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Diffraction of a plane harmonic SH wave by semi-cylindrical layers

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THE PLANE problem of the interaction between a half space, several infinite coaxial cylindrical layers, and an inclusion under the excitation of a plane harmonic SH wave is dealt with by means of the method of wave functions expansion. For one layer and inclusion displacement as a function of space and time is derived explicitly. Furthermore, the three special cases half-space—elastic layer—rigid inclusion, half-space—elastic layer—cavity, and half-space—rigid layer — elastic inclusion are considered.

Rozważono płaskie zagadnienie współdziałania pomiędzy półprzestrzenią, szeregiem nieskończonych, współosiowych warstw walcowych oraz inkluzją, poddanych działaniu płaskiej, harmonicznej fali SH. W przypadku jednej warstwy oraz inlkuzji wyznaczono przemieszczenie jako jawną funkcję zmiennych przestrzennych i czasu. Rozważono następnie trzy przypadki szczególne następujących układów: półprzestrzeń—warstwa sprężysta—sztywna inkluzja, półprzestrzeń—warstwa sprężysta—pustka oraz półprzestrzeń—warstwa sztywna inkluzja sprężysta.

Рассмотрена плоская задача взаимодействия между полупространством, рядом бесконечных слоев и включением подвергнутых действию плоской гармонической волны SH. В случае одного слоя и включения определено перемещение, как явную функцию пространственных переменных и времени. Затем обсуждены три частных случая следующих систем: полупространство — упругий слой — жесткое включение, полупространство — упругий слой — пустота и полупространство — жесткий слой — упругое включение.

1. Introduction

THE MOTION excited by an earthquake is influenced strongly by inhomogeneities, e.g. inclusions or layers of different properties, in the soil if the ratio of the wavelength of the incident seismic wave and a characteristic length of the inclusion is not too large. Inhomogeneities may result in amplification of the surface motion due to the combination of material properties, frequency and angle of the incident wave, and focusing. Heavy and "hard" foundations exhibit smaller amplitudes than the surrounding soil. The knowledge of possible patterns of the surface motion is essential in designing earthquake resistant structures.

To interpret the measured surface motion a mechanical model is needed. The simplest nonhomogeneous model consists of horizontally stratified layers. It is applicable if the depth of the layers is approximately constant and their lateral extension large compared to the depth. In this paper the two-dimensional problem of the interaction of several infinite coaxial semi-cylindrical layers with the half-space under the excitation of a plane harmonic SH wave is considered. The solution gives analytical expressions for the steady state displacement as a function of space and time. Therefrom stress is derived easily. As long as the wavelength of the incident wave is not too small it is not difficult to evaluate the results numerically. For the simpler problem of the diffraction of plane harmonic SH waves by a semi-cylindrical inclusion in a half-space, numerical results were given by TRIFUNAC [1] and by GAMER and PAO [2].

2. Statement of problem and solution

The coordinate system is chosen so that the z-x plane coincides with the free surface of the half-space, the z-axis being the axis of the cylindrical layers. The y-axis shows inside the half-space (Fig. 1). Half-space, layers, and inclusion are considered homogeneous isotropic elastic materials. The radius a_j separates the layer numbered j [1] with density ϱ^j and shear modulus μ^j from the layer numbered $j+1$ with density ϱ^{j+1} and shear modulus μ^{j+1}. The superscript 0 identifies the half-space and $m+1$ the inclusion. A plane harmo-

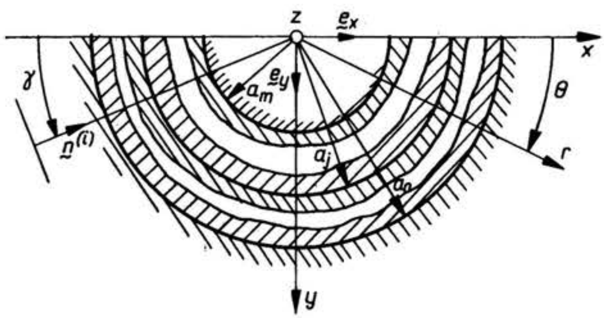

FIG. 1.

nic SH wave incident under the angle of emergence γ [3] hits the first layer and is refracted and reflected at the interface. Refraction and reflection causes vibration of all the layers and the inclusion.

The equation of motion in anti-plane strain

$$u = 0, \quad v = 0, \quad w \neq 0$$

is reduced to the single scalar wave equation

$$\nabla^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}, \tag{2.1}$$

where $c = \sqrt{\mu/\varrho}$ means the velocity of shear waves in the material under consideration. The solution has to comply with the boundary conditions: At "welded" surfaces, displacement and shear stress are continuous. At free surfaces, the shear stress vanishes.

The half-space is excited by an incident (i) plane harmonic wave

$$w^{(i)} = We^{i[k^0(x\cos\gamma - y\sin\gamma)-\omega t]}, \qquad \frac{\omega}{k^j} = c^j, \tag{2.2}$$

propagating along the unit vector

$$\mathbf{n}^{(i)} = \cos\gamma\,\mathbf{e}_x - \sin\gamma\,\mathbf{e}_y,$$

W being the amplitude, k the wave number, and ω the circular frequency. In the steady state the motion of each material point is harmonic in time. The factor $e^{-i\omega t}$ is henceforth omitted.

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(1) Superscripts are probably not misinterpreted as powers.

The incident wave is reflected (r) at the free surface of the half-space $y = 0$ which is equivalent to the superposition of a second plane SH wave propagating along

$$\mathbf{n}^{(r)} = \cos\gamma\, \mathbf{e}_x + \sin\gamma\, \mathbf{e}_y.$$

It is advantageous to use cylindrical coordinates

$$re^{i\theta} = x+iy,$$

since the boundaries are surfaces of constant coordinate in that system. The incident wave has to be expanded into a Fourier series [4, 5]

$$w^{(i)} = W\sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k^0 r)\cos(\theta+\gamma), \tag{2.3}$$

where $J_n(kr)$ designates the Bessel function of the first kind of order n and argument kr. ϵ_n is defined as

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, 3 \dots . \end{cases}$$

The sum of incident and reflected waves which is no longer a plane wave is

$$w^{(i)} + w^{(r)} = 2W\sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k^0 r)\cos n\gamma \cos n\theta. \tag{2.4}$$

To find expressions for the scattered wave and the vibration of the layers and the inclusion the equation of motion, for time harmonic displacement $w(r, \theta)$ the Helmholtz equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + k^2 w = 0, \tag{2.5}$$

is considered once more. By separation one gets as suitable solutions the wave functions

$$J_n(kr)\cos n\theta, \quad Y_n(kr)\cos n\theta,$$

which satisfy for integer separation constant n the condition of vanishing shear stress $\sigma_{\theta z}$ on the free surface $\theta = 0$ and $\theta = \pi$. $Y_n(kr)$ is the Bessel function of the second kind.

The wave functions (multiplied by $e^{-i\omega t}$) mean standing waves with nodal lines in radial and circumferential direction. The displacement in the layers is a combination of such standing waves which is, generally, not a standing wave. Since bounded the displacement of the inclusion does not contain the terms $Y_n(kr)\cos n\theta$. A wave travelling outward in radial direction, e.g., the wave scattered by the first layer, is represented by

$$H_n^{(1)}(kr)\cos n\theta,$$

where

$$H_n^{(1)}(kr) = J_n(kr) + iY_n(kr)$$

is the Hankel function of the first kind(²).

(²) Since the Hankel function of the second kind is not used, the superscript (1) is omitted in the following.

The displacement in the half-space, layers, and inclusion is then, respectively,

$$
\begin{aligned}
w^0 &= 2W\sum_{n=0}^{\infty}[\varepsilon_n i^n J_n(k^0 r)\cos n\gamma + A_n H_n(k^0 r)]\cos n\theta,\\
&\vdots\\
w^j &= -2W\sum_{n=0}^{\infty}[C_n^j J_n(k^j r)+D_n^j Y_n(k^j r)]\cos n\theta,\\
&\vdots\\
w^{m+1} &= -2W\sum_{n=0}^{\infty}C_n^{m+1}J_n(k^{m+1}r)\cos n\theta.
\end{aligned}
\tag{2.6}
$$

The factor -2 in w^1 to w^{m+1} is arbitrary.

The conditions of continuity of displacement and shear stress

$$
\left.\begin{aligned}
w^j &= w^{j+1}\\
\mu^j\frac{\partial w^j}{\partial r} &= \mu^{j+1}\frac{\partial w^{j+1}}{\partial r}
\end{aligned}\right\} r = a_j
\tag{2.7}
$$

give the $2(m+1)n$, $n \to \infty$, unknown complex constants A_n, C_n^j, D_j^n. Equating the coefficients of $\cos n\theta$, one finds the following system of equations ([3]):

$$
\begin{aligned}
&H_n(k^0a_0)A_n + J_n(k^1a_0)C_n^1 + Y_n(k^1a_0)D_n^1 = -\varepsilon_n i^n J_n(k^0a_0)\cos n\gamma,\\
&H_n'(k^0a_0)A_n + \frac{\mu^1k^1}{\mu^0k^0}J_n'(k^1a_0)C_n^1 + \frac{\mu^1k^1}{\mu^0k^0}Y_n'(k^1a_0)D_n^1 = -\varepsilon_n i^n J_n'(k^0a_0)\cos n\gamma,\\
&\vdots\\
&J_n(k^ja_j)C_n^j + Y_n(k^ja_j)D_n^j - J_n(k^{j+1}a_j)C_n^{j+1} - Y_n(k^{j+1}a_j)D_n^{j+1} = 0,\\
&J_n'(k^ja_j)C_n^j + Y_n'(k^ja_j)D_n^j - \frac{\mu^{j+1}k^{j+1}}{\mu^jk^j}J_n'(k^{j+1}a_j)C_n^{j+1} - \frac{\mu^{j+1}k^{j+1}}{\mu^jk^j}Y_n'(k^{j+1}a_j)D_n^{j+1} = 0,\\
&\vdots\\
&J_n(k^ma_m)C_n^m + Y_n(k^ma_m)D_n^m - J_n(k^{m+1}a_m)C_n^{m+1} = 0,\\
&J_n'(k^ma_m)C_n^m + Y_n'(k^ma_m)D_n^m - \frac{\mu^{m+1}k^{m+1}}{\mu^mk^m}J_n'(k^{m+1}a_m)C_n^{m+1} = 0.
\end{aligned}
\tag{2.8}
$$

Since $J_n(0) = 0$ for $n = 1, 2, 3 \ldots$ the motion of the centre point of the inclusion is independent of the angle of emergence.

If there exists only one layer the explicit solution is

$$
A_n = \frac{\Delta_n^{(1)}}{\Delta_n},\quad C_n^1 = \frac{\Delta_n^{(2)}}{\Delta_n},\quad D_n^1 = \frac{\Delta_n^{(2)}}{\Delta_n},\quad C_n^2 = \frac{\Delta_n^{(4)}}{\Delta_n}
\tag{2.9}
$$

with

$$
\begin{aligned}
\Delta_n &= B_n^{(1)}B_n^{(2)} - B_n^{(3)}B_n^{(4)},\\
\Delta_n^{(1)} &= B_n^{(5)}B_n^{(2)} - B_n^{(6)}B_n^{(4)},\\
\Delta_n^{(2)} &= B_n^{(7)}B_n^{(2)},\\
\Delta_n^{(3)} &= -B_n^{(7)}B_n^{(4)},\\
\Delta_n^{(4)} &= B_n^{(7)}B_n^{(8)},
\end{aligned}
$$

([3]) ′means derivative with respect to the argument.

where

$$B_n^{(1)} = \frac{\mu^1 k^1}{\mu^0 k^0} H_n(k^0 a_0) J_n'(k^1 a_0) - H_n'(k^0 a_0) J_n(k^1 a_0),$$

$$B_n^{(2)} = -\frac{\mu^2 k^2}{\mu^1 k^1} Y_n(k^1 a_1) J_n'(k^2 a_1) + Y_n'(k^1 a_1) J_n(k^2 a_1),$$

$$B_n^{(3)} = \frac{\mu^1 k^1}{\mu^0 k^0} H_n(k^0 a_0) Y_n'(k^1 a_0) - H_n'(k^0 a_0) Y_n(k^1 a_0),$$

$$B_n^{(4)} = -\frac{\mu^2 k^2}{\mu^1 k^1} J_n(k^1 a_1) J_n'(k^2 a_1) + J_n'(k^1 a_1) J_n(k^2 a_1),$$

$$B_n^{(5)} = \varepsilon_n i^n \cos n\gamma \left[-\frac{\mu^1 k^1}{\mu^0 k^0} J_n(k^0 a_0) J_n'(k^1 a_0) + J_n'(k^0 a_0) J_n(k^1 a_0) \right],$$

$$B_n^{(6)} = \varepsilon_n i^n \cos n\gamma \left[-\frac{\mu^1 k^1}{\mu^0 k^0} J_n(k^0 a_0) Y_n'(k^1 a_0) + J_n'(k^0 a_0) Y_n(k^1 a_0) \right],$$

$$B_n^{(7)} = \varepsilon_n i^n \cos n\gamma [-H_n(k^0 a_0) J_n'(k^0 a_0) + H_n'(k^0 a_0) J_n(k^0 a_0)],$$

$$B_n^{(8)} = J_n(k^1 a_1) Y_n'(k^1 a_1) - J_n'(k^1 a_1) Y_n(k^1 a_1).$$

The general result implies a rigid layer or inclusion as a limiting case of the elastic material for $\mu \to \infty$ and, on the other hand, the inclusion is replaced by a cavity for $\mu = 0$ [5]. In the following three special cases are dealt with.

3. Special cases

Case I. *Half-space—elastic layer—rigid inclusion*. The displacement of the rigid inclusion

$$w^2 = -2WC_0^2 \tag{3.1}$$

is governed by Newton's second law

$$\varrho^2 \frac{\pi}{2} a_1^2 \ddot{w}^2 = a_1 \int_0^\pi \sigma_{rz}^1(a_1) d\theta. \tag{3.2}$$

Unknown are A_n, C_n^1, D_n^1, C_0^2. The conditions of continuity of displacement and stress at $r = a_0$ yield the first two equations (2.8) as before. At $r = a_1$ displacement has to be continuous and independent of θ.

That means for $n = 1, 2, 3 \ldots$ the third equation

$$J_n(k^1 a_1) C_n^1 + Y_n(k^1 a_1) D_n^1 = 0. \tag{3.3}$$

The solution is given by (2.9) and

$$\Delta_n = \frac{\mu^1 k^1}{\mu^0 k^0} H_n(k^0 a_0) B_n^{(11)} - H_n'(k^0 a_0) B_n^{(12)},$$

$$\Delta_n^{(1)} = \varepsilon_n i^n \cos n\gamma \left[-\frac{\mu^1 k^1}{\mu^0 k^0} J_n(k^0 a_0) B_n^{(11)} + J_n'(k^0 a_0) B_n^{(12)} \right],$$

$$\Delta_n^{(2)} = B_n^{(7)} Y_n(k^1 a_1),$$

$$\Delta_n^{(3)} = -B_n^{(7)} J_n(k^1 a_1)$$

with the abbreviations

$$B_n^{(11)} = J_n'(k^1a_0)Y_n(k^1a_1) - Y_n'(k^1a_0)J_n(k^1a_1),$$
$$B_n^{(12)} = J_n(k^1a_0)Y_n(k^1a_1) - Y_n(k^1a_0)J_n(k^1a_1).$$

For $n = 0$ the additional two equations are

$$\begin{aligned} &J_0(k^1a_1)C_0^1 + Y_0(k^1a_1)D_0^1 - \qquad C_0^2 = 0, \\ &J_0'(k^1a_1)C_0^1 + Y_0'(k^1a_1)D_0^1 + \frac{1}{2}\frac{\varrho^2}{\varrho^1}k^1a_1C_0^2 = 0. \end{aligned} \tag{3.4}$$

From the above equations and the first two (2.8) follows

$$\begin{aligned} \Delta_0 &= B_0^{(1)}B_0^{(9)} - B_0^{(3)}B_0^{(10)}, \\ \Delta_0^{(1)} &= B_0^{(5)}B_0^{(9)} - B_0^{(6)}B_0^{(10)}, \\ \Delta_0^{(2)} &= B_0^{(7)}B_0^{(9)}, \\ \Delta_0^{(3)} &= -B_0^{(7)}B_0^{(10)}, \\ \Delta_0^{(4)} &= B_0^{(7)}B_0^{(8)}, \end{aligned}$$

where

$$B_0^{(9)} = \frac{1}{2}\frac{\varrho^2}{\varrho^1}k^1a_1Y_0(k^1a_1) + Y_0'(k^1a_1),$$
$$B_0^{(10)} = \frac{1}{2}\frac{\varrho^2}{\varrho^1}k^1a_1J_0(k^1a_1) + J_0'(k^1a_1).$$

The motion of the inclusion does not depend on the angle of emergence.

Case II. *Half-space—elastic layer—cavity*. The unknowns A_n, C_n^1, and D_n^1 have to be determined. At $r = a_0$ displacement and stress must be continuous. The first two equations of the general system (2.8) apply. At $r = a_1$ the stress σ_{rz} vanishes which gives

$$J_n'(k^1a_1)C_n^1 + Y_n'(k^1a_1)D_n^1 = 0, \tag{3.5}$$

and therefrom the solution is

$$\begin{aligned} \Delta_n &= \frac{\mu^1k^1}{\mu^0k^0}H_n(k^0a_0)B_n^{(13)} - H_n'(k^0a_0)B_n^{(14)}, \\ \Delta_n^{(1)} &= \varepsilon_n i^n \cos n\gamma\left[-\frac{\mu^1k^1}{\mu^0k^0}J_n(k^0a_0)B_n^{(13)} + J_n'(k^0a_0)B_n^{(14)}\right], \\ \Delta_n^{(2)} &= B_n^{(7)}Y_n'(k^1a_1), \\ \Delta_n^{(3)} &= -B_n^{(7)}J_n'(k^1a_1) \end{aligned}$$

with

$$B_n^{(13)} = J_n'(k^1a_0)Y_n'(k^1a_1) - Y_n'(k^1a_0)J_n'(ik^1a_1),$$
$$B_n^{(14)} = J_n(k^1a_0)Y_n'(k^1a_1) - Y_n(k^1a_0)J_n'(k^1a_1).$$

Case III. *Half-space—rigid layer—elastic inclusion*. The rigid body displacement of the layer is designated by

$$w^1 = -2WC_0^1. \tag{3.6}$$

It moves according to Newton's second law

$$\varrho^1\frac{\pi}{2}(a_0^2 - a_1^2)\ddot{w}^1 = a_0\int_0^\pi \sigma_{12}^0(a_0)d\theta - a_1\int_0^\pi \sigma_{rz}^2(a_1)d\theta. \tag{3.7}$$

Unknown are A_n, C_0^1, and C_n^2. Continuity of displacement at $r = a_0$ and $r = a_1$ means that $w^0(a_0)$ and $w^2(a_1)$ do not depend on θ.

Therefore

$$A_n = -\varepsilon_n i^n \cos n\gamma \frac{J_n(k^0a_0)}{H_n(k^0a_0)}, \quad C_n^2 = 0, \quad n = 1, 2, 3 \ldots$$

The complete system of equations for $n = 0$ is

$$\begin{aligned}
&H_0(k^0a_0)A_0 + && C_0^1 && = -J_0(k^0a_0),\\
&H_0'(k^0a_0)A_0 - \frac{1}{2}\frac{\varrho^1}{\varrho^0}k^0\frac{a_0^2-a_1^2}{a_0} && C_0^1 + \frac{\mu^2k^2}{\mu^0k^0}\frac{a_1}{a_0}J_0'(k^2a_1)C_0^2 && = -J_n'(k^0a_0),\\
& && C_0^1 - \qquad J_0(k^2a_1)C_0^2 && = 0
\end{aligned} \tag{3.8}$$

with the solution

$$\begin{aligned}
\Delta_0 &= H_0(k^0a_0)B_0^{(15)} - H_0'(k^0a_0)J_0(k^2a_1),\\
\Delta_0^{(1)} &= -J_0(k^0a_0)B_0^{(15)} + J_0'(k^0a_0)J_0(k^2a_1),\\
\Delta_0^{(2)} &= B_0^{(7)}J_0(k^2a_1),\\
\Delta_0^{(4)} &= B_0^{(7)}
\end{aligned}$$

where

$$B_0^{(15)} = \frac{\mu^2k^2}{\mu^0k^0}\frac{a_1}{a_0}J_0'(k^2a_1) - \frac{1}{2}\frac{\varrho^1}{\varrho^0}k^0\frac{a_0^2-a_1^2}{a_0}J_0(k^2a_1).$$

Neither the motion of the rigid layer nor the motion of the elastic inclusion is influenced by the angle of emergence. The displacement of the inclusion exhibits the pattern of a standing wave with nodal cylinders. This fact is of interest to earthquake engineering, because structures centered at such a nodal line are excited into torsional oscillations [1].

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