\in\langle -T_0,0\rangle} |f_l(t)|^2\Big\}^{1/2}\Big] + \sup_{t\geq 0}\|K(t)\|_2 \cdot \|f(0)\|_2, \tag{3.2}$$

the proof of the following theorem is readily obtained:

THEOREM 5. *If the assumptions* (a), (b) *of Theorem* 3 *are satisfied, and if there exists such a number* t_1 *that*

(1) $P_1(t)$ is a matrix of functions continuous and bounded for $t \in \langle t_1, \infty)$;

(2) $\sup_{t\geq 0} \int_0^t \|K(t-\tau)P_1(\tau+t_1)\|_2 d\tau < 1$;

(3) $a_{plm_1\cdots m_p}(t)$ *are continuous and bounded functions for* $t \in \langle t_1, \infty)$;

(4) *the radius of convergence of the series*

$$\sum_{p=2}^{\infty} \Big\{\sup_{t\geq t_1} \sum_{l=1}^{M} \sum_{m_1=1}^{M} \cdots \sum_{m_p=1}^{M} |a_{plm_1\cdots m_p}(t)|^2\Big\}^{1/2} \eta^p$$

is positive, then the zero solution of the Eq. (3.1) *is asymptotically and globally stable.*

4. Concluding remarks

It is easily observed that the equations describing the motion of a broad class of mechanical systems [in which x might be interpreted as deviations from the position of equilibrium, or velocities] may be reduced to the form of the Eq. (2.1) [or, in particular, the Eq. (1.1)]. The theorems formulated in this paper enable us, from the practical point of view, to determine not only the conditions of stability of the zero solution (position of equilibrium) of such systems in the Lapunov sense, but also to determine effectively the allowable deviations of the initial conditions ensuring the boundedness of solutions, or even their convergence to zero, and also to estimate the corresponding solutions.

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