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# On the matching of solutions for unsteady transonic nozzle flows

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MANY solutions have been presented for two-dimensional transonic nozzle flows. Two of the more interesting of these solutions are those by TOMOTIKA and TAMADA (1950) and SZANIAWSKI (1965). However, it has not been made clear under what conditions either solution is valid. In this paper the methods of matched asymptotic expansions are used to derive the Szaniawski power series systematically and to show that this solution should be considered an outer solution which may not be uniformly valid as the throat is approached. The inner throat region is governed by the non-linear transonic equation which admits as one class of solutions, similarity solutions. The analysis is made using the nonsteady inviscid equations of motion, with steady flow results being derivable as a special case. As an example, a similarity solution for unsteady transonic flow with an infinitesimally thin shock wave, found by ADAMSON and RICHEY (1972) is used as an inner solution; through matching, conditions on the outer solution are obtained, illustrating the overall kind of problem to which the similarity solutions correspond.

Istnieje wiele rozwiązań dotyczących dwuwymiarowych przepływów w dyszach. Do bardziej interesujących należą dwa rozwiązania przedstawione przez TOMOTIKA i TAMADA (1950) oraz przez SZANIAWSKIEGO (1965). Nie jest jednak jasne, w jakich warunkach rozwiązania te są słuszne. W niniejszej pracy przedstawiono metody kojarzenia rozwinięć asymptotycznych, pozwalające wyprowadzić w sposób systematyczny szeregi potęgowe Szaniawskiego oraz wykazać, że rozwiązanie to należy rozpatrywać jako rozwiązanie zewnętrzne, które może tracić ważność przy zbliżaniu się do gardzieli. Wewnętrzny obszar gardzieli opisany jest przez nieliniowe równanie okołodźwiękowe, dopuszczające rozwiązania samopodobne jako jedną z klas rozwiązań. Analizę wykonano za pomocą nieustalonych równań ruchu, z których można, jako przypadek szczególny, uzyskać przepływy ustalone. W przykładzie zastosowano jako rozwiązanie wewnętrzne rozwiązanie samopodobne, otrzymane przez ADAMSONA i RICHEY'A (1972) i dotyczące nieustalonego przepływu okołodźwiękowego z nieskończenie cienką falą uderzeniową; drogą zszywania otrzymano warunki dla rozwiązania zewnętrznego.

Существует много решений, касающихся двумерных околозвуковых течений в соплях. К более интересным принадлежат два решения, представленные Томотика и Тамада (1950) и Шанявским (1965). Однако не ясно при каких условиях эти решения справедливы. В настоящей работе представлены методы сращивания асимптотических разложений, позволяющие вывести систематическим образом степенные ряды Шанявского и показать, что это решение следует рассматривать как внешнее решение, которое может терять важность при приближении к горловине. Внутренняя область горловины описана нелинейным околозвуковым решением, которое допускает автомодельное решение как один из классов решений. Анализ проведен с помощью неустановившихся уравнений движения, из которых, как частный случай, можно получить установившиеся течения. В примере применено, как внутреннее решение, автомодельное решение полученное Адамсоном и Ричи (1972) и касающиеся неустановившиегося околозвукового течения с бесконечно тонкой ударной волной; путем сшития получены условия для внешнего решения.

## 1. Introduction

THE STUDY of two-dimensional transonic nozzle flows has generally been carried out by searching for similarity solutions which give nozzle-like flows. For example, TOMOTIKA and TAMADA [1] introduce particular similarity transformations in the well-known transonic small-disturbance equation, and then are able to obtain similarity solutions for nozzle flows as well as for other transonic flow problems. This transformation has

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been employed also by SICHEL [2] in his study of nozzle flows with shock waves, and has been extended to cover unsteady transonic channel flows both without and with shock waves by ADAMSON [3] and ADAMSON and RICHEY [4], respectively. FRANKL [5] and GUDERLEY [6] have proposed a different kind of similarity transformation, which has been compared with that of TOMOTIKA and TAMADA in a particular application by SZANIAWSKI [7].

On the other hand, an entirely different approach has been presented by SZANIAWSKI [8] in several papers, with the basic ideas being exemplified in the cited reference. For steady transonic nozzle flows, SZANIAWSKI expands the perturbation velocity potential in an assumed power series in the transverse coordinate, and this series is substituted into the general potential equation and boundary conditions. The method has been extended to flows with shock waves by KOPYSTYŃSKI and SZANIAWSKI [9].

The most interesting aspect of the solution given by SZANIAWSKI, when compared to the similarity solutions, is that arbitrary wall shapes may be considered; that is, this is a direct method. Clearly, similarity solutions are not general, from the viewpoint that one does not obtain a general solution from which specific solutions may be constructed by the application of boundary conditions. Instead one obtains self-similar solutions which satisfy only very special boundary conditions which may or may not correspond to any given physical problem. Hence, while similarity solutions are extremely valuable in providing instructive results with a minimum of computational effort, they do suffer from this lack of applicability insofar as general nozzle shapes are concerned. The direct method proposed by SZANIAWSKI therefore would appear to hold a distinct advantage over the similarity method. However, it has not been made clear under what conditions or restrictions either solution is valid. It is the purpose of this paper, using the methods of matched asymptotic expansions, to illustrate how the SZANIAWSKI power series may be derived in a systematic fashion, and to show that this solution should be considered as an outer solution which may not be uniformly valid as the nozzle throat is approached. The inner throat region will be shown to be governed by the non-linear transonic equations, which admit as one class of solutions those similarity solutions referred to previously.

The analysis is performed using the general equations of motion for nonsteady inviscid flow, with the steady-flow results derivable as a special case. As an example of the use of the analysis, a similarity solution for unsteady transonic flow with a thin shock wave, found previously [4], is used as an inner solution; through matching, conditions on the outer solution are obtained, illustrating the general kind of flow problem to which the similarity solutions correspond.

The flow is assumed to be two-dimensional, compressible, transonic and irrotational at least up to the orders for which calculations are made; the gas is assumed to follow the perfect gas law and to have constant specific heats.

## 2. Derivation of equations

In the problem considered, small perturbations are superimposed on a steady, sonic, irrotational, two-dimensional stream flowing in the $X$ direction. The dimensionless independent variables $X$, $Y$, and $T$ are referred to $\overline{L}$ for the space variables and $\overline{L}/\overline{a}^*$ for the time. Here, $\bar{L}$ is the throat half-width and $\bar{a}^*$ is the critical sound speed, the bar indicating dimensional quantities (Fig. 1). The velocity potential $\Phi(X, Y, T)$ is made dimensionless with respect to $\bar{L}\bar{a}^*$. Then Bernoulli's equation and the so-called gas dynamic equation may be written as follows:

$$\Phi_T+\frac{a^2}{\gamma-1}+\frac{1}{2}(\Phi_X^2+\Phi_Y^2)=\frac{(\gamma+1)}{2(\gamma-1)}, \tag{2.1}$$

$$(a^2-\Phi_X^2)\Phi_{XX}+(a^2-\Phi_Y^2)\Phi_{YY}-\Phi_{TT}-2\Phi_{XY}\Phi_X\Phi_Y-2\Phi_X\Phi_{XT}-2\Phi_Y\Phi_{YT}=0,$$

where $a$ is the dimensionless speed of sound, referred to $\bar{a}^*$, and $\gamma$ is the ratio of specific heats. The constant right-hand side of Eq. $(2.1)_1$ arises because the undisturbed stream is at uniform sonic velocity.

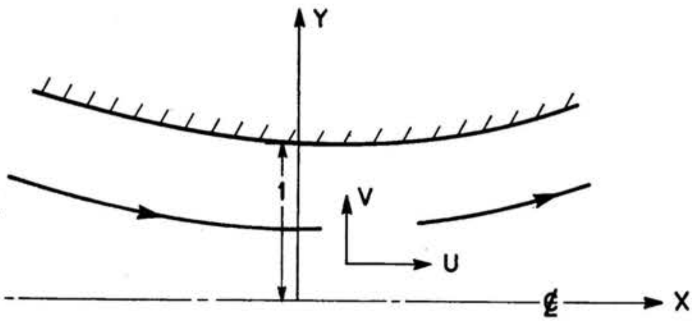


FIG. 1. Sketch of nozzle flow coordinate system.

Since one may wish to consider various ranges of the space and time coordinates, it may in general be appropriate to introduce other non-dimensional variables, say $x$, $y$, and $t$, such that

$$\begin{aligned} X &= \delta x, \quad \delta = \bar{L}_x/\bar{L}, \quad x = O(1),\\ Y &= \varepsilon y, \quad \varepsilon = \bar{L}_y/\bar{L}, \quad y = O(1),\\ T &= \tau t, \quad \tau = \bar{T}_{ch}\Big/\left(\frac{\bar{L}}{\bar{a}^*}\right), \quad t = O(1). \end{aligned} \tag{2.2}$$

In Eqs. (2.2), $\bar{L}_x$ and $\bar{L}_y$ are fictitious lengths of the order of the physical extent of the transonic region under consideration, in the $X$ and $Y$ directions, respectively. Hence $x$ and $y$, which are coordinate distances made dimensionless with respect to $\bar{L}_x$ and $\bar{L}_y$, respectively, are of order unity in this region. Similarly, $t$ is the time made dimensionless with respect to $\bar{T}_{ch}$, the characteristic time associated with the disturbance, and is of order unity.

The velocity potential is written as an asymptotic expansion in terms of a small parameter $E$, which may be chosen as a typical value of the non-dimensional perturbation velocity. The first term describes a uniform sonic stream, and in the examples to be considered, the next few terms proceed in integral powers of $E$:

$$\Phi(X, Y, T)=\delta[x+E\phi_1(x,y,t)+E^2\phi_2(x,y,t)+\ldots]. \tag{2.3}$$

Then the velocity components in the $X$ and $Y$ directions are, respectively,

$$U = \Phi_X = 1+E\phi_{1x}+E^2\phi_{2x}+\ \dots = 1+Eu_1+E^2u_2+\ \dots \tag{2.4}$$
$$V = \Phi_Y = \frac{\delta}{\varepsilon}[E\phi_{1y}+E^2\phi_{2y}+\ \dots] = \frac{\delta}{\varepsilon}[Ev_1+E^2v_2+\ \dots].$$

Thus $U-1 = O(E)$.

The three important possibilities for the order of the non-dimensional characteristic time $\tau$ are $\tau \ll 1$, $\tau = O(1)$, and $\tau \gg 1$. Each of these cases can be shown to imply a different first approximation to the Eq. $(2.1)_2$. In this paper, the "slowly varying" time regime $\tau \gg 1$ will be considered. In view of the expansion for $\Phi$, Eq. (2.3), it is not difficult to show that for $\tau \gg 1$ a time derivative appears earliest, i.e., in the equation for $\phi_2$, when $\tau = O(E^{-1})$. Thus, here we will consider the case where

$$\tau = (kE)^{-1}, \tag{2.5}$$

where $k$ is a constant of order unity.

The parameter $\varepsilon$ is taken to be of order unity because of the physical problem considered. That is, the transonic region covers the whole stream in the $Y$ direction from wall to wall. Here, then, we set

$$\varepsilon = 1. \tag{2.6}$$

It will be seen that the order of magnitude to be chosen for $\delta$ will depend on the flow region of interest, and therefore cannot yet be determined. However, since the physical problems to be considered here are such that the flow region is always contained in a distance of a few throat diameters upstream or downstream of the throat, $\delta$ is at most of order unity.

If Eqs. (2.2), (2.3) and (2.6) are substituted into Eq. (2.1) and Eq. $(2.1)_1$ is then substituted into Eq. $(2.1)_2$, one obtains the governing equation for the velocity potential. Thus,

$$-E^2(\gamma+1)\phi_{1x}\phi_{1xx}+\delta^2\{E\phi_{1yy}+E^2[\phi_{2yy}-(\gamma-1)\phi_{1x}\phi_{1yy}]\} -2\delta^2E^2\phi_{1y}\phi_{1xy}-2\frac{\delta E}{\tau}\phi_{1xt}+\ \dots = 0, \tag{2.7}$$

where only terms to order $E^2$ are retained to illustrate the method without adding undue complexity. It is seen from Eqs. (2.7) and (2.5), that once the relationship between $\delta$ and $E$ has been given, one can find the governing equations for the potential functions $\phi_1, \phi_2, \dots$.

The wall boundary condition for unsteady channel flow has been discussed previously in [3]. Along a wall which may in general be moving with time,

$$Y_w = F(X, T). \tag{2.8}$$

Since the flow velocity is nearly sonic everywhere in the region of interest, the function $F(X, T)$ must be nearly constant, and so Eq. (2.8) can be rewritten, in the coordinates introduced by Eq. (2.2), as

$$y_w = 1+wf(x, t), \tag{2.9}$$

where $f = 0$ at the throat, and $w \ll 1$. Along the wall $y-y_w = 0$ and the Eulerian derivative of $y-y_w$ is also zero. After substituting the changes of variable, the velocity expansions, and the wall definition, Eq. (2.9), into the Eulerian derivative, one finds that, at the wall,

$$\frac{w}{\tau}f_t-[1+E\phi_{1x}+E^2\phi_{2x}+\ ...]wf_x+E\delta[\phi_{1y}+E\phi_{2y}+\ ...] = 0. \tag{2.10}$$

If Eq. (2.5) is now used for $\tau$, it is seen that the term involving $f_t$ is smaller by a factor $E$ than the term involving $f_x$. Therefore, to this order the usual steady state result holds. That is, even though the flow in unsteady and the wall might be moving, the wall is instantaneously a streamline to this order. This result, of course, is true only because we are considering the case $\tau \gg 1$.

## 3. Solutions

### 3.1. Szaniawski series solutions ($\delta = O(1)$)

We first consider cases for which $\delta = 1$. Physically, since $\varepsilon = (\bar{L}_y/\bar{L}) = 1$, this means that the flow region under consideration has an axial length of the order of the throat diameter. Then from Eq. (2.7) with Eq. (2.5) for $\tau$,

$$\begin{gathered}\phi_{1yy} = 0,\\ \phi_{2yy}-(\gamma+1)\phi_{1x}\phi_{1xx}-2\phi_{1y}\phi_{1xy}-2k\phi_{1xt} = 0.\end{gathered} \tag{3.1}$$

If we specify that symmetrical channels are to be considered, so that $V(X, 0, T) = 0$ to all orders, then integration of Eqs. (3.1) gives

$$\begin{gathered}\phi_1 = \phi_1(x, t),\\ \phi_2 = ((\gamma+1)\phi_{1x}\phi_{1xx}+2k\phi_{1xt})\frac{y^2}{2}+h(x, t),\end{gathered} \tag{3.2}$$

where $h(x, t)$ is an arbitrary function of integration.

Next, from Eq. (2.10), for $\delta = 1$ and $\tau = (kE)^{-1}$, the boundary conditions may be derived; since $\phi_{1y} = 0$,

$$w = E^2, \tag{3.3}$$

$$\phi_{2y}(x, \pm 1, t) = \pm f_x(x, t), \tag{3.4}$$

where the upper and lower signs refer to the upper and lower walls, respectively, and $\phi_{2y}$ has been expanded in Taylor series about $y = \pm 1$. Then, substituting Eqs. $(3.2)_2$ and (2.9) into Eq. (3.4), one can show that

$$\phi_{1x}^2 = \frac{2}{(\gamma+1)}f+H(t)-\frac{4k}{(\gamma+1)}\phi_{1t}, \tag{3.5}$$

where $H(t)$ is an arbitrary function of time. In steady flow, where $\phi_{1t} = 0$, $H =$ constant is seen to be the value of the perturbation velocity at the throat, where $f = 0$. In this case, the solution for $\phi_{1x} = u_1$ is given completely by Eq. (3.5). For unsteady flow, it is convenient to consider a derivative of Eq. (3.5), written in terms of $u_1 = \phi_{1x}$. Thus

$$\frac{2k}{(\gamma+1)}u_{1t}+u_1 u_{1x} = \frac{1}{(\gamma+1)}f_x \tag{3.6}$$

and, it is seen that Eq. (3.6) gives the rate of change of $u_1$ along characteristics

$$\frac{dx}{dt} = \frac{\gamma+1}{2k}u_1 \tag{3.7}$$

in terms of the instantaneous wall slope.

The exact equations for one-dimensional, nonsteady, isentropic flow can be integrated to give, in the present notation, $U \pm 2a/(\gamma-1) =$ constant along $dX/dT = U \pm a$. Thus, in a region where $U \approx a \approx 1$, small disturbances are carried downstream at a speed $U+a \approx 2$ and upstream (with respect to the fluid) at a much lower speed $U-a$. If we were to assume a simple wave such that $U+2a/(\gamma-1) =$ constant (no disturbances created upstream), changes $\Delta U$ and $\Delta a$ would be related by $\Delta a = -\frac{1}{2}(\gamma-1)\Delta U$, and so $U-a = \frac{1}{2}(\gamma+1)\Delta U$. If we set $\Delta U \sim Eu_1$ and $T = \tau t = (kE)^{-1}t$, the two families of characteristics become, in the limit as $E \to 0$, $t =$ constant and $dx/dt = (\gamma+1)u_1/(2k)$. The latter is identical to Eq. (3.7) and so the present formulation for $\phi_1$ appears to retain only the disturbances moving upstream relative to the fluid, since their absolute speed of propagation is small compared with the sound speed. The disturbances moving rapidly downstream are lost because they remain in the region of interest for a time which is very short in comparison with $\tau = (kE)^{-1}$.

A solution $u_1$ to Eq. (3.6) can be found for a given wall shape $f$ if an initial condition is specified. The potential $\phi_1$ is obtained by integration over $x$, and therefore contains an arbitrary function of $t$. This function of $t$ is, however, related to $H(t)$, in the sense that the combination $H(t)-4k\phi_{1t}/(\gamma+1)$ appearing in Eq. (3.5) is determined by the solution for $u_1$, and thus by the initial conditions. For the very simple steady parabolic wall shape, where $f =$ constant $\cdot x^2$, analytic solutions may be found; for other more complicated wall shapes, it appears that numerical methods must be employed. In either event, and even for the case where the walls move with time, the procedure seems to be relatively simple. On the other hand, because of the fact that only one family of characteristics is present, and thus that no upstream or downstream boundary conditions can be applied, it may be that the types of flows which can be studied in this time regime are limited. More detailed analysis is necessary to resolve this point.

If Eq. (3.5) is used to calculate the relevant terms in $\phi_2$, Eq. $(3.2)_2$, the second-order velocities may be written as follows:

$$\begin{aligned} u_2 &= \phi_{2x} = f_{xx}\frac{y^2}{2}+h_x, \\ v_2 &= \phi_{2y} = f_x y. \end{aligned} \tag{3.8}$$

If the third-order velocity potential function $\phi_3$ is found and boundary conditions involving $\phi_{3y}$ at the wall are used, an equation involving $h(x, t)$ results. This equation may be written as follows:

$$(3.9)\quad \left(\frac{2k}{\gamma+1}\right)h_t+\phi_{1x}h_x = -\frac{\phi_{1x}}{6}\left\{f_{xx}+(2\gamma-3)\phi_{1x}^2+\frac{6k(\gamma-1)}{(\gamma+1)}\phi_{1t}\right\}$$
$$-\frac{kH'}{4}x-\frac{2k}{(\gamma+1)}f_{xt}+\frac{k}{2(\gamma+1)}\left[\int\limits^{x}\left(\left(\frac{3-\gamma}{2}\right)\phi_{1x}^2-f\right)dx\right]_t+A(t).$$

Thus, the right-hand side of Eq. (3.9) is known except for the function of time $A(t)$, which can be incorporated in $h_t$; $h(x, t)$ is one of the terms in $\phi_2$, and $\phi_2$ may contain an arbitrary function of time. It is seen that Eqs. (3.6) and (3.9) have the same characteristics. Finally, it should be noted that it is the derivative, $h_x$, which is required for the solution for $U$.

Derivation of the higher-order solutions would present no difficulties; they simply become more complex. Further, if the solutions given in Eqs. (3.5), (3.8) and (3.9) are written specifically for steady flow, it is seen that they agree precisely with those solutions given by SZANIAWSKI [8], found by use of an assumed power series. The present method simply provides a systematic method of derivation, and allows a precise ordering of each of the terms relative to each other, as well as to comparable terms in different solutions, as will be seen.

### 3.2. Existence of inner region

The question now arises as to whether the solutions presented in the previous section are uniformly valid throughout the transonic region. In order to answer this question, it is convenient to write the solution found so far for the $x$-component of the velocity, as follows:

$$(3.10)\qquad U = 1+E\phi_{1x}+E^2\left[f_{xx}\frac{y^2}{2}+h_x\right]+\ldots,$$

where Eqs. (3.5) and (3.8) have been used for $\phi_{1x}$ and $\phi_{2x}$. At this point, the throat region for a flow which goes through or very near sonic velocity is chosen for study. The throat is taken to be at $x = 0$, and furthermore it is assumed for this example calculation that the point of minimum area always remains at $x = 0$, even though the wall shape may vary with time. From Eq. (2.9), these conditions require

$$(3.11)\qquad f(0, t) = f_x(0, t) = 0.$$

From Eq. (3.10), it is seen that since $\phi_{1x} \to 0$ at or near the throat, and $f_{xx}$ does not in general go to zero in this region, it is possible that the first perturbation term can decrease in magnitude as $x \to 0$ until it becomes of the order of the second perturbation term. Thus a non-uniformity may exist. In order to investigate this further, an inner region is postulated wherein the two terms are of the same order. Furthermore, it is assumed that in this inner region, the velocity can be expanded in terms of new variables defined in the manner of equations (2.2):

$$(3.12)\qquad X = \delta^*x^*,\quad Y = y,\quad T = \tau^*t^*,$$
$$U = 1+E^*\phi_{1x^*}^*+\ldots = 1+E^*u_1^*+\ldots.$$

Thus only the $x$ coordinate is stretched, since the inner region is very thin but still extends from wall to wall in the $y$ direction. It might be anticipated that $\tau^*$ and $\tau$ will be of the same order, but at this point it is not necessary to make this assumption.

The inner region corresponds to a limit process as $E \to 0$ with $x^*$ and $y$ fixed. In this inner region the quantities $E\phi_{1x}$, $E^2\phi_{2x}$, and $E^*\phi^*_{1x^*}$ are of the same order. That is, since $u_1^* = O(1)$, in the inner region

$$E\phi_{1x} = O(E^*), \quad E^2\phi_{2x} = O(E^*). \tag{3.13}$$

Using Eq. $(3.2)_2$, we assume that $\phi_{2x}$ is of the same order as $(\phi_{1x}\phi_{1xx})_x$. That is, we assume that $\phi_{1xt}$ and $h_x$ are at most of the same order as $\phi_{1x}\phi_{1xx}$ as $x \to 0$. Finally, since $x = \delta^* x^*$, one finds that for $x^* = O(1)$

$$\phi_{2x} = O\left[\left(\frac{E^*}{E\delta^*}\right)^2\right] \tag{3.14}$$

and, using Eq. (3.13), that $\delta^{*2} = O(E^*)$. These results are obtained by assuming that the ratio of $E^2\phi_{2x}$ to $E\phi_{1x}$ does not remain small, but without any further assumption concerning the form of $\phi_{1x}$ as $x \to 0$. For convenience, here we set

$$\delta^{*2} = (\gamma+1)E^*. \tag{3.15}$$

Next, $\tau^*$ is assumed of order $E^{*-n}$:

$$\tau^* = (\gamma+1)^{-1/2}E^{*-n}k^{-1}. \tag{3.16}$$

If one now considers Eq. (2.7) as applied to the inner region (i.e., replace $E$ by $E^*$, $x$ by $x^*$, etc.) and employs Eqs. (3.15) and (3.16) for $\delta^*$ and $\tau^*$, then it is easily shown that the first-order potential equation is

$$\phi^*_{1x^*}\phi^*_{1x^*x^*} - \phi^*_{1yy} + 2k\phi^*_{1x^*t^*} = 0 \tag{3.17}$$

if $n = 1/2$ in Eq. (3.16). That is, if $n > 1/2$, the unsteady term does not appear in Eq. (3.17), and we have already tentatively excluded $n < 1/2$ by our assumptions concerning $\phi_{2x}$.

Equation (3.17) is the well known non-linear transonic equation, written here for unsteady flow; the relative orders of the $x$ and $y$ coordinates in the inner region are, of course, precisely the same as that found in any transonic-flow problem in which this equation holds. In the present problem, thus, it is clear that the Szaniawski type of solution should be considered as an outer transonic solution, described by linear equations. As the velocity gets closer and closer to sonic velocity, there is an inner region in which the non-linear equation must be satisfied. It is of interest, at this point, to consider known inner solutions and through matching, to ascertain to what kind of outer flow solutions they correspond; this is done in the next section.

### 3.3. Inner solutions ($\delta^* = O(E^{*1/2})$)

As mentioned previously, there are several known solutions to the transonic nozzle problem in the regime governed by the non-linear Eq. (3.17). In general, they are similarity solutions and therefore valid only for specific wall shapes. However, in view of the above analysis, this is not a serious drawback because the extent of the inner region is very small compared to the throat diameter. In this section the unsteady similarity solutions given by ADAMSON and RICHEY [4] for flows with infinitesimally thin shocks imbedded in them are used as examples. The basic similarity transformation was introduced by TOMOTIKA and TAMADA [1] and extended for unsteady flow in Refs. [3] and [4]; since the details of the calculation are given in these references, only a very brief review of the important ideas is given here.

The transformation applied to Eq. (3.17) is,

$$s^* = x^* + by^2 + \beta(t^*),$$
$$u_1^* = z(s^*) + 4b^2y^2 - 2k\beta', \tag{3.18}$$

where $\beta(t^*)$ is an arbitrary function of time, the prime on $\beta$ indicates differentiation with respect to $t^*$, and $b$ is an arbitrary constant. From the irrotationality condition, then, $v_1^*$ may be derived as

$$v_1^* = y\left\{2bz + 8b^2x^* + \frac{8b^3y^2}{3} + 8b^2\beta - 4k^2\beta''\right\} \tag{3.19}$$

and from Eq. (3.17), one obtains the governing equation for $z$

$$zz' + (z' - 4b)(z' + 2b) = 0, \tag{3.20}$$

where the prime denotes differentiation with respect to $s^*$. Equation (3.20) has the solution [1]

$$(z - 4bs^*)^2(z + 2bs^*) = \frac{\alpha^3}{4b^3}, \tag{3.21}$$

where $\alpha$ is a constant of integration which characterizes the inviscid solution curves. That is, $\alpha$ = constant along a given solution curve. Equation (3.21) is the solution given by TOMOTIKA and TAMADA [1] for steady flow; thus the transformation given by Eq. $(3.18)_1$ allows one to study the unsteady counterpart of these nozzle flows.

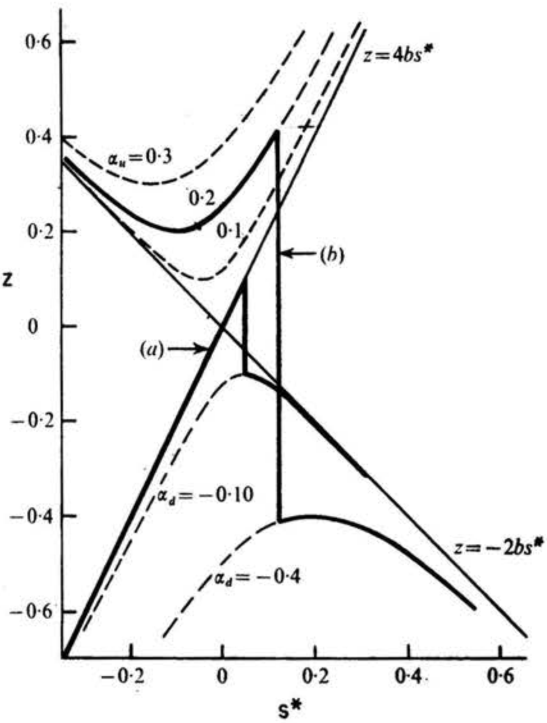


FIG. 2. $z$ vs $s^*$ showing inviscid inner solutions for various $\alpha$ (dashed) and solutions for flows with infinitesimally thin shocks. Solution (a), shocks in accelerating flow; solution (b), shock in decelerating flow. (From reference [4]).

As shown in Ref. [4], it is possible to consider flows with infinitesimally thin shocks such that the flow is inviscid up to and behind the shock; in the $z$, $s^*$ plane, the shock solution joins solutions characterized by different values of $\alpha = \alpha_u$ upstream and $\alpha = \alpha_d$ downstream of the shock. Typical solutions in the $z$, $s^*$ plane are shown in Fig. 2 (from reference [4]). The physical meaning of the solutions in this plane is given by noting in Eq. $(3.18)_2$ that along the $x^*$ axis of a steady flow, $u_1^* = z$.

In Fig. 2 it is seen that for $\alpha = \alpha_d < 0$ the inviscid-flow solutions all lie below $z = 0$ and hence are for subsonic flows that accelerate and then decelerate; for $\alpha = \alpha_u > 0$, $z > 0$ and so these solutions are for supersonic flows which decelerate and then accelerate. For $\alpha = 0$ there are two special solutions; for $z = 4bs^*$, the flow accelerates from subsonic to supersonic velocities, while for $z = -2bs^*$, the reverse occurs. Two solutions for flows with imbedded shocks are shown. Curve (a) shows a flow which begins as a simple accelerating nozzle flow, and then shocks to a decelerating subsonic nozzle flow. Curve (b) shows a flow which begins as a decelerating nozzle flow, goes through a minimum and begins to accelerate, and then shocks to a decelerating subsonic nozzle flow. It can be shown [4] that the shock position is a function of $\alpha_u$ and $\alpha_d$. For example, for curve (a), where $\alpha_u = 0$, the shock position, $s_0^*$, is

$$s_0^* = -\frac{\alpha_d}{8b^2}. \tag{3.22}$$

It is seen in Fig. 2 that both types of solution (curve (a) and curve (b)) become asymptotic to $z = -2bs^*$ as $s^*$ becomes large and positive, but that for $s^*$ large and negative, the upstream solution is either $z = 4bs^*$ or asymptotic to $z = -2bs^*$. Hence it is necessary, for later matching purposes, only to find the asymptotic form for the inviscid solutions which are near $-2bs^*$. This can be shown to be [4]

$$z = -2bs^* + \frac{\alpha^3}{144b^5 s^{*2}} + \dots, \tag{3.23}$$

where $\alpha = \alpha_u$ for $s^*$ large and negative and $\alpha = \alpha_d$ for $s^*$ large and positive.

With the above relations, one can find the complete solution in the inner region, for steady or unsteady flow, either with or without shock waves. It should be clearly understood that they are only one class of solutions to Eq. (3.17) for nozzle flows; there are many others. On the other hand, such similarity solutions are an important class of solutions because they give so much information with relatively little computational effort. Hence it is worthwhile to investigate those conditions under which such solutions hold. This can be done by studying the possibility of matching inner and outer solutions.

### 3.4. Matching inner and outer solutions

In view of the fact that the throat is at $x = 0$ in the outer variables, matching is performed in the limit as $|x| \to 0$ for the outer solutions, and as $|x^*| \to \infty$ for the inner solutions. In order to complete the demonstration of consistency between inner and outer solutions, the possibility of matching should be shown both upstream and downstream of the throat. Here, in the interests of brevity, the inner solutions will be matched only with the downstream outer solutions, this being sufficient to obtain the information desired. In addition, only the velocity components will be matched because all remaining variables can be calculated from them.

The outer solutions are written using Eqs. (2.4), $(3.2)_1$ and (3.8) and expanding them about $x = 0$. We consider the important case for which $\phi_1$ and $\phi_2$ possess at least a few derivatives with respect to $x$, as $x \to 0$. Thus,

$$U = 1+E\{\phi_{1x}(0,t)+x\phi_{1xx}(0,t)+\ldots\} + E^2\left\{f_{xx}(0,t)\frac{y^2}{2}+h_x(0,t)+x\left[f_{xxx}(0,t)\frac{y^2}{2}+h_{xx}(0,t)\right]+\ldots\right\}+\ldots \tag{3.24}$$

$$V = E^2 y\{f_x(0,t)+xf_{xx}(0,t)+\ldots\}+\ldots$$

The expansions for $U$ and $V$ for $x^* \gg 1$ (and thus by Eq. $(3.18)_1$, for $s^* \gg 1$) are found by substituting Eq. (3.23) into Eq. $(3.18)_2$ and (3.19). They then may be written in outer variables by using $x^* = x/\delta^*$ and $\delta^{*2} = (\gamma+1)E^*$. The resulting equations are

$$\begin{aligned} U &= 1-E^{*1/2}\frac{2bx}{\sqrt{\gamma+1}}+E^*\left\{2b^2y^2-2b\left(\beta+\frac{k}{b}\beta'\right)\right\}+\ldots, \\ V &= E^*4b^2xy+\ldots, \end{aligned} \tag{3.25}$$

where only the first term of Eq. (3.23) has been used in the above. Comparing Eqs. (3.24) and (3.25) term by term, it is seen that

$$\begin{aligned} &E^{*1/2} = E, \quad \phi_{1x}(0,t) = 0, \quad \phi_{1xx}(0,t) = -\frac{2b}{\sqrt{\gamma+1}}, \\ &f_{xx}(0,t) = 4b^2, \quad h_x(0,t) = -2b\left(\beta+\frac{k}{b}\beta'\right), \quad f_x(0,t) = 0. \end{aligned} \tag{3.26}$$

First of all it should be noted that, since $\alpha_d$ is not found in any term, the shock position is not given to this order of approximation. In fact, from Eq. (3.23), the first term in $U$ which would involve $\alpha$ is of order $E^*x^{*-2} = O(E^{*2}) = O(E^4)$, where $x^* = O(E^{*-1/2})$. Therefore, information from downstream which sets the shock position arises from fourth-order outer terms.

From Eqs. $(3.26)_1$, (3.16), and (2.5) one can show that $\tau^* = O(\tau)$ if $n = 1/2$, and this is chosen to be the case here. Therefore it is consistent with the present solutions to include the unsteady term in Eq. (3.17).

Equations $(3.26)_2$ and $(3.26)_3$ indicate that the outer velocity in the vicinity of the sonic line is linear in $x$. Equations $(3.26)_4$ and $(3.26)_6$ indicate that the outer wall is therefore parabolic in the vicinity of the throat. Finally, Eq. $(3.26)_5$ shows that the function of time, $\beta$, is prescribed by $h_x$, and thus by the initial conditions imposed on the outer, downstream solutions.

## 4. Discussion

It is clear from the above, that if one were to match the inner solution with the upstream outer solutions, results essentially the same as those shown in Eq. (3.26) would result. That is, since the inner function $z$ either is asymptotic to $-2bs^*$ or equals $4bs^*$, the only differences would be in the constants. On the other hand, Eq. $(3.26)_5$ which involves $\beta$ requires very careful attention because there is only one function $\beta$, with two matching conditions, involving possibly two different initial conditions, one upstream and one downstream of the throat. This interaction between upstream and downstream conditions could be very complex; studies with very simple initial conditions are being made.

The essential result of the above matching is that the only restriction imposed on the outer solution by matching with the given inner similarity solution is that the outer wall shape must become parabolic as the throat is approached. Otherwise, there is no restriction on wall shape. It also seems apparent, from Eq. (3.26), that one could formulate a problem involving stationary walls, with nonsteady flow. However, it is not all clear whether arbitrary oscillations could be imposed on the flow. This point also deserves further study. In this regard, it is worth noting that wall shapes which vary with time do affect $\beta$, since $f_{xt}$ is found in the equation for $h$, Eq. (3.9), and $h_x(0, t)$ is the forcing function for the equation for $\beta$, Eq. $(3.26)_5$.

Although attention here has been focused on the throat region as a region of possibly nonuniform validity of the outer solutions, it is worth mentioning that an inner region is also necessary when a shock exists in the outer region. That problem is presently being studied.

It is believed that systematic derivation of the Szaniawski type of solutions presented here places them in the proper perspective and indicates that they are of fundamental importance in the study of both steady and unsteady transonic flows.

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