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Algebraic properties of nonhomogeneous equations of magnetohydrodynamics in the presence of gravitational and Coriolis forces. Examples of solutions — simple states

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THE PURPOSE of this paper is an algebraic analysis of M.H.D. equations and a physical interpretation of some classes of solutions called simple states. An algebraisation of M.H.D. equations according to [6] is made. All considerations are made at a fixed point of a hodograph space. Simple homogeneous and nonhomogeneous elements being the tangent mapping of simple waves (for a basical homogeneous system) and simple states (for a nonhomogeneous system) respectively were found. Then algebraic properties of simple homogeneous and nonhomogeneous elements are analysed. On this base, the dimension of a tensor space generated by simple elements is determined. It allows to classify the type of M.H.D. equations. At the end of the paper a physical interpretation of solutions is made.

Celem niniejszej pracy jest algebraiczna analiza równań M.H.D. a następnie interpretacja fizyczna pewnych klas rozwiązań, zwanych stanami prostymi. Dokonuje się algebraizacji równań M.H.D. zgodnie z pracą [6]. Wszystkie rozważania są przeprowadzane w ustalonym punkcie przestrzeni hodografu. Poszukuje się elementów prostych jednorodnych i niejednorodnych, będących odwzorowaniami stycznymi odpowiednio — fal prostych (dla wyjściowego układu jednorodnego) i stanów prostych (dla układu niejednorodnego). Z kolei analizuje się własności algebraiczne elementów prostych jednorodnych i niejednorodnych. Na tej poddstawie wyznacza się wymiar przestrzeni tensorowej generowanej przez obliczone elementy proste, co pozwala sklasyfikować typ równań M.H.D. Na koniec dokonuje się pewnej analizy fizycznej otrzymanych rozwiązań.

Целю этой работы является алгебраический анализ уравнений магнетогидродинамики а затем физическая интерпретация некоторых класс решений — так называемых простых состояний. Производится алгебраизация уравнений м.г.д. согласно работе [6]. Все рассуждения проводятся в фиксированной точке пространства годографа. Исследуется простые однородные и неоднородные элементы, являющиеся касательными отображениями соответственно—простых волн (для исходной однородной системы) и простых состояний (для неоднородной системы). Затем анализируется алгебраические свойства простых однородных и неоднородных элементов. На этой основе определяется размер тенсорного пространства, ґенерированного вычисленными простыми элементами, что позваляет произвести классификацию уравнений м.г.д. На конец проводится некоторый физический анализ полученных решений.

Notations

$\mathscr{E}$ physical space,
$\mathscr{H}$ hodograph space,
$\mathscr{K}$ vector space of solutions of the homogeneous system,
$\mathscr{L}$ hyperplane of solutions of the nonhomogeneous system,
Q linear subspace,
u coordinates of $\mathscr{H}$,
γ characteristic vector from $\mathscr{H}$,
$\tilde{\gamma}, \underset{N}{\gamma}$ noncharacteristic vector from $\mathscr{H}$,

$x = (t, \bar{x})$	coordinates of $\mathscr{E}$
λ	characteristic covector from $\mathscr{E}^*$,
$\underset{N}{\lambda}$	noncharacteristic covector from $\mathscr{E}^*$,
R, s	Riemann invariants,
$\delta = (\lambda_0 + \bar{v}\bar{\lambda})$	velocity of wave and state regard to a moving media,
ϱ	density of fluid,
p	pressure of fluid,
$\bar{v}$	velocity of fluid,
$\bar{g}$	gravitation field,
$\bar{E}$	electric field,
$\bar{H}$	magnetic field,
$\bar{j}$	electric current,
q	electrical charge density,
σ	electrical conductivity,
φ	gravitational potential,
k	gravitational constant,
$\bar{\Omega}$	angular velocity of fluid,
$\bar{A}$	direction of propagation of the state.

1. Basical equations

IN THIS paper we will deal with an analysis of nonhomogeneous equations of magnetohydrodynamics from the point of view of a generalised Riemmann invariants method described in papers [1–7, 10–14]. After an algebraisation of those equations (Sec. 2) we will consider whether they admit existence of Riemann invariants. Then we will construct the simplest solutions i.e. simple states (Sec. 5).

We will consider the classical equations of magnetohydrodynamics describing a movement of a fluid conducting medium and placed in a magnetic field in a presence of gravitational and Coriolis forces. We take into account a one-component nonviscous fluid having a finite electrical and thermal conductivity. Under the above assumptions the investigated equations form a quasi-linear system. In the noninertial system they are of the form:

$$\begin{gathered}
\varrho\left\{\frac{\partial \bar{v}}{\partial t}+(\bar{v}\nabla)\bar{v}\right\}+\nabla p=\bar{j}\times\bar{H}+\varrho\bar{g}-2\varrho\bar{\omega}\times\bar{v},\\
\frac{\partial\varrho}{\partial t}+\operatorname{div}(\varrho\bar{v})=0,\qquad \frac{dp}{dt}-f\frac{d\varrho}{dt}=0,\\
\Delta\varphi=4\pi k\varrho,\qquad \text{where}\quad \bar{g}=-\nabla\varphi,\\
\operatorname{rot}\bar{H}=4\pi\bar{j},\qquad \text{where}\quad \bar{j}=\sigma(\bar{E}+\bar{v}\times\bar{H}),\\
\operatorname{rot}\bar{E}=-\frac{\partial\bar{H}}{\partial t},\qquad \operatorname{div}\bar{H}=0\qquad \operatorname{div}\bar{E}=4\pi q,
\end{gathered}\tag{1.1}$$

where the following notations have been introduced: ϱ — density of fluid, p — pressure of the fluid, $\bar{g}$ — gravitational field, $\bar{v}$ — velocity of the fluid, $\bar{E}$ — electric field, $\bar{H}$ — magnetic field, $\bar{j}$ — density of the electric current, φ — gravitational potential, $\bar{\omega}$ — angular velocity of the fluid, q — electrical charge density, σ — electrical conductivity, k — gravitational constant. A substantial derivative will be denoted $\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{v}\nabla)$.

In order to get (1.1) we have coupled together:

electrodynamical equations,

gravitational field equations,

and hydrodynamical equations with the presence of gravitational and Coriolis forces. We reduce the Poisson equation (1.1d) to the first order equations:

$$\mathrm{rot}\,\bar{g} = 0, \quad \mathrm{div}\,\bar{g} = 4\pi k\varrho \tag{1.2}$$

to be able to apply the Riemann method [6, 7]. Consequently we obtain a system of 14 equations of the first order with respect to 14 unknown functions. The Eq. (1.1g) is a consequence of Eqs. (1.1e, f) and the Eq. (1.1h) will be treated as an additional condition for the distribution of the electrical charge density.

Following conservation laws correspond to (1.1):

the energy conservation law:

$$\frac{\partial}{\partial t}\left\{\varrho\left(\frac{\bar{v}^2}{2}+U\right)+\frac{\bar{H}^2}{8\pi}\right\}+\mathrm{div}\left\{\varrho\bar{v}\left(\frac{\bar{v}^2}{2}+W\right)+\frac{1}{4\pi}\bar{E}\otimes\bar{H}-\varkappa\nabla T\right\} = \varrho\bar{v}\cdot\bar{g} \tag{1.3}$$

the momentum conservation law:

$$\frac{\partial}{\partial t}(\varrho\bar{v})+\mathrm{div}\left\{(p\hat{\delta}+\varrho\bar{v}\otimes\bar{v})+\frac{1}{4\pi}\left(\frac{1}{2}\hat{\delta}\bar{H}^2-\bar{H}\otimes\bar{H}\right)\right\} = \varrho(\bar{g}-\bar{\Omega}\times\bar{v}). \tag{1.4}$$

Under our assumptions the heat transport equation is of the form:

$$\varrho T\left\{\frac{\partial S}{\partial t}+(\bar{v}\nabla)S\right\} = \mathrm{div}(\varkappa\nabla T)+\frac{(\mathrm{rot}\,\bar{H})^2}{16\pi^2\sigma}, \tag{1.5}$$

where S, U, W are entropy, internal energy and enthalpy of the fluid mass unit respectively and $\varkappa$ is thermal conductivity.

2. Algebraic properties of nonhomogeneous M.H.D. equations

2.1. Simple elements

Basing on literature [1–7, 10–14] concerning the method of Riemann invariants applied to differential equations we accept the following notations. A physical (Euclidean) space $\mathscr{E} = \mathscr{R}^4$ is a classical spacetime. Each point of $\mathscr{E}$ has coordinates $(t, \bar{x})$. The space of unknown functions i.e. the hodograph space is denoted by $\mathscr{H} \subset \mathscr{R}^{14}$. Each point of $\mathscr{H}$ has coordinates $(\varrho, p, \bar{g}, \bar{v}, \bar{E}, \bar{H})$. Points of the dual space $\mathscr{E}^*$ we call covectors and use for them a symbol $\lambda = (\lambda_0, \bar{\lambda})$, where: $\bar{\lambda} \in \mathscr{R}^3$. Points of the tangent space $T\mathscr{H}$ are called vectors and they are denoted as $\gamma = (\gamma_\varrho, \gamma_p, \bar{\gamma}_g, \bar{\gamma}, \bar{e}, \bar{h})$ where: $\varrho \rightleftharpoons \gamma_\varrho$, $p \rightleftharpoons \gamma_p$, $\bar{g} \rightleftharpoons \bar{\gamma}_g$ $\bar{v} \rightleftharpoons \bar{\gamma}$, $\bar{E} \rightleftharpoons \bar{e}$, $\bar{H} \rightleftharpoons \bar{h}$. Simple elements for nonhomogeneous equations of magnetohydrodynamics (1.1) are defined by algebraic equations. These equations are of the form:

$$\begin{aligned}
&\varrho\delta|\bar{\lambda}|\bar{\gamma}+\gamma_p\bar{\lambda} = \bar{j}\times\bar{H}+\varrho\bar{g}-\varrho\bar{\Omega}\times\bar{v},\\
&\gamma_\varrho\delta|\bar{\lambda}|+\varrho\bar{\gamma}\cdot\bar{\lambda} = 0, \quad \gamma_p\delta|\bar{\lambda}|-f\gamma_\varrho\delta|\bar{\lambda}| = 0,\\
&\bar{\lambda}\times\bar{\gamma}_g = 0, \quad \bar{\gamma}_g\cdot\bar{\lambda} = 4\pi k\varrho,\\
&\bar{\lambda}\times\bar{h} = 4\pi\bar{j}, \quad \bar{\lambda}\times\bar{e} = -\lambda_0\bar{h}, \quad \bar{h}\cdot\bar{\lambda} = 0,
\end{aligned} \tag{2.1}$$

where the followoing notations are introduced $\bar{\Omega}:=2\bar{\omega}$ and

$$\delta|\bar{\lambda}|:=\lambda_0+\bar{v}\cdot\bar{\lambda} \tag{2.2}$$

The function $\delta|\bar{\lambda}|$ has a physical meaning. It describes the (group) velocity of a disturbance propagation relative to the fluid.

When we use a language of simple elements then the conservation laws corresponding to (2.1) and the equation of heat transport are as follows:

$$\begin{aligned}
\delta|\bar{\lambda}|\left\{\frac{\bar{v}^2}{2}\gamma_\rho+\varrho\bar{v}\cdot\bar{\gamma}\right\}+\lambda_0\left\{U\gamma_\rho+\varrho u+\frac{\bar{H}\bar{h}}{4\pi}\right\}+\bar{v}\cdot\bar{\lambda}\{W\gamma_\rho+\varrho w\}+\varrho\bar{\gamma}\cdot\bar{\lambda}\left\{\frac{\bar{v}^2}{2}+W\right\}\\
+\frac{1}{4\pi}\{\bar{H}\cdot\bar{\lambda}\times\bar{e}-\bar{E}\cdot\bar{\lambda}\times\bar{h}\}-\varkappa\bar{\gamma}_\tau\cdot\bar{\lambda}=\varrho\bar{g}\cdot\bar{v}+\bar{j}^2\sigma^{-1},\\
\delta|\bar{\lambda}|\{\bar{v}\gamma_\rho+\varrho\bar{\gamma}\}+\gamma_p\bar{\lambda}+\varrho\bar{\gamma}\cdot\bar{\lambda}\bar{v}+\frac{1}{4\pi}(\bar{H}\cdot\bar{h}\bar{\lambda}-\bar{H}\cdot\bar{\lambda}\bar{h})=\varrho(\bar{g}-\bar{\Omega}\times\bar{v}),\\
\varrho T\delta|\bar{\lambda}|\gamma_s=\varkappa\bar{\gamma}_\tau\cdot\bar{\lambda}+\bar{j}^2\sigma^{-1},\quad \bar{\lambda}\times\bar{\gamma}_\tau=0.
\end{aligned} \tag{2.3}$$

where:

$$S\rightleftharpoons\gamma_s,\quad U\rightleftharpoons u,\quad W\rightleftharpoons w,\quad \nabla T:=\bar{\tau}\rightleftharpoons\bar{\gamma}_\tau$$

The system of Eqs. (2.1) for the nonstationary case is a system of 15 equations. For the stationary case it is a system of 16 equations as one has to consider in addition the Eq. (2.1h). This equation in the former case was a linearly dependent one. In both cases (2.1) is a system with 14 unknown functions. It is an algebraic system of linear nonhomogeneous equations with respect to a vector γ. According to the Kronecker-Capella theorem a nonzero solution γ of (2.1) exists if and only if the following equations are satisfied:

$$\begin{aligned}
&\delta_{E_{N_1}}=\delta|\bar{\lambda}|=0\text{ — entropic velocity and the condition: }\bar{j}\times\bar{H}+\varrho(\bar{g}-\bar{\Omega}\times\bar{v})=0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\gamma_p=0.\\
&\delta_{E_{N_2}}=\delta|\bar{\lambda}|=0\text{ — entropic velocity and the condition: }\bar{j}\cdot(\bar{g}-\bar{\Omega}\times\bar{v})=0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\gamma_p\neq 0,\\
&\delta_{A_N}=\delta|\bar{\lambda}|=\varepsilon|\bar{\lambda}|\sqrt{f}\text{ — acoustic velocity, where }\varepsilon=\pm 1,\\
&\delta_{M_N}=\delta|\bar{\lambda}|=\begin{cases}\neq 0\\ \neq\varepsilon|\bar{\lambda}|\sqrt{f}\end{cases}\text{ — sub-supersonic velocities.}
\end{aligned} \tag{2.4}$$

According to (2.2) the Eqs. (2.4a) and (2.4b) define nonhomogeneous entropic elements. These elements correspond to the simple entropic states E_{N_1} and E_{N_2} respectively. The Eq. (2.4c) determines a nonhomogeneous acoustic element which corresponds to the simple acoustic state A_N. The Eq. (2.4d) defines a nonhomogeneous magnetohydrodynamic element M_N. From the definition (2.2) it follows that the velocity of the entropic state E_N relatively to the fluid is equal to zero. This state propagates together with the fluid and not relatively to it. The velocity of the propagation of the acoustic state A_N relatively to the medium is equal to the sound velocity: $\sqrt{f}=\sqrt{dp/d\varrho}$. But the magnetohydrodynamical state M_N can propagate relatively to the medium with any speed except for the entropic velocities $\delta_{E_N}=0$ and the acoustic velocities $\delta_{A_N}=\varepsilon|\bar{\lambda}|\sqrt{f}$. The conditions (2.4) determine submanifolds in the hodograph space such that cones of nonhomogeneous simple elements are defined on them. Vectors $\underset{N}{\gamma}$ are elements of subspaces tangent to those submanifolds.

In the further course of our considerations we will deal with an analysis of the homogeneous system (2.1). We shall determine homogeneous simple elements which will enable us later to construct some general classes of solutions. These solutions will be non-linear superpositions of simple waves with the simple states [6, 7]. An analysis of the homogeneous system will also allow us to classify the type of the basic nonhomogeneous system. That will be discussed in the Sec. 4.

The homogeneous system (2.1) has zero solutions with respect to $\bar{h}, \bar{\gamma}_g$. Hence it can be reduced to the following system:

$$(2.5)\qquad \begin{aligned} \varrho\delta|\bar{\lambda}|\bar{\gamma}+\bar{\lambda}\gamma_p &= 0,\\ \gamma_\varrho\delta|\bar{\lambda}|+\varrho\bar{\gamma}\cdot\bar{\lambda} &= 0,\\ \gamma_p\delta|\bar{\lambda}|-f\gamma_\varrho\delta|\bar{\lambda}| &= 0,\\ \bar{e}\times\bar{\lambda} &= 0. \end{aligned}$$

Because of an elliptical character of the homogeneous Poisson equations the gravitational force is treated here as a nonconservative force. It is a formal procedure enforced by the Riemann invariants method applied here (we recall that the Laplace equation has no simple elements). The case of the magnetic field $\bar{h}$ is quite analogous. The system (2.5) is a system of 8 equations with 8 unknown functions. It is a linear homogeneous system with respect to the vector γ. Consequently the nonzero solution γ exists if and only if the characteristic determinant of the system vanishes:

$$(2.6)\qquad \delta^4|\bar{\lambda}|^4(\delta^2|\bar{\lambda}|^2-f\bar{\lambda}^2) = 0.$$

The Eq. (2.6) has two kinds of solutions for the function $\delta|\bar{\lambda}|$. They are:

$$(2.7)\qquad \begin{aligned} \delta_E &= \delta|\bar{\lambda}| = 0 \text{ — entropic velocity},\\ \delta_A &= \delta|\bar{\lambda}| = \varepsilon|\bar{\lambda}|\sqrt{f} \text{ — acoustic velocity}. \end{aligned}$$

The Eq. (2.7a) defines homogeneous entropic elements which correspond to the simple entropic waves E. The Eq. (2.7b) determines homogeneous acoustic elements corresponding in turn to the simple acoustic waves A.

It is worthwhile to remark that the homogeneous equations (2.6) written in the language of simple elements don't allow the Alfvén waves which can be observed experimentally. It is known that the Alfvén waves correspond to integral elements of the higher order [9]. More detailed physical interpretation of velocities, simple states and simple waves will be given in the Sec. 5.

Simple homogeneous and nonhomogeneous elements will be presented explicite in the following paragraphs of the Sec. 2. Afterwards their properties and relations between them will be described. Finally a classification of the equations (2.1) based on an analysis of these elements will be given.

Some denotations and definitions useful for further considerations will be introduced. Simple elements will be denoted as follows: $\gamma\otimes\lambda$ for the homogeneous system and $\underset{N}{\gamma}\otimes\underset{N}{\lambda}$

for the nonhomogeneous system. Let $C(\gamma(u))$ denote a cone of vectors from the hodograph space fixed in a point $u_0 = (\varrho, p, \bar{g}, \bar{v}, \bar{E}, \bar{H}) \in \mathscr{H}$. A bundle of cones will be introduced as follows:

$$(2.8) \quad C(\gamma) = (C(\gamma), \pi, \mathscr{H}) \quad \text{where} \quad C(\gamma) = \bigcup_{u \in \mathscr{H}} C(\gamma(u)) \quad \text{and} \quad \pi: C(\gamma) \to \mathscr{H}$$

A bundle of cones of covectors $C(\lambda(u)) \subset \mathscr{E}^*$ can be introduced similarly

$$(2.9) \quad C(\lambda) = (C(\lambda), \tilde{\pi}, \mathscr{H}) \quad \text{where} \quad C(\lambda) = \bigcup_{u \in \mathscr{H}} C(\lambda(u)) \quad \text{and} \quad \tilde{\pi}: C(\lambda) \to \mathscr{H}$$

We will use different kinds of cones $C(\alpha)$ as α will vary over $E, A, E_{N_1}, \ldots$ etc. $C^k(\gamma(u))$ is a map assigning to each vector $\gamma \in C(\gamma(u))$ a system of k linearly independent covectors $\lambda \in C(\lambda(u))$. Similarly, $C^p(\lambda(u))$ is a map assigning to each covector $\lambda \in C(\lambda(u))$ a system of p linearly independent vectors $\gamma \in C(\gamma(u))$. By Q_1 we will denote the linear space generated by all homogeneous simple elements i.e. $Q_1 = \{\gamma_k \otimes \lambda^k\}$ and by $\mathscr{L}_1$ we will denote the hyperplane spanned by nonhomogeneous simple elements of the form:

$$(2.10) \qquad \mathscr{L}_1 = \sum_{s=1}^{m} \mu^s \underset{N}{\gamma_s} \otimes \underset{N}{\lambda^s} \quad \text{where:} \quad \sum_{s=1}^{m} \mu^s = 1$$

$Q_1(\alpha)$ and $\mathscr{L}_1(\alpha)$ will denote the space and hyperplane respectively generated by simple elements of the type α(e.g., E, A, E_N and so on).

Simple elements correspond to the previously introduced velocities (Eqs. (2.4), (2.7)). Those elements can be obtained from Eqs. (2.1) or from (2.5). All considerations concerning the simple elements will be made in a fixed point of the hodograph space i.e. for fixed $u_0 = (\varrho, p, \bar{g}, \bar{v}, \bar{E}, \bar{H})$.

2.2. The homogeneous entropic elements E

We obtain the following system of algebraic equations which defines simple homogeneous elements corresponding to entropic velocities E using the condition (2.7a) in the equations (2.5):

$$(2.11) \qquad \gamma_\varrho \text{ — arbitrary function, } \gamma_p = 0,\ \bar{\gamma}_g \equiv 0,\ \bar{\gamma} \cdot \bar{\lambda} = 0,\ \bar{e} \times \bar{\lambda} = 0,\ \bar{h} \equiv 0.$$

Solving the above system we obtain three kinds of solutions:

$$(2.12) \qquad \gamma_{E_1} = (\gamma_\varrho, 0, \bar{0}, \bar{\gamma}, \bar{e}, \bar{0}), \quad \lambda_{E_1} = (-\bar{v} \cdot \bar{\lambda}, \bar{\lambda}),$$

where $\bar{\gamma} \cdot \bar{\lambda} = 0$, $\bar{e} \times \bar{\lambda} = 0$,

$$(2.13) \qquad \gamma_{E_2} = (\gamma_\varrho, 0, \bar{0}, \bar{\gamma}, \bar{0}, \bar{0}), \quad \underset{i}{\lambda_{E_2}} = (-\bar{v}\bar{\lambda}_i, \bar{\lambda}_i), \quad i = 1, 2$$

where $\bar{\gamma} \times (\bar{\lambda}_1 \times \bar{\lambda}_2) = 0$, $\bar{e} = 0$

$$(2.14) \qquad \gamma_{E_3} = (\gamma_\varrho, 0, \bar{0}, \bar{0}, \bar{0}, \bar{0}), \quad \lambda_{E_3} = (-\bar{v}\bar{\lambda}, \bar{\lambda}).$$

Thus on the basis of (2.12) we confirm that the covector λ is a vector lying in the plane perpendicular to the vector $\bar{\gamma}$ and parallel to the vector $\bar{e}$. It follows additionally from the expression (2.12) that the cone of the covectors $C(\lambda(E_1))$ is generated by the vector

$(-\bar{v}\cdot\bar{\lambda}, \bar{\lambda})$ and is a three-dimensional hyperplane including the zero of the space $\mathscr{E}^*$. This hyperplane is inclined to the axis λ_0 at an angle φ such that:

$$\cos\varphi = -\sqrt{\frac{\bar{v}^2}{1+\bar{v}^2}}$$

and it crosses the plane $\lambda_0 = 0$ along the line perpendicular to the vector $\bar{v}$. It follows from above considerations that if $\bar{v}^2$ decreases, then the cone $C(\lambda(E_1))$ deflectes from the axis λ_0 and if $\bar{v}^2$ increases — then the obtuseness of the cone decreases.

The cone of the characteristic vectors $C(\gamma(E_1))$ from the space $T\mathscr{H}$ is determined by conditions:

(2.15) γ_ϱ — arbitrary function, $\gamma_p = 0$, $\gamma_g \equiv 0$, $\bar{\gamma} = \bar{\beta}\times\bar{\lambda}$, $\bar{e} = \alpha\bar{\lambda}$, $\bar{h} \equiv 0$

where $\bar{\beta}$ — arbitrary vector, α — arbitrary function.

Characteristic vectors from the space $T\mathscr{H}$ for the homogeneous system (2.5) have not (as was previously mentioned) any components in the subspace $(\bar{h}, \bar{\gamma}_g)$. Thus for the fixed $\bar{\lambda}$, $C(\gamma(E_1))$ is a four-dimensional hyperplane in the eight-dimensional space $T\mathscr{H}$. In the case (2.12) it can easily by checked that we have $C^1(\gamma(E_1))$, because to each covector $\lambda \in C(\lambda(E_1))$ corresponds one linearly independent vector $\gamma \in C(\gamma(E_1))$ (by the fact that $\bar{e}||\bar{\lambda}$). However we have $C^2(\lambda(E_1))$ as well as $C^2(\bar{\lambda}(E_1))$ because there exist two linearly independent characteristic covectors $\lambda \in C(\lambda(E_1))$ for the fixed vector $\gamma \in C(\gamma(E_1))$.

We will investigate now the dimension of the tensor subspace generated by homogeneous simple entropic elements $E_1 : Q_1(E_1) = \{\gamma_{E_1}\otimes\lambda_{E_1}\}$. The part of the information which says what part of the whole space of the integral simple elements does this subspace constitute will be essential for us (see Sec. 4). The dimension of the space $Q_1(E_1)$ is determined only by elements of the form:

$$(\gamma_\varrho, \bar{\gamma}, \alpha\,\bar{\lambda})\otimes\bar{\lambda}, \quad \bar{\gamma}\cdot\bar{\lambda} = 0 \tag{2.16}$$

because the expressions:

$$(\gamma_\varrho, \bar{\gamma}, \alpha\bar{\lambda})\,(-\bar{v}\cdot\bar{\lambda}) = -v_i\,\{(\gamma_\varrho, \bar{\gamma}, \alpha\bar{\lambda})\lambda_i\},\ \bar{\gamma}\cdot\bar{\lambda} = 0 \tag{2.17}$$

are linearly dependent on (2.16). The term $(-\bar{v}\cdot\bar{\lambda})$ determines the inclination of linear subspaces only. It can easily be shown that dimension of the space generated by the elements of the form (2.16) is 17, thus: $\dim Q_1(E_1) = 17$.

In the case (2.13) the homogeneous simple element E_2 is generated by the vectors:

$$\gamma = (\gamma_\varrho, 0, \bar{0}, \varkappa\bar{\lambda}_1\times\bar{\lambda}_2, \bar{0}, \bar{0}), \quad \lambda_i = (-\bar{v}\bar{\lambda}_i, \bar{\lambda}_i), \quad i = 1, 2 \tag{2.18}$$

where $\varkappa$ is an arbitrary function.

The cones $C(\lambda(E_2))$ are of the same form as in the case E_1, and $C(\gamma(E_2))$ is a plane generated by two linearly independent vectors: $(1, 0, \bar{0}, \bar{0}, \bar{0}, \bar{0})$, $(0, 0, \bar{0}, \bar{\lambda}_1\times\bar{\lambda}_2, \bar{0}, \bar{0})$. In that case we have $C^2(\gamma(E_2))$ and $C^2(\lambda(E_2))$ as well as $C^2(\bar{\lambda}(E_2))$ and

$$\dim Q_1(E_2) = 9. \tag{2.19}$$

Let us consider now the case (2.14). The cone $C(\lambda(E_3))$ is generated by the vectors $(-\bar{v}\cdot\bar{\lambda}, \bar{\lambda})$, where $\bar{\lambda}$ is an arbitrary vector. So $C(\lambda(E_3))$ is a three-dimensional linear space generated by the vectors $(-v_i, e_j\delta_{ij})$, $i = 1, 2, 3$ where e_i is an i-versor of the orthogonal canonical base in the space $\mathscr{R}^3$. In this case the cone of characteristic vectors $C(\gamma(E_3))$

from the space $T\mathscr{H}$ is a line of the direction independent of the hodograph space point $u \in \mathscr{H}$. As it follows from the expression (2.14) we have here $C^3(\gamma(E_3))$ and $C^1(\lambda(E_3))$ as well as $C^1(\bar{\lambda}(E_3))$. The dimension of the tensor space generated by homogeneous entropic elements E_3 is $\dim Q_1(E_3) = 3$.

Moreover it holds the following relations between entropic elements:

$$C(\gamma(E_3)) \subset C(\gamma(E_2)) \subset C(\gamma(E_1)) \quad \text{and} \quad C(\lambda(E_1)) = C(\lambda(E_2)) \subset C(\lambda(E_3)). \tag{2.20}$$

Recapitulating the obtained results it can easily be stated, that the dimension of the tensor space generated by all homogeneous entropic elements E is $\dim Q_1(E) = 17$.

2.3. The homogeneous acoustic elements A

Adjoining the condition (2.7b) in the equations (2.5) we obtain the following algebraic equation system which defines homogeneous simple acoustic elements A:

$$\begin{aligned} &\varepsilon\varrho\sqrt{f}\,|\bar{\lambda}|\bar{\gamma}+\gamma_p\bar{\lambda} = 0, \\ &\varepsilon\sqrt{f}\,|\bar{\lambda}|\gamma_\varrho+\varrho\bar{\gamma}\cdot\bar{\lambda} = 0, \\ &\gamma_p - f\gamma_\varrho = 0, \quad \bar{e}\times\bar{\lambda} = 0, \quad \bar{\gamma}_g \equiv 0, \quad \bar{h} \equiv 0. \end{aligned} \tag{2.21}$$

We obtain the following form of characteristic simple acoustic elements from the equations (2.21):

$$\gamma_A = \left(\gamma_\varrho, f\gamma_\varrho, \bar{0}, -\varepsilon\sqrt{f}\,\frac{\gamma_\varrho}{\varrho}\,\frac{\bar{\lambda}}{|\bar{\lambda}|}, \bar{e}, \bar{0}\right), \quad \lambda_A = \left(\varepsilon|\bar{\lambda}|\sqrt{f}-\bar{v}\cdot\bar{\lambda}, \bar{\lambda}\right), \tag{2.22}$$

where $\bar{e}\times\bar{\lambda} = 0$, $\varepsilon = \pm 1$.

Also the characteristic vectors from the space $T\mathscr{H}$ have no components in the subspace $(\bar{h}, \bar{\gamma}_g)$.

It follows from the equations (2.22) that the covector $\bar{\lambda}$ is parallel to the vector $\bar{e}$. Also it is obvious that the covector cone $C(\lambda(A))$ is generated by the vectors $(\varepsilon|\bar{\lambda}|\sqrt{f}-\bar{v}\bar{\lambda}, \bar{\lambda})$. It follows from the covector form λ that $C(\lambda(A))$ is a three-dimensional cone in the four-dimensional space. The intersection of that cone with the hyperplane: $\lambda_0 = \varepsilon|\bar{\lambda}|\sqrt{f}-\bar{v}\cdot\bar{\lambda} = \text{const}$ yields the ellipsoidal surface. It follows from the above considerations that if $\varepsilon > 0$ and $\bar{v}\cdot\bar{\lambda} < 0$ the generator of the cone is inclined to the cone axis at the minimal angle and if $\bar{\lambda}\|\bar{v}$ the obtuseness angle of the cone becomes maximal. Namely, we have two kinds of cones: $C(\lambda(A^+))$ and $C(\lambda(A^-))$ and the following relation holds:

$$C(\lambda(A^+)) \cap C(\lambda(A^-)) = C(\lambda(E_1)).$$

The cone of the characteristic vectors $C(\gamma(A^\varepsilon))$ from the space $T\mathscr{H}$ is determined by the condition (2.22) from which it follows that here — as in the above case — we have two kinds of cones: $C(\gamma(A^+))$ and $C(\gamma(A^-))$ connected by the relation:

$$\lim_{f\to 0}\{C(\gamma(A^+))\cap C(\gamma(A^-))\} \subset C(\gamma(E_1)).$$

The cone $C(\gamma(A^\varepsilon))$ is spanned on the vectors:

$$\left(1, f, \bar{0}, -\varepsilon\frac{\sqrt{f}}{\varrho}\,\frac{\bar{\lambda}}{|\bar{\lambda}|}, \bar{0}, \bar{0}\right), \quad (0, 0, \bar{0}, \bar{0}, \bar{\lambda}, \bar{0}).$$

So it forms the two-dimensional hyperplane in the space $T\mathscr{H}$ (for the fixed λ). In the case of the homogeneous acoustic elements (2.22) it can be easily stated that we have $C^1(\gamma(A))$ and $C^2(\bar{\lambda}(A))$ as well as $C^2(\lambda(A))$. The equations (2.21) do not allow any non-planar simple acoustic waves A as in case of hydrodynamical equations [13–14]. It can be shown that the dimension of tensor space generated by homogeneous elements $\gamma_A \otimes \lambda_A$ is equal to 17, so $\dim Q_1(A) = 17$.

2.4. The nonhomogeneous simple entropic elements E_N

It follows from (2.4a, b) for the basic system that we obtain two kinds of simple entropic elements E_{N_1}, E_{N_2}. Using the condition (2.4a) in equations (2.1) we obtain the following system of the algebraic equations which defines nonhomogeneous simple entropic elements E_{N_1}:

$$\begin{gathered}\gamma_p \bar{\lambda} - \bar{j} \times \bar{H} - \varrho \bar{g} + \varrho \bar{\Omega} \times \bar{v} = 0,\\ \bar{\gamma} \cdot \bar{\lambda} = 0, \quad \bar{\lambda} \times \bar{\gamma}_g = 0, \quad \bar{\gamma}_g \cdot \bar{\lambda} = 4\pi k \varrho,\\ \bar{\lambda} \times \bar{h} = 4\pi \bar{j}, \quad \bar{\lambda} \times \bar{e} = -\lambda_0 \bar{h}, \quad \bar{h} \cdot \bar{\lambda} = 0,\end{gathered} \tag{2.23}$$

and the condition

$$\bar{j} \times \bar{H} + \varrho(\bar{g} - \bar{\Omega} \times \bar{v}) = 0, \quad \gamma_p = 0. \tag{2.24}$$

Solving the above system we obtain the following form of the elements E_{N_1}:

$$\begin{gathered}\gamma_{E_{N_1}} = \left\{\gamma_\rho, 0, \frac{4\pi k \varrho}{(\bar{\alpha} \times \bar{j})^2} \bar{\alpha} \times \bar{j}, \bar{\beta} \times (\bar{\alpha} \times \bar{j}), a\bar{\alpha} \times \bar{j} + \frac{4\pi \lambda_0}{(\alpha \times j)^2} \bar{j}, \frac{-4\pi}{(\bar{\alpha} \times \bar{j})^2} (\bar{\alpha} \times \bar{j}) \times \bar{j}\right\},\\ \lambda_{E_{N_1}} = \{-\bar{v} \cdot \bar{\alpha} \times \bar{j}, \bar{\alpha} \times \bar{j}\}.\end{gathered} \tag{2.25}$$

The vectors $\gamma_{E_{N_1}}$ from the space $T\mathscr{H}$ for the nonhomogeneous system (2.23) must lay in the subspace $T\mathfrak{M}_1 \subset T\mathscr{H}$ which is tangent to the submanifold $\mathfrak{M}_1$ defined by the Kronecker-Capella theorem where:

$$\mathfrak{M}_1 = \{u \in \mathscr{H} : \bar{j} \times \bar{H} + \varrho(\bar{g} - \bar{\Omega} \times \bar{v}) = 0\}. \tag{2.26}$$

It follows from the system (2.23) that the covector $\bar{\lambda}$ is a vector laying in the plane perpendicular to the vector $\bar{\gamma}$ and parallel to the vector $\bar{\gamma}_g$. As it is easy to see from the expression (2.25) the cone of covectors $C(\lambda(E_{N_1}))$ constitutes a plane in the four-dimensional space including the zero of the space $\mathscr{E}^*$. This plane is inclined to the axis λ_0 at an angle φ such that:

$$\cos\varphi = -\sqrt{\frac{\bar{v}^2}{1+\bar{v}^2}}$$

and intersects the plane $\lambda_0 = 0$ along the line perpendicular to the vector $\bar{v}$. It follows from the above that if $\bar{v}^2$ decreases then the generator of the cone $C(\lambda(E_{N_1}))$ deflectes from the axis λ_0 and if $\bar{v}^2$ increases then the obtuseness angle of the cone decreases.

Considering the expressions (2.25) for nonhomogeneous simple elements together with the expression (2.12) for the homogeneous simple elements E_{N_1} we can state that the covector $\lambda_{E_{N_1}}$ is a characteristic covector. Therefore according to considerations in paper [7] the vector $\gamma_{E_{N_1}}$ can be represented as the sum of the characteristic vector γ_{E_1} and noncharacteristic vector $\tilde{\gamma}_{E_{N_1}}$, namely:

$$(2.27)\qquad \gamma_{E_{N_1}} = \gamma_{E_1}\Big|_{\bar{\lambda}=\bar{\alpha}\times\bar{j}} + \tilde{\gamma}_{E_{N_1}} = \begin{bmatrix} \gamma_\rho \\ 0 \\ \bar{0} \\ \bar{\beta}\times(\bar{\alpha}\times\bar{j}) \\ a\bar{\alpha}\times\bar{j} \\ \bar{0} \end{bmatrix} + \frac{4\pi}{(\bar{\alpha}\times\bar{j})^2}\begin{bmatrix} 0 \\ 0 \\ k\varrho\bar{\alpha}\times\bar{j} \\ \bar{0} \\ -\bar{v}\cdot\bar{\alpha}\times\bar{j}\,\bar{j} \\ -(\bar{\alpha}\times\bar{j})\times\bar{j} \end{bmatrix}.$$

This form of $\gamma_{E_{N_1}}$ becomes important when we construct solutions for (2.23). It leads to the weaken conditions of the integrability, see [7]. Consequently it leads to solutions containing arbitrary functions. For the fixed $\bar{\lambda}$, the cone of vectors $C(\gamma(E_{N_1}^{k}))$ is a four-dimensional hyperplane spanned by the vectors:

$$(1, 0, \bar{0}, \bar{0}, \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{e}_1\times(\bar{\alpha}\times\bar{j}), \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{e}_2\times(\bar{\alpha}\times\bar{j}), \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{0}, \bar{\alpha}\times\bar{j}, \bar{0}),$$

where $\bar{e}_i$ are versors satisfying $\langle\bar{e}_i, \bar{\alpha}\times\bar{j}\rangle = 0$, $i = 1, 2$ in $T\mathscr{H}$. It can be easily checked that γ_{E_1} and $\tilde{\gamma}_{E_{N_1}}$ are orthogonal $\langle\gamma_{E_1}, \tilde{\gamma}_{E_{N_1}}\rangle = 0$. Thus the hyperplane $C(\gamma(E_{N_1}))$ is shifted from the zero of the coordinate system by the segment:

$$|\tilde{\gamma}_{E_{N_1}}| = \frac{-4\pi}{(\bar{\alpha}\times\bar{j})^2}|(-k\varrho(\bar{\alpha}\times\bar{j}), \bar{v}\cdot\bar{\alpha}\times\bar{j}\,\bar{j}, (\bar{\alpha}\times\bar{j})\times\bar{j})|.$$

The form of $\gamma_{E_{N_1}}$ given by (2.27) shows that the formulae (2.25) induces $C^4(\lambda(E_{N_1}))$ as well as $C^4(\bar{\lambda}(E_{N_1}))$ and $C^1(\gamma(E_{N_1}))$. It follows from the character of the Poisson equation (2.23) we have accepted here. In consequence it excludes some kind of solutions(1). To end with the case of E_{N_1} we will derive dimension of the hyperplane $\mathscr{L}_1(E_{N_1}) = \{\gamma_{E_{N_1}}\otimes\lambda_{E_{N_1}}\}$ generated by nonhomogeneous simple entropic elements E_{N_1}. This dimension is determined only by elements of the form:

$$(2.28)\qquad \gamma_{E_{N_1}}\otimes(\bar{\alpha}\times\bar{j})$$

because the expressions:

$$(2.29)\qquad \gamma_{E_{N_1}}(-\bar{v}\cdot\bar{\alpha}\times\bar{j}) = -v_i\{\gamma_{E_{N_1}}(\bar{\alpha}\times\bar{j})_i\}$$

are linearly dependent on (2.28). The term $\{-\bar{v}\cdot\bar{\alpha}\times\bar{j}\}$ determines an inclination of the linear subspace only. It can be easily shown that the dimension of the hyperplane $\mathscr{L}_1(E_{N_1})$ generated by elements (2.25) is:

$$(2.30)\qquad \dim\mathscr{L}_1(E_{N_1}) = 12.$$

(1) We exclude here the solutions describing nonplanar simple wave propagating in the nonhomogeneous medium with anisotropy in direction of the noncharacteristic covector $\underset{N}{\lambda}$. These remarks concern also cases of simple elements we are going to consider later on.

Let us also notice that (2.25) and (2.12) yield:

$$\dim \mathscr{L}_1(E_{N_1}) = \dim Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\alpha}\times\bar{j}\\ \mathfrak{M}_1}}\Big) + \dim \mathscr{L}_1(\tilde{E}_{N_1}), \tag{2.31}$$

where

$$Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\alpha}\times\bar{j}\\ \mathfrak{M}_1}}\Big) = \Big\{\gamma_{E_1}\Big|_{\substack{\bar{\lambda}=\bar{\alpha}\times\bar{j}\\ \mathfrak{M}_1}} \otimes \lambda_{E_{N_1}}\Big\}, \qquad \mathscr{L}_1(E_{N_1}) = \{\tilde{\gamma}_{E_{N_1}} \otimes \lambda_{E_{N_1}}\}.$$

As the cone $C(\tilde{\gamma}(E_{N_1}))$ is a line in the space $T\mathscr{H}$ with a directional vector:

$$(0, 0, k\varrho\bar{\alpha}\times\bar{j}, \bar{0}, -\bar{v}\cdot\bar{\alpha}\times\bar{j}\bar{j}, \bar{j}\times(\bar{\alpha}\times\bar{j})), \quad \text{so} \quad \dim \mathscr{L}_1(\tilde{E}_{N_1}) = 3.$$

Therefore $\dim Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\alpha}\times\bar{j}\\ \mathfrak{M}_1}}\Big) = 9$.

The Eqs. (2.1) together with the condition (2.4b) gives us a system of algebraic equations defining nonhomogeneous simple entropic elements E_{N_2}:

$$\begin{gathered} \gamma_p\bar{\lambda} - \bar{j}\times\bar{H} - \varrho\bar{g} + \varrho\bar{\Omega}\times\bar{v} = 0, \\ \bar{\gamma}\cdot\bar{\lambda} = 0, \quad \bar{h}\cdot\bar{\lambda} = 0, \quad \bar{\lambda}\times\bar{\gamma}_g = 0 \\ \bar{\gamma}_g\cdot\bar{\lambda} = 4\pi k\varrho, \quad \bar{\lambda}\times\bar{h} = 4\pi j, \quad \bar{\lambda}\times\bar{e} = -\lambda_0\bar{h}, \end{gathered} \tag{2.32}$$

and the condition:

$$\bar{j}\cdot(\bar{g} - \bar{\Omega}\times\bar{v}) = 0. \tag{2.33}$$

When we solve the above system we obtain the following form of elements E_{N_2}:

$$\gamma_{E_{N_2}} = \begin{bmatrix} 1 \\ \gamma_p \\ \dfrac{4\pi k\varrho}{\bar{\chi}^2}\bar{\chi} \\ \bar{\alpha}\times\bar{\chi} \\ a\bar{\chi} - \dfrac{4\pi\bar{v}\cdot\bar{\chi}}{\bar{\chi}^2}\bar{j} \\ \dfrac{-4\pi}{\bar{\chi}^2}\bar{\chi}\times\bar{j} \end{bmatrix}, \tag{2.34}$$

$$\lambda_{E_{N_2}} = (-\bar{\chi}\cdot\bar{v}, \bar{\chi})$$

where notation: $\bar{\chi} = j\times\bar{H} + \varrho(\bar{g} - \bar{\Omega}\times\bar{v})$ is introduced. The Kronecker-Capella condition:

$$\mathfrak{M}_2 = \{u \in \mathscr{H} : \bar{j}\cdot\bar{\chi} = 0\} \tag{2.35}$$

defines the subspace $T\mathfrak{M}_2$ of vectors $\gamma_{E_{N_2}}$.

From (2.32) it follows that $\bar{\lambda}$ is orthogonal to $\bar{\gamma}$ and is parallel to $\bar{\gamma}_g$. The cone of covectors $C(\lambda(E_{N_2}))$ is a line in the four-dimensional space with a directional vector: $\{-\bar{v}\cdot\bar{\chi}, \bar{\chi}\}$.

This line contains the zero of the space $\mathscr{E}^*$. It is inclined to the axis λ_0 at an angle φ such that:

$$\mathrm{tg}\varphi = -\frac{|\bar{\chi}|}{\bar{v}\cdot\bar{\chi}}.$$

In the stationary case i.e. when $\bar{v}\cdot\bar{\chi} = 0$ the angle φ becomes equal to $\pi/2$. But when the covector λ doesn't depend on the point of the physical space i.e. when $\bar{\chi} = \bar{0}$ then the angle φ is equal to zero. (2.34) shows the covector $\lambda_{E_{N_2}}$ is a characteristic covector. Consequently the vector $\gamma_{E_{N_2}}$ can be represented in the form:

$$\gamma_{E_{N_2}} = \gamma_{E_1} + \tilde{\gamma}_{E_{N_2}} = \begin{bmatrix} \gamma_\rho \\ 0 \\ \bar{0} \\ \bar{\alpha}\times\bar{\chi} \\ \alpha\bar{\chi} \\ \bar{0} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{4\pi k\varrho}{\bar{\chi}^2}\bar{\chi} \\ \bar{0} \\ \frac{-4\pi\bar{v}\cdot\bar{\chi}}{\bar{\chi}^2}\bar{j} \\ \frac{-4\pi}{\bar{\chi}^2}\bar{\chi}\times\bar{j} \end{bmatrix} \tag{2.36}$$

The cone of tangent vectors $C(\gamma(E_{N_2}))$ is defined by the expression (2.36). Hence it is a four-dimensional hyperplane in the fourteen-dimensional space $T\mathscr{H}$ spanned by vectors:

$$(1, 0, \bar{0}, \bar{0}, \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{e}_1\times\bar{\chi}, \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{e}_2\times\bar{\chi}, \bar{0}, \bar{0}),$$
$$(0, 0, \bar{0}, \bar{0}, \bar{\chi}, \bar{0}),$$

where $\bar{e}_i$ are versors such that: $\langle\bar{e}_i, \bar{\chi}\rangle = 0$, $i = 1, 2$. It can be easily checked that $\langle\gamma_{E_1}, \tilde{\gamma}_{E_{N_2}}\rangle = 0$. Therefore the hyperplane $C(\gamma(E_{N_2}))$ is shifted from the zero of the coordinate system by the segment:

$$|\tilde{\gamma}_{E_{N_2}}| = \left|\left(1, \frac{4\pi k\varrho}{\bar{\chi}^2}\bar{\chi}, \frac{-4\pi\bar{v}\cdot\bar{\chi}}{\bar{\chi}^2}\bar{j}, \frac{-4\pi}{\bar{\chi}^2}\bar{\chi}\times\bar{j}\right)\right|$$

The form of $\gamma_{E_{N_2}}$ given by (2.36) shows that the formulae (2.34) induces $C^4(\lambda(E_{N_2}))$ as well as $C^4(\bar{\lambda}(E_{N_2}))$ and $C^1(\gamma(E_{N_2}))$.

The dimension of the hyperplane $\mathscr{L}_1(E_{N_2}) = \{\gamma_{E_{N_2}}\otimes\lambda_{E_{N_2}}\}$ generated by nonhomogeneous simple entropic elements E_{N_2} is determined only by the elements of the form:

$$\gamma_{E_{N_2}}\otimes\bar{\chi}. \tag{2.37}$$

It is so because expressions

$$\gamma_{E_{N_2}}\{-\bar{v}\cdot\bar{\chi}\} = -v_i\{\gamma_{E_{N_2}}\chi_i\}$$

are linearly dependent on (2.37). The term $\{-\bar{v}\cdot\bar{\chi}\}$ determines an inclination of the linear subspace only. It can be easily shown that the dimension of the hyperplane generated by the elements (2.34) is:

$$\dim \mathscr{L}_1(E_{N_2}) = 5. \tag{2.38}$$

Let us mention that (2.34) and (2.12) yield:

$$\dim \mathscr{L}_1(E_{N_2}) = \dim Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\chi}\\ \mathfrak{M}_2}}\Big)+\dim \mathscr{L}_1(\tilde{E}_{N_2}),$$

where

$$Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\chi}\\ \mathfrak{M}_2}}\Big) = \Big\{\gamma_{E_1}\Big|_{\substack{\bar{\lambda}=\bar{\chi}\\ \mathfrak{M}_2}}\otimes\lambda_{E_{N_2}}\Big\},\qquad \mathscr{L}_1(\tilde{E}_{N_2}) = \{\tilde{\gamma}_{E_{N_2}}\otimes\lambda_{E_{N_2}}\}.$$

As the cone $C(\tilde{\gamma}(E_{N_2}))$ is a line in the space $T\mathscr{H}$ with a directional vector:

$$\left(0, 1, \frac{4\pi k\varrho}{\bar{\chi}^2}\bar{\chi}, \bar{0}, \frac{-4\pi\bar{v}\cdot\bar{\chi}}{\bar{\chi}^2}\bar{j}, \frac{-4\pi}{\bar{\chi}^2}\bar{j}\times\bar{\chi}\right)$$

so $\dim \mathscr{L}_1(\tilde{E}_{N_2}) = 1$. Therefore $\dim Q_1\Big(E_1\Big|_{\substack{\bar{\lambda}=\bar{\chi}\\ \mathfrak{M}_2}}\Big) = 4$.

2.5. The nonhomogeneous simple acoustic element A_N.

Including the condition (2.32) in Eqs. (2.1) we obtain the following system of algebraic equations defining nonhomogeneous simple acoustic elements A_N:

$$\begin{aligned}
&\varepsilon|\bar{\lambda}|\varrho\sqrt{f}\bar{\gamma}+\gamma_{\bar{g}}\bar{\lambda}-\bar{j}\times\bar{H}-\varrho(\bar{g}-\bar{\Omega}\times\bar{v}) = 0,\\
&\varepsilon|\bar{\lambda}|\sqrt{f}\gamma_\rho+\varrho\bar{\gamma}\cdot\bar{\lambda} = 0,\\
&\varepsilon|\bar{\lambda}|\sqrt{f}\gamma_p-f\varepsilon|\bar{\lambda}|\sqrt{f}\gamma_\rho = 0,\\
&\bar{\lambda}\times\bar{\gamma}_g = 0,\quad \bar{\gamma}_g\cdot\bar{\lambda} = 4\pi k\varrho,\\
&\bar{\lambda}\times\bar{h} = 4\pi\bar{j},\quad \bar{\lambda}\times\bar{e} = -\lambda_0\bar{h},\quad \bar{h}\cdot\bar{\lambda}=0.
\end{aligned}\tag{2.39}$$

A solution of (2.39) yields the following form of elements A_N:

$$\begin{aligned}
\gamma_{A_N} = \Big(&\gamma_\rho, f\gamma_\rho, \frac{4\pi k\varrho}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times\bar{\chi}, \frac{1}{\varepsilon\varrho\sqrt{f}|\bar{j}\times\bar{\chi}|}(\bar{\chi}-f\gamma_\rho\bar{j}\times\bar{\chi}), a\bar{j}\times\bar{\chi}+\\
&+\frac{4\pi(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi})}{(\bar{j}\times\bar{\chi})^2}\bar{j}, \frac{4\pi}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times(\bar{j}\times\bar{\chi})\Big)
\end{aligned}\tag{2.40}$$

$$\lambda_{A_N} = (\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi}, \bar{j}\times\bar{\chi}),$$

where γ_ρ, a — arbitrary functions. We have introduced here a notation $\bar{\chi} := \{\bar{j}\times\bar{H}+ +\varrho(\bar{g}-\Omega\times\bar{v})\}$. It follows from (2.39) that the covector $\bar{\lambda}$ is parallel to the vector $\bar{\gamma}_g$. The cone of covectors $C(\lambda(A_N))$ is a two-dimensional cone in the four-dimensional space. An intersection of this cone with the hyperplane $\lambda_0 = \varepsilon|\bar{\lambda}|\sqrt{f}-\bar{v}\cdot\bar{\lambda} = \text{const}$ gives us an elipsoidal surface. When $\varepsilon > 0$ and $\bar{v}\cdot\bar{j}\times\bar{\chi} > 0$ then a generator of the cone is inclined to the axis of the cone at the minimal angle. At the second hand when $\bar{j}\times\bar{\chi}\|\bar{v}$ then this angle becomes maximal. As in the case of homogeneous acoustic elements A we have here two kinds of cones: $C(\lambda(A_N^+))$ and $C(\lambda(A_N^-))$. They are related by: $C(\lambda(A_N^+)) \cap \cap C(\lambda(A_N^-)) \subset C(\lambda(E_{N_1}))$. As before the covector λ_{A_N} is a characteristic covector. This enables us to present the vector γ_{A_N} in the form:

$$(2.41)\qquad \gamma_{A_N} = \gamma_A + \tilde{\gamma}_{A_N} = \begin{bmatrix} \gamma_\rho \\ f\gamma_\rho \\ \bar{0} \\ -\varepsilon\sqrt{f}\,\dfrac{\gamma_\rho}{\varrho}\,\dfrac{\bar{j}\times\bar{\chi}}{|\bar{j}\times\bar{\chi}|} \\ a\bar{j}\times\bar{\chi} \\ \bar{0} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{4\pi k\varrho}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times\bar{\chi} \\ \dfrac{\bar{\chi}}{\varepsilon\varrho\sqrt{f}|\bar{j}\times\bar{\chi}|} \\ \dfrac{4\pi(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi})}{(\bar{j}\times\bar{\chi})^2}\bar{j} \\ \dfrac{4\pi}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times(\bar{j}\times\bar{\chi}) \end{bmatrix}.$$

The cone of tangent vectors $C(\gamma(A_N^\varepsilon))$ in $T\mathscr{H}$ is described by conditions (2.40). We have here also two kinds of cones $C(\gamma(A_N^+))$ and $C(\gamma(A_N^-))$. They are related by:

$$\lim_{f\to 0}\{C(\gamma(A_N^+)) \cap C(\gamma(A_N^-))\} \subset C(\gamma(E_{N_1}))$$

It follows from (2.41) that for the fixed $\bar{\lambda}$ the cones $C(\gamma(A_N^\varepsilon))$ are spanned by the vectors:

$$\left(1, f, \bar{0}, \frac{-\sqrt{f}\bar{j}\times\bar{\chi}}{\varepsilon\varrho|\bar{j}\times\bar{\chi}|}, \bar{0}, \bar{0}\right)$$
$$(0, 0, \bar{0}, \bar{0}, \bar{j}\times\bar{\chi}, \bar{0})$$

Consequently they are two-dimensional hyperplanes fixed at the points:

$$\left(0, 0, \frac{4\pi k\varrho}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times\bar{\chi}, \frac{\bar{\chi}}{\varepsilon\varrho\sqrt{f}|\bar{j}\times\bar{\chi}|}, \frac{4\pi(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi})}{(\bar{j}\times\bar{\chi})^2}\bar{j}, \frac{4\pi}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times(\bar{j}\times\bar{\chi})\right)$$

It can be easily checked that $\langle\gamma_A, \tilde{\gamma}_{A_N}\rangle = 0$. So the hyperplane $C(\gamma(A_N))$ is shifted from the zero of the coordinate system by the segment:

$$|\tilde{\gamma}_{A_N}| = \left|\left(\frac{4\pi k\varrho}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times\bar{\chi}, \frac{\bar{\chi}}{\varepsilon\varrho\sqrt{f}|\bar{j}\times\bar{\chi}|}, \frac{4\pi(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi})}{(\bar{j}\times\bar{\chi})^2}\bar{j}, \frac{4\pi}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times(\bar{j}\times\bar{\chi})\right)\right|$$

The form of γ_{A_N} given by (2.41) shows that the formulae (2.40) induces $C^2(\lambda(A_N))$ as well as $C^2(\bar{\lambda}(A_N))$ and $C^1(\gamma(A_N))$. The dimension of the hyperplane $\mathscr{L}_1(A_N)$ generated by nonhomogeneous elements (2.40) is

$$(2.42)\qquad \dim\mathscr{L}_1(A_N) = 15.$$

Let us notice that (2.40) and (2.22) yields

$$\dim\mathscr{L}_1(A_N) = \dim Q_1(A|_{\bar{\lambda}=\bar{j}\times\bar{\chi}}) + \dim\mathscr{L}_1(\tilde{A}_N),$$

where

$$Q_1(A|_{\bar{\lambda}=\bar{j}\times\bar{\chi}}) = \{\gamma_A|_{\bar{\lambda}=\bar{j}\times\bar{\chi}} \otimes \lambda_{A_N}\}, \quad \mathscr{L}_1(\tilde{A}_N) = \{\tilde{\gamma}_{A_N} \otimes \lambda_{A_N}\}.$$

As the cone $C(\tilde{\gamma}(A_N))$ is a line in $T\mathscr{H}$ with a directional vector:

$$\left(0, 0, \frac{4\pi k\varrho}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times\bar{\chi}, \frac{\bar{\chi}}{\varepsilon\varrho\sqrt{f}|\bar{j}\times\bar{\chi}|}, \frac{4\pi(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi})}{(\bar{j}\times\bar{\chi})^2}\bar{j}, \frac{4\pi}{(\bar{j}\times\bar{\chi})^2}\bar{j}\times(\bar{j}\times\bar{\chi})\right)$$

so $\dim \mathscr{L}_1(\tilde{A}_N) = 5$. Hence $\dim Q_1(A|_{\bar{\lambda}=\bar{j}\times\bar{\chi}}) = 10$.

2.6. Nonhomogeneous simple magnetohydrodynamic element M_N

Equations (2.1) together with the condition (2.4d) yield the following system of algebraic equations:

$$\begin{gathered}
\varrho\delta\bar{\gamma}+\gamma_p\bar{\lambda}-\bar{j}\times\bar{H}+\varrho(\bar{\Omega}\times\bar{v}-\bar{g}) = 0, \\
\gamma_\rho\delta+\varrho\bar{\gamma}\cdot\bar{\lambda} = 0, \quad \gamma_p\delta-f\delta\gamma_\varrho = 0, \\
\bar{\lambda}\times\bar{\gamma}_g = 0, \quad \bar{\gamma}_g\cdot\bar{\lambda} = 4\pi k\varrho, \quad \bar{\lambda}\times\bar{h} = 4\pi\bar{j}, \\
\bar{\lambda}\times\bar{e} = -\lambda_0\bar{h}, \quad \bar{h}\cdot\bar{\lambda} = 0, \quad \bar{j} = \sigma(\bar{E}+\bar{v}\times\bar{H}).
\end{gathered} \tag{2.43}$$

This system defines a nonhomogeneous simple magnetohydrodynamic element M_N. This element M_N is of the form:

$$\gamma_{M_N} = \begin{bmatrix}
\dfrac{\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2} \\[2ex]
\dfrac{-f\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2} \\[2ex]
\dfrac{4\pi k\varrho}{(\bar{\alpha}\times\bar{j})^2}\bar{\alpha}\times\bar{j}, \\[2ex]
\dfrac{1}{\varrho\delta}\left[\bar{\chi}+\dfrac{f\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2}\bar{\alpha}\times\bar{j}\right] \\[2ex]
a\bar{\alpha}\times\bar{j}+\dfrac{4\pi\lambda_0}{(\bar{\alpha}\times\bar{j})^2}\bar{j} \\[2ex]
\dfrac{-4\pi}{(\bar{\alpha}\times\bar{j})^2}(\bar{\alpha}\times\bar{j})\times\bar{j}
\end{bmatrix}, \quad \lambda_{M_N} = \begin{pmatrix} \delta-\bar{v}\cdot\bar{\alpha}\times\bar{j} \\ \bar{\alpha}\times\bar{j} \end{pmatrix}, \tag{2.44}$$

where

$$\bar{\chi} = \bar{j}\times\bar{H}+\varrho(\bar{g}-\bar{\Omega}\times\bar{v}), \quad \varepsilon = \pm 1, \quad \delta = \begin{cases} \neq 0 \\ \neq \varepsilon|\bar{\alpha}\times\bar{j}|\sqrt{f}. \end{cases}$$

It follows from (2.43) that the covector $\bar{\lambda}$ is parallel to the vector $\bar{\gamma}_g$. From the expression (2.44) for the covector λ we see that

$$C(\lambda(M_N)) = \mathscr{E}^* - (P \cap S)$$

where P is a plane in $\mathscr{E}^*$ generated by the vector $(-\bar{v}\cdot\bar{\alpha}\times\bar{j}, \bar{\alpha}\times\bar{j})$ and including the zero of $\mathscr{E}^*$ (see $2.3-E_{N_1}$); S — is a two-dimensional cone with an eliptical base. This cone is generated by the vector $(\varepsilon\sqrt{f}|\bar{j}\times\bar{\chi}|-\bar{v}\cdot\bar{j}\times\bar{\chi}, \bar{j}\times\bar{\chi})$ in $\mathscr{E}^*$(see $2.4-A_N$).

Thus $C(\lambda(M_N))$ is an open set and consequently the simple element M_N is also an open set in the space $T\mathscr{H}$.

For the fixed covector $\bar{\lambda}$ a cone of noncharacteristic vectors $C(\gamma(M_N))$ is a line in $T\mathscr{H}$ with the directional vector $(0, 0, \bar{0}, \bar{0}, \bar{\alpha}\times\bar{j}, \bar{0})$. This line contains a point:

$$\begin{bmatrix} \dfrac{\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2} \\ \dfrac{-f\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2} \\ \dfrac{4\pi k\varrho}{(\bar{\alpha}\times\bar{j})^2}\bar{\alpha}\times\bar{j} \\ \dfrac{1}{\varrho\delta}\left\{\bar{\chi}+\dfrac{\bar{\chi}\cdot\bar{\alpha}\times\bar{j}}{\delta^2-f(\bar{\alpha}\times\bar{j})^2}\bar{\alpha}\times\bar{j}\right\} \\ \dfrac{4\pi\lambda_0}{(\bar{\alpha}\times\bar{j})^2}\bar{j} \\ \dfrac{-4\pi}{(\bar{\alpha}\times\bar{j})^2}(\bar{\alpha}\times\bar{j})\times\bar{j} \end{bmatrix} \tag{2.45}$$

The nonhomogeneous simple element M_N is really a noncharacteristic element. It follows from (2.44), (2.12) and (2.22). The form of given by (2.44) shows that the formulae (2.45) induces $C^1(\lambda(M_N))$ as well as $C^1(\bar{\lambda}(M_N))$ and $C^1(\gamma(M_N))$. Thus there exists a one-to-one correspondence between λ_{M_N} and γ_{M_N}, $(\lambda_{M_N}\rightleftharpoons\gamma_{M_N})$. It can be shown that the dimension of hyperplane $\mathscr{L}_1(M_N)$ generated by elements $\gamma_{M_N}\otimes\lambda_{M_N}$ is equal:

$$\dim\mathscr{L}_1(M_N) = 16. \tag{2.46}$$

3. Topology of simple homogeneous and nonhomogeneous elements

In this chapter we are going to present relations between simple integral elements. The knowledge of boundary translations between these elements is helpful in a construction of some classes of solutions. They describe an interaction of simple waves and simple states in terms of Riemann invariants. In fact, let us suppose for a moment that integrability conditions imply the existence of boundary translations. Then one of the solutions degenerates to another one and there is no sense to look for any interactions between them. Topologically it means that the cones of simple elements in the space $\mathscr{E}\times\mathscr{H}$ have a non-empty intersection.

3.1. Relations between homogeneous simple elements

In our description of relations between homogeneous simple elements we will use a theorem given in [16].

THEOREM 1. *Let α, β numerate simple elements. Suppose that some limitations on elements of the matrix A imply $\delta_\alpha \to \delta_\beta$. Then under the same limitations the tensor space $Q_1(\alpha)$ is a tensor subspace of $Q_1(\beta)$.*

Theorem 1 together with (2.22) and (2.12) yield corollary:

$$\delta_A = 0 \Leftrightarrow f = 0. \tag{3.1}$$

Proof. If $f = 0$ then $C(\gamma(A))$ are generated by vectors $\gamma = (\gamma_\rho, 0, \bar{0}, \bar{0}, a\,\bar{\lambda}, \bar{0})$ and $C(\lambda(A))$ by $\lambda = (-\bar{v}\cdot\bar{\lambda}, \bar{\lambda})$ respectively. Thus (3.3) yields the entropic elements E_1 of the form:

$$(\gamma_\rho, 0, \bar{0}, \bar{\gamma}, a\,\bar{\lambda}, \bar{0}) \otimes (-\bar{v}\,\bar{\lambda}, \bar{\lambda}) \text{ where } \bar{\gamma}\cdot\bar{\lambda} = 0. \tag{3.2}$$

One can easily observe that

$$\dim Q_1(A|_{f=0}) = 9 < \dim Q_1(E_1) \quad \text{and} \quad Q_1(A|_{f=0}) \subset Q_1(E_1)$$

So the acoustic elements become entropic when the velocity of the sound approaches zero. Q.E.D.

3.2. Relations between nonhomogeneous simple elements

Let $\delta|\bar{\lambda}|$ be eigenvalues of nonhomogeneous simple elements. We have $\delta|\bar{\lambda}| = 0$ for entropic elements and $\delta|\bar{\lambda}| = \varepsilon|\bar{\lambda}|\sqrt{f}$ for acoustic ones respectively. Consequently covectors λ are characteristic. It follows from the form of nonhomogeneous simple entropic elements E_{N_1} and E_{N_2} that there is no boundary translation between them. In fact, the component of E_{N_1} in the direction γ_p is different from zero while the component of E_{N_2} vanishes. The second reason follows from the fact that the Kronecker-Capella condition for the element E_{N_2} doesn't hold i.e. $\lim\limits_{\bar{\chi}\to 0} E_{N_2}$ doesn't exist. Therefore we have two complementary kinds of entropic elements in the case. Acoustic elements A_N don't pass into E_{N_1} because the Kronecker-Capella condition $\bar{\chi} = 0$ doesn't hold for the element A_N. There is also no boundary translation between acoustic elements A_N and entropic ones of the type E_{N_2}. It follows from the fact that the condition $f \to 0$ leads to $\gamma_p = 0$. Consequently we get here a contradiciton with the form of the element E_{N_2} (as $\gamma_p \neq 0$). There is no boundary translation between simple elements M_N and other integral elements. In fact, elements M_N are noncharacteristic while other elements have characteristic covectors λ.

3.3. Relations between simple homogeneous and nonhomogeneous elements

Boundary translations from simple homogeneous elements to nonhomogeneous ones can be reduced to shifts of subspace $Q_1(\alpha)$ on the noncharacteristic vector $\tilde{\gamma}_\alpha$. The shift operator can be defined as follows:

$$P_\alpha : \mathscr{R}^n \otimes \mathscr{R}^k \to \mathscr{R}^n \otimes \mathscr{R}^k \text{such that} \bigwedge_{e\otimes\tau\in\mathscr{R}^n\otimes\mathscr{R}^k} P_\alpha(e\otimes\tau) = (e+\alpha)\otimes\tau \in \mathscr{R}^n \otimes \mathscr{R}^k \tag{3.3}$$

We shall consider a field of homogeneous simple elements on submanifolds determined by the Kronecker-Capella condition in the hodograph space. We shall denote it as follows:

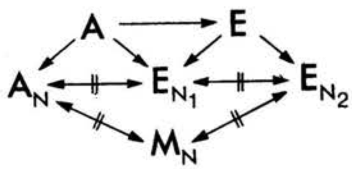
$$Q_1(\alpha)|_{\mathfrak{M}} := \{\gamma(u)\otimes\lambda(u) \in Q_1(\alpha) : u \in \mathfrak{M} \wedge \gamma(u) \in T_u\mathfrak{M}\} \tag{3.4}$$

Boundary translations from simple homogeneous elements to nonhomogeneous elements are possible in the following cases:

$$
\begin{aligned}
&P_{\tilde{\gamma}_{E_{N_1}}}\big(\lim_{\bar{\lambda}\to\bar{\alpha}\times\bar{j}} Q_1(E_1)|_{m_1}\big) = \mathscr{L}_1(E_{N_1}),\\
&P_{\tilde{\gamma}_{E_{N_2}}}\big(\lim_{\bar{\lambda}\to\bar{\chi}} Q_1(E_1)|_{m_2}\big) = \mathscr{L}_1(E_{N_2}),\\
&P_{\tilde{\gamma}_{A_N}}\big(\lim_{\bar{\lambda}\to\bar{j}\times\bar{\chi}} Q_1(A)\big) = \mathscr{L}_1(A_N),\\
&P_{\tilde{\gamma}_{E_{N_1}}}\big(\lim_{\substack{\bar{\lambda}\to\bar{\alpha}\times\bar{j}\\ f\to 0}} Q_1(A)|_{m_1}\big) = \mathscr{L}_1(E_{N_1}).
\end{aligned}
\tag{3.5}
$$

There is no boundary translation between simple elements A and E_{N_1} as well as between E_1 and A_N. Obviously there is no translation from simple homogeneous elements to nonhomogeneous elements M_N. It is so because M_N are noncharacteristic.

Therefore relations between simple elements can be described in a form of the diagram:



$\longrightarrow$ denotes boundary translation, $\nleftrightarrow$ denotes absence of boundary translation.

4. Classification of the basic equations of magnetohydrodynamics

We are going to deal with a classification of the basic equations from the point of view of simple integral elements. The principles of this classification were presented in the works [6, 7, 10]. This classification applied to the homogeneous part of Eqs. (2.5) gives us an information about hyperbolicity of the basic system as well as about the wave or non-wave character of solutions. When we know the type of the nonhomogeneous system we can decide if simple integral elements span the whole space of integral elements (this means simple elements give us the whole set of solutions).

4.1. The space Q_1

From the preceding algebraic analysis of the basic system (2.1) it follows that the characteristic polynomial (2.6) has three different eigenvalues:

$$
\begin{aligned}
&\delta_E = 0 \text{ with the multiplicity four,}\\
&\left.\begin{aligned}\delta_A+ &= +|\bar{\lambda}|\sqrt{f}\\ \delta_A- &= -|\bar{\lambda}|\sqrt{f}\end{aligned}\right\} \text{ with the multiplicity two.}
\end{aligned}
\tag{4.1}
$$

Those eigenvalues correspond to eigenvectors which generate a six-dimensional space. Hence they do not span the whole hodograph space $\mathscr{H}^{14}$. In other words — this system is not quite hyperbolic. Let Q_1 be a tensor space generated by all homogeneous simple elements. Its dimension is determined only by the expression:

$$
\{\gamma_{E_1}, \gamma_A\} \otimes \bar{\lambda} = Q_1. \tag{4.2}
$$

Indeed, the space spanned by tensors:

$$\{\gamma_{E_1} \otimes \lambda_{0E_1}, \gamma_A \otimes \lambda_{0A}\} = \mathscr{P} \tag{4.3}$$

is linearly dependent to the space $Q_1(\mathscr{P} \subset Q_1)$. It follows from (4.2) that

$$\dim Q_1 = 21. \tag{4.4}$$

Dimension of the space $\mathscr{K}(x_0, u_0)$ of homogeneous integral elements is equal

$$\dim \mathscr{K}(x_0, u_0) = 42. \tag{4.5}$$

This proves our previous conclusion. The system (1.1) has integral elements which are not linear combinations of homogeneous simple elements. That means (1.1) is a mixed type system.

4.2. The hyperplane $\mathscr{L}_1$

Dimension of the hyperplane $\mathscr{L}_1$ generated by all nonhomogeneous simple elements is equal

$$\dim \mathscr{L}_1 = \dim\{\mu^1\gamma_{E_{N_1}} \otimes \lambda_{E_{N_1}} + \mu^2\gamma_{E_{N_2}} \otimes \lambda_{E_{N_2}} + \mu^3\gamma_{A_N} \otimes \lambda_{A_N} + \mu^4\gamma_{M_N} \otimes \lambda_{M_N}\}, \tag{4.6}$$

where $\sum_{s=1}^{4} \mu^s = 1$.

This dimension is determined by a linear space generated by tensors:

$$a_{ij}\{(1-(\mu^2+\mu^3+\mu^4))\gamma^i_{E_{N_1}} \otimes \lambda^j_{E_{N_1}} + \mu^2\gamma^i_{E_{N_2}} \otimes \lambda^j_{E_{N_2}} + \mu^3\gamma^i_{A_N} \otimes \lambda^j_{A_N} + \mu^4\gamma^i_{M_N} \otimes \lambda^j_{M_N}\} \tag{4.7}$$

where coefficients μ^i, $i = 1, 2, 3$ are arbitrary. If we put successively $\mu^i = 1$ for $i = j$ and $\mu^i = 0$ for $i \neq j (j = 1, 2, 3)$ in (4.2) then we get a system of equations:

$$\begin{aligned} a_{ij}\gamma^i_{E_{N_1}} \otimes \lambda^j_{E_{N_1}} &= 0, \quad a_{ij}\gamma^i_{E_{N_2}} \otimes \lambda^j_{E_{N_2}} = 0 \\ a_{ij}\gamma^i_{A_N} \otimes \lambda^j_{A_N} &= 0, \quad a_{ij}\gamma^i_{M_N} \otimes \lambda^j_{M_N} = 0 \end{aligned} \tag{4.8}$$

Thus the dimension of the hyperplane $\mathscr{L}_1(x_0, u_0)$ generated by nonhomogeneous simple elements for the system (2.1) is equal:

$$\dim \mathscr{L}_1 = 37. \tag{4.9}$$

Dimension of the hyperplane $\mathscr{L}(x_0, u_0)$ of nonhomogeneous integral elements is $\dim \mathscr{L} = \dim \mathscr{K} = 42$. Thus the system (2.1) has integral elements which are not linear combinations of nonhomogeneous simple integral elements. In other words nonhomogeneous simple elements do not span the whole space of integral elements. So the system is not of the type $\mathscr{L}_1$.

Finally it is worthwile to consider a structure of the hyperplane $\mathscr{L}_1$. It contains two disconnected hyperplanes

$$[\mathscr{L}_1(E_{N_1}) \cup \mathscr{L}_1(E_{N_2}) \cup \mathscr{L}_1(A_N)] \cap \mathscr{L}_1(M_N) = 0. \tag{4.10}$$

Of course it is related to the fact that the nonhomogeneous simple entropic elements E_{N1}, E_{N2} and acoustic A_N correspond to characteristic surfaces, while the complementary nonhomogeneous simple element M_N is noncharacteristic. Using (4.10) it is easy to compute the dimension of the hyperplane generated by nonhomogeneous simple elements which have characteristic covectors λ;

$$\dim\{\mathscr{L}_1(E_{N_1}) \cup \mathscr{L}_1(E_{N_2}) \cup \mathscr{L}_1(A_N)\} = \dim \mathscr{L}_1 \ominus \dim \mathscr{L}_1(M_N) = 19. \tag{4.11}$$

Solutions constructed from elements of this hyperplane will be contained in the area of hyperbolicity(²) of the nonhomogeneous system (2.1) while the solution constructed from elements M_N will be contained within the ellipticity area.

5. Examples of solutions — simple states

Now, we shall present the simplest solutions of the basic equation system (1.1). These solutions are induced in some way by nonhomogeneous simple elements E_N, A_N, M_N. The method we use here was also presented in [6, 7]. According to the terminology introduced there the simplest solutions of the nonhomogeneous system will be called simple states. Let us recall that solutions of a homogeneous system are called simple waves and they correspond to waves in the physical sense. However simple states don't describe wave phenomena sometimes. It happens when covector λ corresponding to a nonhomogeneous simple element is noncharacteristic one. The covector λ can be treated as an analogue of a wave vector $(\omega, \bar{k})$ determining the velocity and direction of state propagation in the case of a simple wave. Our involutivity conditions yield that covectors λ are independent

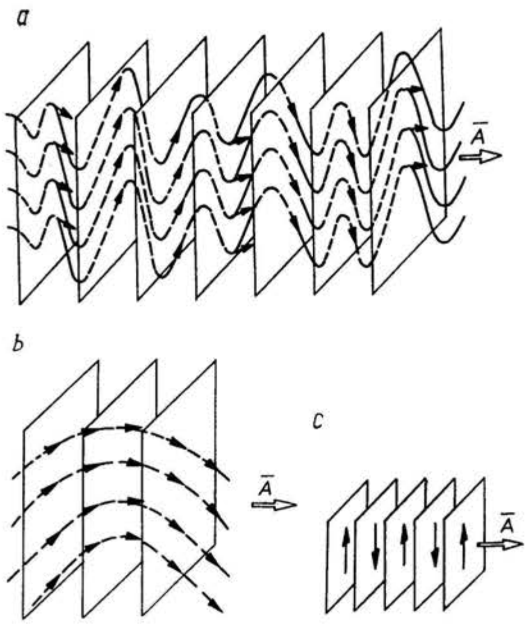


Fig. 1. The examples of the flow of the fluid described by simple states.

(²) According to J. HADAMARD [8] characteristic surface is a surface on which the characteristic determinant of the system vanishes. The elipticity area is the complement of the characteristic surface.

from the Riemann invariant. Therefore they have a constant direction in the physical space. When those conditions are satisfied we can present our solution in the form $u = u(s)$, where s is a Riemann invariant: $s = \lambda_\nu x^\nu \equiv c_1 t + \bar{A}\cdot\bar{x}$ (here $\bar{A}$ is a direction of the state propagation). This implies the principal properties of simple states. They are one-dimensional solutions in the physical space $\mathscr{E}^3$ which are constant on a family of parallel planes. They can propagate with a constant velocity ($v_f = \lambda_0/|\bar{\lambda}| = \text{const} = c_1$) in the direction perpendicular to them. Those solutions are functions of the class C^1 and they describe flows of the laminar type. Examples are illustrated on the Fig. 1a–c. In many cases a profile of the state is not uniquely determined by the obtained solution. There exists here some arbitrareness connected with a freedom of choice of some functions and constants. The above remarks concern all simple states. The more detailed properties of them will be discussed later.

5.1. Simple entropic states E_N

There exist two nonhomogeneous simple entropic elements E_{N_1}, E_{N_2}. So if integrability conditions [6] are satisfied we obtain two kinds of entropic simple states. The entropic state E_{N_1} corresponding to the simple element (2.25) is of the form:

$$\varrho(s) = \frac{\sigma C_2^2}{\bar{\Omega}\cdot\bar{A}}\frac{\bar{e}\cdot\bar{\alpha}_{;s}}{\bar{e}\cdot\bar{\alpha}_{,s}\times\bar{A}}, \quad p_0 = \text{const}, \quad \bar{g}(s) = \xi(s)\bar{A}+\bar{B},$$

$$\bar{v}(s) = \frac{-1}{c_2}\left\{\frac{\bar{\alpha}_{,s}\times\bar{A}}{4\pi\sigma}+\bar{k}\right\}-c_1\bar{A}, \quad \bar{E}(s) = -\frac{\bar{\alpha}\cdot\bar{k}}{c_2}\bar{A}+c_1\bar{A}+\bar{k}\times\bar{A}, \tag{5.1}$$

$$\bar{H}(s) = \bar{\alpha}(s)\times\bar{A}+c_2\bar{A} \quad \bar{j}(s) = \frac{\bar{\alpha}_{,s}}{4\pi}$$

where

$$|\bar{A}| = 1, \bar{e} = \left\{\bar{B}+c_1\bar{\Omega}\times\bar{A}+\frac{1}{c_2}\bar{\Omega}\times\bar{k}\right\}, \quad \bar{\Omega}\cdot\bar{A}\neq 0,, \quad \bar{k}\cdot\bar{A} = 0,$$

$$\bar{B}\cdot\bar{A} = 0, \bar{\alpha}(s)\cdot\bar{A} = 0, \eta = \frac{-4\pi\sigma c_2}{\bar{\Omega}\cdot\bar{A}}, \quad \varepsilon = \pm 1,$$

$$\bar{\alpha}(s) = \varepsilon\frac{\eta}{2}|\bar{e}|\begin{pmatrix}\int\cos\varphi(s)ds\\ \int\sin\varphi(s)ds\\ 0\end{pmatrix}$$

$$\xi(s) = \frac{(\bar{\alpha}^2)_{,s}\bar{e}\cdot\bar{\alpha}_{,s}\times\bar{A}\bar{\Omega}\cdot\bar{A}}{8\pi\sigma c_2^2\bar{\alpha}_{;s}\cdot\bar{e}} - \frac{1}{c_2}\left(\bar{\Omega}\times\bar{k}\cdot\bar{A}-\frac{\bar{\alpha}_{,s}\cdot\bar{\Omega}}{4\pi\sigma}\right).$$

The function $\varphi(s)$ is described by the differential — integral equation:

$$-\frac{4\pi\sigma k c_2^2}{\bar{\Omega}\cdot\bar{A}}\frac{\bar{e}\cdot\bar{\alpha}_{,s}}{\bar{e}\cdot\bar{\alpha}_{,s}\times\bar{A}}+\frac{1}{c_2}\frac{\bar{\alpha}_{,ss}\cdot\bar{\Omega}}{4\pi\sigma}+\frac{\bar{\Omega}\cdot\bar{A}}{8\pi\sigma c_2^2}\left(\frac{\bar{e}\cdot\bar{\alpha}_{,s}\times\bar{A}(\bar{\alpha}^2)_{,s}}{\bar{e}\cdot\bar{\alpha}_{,s}}\right)_{,s} = 0 \tag{5.2}$$

The simple state E_{N_2} differs from the state E_{N_1} in the expressions for the pressure $p(s)$ and in a function $\xi(s)$:

$$(5.3)\qquad \begin{aligned} p(s) &= s+p_0, \\ \xi(s) &= \frac{\bar{\Omega}\cdot\bar{A}}{\sigma c_2^2}\frac{\bar{e}\cdot\bar{\alpha}_{,s}\times\bar{A}}{\bar{e}\cdot\bar{\alpha}_{,s}}\left(1+\frac{(\bar{\alpha}^2)_{,s}}{8\pi}\right)-\frac{1}{c_2}\left(\bar{\Omega}\times\bar{k}\cdot\bar{A}-\frac{\bar{\alpha}_{,s}\cdot\bar{\Omega}}{4\pi\sigma}\right) \end{aligned}$$

Moreover — unlike for the state E_{N_1} — we do not have here any restriction on the function $\varphi(s)$ which is an arbitrary one.

Let us notice that solutions are not defined when $\bar{\Omega}\cdot\bar{A} = 0$. This excludes the direction of propagation of the state E_N perpendicular to the direction $\bar{\Omega}$. The most characteristic properties of entropic states E_N are connected with the Kronecker-Capella condition (2.4ab). In this case the condition (2.4ab) is equivalent to the fact that the operator d/dt vanishes everywhere. It means that the particular physical quantities are changing in the moving fluid element and consequently that the planes of constancy of the solution are stationary relative to the medium. Thus the state propagates together with the fluid and not relatively to it. The second important consequence of the condition (2.4ab) is that in the Euler equation (1.1) the inertion forces are equal to zero. Thus the forces acting on the fluid element are in balance. Hence we get the fluid elements move without accelerations in the case of entropic simple states. The momentum of the system is conserved:

$$(5.4)\qquad \frac{\partial}{\partial t}(\varrho\bar{v})+\operatorname{div}\left\{(2p\hat{\delta}+\varrho\bar{v}\otimes\bar{v})+\frac{1}{2\pi}\left(\frac{1}{2}\bar{H}^2\hat{\delta}-\bar{H}\otimes\bar{H}\right)\right\} = 0.$$

Observe the expression for the tensor of the magnetohydrodynamics momentum stream differs in this case from the tensor of the momentum stream in the equation (5.4) in expressions:

$$\nabla p+\frac{1}{4\pi}\left(\frac{1}{2}\nabla\bar{H}^2-(\bar{H}\nabla)\bar{H}\right).$$

From the condition $d/dt = 0$ it follows directly that the circulation of the velocity along the closed fluid contour stays unchanged in time (hence the Kelvin theorem is identicaly satisfied in our case). From rot $\bar{v}(s)\neq 0$ it follows that the solution (5.1) allows the vortex of the fluid. We have div $\bar{v}(s) = 0$ for each entropic state. It means that the considerated fluid is noncompressionable in spite of the density changes from point to point. In other words the density distribution is arbitrary but constant in a given fluid element.

The magnetic force $\bar{F}_M$ acting here on the fluid element will be of the form:

$$(5.5)\qquad \bar{F}_M = \frac{1}{4\pi}\left\{-\frac{(\bar{\alpha}^2)_{,s}}{2}\bar{A}+c_2\bar{\alpha}_{,s}\times\bar{A}\right\}.$$

The first term in the expression (5.5) affects perpendicularly the fluid element $\left(\text{as it is of the form}\left(\nabla\frac{\bar{H}^2}{2}-(H_i\nabla)H_i\right)\right)$. Hence it can only compress or stretch the fluid. It gives a contribution to the pressure. The second term(which is of the form: $(H_i\nabla)H_j$ $i\neq j$) yields that the fluid element is under the action of torsion forces. They crook its trajectory. However it is possible to conserve the constant direction $\bar{A}$ of disturbance propagation in the physical space. In fact, it follows from our earlier considerations that the Coriolis force compensates the torsional contribution of the magnetic force.

It follows from the expression (5.1) that the oscillation planes of the magnetic field $\overline{H}$ and electric field $\overline{E}$ are orthogonal. So the entropic states E_N have a character similar to that of plane waves in electrodynamics.

From (5.1) we can deduce that the currents $\overline{j}$ are always contained in the plane of constancy of the solution (i.e. they are perpendicular to $\overline{A}$). Moreover the length of the vector $|\overline{j}|$ is a constant one. Consequently we have the constant dissipation for Joule heat in our system:

$$\sigma^{-1}\overline{j}^2 = \frac{(\overline{\alpha}_{,s})^2}{16\pi\sigma} = \text{const}. \tag{5.6}$$

The heat transport equation (1.5) together with (5.6) yield the determination of the temperature distribution form in the concerned area:

$$T = Ax^2+(2Ac_1t+D_1)x+c_1(Ac_1t^2+D_1t)+D_2$$

where A, D_1, D_2 are arbitrary constants.

Using the known thermodynamical relations it is easy to check that in the case of E_N states the entropy is a function (arbitrary) of the temperature only: $S = S(T)$. The internal energy and enthalpy for the state E_{N_1} is of the form:

$$U = S(T)-T\frac{\partial S}{\partial T}-\frac{P_0}{\varrho}, \tag{5.7}$$

$$\frac{\partial H}{\partial T} = -T\frac{\partial^2 S}{\partial T^2}$$

so that $H = H(T)$ — enthalpy depends on the temperature only. In the case of the state E_{N_2} we have respectively:

$$\begin{aligned} U &= S(T)-T\frac{\partial S}{\partial T}+\int\frac{P}{\varrho^2}d\varrho+U_0, \\ H &= S(T)-T\frac{\partial S}{\partial T}+\int\frac{dp}{\varrho}+H_0. \end{aligned} \tag{5.8}$$

If we substitute the expression (5.8) into the equation of energy conservation (1.4) then we see that the gravitation doesn't influence the fluid movement. However the density change implies the movement causes the changes of the gravitational field.

We shall show now the relations between hydrodynamic and electordynamic phenomena. The following relation on the kinetic energy can be deduced from (1.1):

$$\frac{\partial}{\partial t}\left(\varrho\frac{\overline{v}^2}{2}\right) = -\operatorname{div}\left\{\varrho\frac{\overline{v}^2}{2}\overline{v}\right\}-\overline{v}\cdot\nabla p+\overline{v}\cdot\{\overline{F}_M+\overline{F}_g+\overline{F}_c\}, \tag{5.9}$$

where

$$\overline{F}_M = \overline{j}\times\overline{H}, \quad \overline{F}_g = \varrho\overline{g}, \quad \overline{F}_c = -\varrho\overline{\Omega}\times\overline{v}.$$

Consequently the change of the kinetic energy of the fluid in a given volume is caused by the work done by the pressure forces and the sum of the forces: magnetic $\overline{F}_M$, gravitational $\bar{F}_g$ and Coriolis $\bar{F}_c$ and of course by the energy outflow from the area under consideration. In the case of entropic states E_N the kinetic energy is conserved. In fact: ∇_P, $\bar{F}_M$, $\bar{F}_g$, $\bar{F}_c$ are compensated. However no dissipative effect occurs. Only the kinetic energy stream influences the change of the kinetic energy in the investigated fluid element.

The change of magnetic energy in time can be expressed as follows:

$$\frac{\partial}{\partial t}\left(\frac{\bar{H}^2}{8\pi}\right) = \frac{1}{4\pi}\operatorname{div}\{\bar{E}\times\bar{H}\} - \sigma^{-1}\bar{j}^2 - \bar{v}\cdot\bar{j}\times\bar{H}. \tag{5.10}$$

The first term describes the energy stream (i.e. the Poynting vector). The second term describes the velocity of magnetic energy changes into the Joule heat and the third term — work of the medium against magnetic field forces. When we substitute (5.1) into (5.10) we get that the changes of electromagnetic energy are dependent on kinetic phenomena. Consequently it follows that the fluid movement influences the change of the magnetic field. However electromagnetic phenomena do not influence kinetic phenomena.

5.2. Simple acoustic state A_N

The simple acoustic state A_N corresponding to simple elements (2.24) is of the form:

$$\begin{gathered} \varrho_0 = \text{const}, \quad P_0 = f\varrho_0, \quad \bar{g} = (cs+D)\bar{A}, \\ \bar{v}(s) = s\bar{A}_1 + \bar{A}_2, \quad \bar{E}(s) = s\bar{B}_1 + \bar{B}_2, \quad \bar{H}(s) = s\bar{e}\times\bar{A} + c_2\bar{A}, \quad \bar{j} = \frac{\bar{e}}{4\pi} \end{gathered} \tag{5.11}$$

where

$$|\bar{A}| = 1, \quad \bar{B}_1\bar{A} = 0, \quad \bar{A}_1\cdot\bar{A} = 0, \quad \bar{e}\cdot\bar{A} = 0.$$

This state differs from the previously described ones first of all in the fact that the planes of the constancy of the solutions dislocate relatively to the medium with the constant velocity of sound.

Moreover the direction of the state $\bar{A}$ propagation is distinguished in the space $\mathscr{E}$. In fact, it has to be orthogonal to $\bar{\Omega}$ as well as to the sum of magnetic and Coriolis forces. It is a characetristic fact for the acoustic state A_N that the density ϱ_0, pressure p_0 and current $\bar{j}$ remain constant. The solution allows also a fluid vortex (rot $\bar{v}(s) \neq 0$). However the velocity circulation along an arbitrary closed contour is not conserved. The magnetic force acting on medium is of the form (5.5) — just as in the previously described case. The contribution causing the torsion of the fluid element is compensated by the inertion and the Coriolis forces (this yields the constant direction of state propagation is conserved). Directions of the magnetic and the electric field are orthogonal like in the case of entropic states. We have ϱ_0 = const, p_0 = const, consequently entropy is only the temperature function: $S = S(T)$. Likewise in entropic states dissipation for the Joule heat is constant ($\bar{j}$ = const.) Therefore the heat transport equation is of the form:

$$\varrho_0 T\frac{\partial S}{\partial T}\frac{\partial T}{\partial t} = \varkappa\Delta T + \text{const.} \tag{5.12}$$

The temperature distribution in the investigated area depends on the adopted model of entropy changes. It can be easily checked that the internal energy and enthalpy of the acoustic state depend on the temperature only.

$$U = \int T \frac{\partial S}{\partial T} + U_0, \quad H = \int T \frac{\partial S}{\partial T} + H_0 \tag{5.13}$$

When we substitute (5.13a, 5.13b) into the energy conservation equation (1.4) then we get that the gravitation influences the fluid movement but there is no inverse influence. When we substitute (5.11) in the expression on the kinetic energy change we can easily see the changes of the magnetic field influence the quantity of kinetic energy changes (an expression $\bar{v} \cdot \bar{F}_M \neq 0$ is the coupling term here). Moreover an analysis of the expression on the magnetic energy change (5.10) for the case A_N shows that this change is influenced by kinetic phenomena (the coupling term: $\bar{v} \cdot \bar{j} \times \bar{H} \neq 0$). So we have here a feedback of kinetic and magnetic phenomena.

5.3. Simple magnetohydrodynamic state M_N

The simple state M_N corresponding to the simple elements (2.40) is of the form:

$$\begin{gathered}
\varrho = \frac{\varrho_0}{4\pi k \delta}, \quad p = \frac{f\varrho_0}{4\pi k\delta} + p_0, \quad \bar{g}(s) = \xi(s)\bar{A} + \bar{B}, \\
\bar{v}(s) = \frac{1}{c_2}\left\{\delta\bar{\alpha} \times \bar{A} - \frac{1}{4\pi\sigma}\bar{\alpha}_{,s} \times \bar{A} - \bar{k}\right\} + (\delta - c_1)\bar{A}, \\
\bar{E}(s) = \frac{-1}{c_2}\left\{\bar{k} \times \bar{A} - \frac{1}{4\pi\sigma}\bar{\alpha}_{,s}\right\} \cdot \bar{\alpha} \times \bar{A}\bar{A} + c_1\bar{\alpha} + \bar{k} \times \bar{A}, \\
\bar{H}(s) = \bar{\alpha}(s) \times \bar{A} + c_2\bar{A}, \quad \bar{j}(s) = \frac{\bar{\alpha}_{,s}}{4\pi},
\end{gathered} \tag{5.14}$$

where

$$|\bar{A}| = 1, \quad \bar{B} \cdot \bar{A} = 0, \quad \bar{k} \cdot \bar{A} = 0,$$

$$\xi(s) = \frac{\delta}{\varrho_0}\left\{\varrho_0\,\delta_{,s} + k\left(\frac{\bar{\alpha}^2}{2}\right)_{,s} + f\left(\frac{\varrho_0}{\delta}\right)_{,s}\right\} + \frac{1}{c_2}\left\{\frac{\bar{\alpha}_{,s} \cdot \bar{\Omega}}{4\pi\sigma} - \delta\bar{\alpha} \cdot \bar{\Omega} - \bar{\Omega} \times \bar{k} \cdot \bar{A}\right\}$$

with conditions

$$0 < |\delta| < \sqrt{f} \quad \text{or} \quad |\delta| > \sqrt{f} \quad \text{satisfied}. \tag{5.15}$$

Moreover we have a system of three equations with respect to three functions:

$$\left\{\frac{\delta}{\varrho_0}\left(\delta_{,s}\varrho_0 + k\frac{(\bar{\alpha}^2)_{,s}}{2}\right) + f\varrho_0\left(\frac{1}{\delta}\right)_{,s}\right\}_{,s} - \frac{1}{c_2}\left\{\delta\bar{\alpha} \cdot \bar{\Omega} + \bar{\Omega} \times \bar{k} \cdot \bar{A} + \frac{\bar{\alpha}_{,s} \cdot \bar{\Omega}}{4\pi\sigma}\right\} = \frac{\varrho_0}{\delta}, \tag{5.16}$$

$$\begin{aligned}
\frac{\varrho_0}{c_2}\left\{\delta\bar{\alpha}_{,s} \times \bar{A} + \delta_{,s}\bar{\alpha} \times \bar{A} - \frac{\bar{\alpha}_{,ss} \times \bar{A}}{4\pi\sigma}\right\} = \Bigg\{& kC_2\bar{\alpha}_{,s} \times \bar{A} \\
&+ \frac{\varrho_0}{\delta}\left[\bar{B} - \frac{1}{c_2}\left(\delta\bar{\Omega} \cdot \bar{A}\bar{\alpha} - (\bar{\Omega} \times \bar{k} - \bar{\Omega} \times \bar{k} \cdot \bar{A}\bar{A}) - \frac{\bar{\Omega} \cdot \bar{A}}{4\pi\sigma}\bar{\alpha}_{,s}\right) - (\delta - c_1)\bar{\Omega} \times \bar{A}\right]\Bigg\}.
\end{aligned}$$

This is an involutive system (hence it has solutions).

Likewise in the case of the acoustic state A_N the planes of constancy of the solutions propagate relatively to the medium. It is characteristic however that their propagation velocity is strictly determined and according to the function $\delta(s)$ it can have various sub or supersonic values (5.15). The direction $\bar{A}$ of propagation of the state M_N is orthogonal to the current $\bar{j}$. Fluid vortexes are possible. The velocity circulation on the closed contour is not conserved here. The magnetic field has a form similar to one in the case of entropic and acoustic states. Its torsioning action is compensated by the inertion and Coriolis forces. As before entropy is only a temperature function so the equation of the heat transport is of the form:

$$\varrho T \frac{\partial S}{\partial T} \frac{\partial T}{\partial t} + \varkappa \Delta T + \sigma^{-1} \bar{j}^2 = 0. \tag{5.17}$$

The internal energy and enthalpy are expressed as follows:

$$\begin{aligned} U &= S(T) - T\frac{\partial S}{\partial T} + f \ln \varrho + U_0, \\ H &= S(T) - T\frac{\partial S}{\partial T} + f \ln \varrho + H_0. \end{aligned} \tag{5.18}$$

If we substitute (5.18a, 5.18b) in the Eq. (1.4) we get that the fluid movement influences changes of the gravitational field as well as the gravitational field influences the change of fluid velocity. An analysis of the expression on kinetic and magnetic energy changes shows that the kinetic and magnetic phenomena are mutually coupled. Expressions $\bar{F}_M \cdot \bar{v} \neq 0$, $\bar{v} \cdot \bar{j} \times \bar{H} \neq 0$ are the coupling term here.

Let us notice that the propagation velocity of the state is equal to group velocity for entropic states E_N and acoustic states A_N:

$$v_f = \frac{\lambda_0}{|\bar{\lambda}|} = |\bar{v}_{gr}| = |\nabla_{\bar{\lambda}} \lambda_0|. \tag{5.19}$$

Hence we have no dispersion phenomena here. On the other hand the magnetohydrodynamic state M_N has a noncountable velocity spectrum δ (5.15.). This yields $v_f \neq v_{gr}$. Therefore we have dispersion medium here.

6. Final remarks

Finally let us notice that particular states differ at least in one vector coordinate. In fact, it follows from the correspondence between the covector λ and the wave vector $(c_1, \bar{A})$. Consequently, when various states propagate in the medium in the same direction then their velocities differ. On the other hand when they propagate with the same velocity then the directions are different.

The simple states we have just described form a basis for searching for wider classes of solutions of M.H.D. equations. These solutions are nonlinear superposition of simple waves and simple states. An interaction of this type can be of interest from the physical point of view thus they will be an object of consideration in our subsequent papers.

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