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Analysis of non-linear dynamic systems in the spaces of square integrable functions

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THE DYNAMIC system described by a set of integrale quations is considered on the basis of an analysis of the general operator eqnation in a Banach space. Depending on the various forms of integral operators, the theorems are formulated and proved concerning the conditions under which the set of equations considered possesses a unique solution within the space of vector functions square integrable along the segment $\langle 0, T\rangle$, for $T \leqslant \infty$.

W oparciu o analizę ogólnego równania operatorowego w przestrzeni Banacha rozpatrzono układ dynamiczny, opisany układem równań całkowych. W odniesieniu do różnej postaci operatorów całkowych sformułowano i udowodniono twierdzenia określające warunki, przy spełnieniu których rozważany układ równań ma dokładnie jedno rozwiązanie w przestrzeni wektorfunkcji całkowalnych z kwadratem na odcinku $\langle 0, T\rangle$ dla $T \leqslant \infty$.

Опираясь на анализ общего операторного уравнения в банаховом пространстве рассмотрена динамическая система, описанная системой интегральных уравнений. По отношению к разного вида интегральным операторам сформулированы и доказаны теоремы, определяющие условия, при удовлетворении которым рассматриваемая система уравнений имеет точно одно решение в пространстве вектор-функций интегрируемых с квадратом на отрезке $\langle 0, T\rangle$ для $T \leqslant \infty$.

1. Introduction

THE NOTION of stability, which is usually introduced in connection with dynamic systems described by differential or integro-differential equations [1], is most frequently understood in the Lapunov sense. It may be shown that a qualitative analysis of a wide class of equations governing various dynamic processes may be considerably generalized by introducing the notion of stability in a Banach space. Depending on the choice of space to be considered, conditions of the ordinary or asymptotic Lapunov stability may be determined with respect to the sets of ordinary differential equations [2], as also the conditions according to the generalized definition of Lapunov (integro-differential equations).

One of the most interesting problems which, in addition, yields an accurate and comparatively simple solution with respect to a wide class of equations is that of determining stability conditions in the spaces of square integrable functions. For instance, let us consider the dynamic system described by the equation

$$Fx = z, \tag{1.1}$$

z being an element of a certain Banach space X, and F denoting an operation defined on the elements of X with values from within the same space. Let $x = x(t) = u(t) - u_0$ denote the deviation of a characteristic value of the system from the position of equilibrium, and $z = z(t)$ — the perturbation. Now, in order to determine the stability conditions in the space of continuous and bounded functions, we must ascertain whether the system displaced from the equilibrium position will oscillate about it or even show a tendency to return back to that position, $\lim_{t\to\infty} u(t) = u_0$. Considering the problem in a space of square integrable (in the Lebesgue sense) functions in the interval $<0, \infty)$ and taking into account the inequality (true for arbitrary $T > 0$)

$$\frac{1}{T}\int_0^T u^2(t)\,dt \leqslant \frac{1}{T}\int_0^T [u(t)-u_0]^2 dt + \frac{1}{T}\int_0^T u_0^2\,dt + 2u_0 \sqrt{\frac{1}{T}\int_0^T [u(t)-u_0]^2 dt} = \left\{\sqrt{\frac{1}{T}\int_0^T [u(t)-u_0]^2 dt} + u_0\right\}^2,$$

it may easily be observed that the condition

$$\int_0^\infty |x(t)|^2 dt < \infty$$

yields:

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T |u(t)|^2 dt\right]^{1/2} = u_0. \tag{1.2}$$

This means that the system returns to the equilibrium position asymptotically. Convergence understood in the sense of the Eq. (1.2) allows, by contrast with the ordinary convergence, for arbitrary values of the deviations provided their times of duration are sufficiently small. Determination of such properties of the solutions is of primary importance for numerous models of dynamics of physical processes.

2. Formulation of the problem

A particular case of the Eq. (2.1) has the following form:

$$x - A\sum_{p=2}^{\infty} G_p x^p = z, \tag{2.1}$$

A being a linear bounded operator defined on X with values from within the same space, $A \in (X \to X)$, and $G\ (p = 2, 3, \ldots)$ — analogous multilinear operators, $G_p \in (X^p \to X)$. The symbol $G_p x^p$ is a simplified notation for $G_p(x, x, \ldots, x)$. General analysis of the Eq. (2.1) was presented in [4] where the following existence and uniqueness theorem is given:

THEOREM 1. *If the series* $\sum_{p=2}^{\infty} \|G_p\| y^p$ *has a positive radius of convergence then, there exist such numbers* α *and* β *that for every* z *satisfying the inequality* $\|z\| \leqslant \alpha$ *the Eq.* (2.1) *has* *in the sphere $\bar{K}(0, \beta) \subset X$ exactly one solution $x^* \in X$ which continuously depends on z and constitutes the limit of consecutive approximations:*

$$x_{n+1} = z + A \sum_{p=2}^{\infty} G_p x_n^p .$$

Here the term x_0 may be an arbitrary element of the space X satisfying $\|x_0\| \leqslant y^*$, y^*, being the unique non-negative solution of the equation:

$$y - \|A\| \sum_{p=2}^{\infty} \|G_p\| y^p = \|z\| .$$

Solution x^* satisfies then the estimate

$$\|x^*\| \leqslant y^* \leqslant \beta .$$

From the theorem it follows that for each equation of the form of (2.1) there exists such a continuous function f, defined and non-decreasing along the segment $\langle 0, \alpha \rangle$, that for each z such that $\|z\| \leqslant \alpha$ the inequality $\|x^*\| \leqslant f(\|z\|)$ holds true; here $f(0) = 0$ and $f(\alpha) = \beta$.

In this paper we shall discuss and investigate the properties of the Eq. (2.1) in the case in which X is a space of vector-functions $x(t) = \{x_i(t)\}_{(i=1,\dots,M)}$, square integrable in the Lebesgue sense along the segment $\langle 0, T \rangle$; here, T may be finite or infinite. The space will be denoted by the symbol $L_M^2(0, T)$, and in the case of $T = \infty$—by the symbol L_M^2. The norm of x is given by the formula:

$$\|x\| = \Big\{ \sum_{i=1}^{M} \int_0^T |x_i(t)|^2 dt \Big\}^{1/2} .$$

With respect to the space thus defined, the following multilinear operators will be considered:

$$G_p = \{G_{pi}\}_{(i=1,\dots,M)}, \qquad p = 1, 2, \dots$$

$$G_{pi}(x_1, \dots, x_p) = \sum_{m_1=1}^{M} \dots \sum_{m_p=1}^{M} G_{pim_1\dots m_p}(x_{1m_1}, \dots, x_{pm_p}). \tag{2.2}$$

Operators $G_{pim_1\dots m_p}$ assume one of the following forms:

$$G_{pim_1\dots m_p}(x_{1m_1}, \dots, x_{pm_p}) = \int_0^t \dots \int_0^t k_{pim_1\dots m_p}(t, \tau_1, \dots, \tau_p) x_{1m_1}(\tau_1) \dots x_{pm_p}(\tau_p) d\tau_1 \dots d\tau_p , \tag{2.3}$$

$$G_{pim_1\dots m_p}(x_{1m_1}, \dots, x_{pm_p}) = \prod_{l=1}^{p} \int_0^t x_{lm_l}(t - \tau_l) dh_{pim_l}(\tau_l) , \tag{2.4}$$

$$G_{pim_1\dots m_p}(x_{1m_1}, \dots, x_{pm_p}) = \int_0^t \dots \int_0^t k_{pim_1\dots m_p}(\tau_1, \dots, \tau_p) x_{1m_1}(t-\tau_1) \dots x_{pm_p}(t-\tau_p) d\tau_1 \dots d\tau_p \tag{2.5}$$

integration in the Eq. (2.4) being understood in the Stieltjes sense. With $p = 1$ operator G_p becomes linear, and hence the above formulae contain also the form of the operator A. It is easily seen that with such definitions of the operators Eq. (2.1) takes the form of a system of integral equations. From the Theorem 1 it follows that in order to ensure the existence and uniqueness of solutions of that system it will be sufficient to formulate the conditions under which the operators A and G_p are multilinear operators mapping the space $L^2_M(0, T)$ onto itself (under the assumption that the radius of convergence of $\sum_{p=2}^{\infty} \|G_p\| y^p$ is positive).

3. Analysis of integral operators

From a lemma proved in [4] it follows that the investigation of operators in vector spaces may be reduced to the investigation of scalar operators, thus the latter ones should be considered first of all. It corresponds to the case of a space $L^2_1(0, T)$ which will be denoted, for the sake of brevity, by the symbol $L^2(0, T)$ (if $T = \infty$, then the notation will be L^2).

3.1. Operators in the space $L^2(0, T)$

Let us consider the operator:

$$[G_p(x_1, \ldots, x_p)](t) = \int_0^t \ldots \int_0^t k(t, \tau_1, \ldots, \tau_p) x_1(\tau_1) \ldots x_p(\tau_p) d\tau_1 \ldots d\tau_p. \tag{3.1}$$

THEOREM 2. *If*

$$\int_0^T \int_0^t \ldots \int_0^t |k(t, \tau_1, \ldots, \tau_p)|^2 d\tau_1 \ldots d\tau_p dt < \infty,$$

then the operator (3.1) *transforms* $[L^2(0, T)]^p$ *into* $L^2(0, T)$ (*for* $p = 1, 2, \ldots$), *and*

$$\|G_p\| \leqslant \left\{ \int_0^T \int_0^t \ldots \int_0^t |k(t, \tau_1, \ldots, \tau_p)|^2 d\tau_1 \ldots d\tau_p dt \right\}^{1/2}. \tag{3.2}$$

Proof. Let $p = 1$. If $y(t) = [G_1 x](t) = \int_0^t k(t, \tau) x(\tau) d\tau$,

then

$$\int_0^T |y(t)|^2 dt = \int_0^T \left| \int_0^t k(t, \tau) x(\tau) d\tau \right|^2 dt.$$

From the Buniakovski-Schwarz inequality, it follows that

$$\left| \int_0^t k(t, \tau) x(\tau) d\tau \right|^2 \leqslant \int_0^t |k(t, \tau)|^2 d\tau \cdot \int_0^t |x(\tau)|^2 d\tau,$$

whence

$$\|y\|^2 = \int_0^T |y(t)|^2 dt \leqslant \int_0^T\int_0^t |k(t,\tau)|^2 d\tau\, d\tau \cdot \int_0^T |x(\tau)|^2 d\tau = \int_0^T\int_0^t |k(t,\tau)|^2 d\tau\, dt \|x\|^2,$$

which proves that $G_1 \in (L^2(0,T) \to L^2(0,T))$ and $\|G_1\| \leqslant \left\{\int_0^T\int_0^t |k(t,\tau)|^2 d\tau\, dt\right\}^{1/2}$. In the case of $p > 1$, it suffices to assume

$$y(t) = \int_0^t \dots \int_0^t k(t, \tau_1, \dots, \tau_p) x_1(\tau_1) \dots x_p(\tau_p) d\tau_1 \dots d\tau_p$$

and to apply the Buniakovski-Schwarz inequality p times. This leads to:

$$\|y\|^2 = \int_0^T |y(t)|^2 dt \leqslant \int_0^T \left\{\int_0^t \dots \int_0^t |k(t, \tau_1, \dots, \tau_p)|^2 d\tau_1 \dots d\tau_p \prod_{i=1}^p \int_0^t |x_i(\tau_i)|^2 d\tau_i\right\} dt$$

$$\leqslant \int_0^T\int_0^t \dots \int_0^t |k(t, \tau_1, \dots, \tau_p)|^2 d\tau_1 \dots d\tau_p dt \cdot \prod_{i=1}^p \int_0^T |x_i(t)|^2 dt$$

$$= \int_0^T\int_0^t \dots \int_0^t |k(t, \tau_1, \dots, \tau_p)|^2 d\tau_1 \dots d\tau_p dt \cdot \prod_{i=1}^p \|x_i\|^2,$$

which proves the theorem.

For the operator

$$[G_p(x_1, \dots, x_p)](t) = \prod_{i=1}^p \int_0^t x_i(t-\tau_i) dh_i(\tau_i), \tag{3.3}$$

an analogous theorem may be formulated:

THEOREM 3. *If $H_i(t)$ are functions of bounded variation* ($\operatorname{Var}_{<0,\infty)} h_i < \infty$ *for* $i = 1, \dots, p$), *and if for an arbitrary* $i_0 = 1, \dots, p$ *is fulfilled the condition*

$$\sup_\omega \int_{-\infty}^{\infty} \left| K_{i_0}\left[j\left(\omega - \sum_i \omega_i\right)\right] \prod_i K_i(j\omega_i)\right|^2 d\overline{\omega}_i < \infty, \quad i \neq i_0, \tag{3.4}$$

in which $K_i(s) = \int_0^\infty e^{-st} dh_i(t)$ *is the Laplace-Stieltjes transform of the function* $h_i(t)$, *then the operator* (3.3) *transforms* $[L^2(0,T)]^p$ *into* $L^2(0,T)$, *and*

$$\|G_p\| \leqslant \left\{\sup_\omega \int_{-\infty}^{\infty} \left| K_{i_0}\left[j\left(\omega - \sum_i \omega_i\right)\right] \prod_i K_i(j\omega_i)\right|^2 d\overline{\omega}_i\right\}^{1/2}, \quad i \neq i_0. \tag{3.5}$$

Symbol $d\overline{\omega}_i$ *means that the integrals appearing in the Eqs.* (3.4), (3.5) *are multiple (here of the order* $p-1$). *Such notations will be used throughout this paper in the cases in which no ambiguity could result.*

P r o o f. Let us consider the function

$$y(t) = \prod_{i=1}^{p} y_i(t) = \prod_{i=1}^{p} \int_0^t x_i(t-\tau_i)\,dh_i(\tau_i).$$

If now another function $y^*(t)$ is introduced,

$$y^*(t) = \prod_{i=1}^{p} y_i^*(t) = \prod_{i=1}^{p} \int_{max(0,\,t-T)}^{t} x_i(t-\tau_i)\,dh_i(\tau_i),$$

it may then easily be observed that $y^*(t) = y(t)$ for $t \in \langle 0, T\rangle$ and

$$\int_0^T |y(t)|^2\,dt \leqslant \int_0^\infty |y^*(t)|^2\,dt$$

If $x_i \in L^2(0, T)$, then

$$\int_0^T |x_i(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_{iT}(j\omega)|^2 d\omega \qquad \text{(Parseval's formula)}$$

with $X_{iT}(j\omega) = \int_0^T x_i(t)e^{-j\omega t}dt$. We obtain the equation

$$Y_i^*(j\omega) = \int_0^\infty e^{-j\omega t} \int_{max\,(0,\,t-T)}^{t} x_i(t-\tau_i)\,dh_i(\tau_i)\,dt = \int_0^\infty e^{-j\omega\tau} dh_i(\tau)\cdot\int_0^T x_i(t)e^{-j\omega t}dt$$
$$= K_i(j\omega)X_{iT}(j\omega).$$

By the method of induction, we may prove that for arbitrary $i_0 = 1, \ldots, p$ the following relation holds:

$$Y^*(j\omega) = \int_0^\infty y^*(t)e^{j\omega t}dt = \frac{1}{(2\pi)^{p-1}}\int_{-\infty}^{\infty} Y_{i_0}^*\Big[j\Big(\omega - \sum_i \omega_i\Big)\Big]\prod_i Y_i^*(j\omega_i)\,d\overline{\omega}_i$$

$$= \frac{1}{(2\pi)^{p-1}}\int_{-\infty}^{\infty} K_{i_0}\Big[j\Big(\omega - \sum_i \omega_i\Big)\Big]\prod_i K_i(j\omega_i)X_{i_0T}\Big[j\Big(\omega - \sum_i \omega_i\Big)\Big]\prod_i X_{iT}(j\omega_i)\,d\overline{\omega}_i$$
$$(i \neq i_0).$$

Using now the Buniakovski-Schwarz inequality we establish the following estimate

$$|Y^*(j\omega)|^2 \leqslant \frac{1}{(2\pi)^{p-1}}\int_{-\infty}^{\infty}\Big|K_{i_0}\Big[j\Big(\omega - \sum_i \omega_i\Big)\Big]\prod_i K_i(j\omega_i)\Big|^2 d\overline{\omega}_i$$

$$\times\int_{-\infty}^{\infty}\Big|X_{i_0T}\Big[j\Big(\omega - \sum_i \omega_i\Big)\Big]\Big|^2\prod_i |X_{iT}(j\omega_i)|^2 d\overline{\omega}_i \qquad (i \neq i_0),$$

whence

$$\int_0^T |y(t)|^2 dt \leqslant \int_0^\infty |y^*(t)|^2 dt \leqslant \sup_\omega \int_{-\infty}^{\infty} \left| K_{i_0}\left[j\left(\omega - \sum_i \omega_i\right)\right] \prod_i K_i(j\omega_i) \right|^2 d\overline{\omega}_i$$

$$\times \prod_{k=1}^{p} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_{kT}(j\omega_k)|^2 d\omega_k \right\} \quad i \neq i_0,$$

which proves the theorem.

In the case of a linear operator ($p = 1$)—i.e., the operator

$$[Ax](t) = \int_0^t x(t-\tau) dh(\tau) \tag{3.6}$$

and with $T = \infty$, the inequality (3.4) is transformed (cf. [5]) into the equality

$$\|A\| = \sup_\omega |K(j\omega)|. \tag{3.7}$$

With respect to the last of the operators

$$[G_p(x_1, \ldots, x_p)](t) = \int_0^t \ldots \int_0^t x_1(t-\tau_1) \ldots x_p(t-\tau_p) k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p \tag{3.8}$$

$$= \int_0^t \ldots \int_0^t k(t-\tau_1, \ldots, t-\tau_p) x_1(\tau_1) \ldots x_p(\tau_p) d\tau_1 \ldots d\tau_p,$$

the following theorem may be formulated:

THEOREM 4. *If*

$$\int_0^\infty \ldots \int_0^\infty k|(\tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p < \infty$$

and for arbitrary $i_0 = 1, \ldots, p$, is satisfied the condition

$$\sup_\omega \int_{-\infty}^{\infty} \left| K\left[j\omega_k, j\left(\omega - \sum_i \omega_i\right), j\omega_m \right] \right|^2 d\overline{\omega}_i < \infty, \quad \begin{array}{l} m = i_0+1, \ldots, p, \\ m = i_0+1, \ldots, p, \\ i \neq i_0; \end{array}$$

$K(s_1, \ldots, s_p)$ being the p-dimensional Laplace transform of $k(t_1, \ldots, t_p)$, then the operator (3.8) *transforms $[L^2(0, T)]^p$ into $L^2(0, T)$, and*

$$\|G_p\| \leqslant \left\{ \sup_\omega \int_{-\infty}^{\infty} \left| K\left[j\omega_k, j\left(\omega - \sum_i \omega_i\right), j\omega_m \right] \right|^2 d\overline{\omega}_i \right\}^{1/2}, \quad \begin{array}{l} k = 1, \ldots, i_0-1, \\ m = i+1, \ldots, p, \\ i \neq i_0. \end{array} \tag{3.9}$$

P r o o f of this theorem will be preceded by the following lemma:

LEMMA 1. *If $\bar{v}(t_1, \ldots, t_p)$ is an absolutely integrable function in the region $\{\langle 0, \infty), \ldots, \langle 0, \infty)\}$, and the function $v(t) = \bar{v}(t, \ldots, t)$ is absolutely integrable in the interval $\langle 0, \infty)$, then*

$$V(j\omega) = \frac{1}{(2\pi)^{p-1}} \int_{-\infty}^{\infty} \bar{V}\left[j\omega_k, j\left(\omega - \sum_i \omega_i\right), j\omega_m\right] d\overline{\omega_i},$$

where

$$\bar{V}(j\omega_1, \ldots, j\omega_p) = \int_0^\infty \ldots \int_0^\infty \bar{v}(t_1, \ldots, t_p) e^{-j\sum_{i=1}^{p} \omega_i t_i} dt_1 \ldots dt_p,$$

$$V(j\omega) = \int_0^\infty v(t) e^{-j\omega t} dt.$$

P r o o f of the lemma. Since

$$\bar{v}(t_1, \ldots, t_p) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{V}(j\omega_1, \ldots, j\omega_p) e^{j\sum_{i=1}^{p} \omega_i t_i} d\omega_1 \ldots d\omega_p,$$

then

$$v(t) = \bar{v}(t, \ldots, t) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{V}(j\omega_1, \ldots, j\omega_p) e^{jt\sum_{i=1}^{p} \omega_i} d\omega_1 \ldots d\omega_p.$$

On substituting $\omega = \sum_{i=1}^{p} \omega_i$ and $\omega_{i_0} = \omega - \sum_{i \neq i_0} \omega_i$, we obtain:

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{(2\pi)^{p-1}} \int_{-\infty}^{\infty} \bar{V}\left[j\omega_k, j\left(\omega - \sum_i \omega_i\right), j\omega_m\right] d\overline{\omega_i} \right\} e^{j\omega t} d\omega$$

which, in view of $v(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} V(j\omega) e^{j\omega t} d\omega$, proves the lemma.

P r o o f of Theorem 4. If

$$y(t) = \int_0^t \ldots \int_0^t k(\tau_1, \ldots, \tau_p) x_1(t-\tau_1) \ldots x_p(t-\tau_p) d\tau_1 \ldots d\tau_p,$$

and $x_i \in L^2(0, T)$ for $i = 1, \ldots, p$, then by introducing the functions

$$\bar{v}(t_1, \ldots, t_p) = \int_{max(0,\, t_1 - T)}^{t_1} \ldots \int_{max(0,\, t_p - T)}^{t_p} k(\tau_1, \ldots, \tau_p) x_1(t_1 - \tau_1) \ldots x_p(t_p - \tau_p) d\tau_1 \ldots d\tau_p,$$

$$v(t) = \bar{v}(t, \ldots, t),$$

we may observe that:

1) $v(t) = y(t)$ for $t \in \langle 0, T\rangle$,

2)
$$\bar{V}(j\omega_1, \ldots, j\omega_p) = \int_0^\infty \ldots \int_0^\infty e^{-j\sum_{i=1}^{p}\omega_i t_i} \int_{max(0,\, t_1-T)}^{t_1} \ldots \int_{max(0,\, t_p-T)}^{t_p} k(\tau_1, \ldots, \tau_p) x_1(t_1-\tau_1)$$

$$\ldots x_p(t_p-\tau_p) d\tau_1 \ldots d\tau_p dt_1 \ldots dt_p = \int_0^\infty \ldots \int_0^\infty k(\tau_1, \ldots, \tau_p) e^{-j\sum_{i=1}^{p}\omega_i\tau_i} d\tau_1 \ldots d\tau_p$$

$$\times \int_0^T x_1(t) e^{-j\omega_1 t} dt \ldots \int_0^T x_p(t) e^{-j\omega_p t} dt = K(j\omega_1, \ldots, j\omega_p) \prod_{i=1}^{p} X_{iT}(j\omega_i).$$

Making use of the lemma previously proved, and applying the Buniakovski-Schwarz inequality $p-1$ times, we obtain the estimate:

$$|V(j\omega)|^2 = \left|\frac{1}{(2\pi)^{p-1}} \int_{-\infty}^{\infty} K\Big[j\omega_k, j\Big(\omega - \sum_i{}' \omega_i\Big), j\omega_m\Big] X_{i_0T}\Big[j\Big(\omega - \sum_i{}' \omega_i\Big)\Big]\right.$$

$$\left.\times \prod_i X_{iT}(j\omega_i) d\bar{\omega}_i\right|^2 \leqslant \frac{1}{(2\pi)^{p-1}} \int_{-\infty}^{\infty} \left|K\Big[j\omega_k, j\Big(\omega - \sum_i{}' \omega_i\Big), j\omega_m\Big]\right|^2 \Big| X_{i_0T}$$

$$\times \Big[j\Big(\omega - \sum_i{}' \omega_i\Big)\Big]\Big|^2 d\bar{\omega}_i \prod_i \int_{-\infty}^{\infty} |X_{iT}(j\omega_i)|^2 d\omega_i,$$

whence

$$\int_0^T |y(t)|^2 dt \leqslant \int_0^\infty |v(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |V(j\omega)|^2 d\omega$$

$$\leqslant \sup_\omega \int_{-\infty}^{\infty} \left|K\Big[j\omega_k, j\Big(\omega - \sum_i{}' \omega_i\Big), j\omega_m\Big]\right|^2 d\bar{\omega}_l \prod_{l=1}^{p} \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} |X_{lT}(j\omega_l)|^2 d\omega_l\right\}.$$

From this inequality, it may be concluded that $G_p \in \big([L^2(0, T)]^p \to L^2(0, T)\big)$, and that the estimate (3.9) holds true, which concludes the proof.

With $p = 1$ the operator (3.7) becomes a particular case of the operator (3.6) (it suffices to put $h(t) = \int_0^t k(\tau) d\tau$) and the remarks following the Theorem 3 hold true once more.

3.2. Operators in the space $L^2_M(0, T)$

As already indicated, properties of the operators in vector spaces ($M > 1$) follow from the properties of their component operators in the space $L^2(0, T)$: if all the operators $G_{pim_1 \ldots m_p}$ transform $[L^2(0, T)]^p$ into $L^2(0, T)$, then the operator G_p transforms $[L^2_M(0, T)]^p$ into $L^2_M(0, T)$. The corresponding theorems then have the form:

THEOREM 5. *If the operators* (2.3) *transform* $[L^2(0, T)]^p$ *into* $L^2(0, T)$, *then the operator* (2.2) *transforms* $[L^2_M(0, T)]^p$ *into* $L^2_M(0, T)$, *and*

$$(3.10)\qquad \|G_p\| \leqslant \Big\{\sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_1=1}^{M}\int_0^T\int_0^t\dots\int_0^t |k_{pim_1\dots m_p}(t,\tau_1,\dots,\tau_p)|^2 d\tau_1\dots d\tau_p\Big\}^{1/2}.$$

P r o o f. It will be sufficient to prove the validity of the estimate (3.10). This follows directly from the inequality

$$\|G_p(x_1,\dots,x_p)\|^2 \leqslant \sum_{i=1}^{M}\int_0^T\Big|\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\int_0^t\dots\int_0^t k_{pim_1\dots m_p}(t,\tau_1,\dots,\tau_p)$$
$$\times x_{1m_1}(\tau_1)\dots x_{pm_p}(\tau_p)d\tau_1\dots d\tau_p\Big|^2 dt$$
$$\leqslant 2\sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\int_0^T\int_0^t\dots\int_0^t |k_{pim_1\dots m_p}(t,\tau_1,\dots,\tau_p)x_{1m_1}(\tau_1)\dots x_{pm_p}(\tau_p)d\tau_1\dots d\tau_p|^2 dt$$
$$\leqslant 2\sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\int_0^T\int_0^t\dots\int_0^t |k_{pim_1\dots m_p}(t,\tau_1,\dots,\tau_p)|^2 d\tau_1\dots d\tau_p\, dt$$
$$\times\int_0^T |x_{1m_1}(t)|^2 dt\dots\int_0^T |x_{pm_p}(t)|^2 dt$$
$$\leqslant \sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\int_0^T\int_0^t\dots\int_0^t |k_{pim_1\dots m_p}(t,\tau_1,\dots,\tau_p)|^2 d\tau_1\dots d\tau_p dt$$
$$\times\sum_{m_1=1}^{M}\int_0^T |x_{1m_1}(t)|^2 dt\dots\sum_{m_p=1}^{M}\int_0^T |x_{pm_p}(t)|^2 dt.$$

THEOREM 6. *If the operators* (2.4) *transform* $[L^2(0, T)]^p$ *into* $L^2(0, T)$, *then the operator* (2.2) *transforms* $[L^2_M(0, T)]^p$ *into* $L^2_M(0, T)$, *and*

$$(3.11)\qquad \|G_p\| \leqslant \Big\{\sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\sup_{\omega}\int_{-\infty}^{\infty}\Big|K_{m_0}\Big[j\Big(\omega-\sum_{m_l}\omega_{m_l}\Big)\Big]\prod_{m_l}K_{m_l}(j\omega_{m_l})|^2 d\overline{\omega}_{m_l}\Big\}^{1/2},$$
$$m_l \neq m_0$$

THEOREM 7. *If the operators* (2.5) *transform* $[L^2(0, T)]^p$ *into* $L^2(0, T)$, *then the operator* (2.2) *transforms* $[L^2_M(0, T)]^p$ *into* $L^2_M(0, T)$, *and*

$$\|G_p\| \leqslant \Big\{\sum_{i=1}^{M}\sum_{m_1=1}^{M}\dots\sum_{m_p=1}^{M}\sup_{\omega}\int_{-\infty}^{\infty}\Big|K_{pim_1\dots m_p}\Big[j\omega_k, j\Big(\omega-\sum_{m_l}\omega_l\Big), j\omega_n\Big]\Big|^2 d\overline{\omega}_{m_l}\Big\}^{1/2},$$
$$k = 1,\dots,m_0-1;\ n = m_0+1,\dots,p;\ m_l\neq m_0.$$

P r o o f s of both these theorems are similar to those in the case of Theorem 5. If $p = 2$, then a stronger estimate of the norms may be established:

$$\|G_2\| \leqslant \Big\{\sum_{i=1}^{M}\Big[\sup_{\omega}\int_{-\infty}^{\infty}\|K_{2i}[j\omega_1, j(\omega-\omega_1)\|_2 d\omega_1]^2\Big\}^{1/2}.$$

Here

$$K_{2i}(j\omega_1, j\omega_2) = \{K_{2im_1}(j\omega_1)K_{2im_2}(j\omega_2)\}_{(m_1, m_2=1,\dots,M)} \text{ for operator (2.4),}$$

and

$$K_{2i}(j\omega_1, j\omega_2) = \{K_{2im_1m_2}(j\omega_1, j\omega_2)\}_{(m_1m_2=1,\dots,M)} \text{ for operator (2.5),}$$

the symbol $\| \ \|_2$ denoting the Euclidean norm.

In the linear case ($p = 1$), operator (2.5) becomes a particular case of operator (2.4). The following norm estimate may then be given [5]:

$$\|A\| = \|G_1\| \leqslant \sup_{\omega} \|K(j\omega)\|_2.$$

Here $K(j\omega) = \{K_{im}(j\omega)\}_{(i,m=1,\dots,M)}$, and with $T = \infty$ the inequality is transformed into an equality.

4. Final conclusions

The set of theorems presented in Sec. 3 determines the conditions under which the operators considered transform the spaces of square integrable functions into themselves. From Theorem 1 it follows that if the assumptions of any of these theorems are satisfied, and the radius of convergence of $\sum_{p=2}^{\infty} \|G_p\| y^p$ is positive, then the Eq. (3)—which assumes the form of an integral equation or a set of integral equations—has at a sufficiently "small" z—i.e., at

$$\sum_{i=1}^{M} \int_0^T |z_i(t)|^2 dt\Big\}^{1/2} \leqslant \alpha$$

—the unique solution $x^* \in L_M^2(0, T)$, $M \geqslant 1$, with a norm satisfying the estimate

$$\|x^*\| \leqslant f(\|z\|), \tag{4.1}$$

f being a non-negative continuous function, defined and increasing in the interval $\langle 0, \alpha\rangle$ and such that $f(0) = 0$. The value of α depends on the norms of operators A and G_p, and so—according to the proof of theorem 1 [4]—the better the estimates of those norms the greater will be the permissible values of α. In practical applications both the α and the estimate (4.1) may be evaluated numerically for each particular equation.

Note that the form of the Eq. (2.1) contains a wide class of integral, differential and integro-differential equations. In particular, the following equation, frequently encountered in practice may be reduced to such a form:

$$\lambda x(t) - \int_0^t k_1(t-\tau)x(\tau)d\tau = y(t) + \sum_{p=2}^{\infty} \int_0^t \dots \int_0^t k_p(t-\tau_1, \dots, t-\tau_p)x(\tau_1) \dots x(\tau_p)d\tau_1 \dots d\tau_p. \tag{4.2}$$

This is a non-linear Volterra equation, the linear part of which possesses exactly one solution if, and only if, λ is a regular value [6] (in that case—with respect to the space $L^2(0, T)$) of the linear operator A_1

$$[A_1 x](t) = \int_0^t k_1(t-\tau x(\tau) d\tau.$$

The solution may be represented in the form

$$x(t) = [Ay](t) = \int_0^t k(t-\tau) y(\tau) d\tau.$$

Here $A = (\lambda I - A_1)^{-1}$, and I is the identity operator. A similar procedure applied to the Eq. (4.2) with the substitution $z = Ay$ yields the Eq. (2.1). The same results may also be obtained in a more general case—i.e., for a system of equations and for different forms of the integral operators.

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