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Classes of discontinuous motions in elastic and rate-type materials. One-dimensional case

I. SULICIU (BUCHAREST)

IN THE PAPER, different classes of regulated functions are introduced. Hadamard's theory of wave propagation (when an isolated curve of discontinuity is present) is generalized for the frame of regulated functions (when a countable set of discontinuity directions can meet at a point). Kinematic and dynamic jump conditions are given. For the rate-type materials, necessary conditions for the constitutive equation to be written in the form of a continuously differentiable functional of strain (regulated function) are given. Finally, for motions with second-order discontinuities (generalization of acceleration waves), one proves that on a bounded and closed domain, the number of discontinuity directions at a point is at most two for the elastic case and at most three for the "rate" case; moreover, the set of points where there are effectively two (respectively three) discontinuity directions, is at most countable.

W pracy wprowadzono różne klasy funkcji z nieciągłościami pierwszego rodzaju. Uogólniono teorię propagacji fal Hadamarda, w której występują izolowane krzywe nieciągłości, na układy funkcji z nieciągłościami pierwszego rodzaju, które, dla odmiany, dopuszczają w każdym punkcie istnienie przeliczalnego zbioru kierunków nieciągłości. Wyprowadzono kinematyczne i dynamiczne warunki dla skoków nieciągłości. Dla materiałów typu prędkościowego podano warunki konieczne na wyrażenie równania konstytutywnego w postaci ciągle różniczkowalnego funkcjonału od odkształcenia (będącego funkcją z nieciągłościami pierwszego rodzaju). W zakończeniu, dla klasy ruchów z nieciągłościami drugiego rzędu (uogólnione fale przyspieszenia) udowodniono, że na obszarach ograniczonych i domkniętych istnieją co najwyżej dwa kierunki nieciągłości w każdym punkcie dla przypadku sprężystego i co najwyżej trzy — dla przypadku "prędkościowego". Ponadto zbiór punktów, w których istnieją faktycznie dwa (odpowiednio trzy) kierunki nieciągłości, jest najwyżej przeliczalny.

В работе введены разные классы функций с разрывами первого рода. Теория распространения волн Адамара, в которой выступают изолированные кривые разрыва, обобщена на системы функций с разрывами первого рода, которые в отличие от обычного допускают в каждой точке существование счетного множества направлений разрыва. Выведены кинематические и динамические условия для скачков разрыва. Для материалов скоростного типа даются необходимые условия для выражения определяющего уравнения в виде непрерывно дифференцируемого функционала от деформаций (будущего функцией с разрывами первого рода). В заключение для класса движений с разрывами второго рода (обобщенные волны ускорения) доказано, что в ограниченных и замкнутых областях существуют по крайней мере два направления разрыва в каждой точке для упругого случая и по крайней мере три — для „скоростного" случая. Кроме этого множество точек, в которых существуют фактически два (соответственно три) направления разрыва, является по крайней мере счетным.

1. Introduction

WE CONSIDER here only the motions of a material body $\mathscr{B}$ which can be described with a single spatial coordinate x for the actual coordinate and X ($X \in [a, b]$) for the coordinate in the reference configuration. The notation used throughout this paper follows mainly the notation involved in [1, 2].

The motion of a body $\mathscr{B}$ in a function $x:\bar{D} \to R$ ($\bar{D} = [a, b] \times [t_0, t_1]$) giving the location $x = x(X, t)$ at time t of the material point which had position X in the reference configuration. The stress $T = T(X, t)$ will be here a real number defined as the force per unit area in the reference configuration. When the derivatives

$$F(X,t) = \partial_X x(X,t), \quad \gamma = F-1, \quad \dot{x} = \partial_t x(X,t) = V(X,t), \tag{1.1}$$
$$\ddot{x} = \partial_t^2 x(X,t) = \partial_t V(X,t) = \dot{V}$$

exist, we call them, respectively, the deformation gradient, the strain, the velocity and the acceleration of X at time t.

A motion must obey the law of balance of momentum

$$\partial_t \int_{X_1}^{X_2} \dot{x}(X,t)\varrho_0 dX = \int_{X_1}^{X_2} b(X,t)\varrho_0 dX + T(X_2,t) - T(X_1,t) \tag{1.2}$$

for every pair $X_1, X_2 \in [a, b]$. Here body forces $b(X, t)$ are assumed sufficiently smooth with respect to (X, t), and ϱ_0 is the mass density in the reference configuration.

Let the strains $\gamma: \bar{D} \to R$ take their values as an interval $I \subset R$; then we say that the body $\mathscr{B}$ is elastic if there exists a smooth injection function $g: I \to R$ such that

$$T = g(\gamma). \tag{1.3}$$

We consider now a domain $\mathscr{D}$ of $\gamma\ T$ plane. Let $\varphi, \psi: \bar{\mathscr{D}} \to R$; then a quasilinear first-order constitutive equation of the rate type (see, for example, [1] Sec. 36, and [3] Chap. III) is postulated under the form

$$\dot{T} = \varphi(\gamma, T)\dot{\gamma} + \psi(\gamma, T), \tag{1.4}$$

if $T = T(X, t)$ and $\gamma = \gamma(X, t)$ are differentiable with respect to t.

The first part of Sec. 2 of this paper presents some results from [4, 5] concerning regulated functions defined on the real line. The second part of Sec. 2 introduces the regulated functions defined on plane domains with real values. In Sec. 3, the classical results of HADAMARD [6] (see also, for instance, [7]) are obtained for motions belonging to different classes of regulated functions.

The word "wave" has not been used since the discontinuities of the motion have not, in general, the character of an isolated wave, i.e., a perturbation which propagates through the body and reaches at different times t different material particles X. The motion x is supposed to be a continuous function with respect to (X, t), however, its derivatives v and F (or γ) may have different values at a fixed point $P_0 = (X_0, t_0)$ and for a fixed unit vector e_1 at P_0, depending on the way we reach P_0 along a right or left tangent curve to e_1; these discontinuities will be called first-order discontinuities. This is a generalization for shock waves.

If v and F are continuous but have one-side derivatives with discontinuities of the type described above, then we say the motion has second-order discontinuities; this represents a generalization of acceleration waves.

In both cases the set of discontinuity directions e_1, for a fixed P_0, is at most countable; moreover, if (X, t) varies over a bounded domain, the set of points at which there are at least two discontinuity directions is countable.

The kinematical and dynamical conditions of compatibility have similar forms to those already known [6, 7], but they depend on both the considered point and direction.

In Sec. 4, conditions under which the constitutive equation (1.4) describes a single material are discussed (see also, Sec. 5.2). Many of the proofs given there follow the proof of classical theorems in the theory of differential equations as, for example, can be found in the books [8, 9]; however, more detailed proofs of these results can be found in [10].

The solution obtained is an explicit function of two variables (see Sec. 4), the value of γ at time t and the history of γ up to time t; i.e., it is a function of $\gamma(t)$ and history parameter $\tau(t)$; $\tau(t)$ being a functional of $\gamma:[t_0, t] \to R$. The discontinuities of $T(t)$ are essentially determined by discontinuities of $\gamma(t)$. The history parameter $\tau(t)$ is, roughly speaking, of a class better than $\gamma(t)$.

In the last section are discussed the conditions imposed on a motion with first- and second-order discontinuities in order that it may represent a motion of an elastic or a rate type material body $\mathscr{B}$.

2. Regulated functions

2.1. Regulated functions on real line

In this section we reproduce briefly some results from NICOLESCU [4] and DIEUDONNÉ [5]. We denote by R or R^1 the set of real numbers and I an interval on R with origin in a and with the other extremity in b (a, b may be finite or infinite).

DEFINITION 2.1. *A function $f: I \to R$ will be called a regulated function on I if for any $t \in I$, f has one-side limits*

$$f(t-0) = f^-(t) = \lim_{\substack{s \to t,\, s \in I \\ s < t,\, t \neq a}} f(s)$$

and

$$f(t+0) = f^+(t) = \lim_{\substack{s \to 0,\, s \in I \\ s > t,\, t \neq b}} f(s).$$

We denote by $\mathsf{R}^0(I)$ the set of regulated functions $f: I \to R$.

DEFINITION 2.2. *A function $f: I \to R$ is called a step function on I if there is an increasing finite sequence $\{t_i\}_{0 \leqslant i \leqslant n}$ of points of $\bar{I}$ (closure of I in $\bar{R}$) such that $t_0 = a$, $t_n = b$ and f is constant on each of the open intervals (t_i, t_{i+1}), $i = 0, 1, \ldots, n-1$.*

THEOREM 2.1. *A necessary and sufficient condition for $f:[a, b] \to R$ to be a regulated function is that f be a limit of uniformly convergent sequence of step-functions.*

As a consequence of this theorem one gets that the set of discontinuities of a regulated function is at most countable.

DEFINITION 2.3. *A regulated function $f \in \mathsf{R}^0(I)$ is said to possess one-side derivatives on I if the following limits exist:*

$$\dot{f}_s(t) = \lim_{\substack{s \to t,\, s \in I \\ s < t,\, t \neq a}} \frac{f(s) - f^-(t)}{s - t}, \quad f_d(t) = \lim_{\substack{s \to t,\, s \in I \\ s > t,\, t \neq b}} \frac{f(s) - f^+(t)}{s - t}.$$

We write $f \in R^1(I)$ if $f \in R^0(I), \dot{f}_s, \dot{f}_d \in R^0(I)$ (if, for instance, $a \in I$, we define $\dot{f}_s(a) = \dot{f}_s(a+0)$, so that $\dot{f}_s$ is defined on whole I, etc.). We write $f \in C^{01}(I)$ if $f \in C^0(I)$ and $\dot{f}_s, \dot{f}_d \in R^0(I)$.

The following result belongs to A. Denjoy (see, for instance, NICOLESCU [4], Chap. XVII): If $f:[a, b] \to R$ has in any point of $[a, b]$, one-side derivatives, except on an at most countable set, then in any point of $[a, b]$ but an at most countable set, f has a derivative.

PROPOSITION 2.1. *Let $f, \varphi:[a, b] \to R$ be two functions in $C^{01}[a, b]$, so that $|\dot{f}(s)| \leqslant \dot{\varphi}(s)$ (where $s \in (a, b)$ are the points where both derivatives exist and are equal). Then $|f(b)-f(a)| \leqslant \varphi(b)-\varphi(a)$ (see* DIEUDONNÉ [5], *Chap.* 8).

As a consequence of Proposition 2.1 it is easy to prove the following:

LEMMA 2.1. *Let $f:[a, b] \to R$ be a function in $C^{01}[a, b]$ so that*

$$|\dot{f}_d| \leqslant M|f|, f(a) = 0, M > 0.$$

Then $f(t) \equiv 0$ for any $t \in [a, b]$.

2.2. Regulated functions in the plane

We denote by $R^2 = R \times R$ the Euclidean two-dimensional space and by $D \subset R^2$ a plane domain. Let $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ be standard basis in R^2. We consider another orthonormal basis (e_1, e_2) in R^2 having the same orientation as $(\mathbf{i}, \mathbf{j})$. We introduce the following notations

$$\pi_2 = (P_0, e_1, e_2), \quad \pi_1 = (P_0, e_1), \tag{2.1}$$

i.e., π_2 is the frame formed by vectors e_1, e_2 with the same orientation as $\mathbf{i}, \mathbf{j}$ and with the origin in P_0, and π_1 is the frame formed by vector e_1 with the origin in P_0. Based on e_1, e_2 we introduce the vectors

$$f_1^\varkappa = \varkappa h_1 e_1, \quad f_2^\varkappa = h_1 e_1 + \varkappa h_2 e_2, \tag{2.2}$$

where $\varkappa = +$ or $-$ and $h_1 > 0$, $h_2 > 0$ (i.e., h_i are positive real numbers).

Let P_0 be a point in R^2 and let us consider the following sets:

$$\begin{aligned} &\Delta_h^\varkappa(\pi_2) = \{P, P \in R^2, P = P_0 + \lambda_1 f_1^+ + \lambda_2 f_2^\varkappa; \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 \in (0, 1)\}, \\ &\Delta_h^\varkappa(\pi_1) = \{P, P \in R^2, P = P_0 + \lambda f_1^\varkappa, 0 < \lambda < 1\}. \quad \varkappa = + \text{ or } - \end{aligned} \tag{2.3}$$

$\Delta_h^\varkappa(\pi_2)$ are open right triangles with a vortex in P_0 and a leg of length h_1 in the positive direction of e_1 and the other leg of length h_2 in the positive direction of e_2, for $\Delta_h^+(\pi_2)$, or in the negative direction of e_2, for $\Delta_h^-(P_2)$. $\Delta_h^+(\pi_1)$ represents an open segment along e_1 with starting point in P_0 and with the end point in $P_0 + h_1 e_1$ and similarly for $\Delta_h^-(\pi_1)$.

DEFINITION 2.4. *A function $f:D \to R$ is said to have one-side limits in $P_0 \in D$ in the direction e_1 if for any $\varepsilon > 0$ there are $h_1(\varepsilon) > 0$, $h_2(\varepsilon) > 0$ and the numbers $f_{\pi_2}^+(P_0), f_{\pi_2}^-(P_0)$ such that*

$$|f(P) - f_{\pi_2}^+(P_0)| < \varepsilon \text{ for any } P \in \Delta_{h(\varepsilon)}^+(\pi_2) \cap D,$$

$$|f(Q) - f_{\pi_2}^-(P_0)| < \varepsilon \text{ for any } Q \in \Delta_{h(\varepsilon)}^-(\pi_3) \cap D.$$

DEFINITION 2.5. *We say that a smooth curve* $P(s) = P_0+se_1+0(s)$, $|0(s)|/s \to 0$ *for* $s \to 0$, *is right (left) tangent to* e_1 *in* P_0 *if there is an* $s_0 > 0$ *such that* $P(s)$ *lies on the right (left) side of* e_1 *for* $s \in (0, s_0)$.

The following proposition is obvious.

PROPOSITION 2.2. *If* $f: D \to R$ *has one-side limits in* $P_0 \in D$ *in the direction* e_1, *then these limits are given by*

$$\lim_{s\to 0} f(P(s)) = f^-_{\pi_2}(P_0), \quad \lim_{s\to 0} f(Q(s)) = f^+_{\pi_2}(P_0),$$

where $P(s)$ *is a smooth right tangent curve to* e_1 *in* P_0 *and* $Q(s)$ *is a smooth left tangent curve to* e_1 *in* P_0.

DEFINITION 2.6. *Let* D *be a plane bounded domain. We say that the boundary* ∂D *of* D *is of class* C^{01} *if for each point* $P_0 \in \partial D$ *there exists a ball* B_0 *with center* P_0 *such that* $\partial D \cap B_0$ *can be represented in the form* $t = g(X)$ *or* $X = h(t)$ *with* g *or* h *in* C^{01}.

DEFINITION 2.7. *A function* $f: D \to R$ *is called an* R^2*-regulated function if* f *has one-side limits in the direction* e_1 *in* P_0 *for any* $P_0 \in D$ *and any unit vector* e_1. *The set of all regulated functions on* D *will be denoted by* $\mathsf{R}^0(D)$.

For a fixed $P_0 \in D$, e_1 depends on the angle $\theta \in [0, 2\pi]$ between e_1 and $\mathbf{i}$, $e_1 = e_1(\theta)$ and therefore $f^{\pm}_{\pi_2}(P_0)$ are functions of θ. If $P_0 \in \partial D$ and ∂D is a smooth curve, then θ belongs only to an interval of length π; since f is not defined on the exterior of D, one of its one-side limits at the end of this interval has no sense. If $D \in C^{01}$, then θ may belong to an interval of length greater than zero but smaller than 2π.

PROPOSITION 2.3. *For a fixed* $P_0 \in D$, $f^{\pm}_{\pi_2}(P_0)$ *are* R^1*-regulated functions (regulated functions on the real line) on* $\theta \in [0, 2\pi]$. *In fact, the following relations are valued as*

$$\lim_{\substack{\theta_2\to\theta_1\\ \theta_2<\theta_1}} f^-_{\pi_2(\theta_2)}(P_0) = f^+_{\pi_2(\theta_1)}(P_0), \quad \lim_{\substack{\theta_2\to\theta_1\\ \theta_2<\theta_1}} f^+_{\pi_2(\theta_2)}(P_0) = f^+_{\pi_2(\theta_1)}(P_0),$$

$$\lim_{\substack{\theta_2\to\theta_1\\ \theta_2>\theta_1}} f^+_{\pi_2(\theta_2)}(P_0) = f^-_{\pi_2(\theta_1)}(P_0), \quad \lim_{\substack{\theta_2\to\theta_1\\ \theta_2>\theta_1}} f^-_{\pi_2(\theta_2)}(P_0) = f^-_{\pi_2(\theta_1)}(P_0).$$

The proof follows at once from definitions.

COROLLARY 2.1. *An* R^2*-regulated function admits at any point an at most countable set of discontinuous directions.*

DEFINITION 2.8. *Let* $D \subset R^2$ *be a bounded domain with* ∂D *of class* C^{01}; $\{\Delta_k\}_{k=1,\dots,m}$ *is said to be a partition of* D *if*: (1) Δ_k *are connected domains whose boundaries consist of straight line segments and parts of* ∂D, (2) $\Delta_l \cap \Delta_k = \phi$ *for any* $l, k = 1, \dots, m$, $l \neq k$, *and* (3) $\bigcup_{k=1}^{m} \overline{\Delta}_k = \overline{D}$.

DEFINITION 2.9. *A function* $g: \overline{D} \to R$ *is called a step function on* D *if there is a finite partition* $\{\Delta_k\}_{1\leq k\leq m}$ *of* D *such that*

$$g(P) = C_k = \text{const} \quad \text{for} \quad P \in \Delta_k.$$

Now, we can prove the analog to Theorem 2.1 for compact plane domains.

THEOREM 2.2. *If* $g_n: \overline{D} \to R$ *is a uniformly convergent sequence of step functions to a function* f, *then* f *is an* R^2*-regulated function; conversely, any* R^2*-regulated function on a compact domain is a uniform limit of step functions.*

P r o o f. The first part of the theorem can be proved in a similar way as for Theorem 2.1, so we omit it.

For the second part of the theorem we have to build a sequence of step functions uniformly convergent to the given R^2-regulated function f.

According to Proposition 2.3, for a fixed P_0, $f^{\varkappa}_{\pi_2}(P_0)$ are R^1-regulated functions of the angle $\theta \in [0, 2\pi]$. We choose $\varepsilon = 1/n$ and let θ_0 be fixed; then there are $h_1(\theta_0, n) > 0$, $h_2(\theta_0, n) > 0$ such that

$$|f(P)-f^{+}_{\pi_2(\theta_0)}(P_0)| < \frac{1}{n} \text{ for any } P \in \Delta^{+}_{n}(\pi_2(\theta_0)),$$

$$|f(Q)-f^{-}_{\pi_2(\theta_0)}(P_0)| < \frac{1}{n} \text{ for any } Q \in \Delta^{-}_{n}(\pi_2(\theta_0)).$$

Consider

$$\Delta_n(\pi_2(\theta_0)) = \Delta^{+}_{n}(\pi_2(\theta_0)) \cup \Delta^{-}_{n}(\pi_2(\theta_0)) \cup \{P_0 + se_1(\theta_0)\}, \quad s \in (0, h_1(\theta_0, n)).$$

Now, let $\alpha_n(\theta_0)$ be the angle between the vectors $P_0 + h_1(\theta_0, n)e_1$, $P_0 + h_1(\theta_0, n)e_1 + h_2(\theta_0, n)e_2$ and we take the open intervals $I_n(\theta_0) = (\theta_0 - \alpha_n(\theta_0), \theta_0 + \alpha_n(\theta_0))$; we have $\bigcup_{\theta_0 \in [0, 2\pi]} I_n(\theta_0) \supset [0, 2\pi]$ which implies the existence of $\theta_1, \ldots, \theta_{m_1}$, with $\bigcup_{i=1}^{m_1} I_n(\theta_i) \supset [0, 2\pi]$.

We write the numbers $\theta_k - \alpha_n(\theta_k)$, θ_k, $\theta_k + \alpha_n(\theta_k)$ in increasing order and denote them by

$$0 = \alpha_1 < \alpha_2 < \ldots < \alpha_{m_2} = 2\pi.$$

We have obtained around P_0 a set of triangles of angles $\alpha_{i+1} - \alpha_i$ in the vortex P_0. For any P, Q in one of these open triangles, we have

$$|f(P)-f(Q)| < \frac{2}{n}.$$

We denote by $\Delta_n(P_0) = \operatorname{int} \bigcup_{i=1}^{m_2} \overline{\Delta_n(\pi_2(\alpha_i))}$ and we have $\bigcup_{P_0 \in \overline{D}} \Delta_n(P_0) \supset \overline{D}$, which implies $\bigcup_{k=1}^{m_3} \Delta_n(P_k) \supset \overline{D}$.

Now taking the intersections between the obtained domains $\Delta_n(P_k)$ and between $\Delta_n(P_k)$ and D, we get a set of a connected disjoint open sets denoted by Δ^k_n, $k = 1, \ldots, m(n)$ and $\bigcup_{k=1}^{m(n)} \overline{\Delta^k_n} = \overline{D}$. The boundaries of Δ^k_n, $k = 1, \ldots, m(n)$ consists of straight line segments and parts of the boundary of D. Obviously,

$$|f(P)-f(Q)| < \frac{2}{n} \text{ for any } P, Q \in \Delta^k_n.$$

Consider now the sequence of step functions

$$g_n(P) = \begin{cases} C^k_n & P \in \Delta^k_n \\ f(P) & P \in \partial \Delta^k_n \end{cases} \quad k = 1, \ldots, m(n),$$

where $C^k_n = f(Q)$, and Q is an arbitrary fixed point in Δ^k_n. Then for any $P \in \overline{D}$, $|f(P) - g_n(P)| < \frac{2}{n}$ and therefore $g_n(P) \xrightarrow{\text{u.c.}} f(P)$.

COROLLARY 2.2. *The set of discontinuity points of an R^2-regulated function defined on a compact domain, through which pass at least two discontinuity directions, is at most countable.*

According to Corollary 2.1, the set of discontinuity directions through any point is also at most countable. Of course, the set of discontinuity points in which there is only one discontinuity direction is of continuum power.

Let (e_1^1, e_2^1) and (e_1^2, e_2^2) be two orthonormal bases in R^2 with the same orientation as standard basis $(\mathbf{i}, \mathbf{j})$ and $\alpha \in (0, \pi)$ be the angle between e_1^1, e_1^2. Let two families of smooth curves be

$$
(2.4)\qquad \begin{aligned} P_1(\lambda, s) &= P_0 + \lambda e_1^1 + s e_1^2 + 0_1(s), \qquad \lambda > 0, s > 0 \\ P_2(\lambda, s) &= P_0 + \lambda e_1^1 + s e_1^2 + 0_2(s), \end{aligned}
$$

where $0_1(s)$, $0_2(s)$ are such that, for fixed λ, the curve $P_1(\lambda, s)$ lies on the right of e_1^2 and $P_2(\lambda, s)$ on the left of e_1^2; moreover,

$$
\lim_{s\to 0} \frac{0_1(s)}{s} = \lim_{s\to 0} \frac{0_2(s)}{s} = 0.
$$

Let D be a plane domain and $f: D \to R$ be an R^2-regulated function. For any fixed λ, we have

$$
\begin{aligned} \lim_{s\to 0} f(P_1(\lambda, s)) &= f_{\pi_2^2}^{-}(P_0 + \lambda e_1^1), \\ \lim_{s\to 0} f(P_2(\lambda, s)) &= f_{\pi_2^2}^{+}(P_0 + \lambda e_1^1). \end{aligned}
$$

We prove now that

$$
(2.5)\qquad \begin{aligned} \lim_{\lambda\to 0} f_{\pi_2^2}^{-}(P_0 + \lambda e_1^1) &= f_{\pi_2^1}^{\mp}(P_0), \\ \lim_{\lambda\to 0} f_{\pi_2^2}^{+}(P_0 + \lambda e_1^1) &= f_{\pi_2^1}^{\pm}(P_0), \end{aligned}
$$

where $\pi_2^1 = (P_0, e_1^1, e_2^1)$ and $\pi_2^2 = (P_0, e_1^2, e_2^2)$. We obtain here $f_{\pi_2^1}^{-}(P_0)$ if e_1^2 lies on the right-hand side of e_1^1 and $f_{\pi_2^1}^{+}(P_0)$ if e_1^2 lies on the left-hand side of e_1^1.

First, we need to show that there is a smooth curve $s = g(\lambda)$, with $g(0) = g'(0) = 0$; then we have

$$
(2.6)\qquad P_1(\lambda, g(\lambda)) = P_0 + \lambda e_1^1 + g(\lambda) e_1^2 + 0_1(g(\lambda)),
$$

which is a smooth tangent curve to e_1^1 at P_0 and lies on the right of e_1^1 (if e_2^2 does).

LEMMA 2.2. *Let λ_n and s_n be two sequences of decreasing numbers converging to zero, with the additional property: $s_{n-1}/\lambda_n < 1/n$. Then there is a smooth function $g: [0, \lambda_1] \to R$ with the following properties: $g(\lambda_n) = s_n$, $g(0) = 0$ and $g'(0) = 0$.*

Proof. Consider the sequence of smooth and decreasing functions g_n, $g_n: [0, \lambda_1] \to R$, having the following properties: (1) $g_n(\lambda_i) = s_i$, $i = 1, \ldots, n$, (2) $g_n(\lambda) = 0$, $\lambda \in [0, \lambda_{n+1})$, (3) $g_{n+1}(\lambda) = g_n(\lambda)$, $\lambda \in [0, \lambda_{n+2}] \bigcup [\lambda_n, \lambda_1]$. The sequence g_n is uniformly convergent to a function g, as it is easy to verify, and $g(0) = 0$.

Now, for $\lambda \in [0, \lambda_1]$, we have $g'(\lambda) = \lim_{n\to\infty} g'_n(\lambda)$ and $g'(0) = 0$, because $\lim_{\lambda\to 0} g'_n(\lambda) = 0$ for any n. Let us show that $\lim_{k\to\infty} g'(\tilde{\lambda}_k) = 0$ for any $\tilde{\lambda}_\kappa \to 0$, $\tilde{\lambda}_k > 0$. We have

$$g(\tilde{\lambda}_k)/\tilde{\lambda}_k \leqslant g(\tilde{\lambda}_k)/\lambda_{n_k} \leqslant g(\lambda_{n_k-1})/\lambda_{n_k} = s_{n_k-1}/\lambda_{n_k} < \frac{1}{n_k},$$

where $\lambda_{n_k} \leqslant \tilde{\lambda}_k \leqslant \lambda_{n_k-1}$. Hence $g(\lambda)$ is differentiable for any $\lambda \in [0, \lambda_1]$ and $g'(0) = 0$.

LEMMA 2.3. *In the conditions stated above,* (2.5) *holds.*

Proof. Consider an arbitrarily decreasing sequence $\lambda_n \to 0$, $\lambda_n > 0$, $\lambda_n > \lambda_{n+1}$. It is obvious that if the limits in (2.5) exist for any decreasing sequence, they exist also for any $\lambda_n \to 0$, $\lambda_n > 0$.

For any n we find a $\delta_n > 0$ such that

$$|f^+_{\pi_2^2}(P_0+\lambda_n e_1^1) - f(P_2(\lambda_n, s))| < \frac{1}{n} \text{ for } 0 < s < \delta_n,$$

where $P_2(\lambda_n, s)$ is given by (2.4). We choose $s_n < \delta_n$ such that $s_n < \lambda_{n+1}/(n+1)$ and $s_n < s_{n-1}$. Hence, for any decreasing sequence λ_n, we can find a decreasing sequence s_n which satisfies the conditions of Lemma 2.2. Then we have

$$|f^+_{\pi_2^2}(P_0+\lambda_n e_1^1) - f^-_{\pi_2^2}(P_0)| \leqslant |f^+_{\pi_2^2}(P_0+\lambda_n e_1^1) - f(P_2(\lambda_n, s_n)|$$
$$+|f(P_2(\lambda_n, s_n)) - f^-_{\pi_2^1}(P_0)| \leqslant \frac{1}{n} + |f(P_2(\lambda_n, g(\lambda_n)) - f^-_{\pi_2^1}(P_0)|.$$

As $\lim_{n\to\infty} |f(P_2(\lambda_n, g(\lambda_n)) - f^-_{\pi_2^1}(P_0)| = 0$, if e_1^2 lies on the right of e_1^1, we get one of the second relations of (2.5). The other relations of (2.5) can be proved in a similar way.

DEFINITION 2.10. *Consider $f \in \mathrm{R}^0(D)$, $P_0 \in D$ and (e_1, e_2) an orthonormal basis as above; we say that f has one-side derivatives on the direction e_1 at P_0 if for any smooth right and left tangent curve to e_1: $P(s) = P_0 + se_1 + 0(s)$, $0(s)/s \to 0$ for $s \to 0$, the limits*

$$\lim_{\substack{s\to 0\\ s>0}} \frac{f(P(s)) - f^-_{\pi_2}(P_0)}{s} = \partial^-_{\pi_2} f(P_0), \quad \lim_{\substack{s\to 0\\ s>0}} \frac{f(P(s)) - f^+_{\pi_2}(P_0)}{s} = \partial^+_{\pi_2} f(P_0)$$

exist for right and left curves, respectively, and are independent on curves $P(s)$.

We denote by $\mathrm{R}^1(D)$ the set of all $f \in \mathrm{R}^0(D)$, possessing one-side derivatives for any $P_0 \in D$ and any orthonormal basis (e_1, e_2) which are regulated functions with respect to any $P_0 \in D$ for any fixed (e_1, e_2), and by $C^{01}(D) = C^0(D) \cap \mathrm{R}^1(D)$.

DEFINITION 2.11. *A function $f \in \mathrm{R}^0(D)$ will be called totally regulated on D if for any $P_0 \in D$ and any unit vector e_1 with the origin in P_0 the following limit*

$$\lim_{\substack{\lambda\to 0\\ \lambda>0}} f(P_0 + \lambda e_1) = f^+_{\pi_1}(P_0)$$

exists. The set of these functions will be denoted by $\mathrm{R}^0_T(D)$. Also, we say $f \in \mathrm{R}^1_T(D)$ if $f \in \mathrm{R}^0_T(D) \cap \mathrm{R}^1(D)$ and

$$\lim_{\substack{\lambda\to 0\\ \lambda>0}} \frac{f(P_0+\lambda e_1) - f^+_{\pi_1}(P_0)}{\lambda} = \partial^+_{\pi_1} f(P_0)$$

exists for any $P_0 \in D$ and any e_1, and for e_1 fixed, $\partial^+_{\pi_1} f(P_0)$ is a regulated function with respect to $P_0 \in D$.

Now we can prove one of the most important results of this section.

Generalized Hadamard LEMMA 2.4. (See HADAMARD [6] Sec. 72, and TRUESDELL and TOUPIN [7], Sec. 174). *If $f \in \mathrm{R}^0(D)$ and $\partial^-_{\pi_2} f(P_0)$ exists in P_0, then $\partial^+_{\pi_1} f^-_{\pi_2}(P_0)$ exists and we have*

$$\partial^-_{\pi_2} f(P_0) = \partial^+_{\pi_1} f^-_{\pi_2}(P_0), \tag{2.6}$$

where $\pi_2 = (P_0, e_1, e_2)$, $\pi_1 = (P_0, e_1)$ from (2.1).

Proof. Consider a family of curves

$$P(\lambda, s) = P_0 + \lambda e_1 + s e_1 + 0(s)$$

lying on the right side of e_1, with $0(s)/s \to 0$ for $s \to 0$; we have

$$\left|\frac{f^-_{\pi_2}(P_0+\lambda e_1) - f^-_{\pi_2}(P_0)}{\lambda} - \partial^-_{\pi_2} f(P_0)\right| \leqslant \left|\frac{f\big(P(\lambda, s)\big) - f^-_{\pi_2}(P_0+\lambda e_1)}{\lambda}\right| + \left|\frac{f\big(P(\lambda, s)\big) - f^-_{\pi_2}(P_0)}{\lambda} - \partial^-_{\pi_2} f(P_0)\right|.$$

Let $\lambda_n > 0$, $\lambda_n \to 0$ for $n \to \infty$, $\lambda_{n+1} < \lambda_n$. Then, according to Lemmas 2.2, 2.3, we can find a decreasing sequence $s_n \to 0$, $s_n > 0$, $s_{n-1}/\lambda_n < 1/n$ and a $g(\lambda) = s$ such that $g(\lambda_n) = s_n$ and the curve $Q(\lambda) = P\big(\lambda, g(\lambda)\big)$ is a right tangent curve to τ at P_0 and

$$\left|\frac{f\big(Q(\lambda_n)\big) - f^-_{\pi_2}(P_0+\lambda_n e_1)}{\lambda_n}\right| < \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} \frac{f\big(Q(\lambda_n)\big) - f^-_{\pi_2}(P_0)}{\lambda_n} = \partial^-_{\pi_2} f(P_0),$$

the lemma follows.

COROLLARY 2.3. *If f has one-side derivatives in the e_1 direction at P_0, then*

$$\partial^+_{\pi_1}[f_{\pi_2}(P_0)] = [\partial_{\pi_2} f(P_0)], \tag{2.7}$$

where

$$[f_{\pi_2}(P_0)] = f^+_{\pi_2}(P_0) - f^-_{\pi_2}(P_0),$$
$$[\partial_{\pi_2} f(P_0)] = \partial^+_{\pi_2} f(P_0) - \partial^-_{\pi_2} f(P_0).$$

COROLLARY 2.4. *Let $f \in C^{01}(D)$. Then*

$$\partial^+_{\pi_2} f(P) = \partial^-_{\pi_2} f(P) = \lim_{\substack{\lambda\to 0\\ \lambda>0}} \frac{f(P+\lambda e_1) - f(P)}{\lambda} = \partial^+_{\pi_1} f(P),$$

i.e., a continuous function possessing one-side derivatives, it is differentiable on each direction $\pi_1 = (P, e_1)$.

In what follows, it is convenient to introduce, for a given direction e_1 in a point P, the directional derivative in the direction $-e_1$, i.e.,

$$\partial^-_{\pi_1} f(P_0) = \lim_{\substack{\lambda\to 0\\ \lambda>0}} \frac{f(P-\lambda e_1) - f^-_{\pi_1}(P_0)}{\lambda}, \tag{2.8}$$

where $f^-_{\pi_1}(P_0) = \lim_{\substack{\lambda\to 0\\ \lambda>0}} f(P-\lambda e_1)$. Therefore, in the meaning used in Corollary 2.4, if we take $e_1 = i$ or j, we can introduce the following notation:

$$(2.9) \qquad \partial^{\pm}_X f(P) = \partial^{\pm}_{\pi_1} f(P), \quad \partial^{\pm}_t f(P) = \partial^{\pm}_{\pi_1^1} f(P):$$

COROLLARY 2.5. *According to Denjoy's theorem* (see Sec. 2.1 above) *in condition of corollary* 2.4, *$f(X, t)$ is differentiable with respect to (X, t), for a fixed t, (X), everywhere but on an at most countable set, i.e., $\partial^+_X f(X, t) = \partial^-_X f(X, t)$ and $\partial^+_t f(X, t) = \partial^-_t f(X, t)$ everywhere on a parallel segment to the X axis or to the t axis in $\overline{D}$, except on an at most countable set.*

PROPOSITION 2.4. *Let $f \in C^{01}(D)$. Then*

$$(2.10) \qquad \begin{aligned} (\partial^+_X f)^{\varkappa}_{\pi_2}(P) &= (\partial^-_X f)^{\varkappa}_{\pi_2}(P),\\ (\partial^+_t f)^{\varkappa}_{\pi_2}(P) &= (\partial^-_t f)^{\varkappa}_{\pi_2}(P), \end{aligned} \qquad \varkappa = + \text{ or } -$$

for any $P \in D$ and any $\pi_2 = (P, e_1, e_2)$.

P r o o f. We shall prove only the first relation (2.10) for $\varkappa = -$; the other relations follow quite similarly.

Consider $P \in D$ and $\pi_2 = (P, e_1, e_2)$. Let $\lambda_n \to 0$ be a decreasing sequence and $P_n = P_0 + \lambda_n e_1$. We draw through P_n the straight line $t = t_n$ and denote by $\{a^n_m\}_{n\in N}$ the points on the segment $(X, t_n) \in D$, where $\partial^-_X f(X, t_n) \neq \partial^+_X f(X, t_n)$.

These points form an at most countable set. Then for any λ_n we can find a decreasing sequence $s_n > 0$, $s_{n-1} < \frac{1}{n}\lambda_n$ such that $\partial^+_X f(X+s_n, t_n) = \partial^-_X f(X+s_n, t_n)$. According to Lemma 2.2, there is a smooth function $g(\lambda) = s$ with $g(\lambda_n) = s_n$ and with the property that $P(\lambda) = P + \lambda e_1 + g(\lambda)i$ is a right tangent curve to e_1 at P. The relation follows immediately.

Another useful result is given by

PROPOSITION 2.5. *Suppose $f \in R^0(D)$ and f is continuous on D with respect to t for any fixed X, then*

$$f^+_{\pi_2}(P_0) = f^-_{\pi_2}(P_0) = f^+_{\pi_1}(P_0)$$

for any $\pi_2 = (P_0, e_1, e_2)$, $\pi_1 = (P_0, e_1)$ with $e_1 \neq \pm j$ $(e_1 = \cos\theta i + \sin\theta j)$.

P r o o f. Consider $P_0 + \lambda_n e_1$, $\lambda_n > \lambda_{n+1}$, $\lambda_n \to 0$, $n \to \infty$; using the continuity of f with respect to t, we can write

$$|f(P_0+\lambda_n e_1 + tj) - f(P_0+\lambda_n e_1)| < \frac{1}{n} \quad \text{for} \quad |t-\lambda_n \sin\theta| < \tau_n;$$

choosing $t = s^{\pm}_n$ as in Lemmas 2.2, 2.3, we obtain $t^{\pm} = g^{\pm}(\lambda)$, so that $P_0 + \lambda e_1 + g^-(\lambda)j$ is a smooth curve tangent to e_1 at P_0 on the right and $P_0 + \lambda e_1 + g^+(\lambda)j$ is tangent on the left. Then for $n \to \infty$ and involving the above inequality, we get the announced result.

3. Compatibility conditions

In this section we generalize the classical jump conditions for the case when the functions involved are the elements in different classes of regulated functions. Usually, the conditions of compatibility or jump conditions are divided into two groups: (a) the conditions which must be satisfied by functions describing the geometry of motion only are so-called kinematical jump conditions, (b) the conditions which involve (dynamical) equilibrium of forces are called dynamical jump conditions.

3.1. Kinematical jump conditions

In this subsection the main results are contained in two theorems (Theorem 3.1 and Lemma 3.1). First, the theorem generalizes the kinematical compatibility conditions for shock waves, the second result combined with the first generalize the compatibility conditions for acceleration waves.

THEOREM 3.1. *Let* $f:D \to R$ *be a function of class* $C^{01}(D)$; *then for any* $P_0 \in D$ *and any* $\pi_2 = (P_0, e_1, e_2)$, $e_1 = \cos\theta i + \sin\theta j$, *we have*

$$[(\partial_X^{\pm} f)_{\pi_2}(P_0)]\cos\theta + [(\partial_t^{\pm} f)_{\pi_2}(P_0)]\sin\theta = 0, \tag{3.1}$$

where the signs from the operators $\partial_X^{\pm}$, $\partial_t^{\pm}$ *are indifferent, by the Proposition* 2.4, *and* $[(\partial_X^{\pm} f)_{\pi_2}(P_0)]$ *has the meaning from* (2.7).

Proof. Let $\lambda:[0, s_0] \to R$, $\lambda(s) > 0$ for $s \in (0, s_0)$, $\lambda(s)/s \to 0$ for $s \to 0$ and let $P = P_0 + se_1$, $P^1 = P + \lambda e_1$, $Q = P + \lambda\cos\theta i$, $Q^1 = P + \lambda\sin\theta j$, where $Q(s)$, $Q^1(s)$ move along tangent curves to e_1 at P_0, on the right and left of e_1, respectively. We have

$$\frac{f(P^1)-f(P)}{\lambda} = \frac{f(P^1)-f(Q)}{\lambda} + \frac{f(Q)-f(P)}{\lambda}$$

$$= \partial_t^+ f(Q)\sin\theta + \partial_X^- f(Q)\cos\theta + \Omega_1(\lambda\sin\theta) + \Omega_2(\lambda\cos\theta)$$

$$= \frac{f(P^1)-f(Q^1)}{\lambda} + \frac{f(Q^1)-f(P)}{\lambda}$$

$$= \partial_t^+ f(Q^1)\sin\theta + \partial_X^- f(Q^1)\cos\theta + \bar{\Omega}_1(\lambda\sin\theta) + \bar{\Omega}_2(\lambda\cos\theta),$$

where $\Omega_1(\lambda\sin\theta) \to 0$, etc. for $\lambda \to 0$.

Since $\lambda(s) \to 0$ and $Q(s)$, $Q^1(s) \to P_0$ along e_1 for $s \to 0$, then according to Proposition 2.4, the theorem follows.

Generalized Schwarz LEMMA 3.1. *Let* f *be a function with the properties*: (1) $f \in C^1(D)$, *and* (2) $\partial_X f, \partial_t f \in C^{01}(D)$; *then the following equalities*

$$\big(\partial_t^{\pm}(\partial f)\big)_{\pi_2}^{\varkappa}(P_0) = \big(\partial_X^{\pm}(\partial_t f)\big)_{\pi_2}^{\varkappa}(P_0), \quad \varkappa = + \text{ or } - \tag{3.2}$$

hold for any $P_0 \in D$ *and any* $\pi_2 = (P_0, e_1, e_2)$. *The signs from* $\partial_X^{\pm}$, $\partial_t^{\pm}$ *are indifferent.*

First, we prove

LEMMA 3.2. *Let* P_0 *be a point in D through which there exists one discontinuity direction for the four functions* $\partial_t^{\pm}(\partial_X f)$, $\partial_X^{\pm}(\partial_t f)$; *denote this direction by* e_1, *then the following equalities*:

$$\partial_X^{\pm}(\partial_t f)(P_0) = \partial_t^{\pm}(\partial_X f)(P_0) \quad \text{for} \quad \theta \in \left(0, \frac{\pi}{2}\right),$$

$$\partial_X^{\pm}(\partial_t f)(P_0) = \partial_t^{\pm}(\partial_X f)(P_0) \quad \text{for} \quad \theta \in \left(\frac{\pi}{2}, \pi\right)$$

hold. The signs correspond to each other. If $\theta = 0$ or $\theta = \pi/2$, both equalities hold.

P r o o f. We prove only the first equality. Since there is only one discontinuity direction through P_0, we have

$$\lim_{P_1 \to P_0} \partial_X^-(\partial_t f)(P_1) = \partial_X^-(\partial_t f)(P_0)$$

and

$$\lim_{P_1 \to P_0} \partial_t^+(\partial_X f)(P_1) = \partial_t^+(\partial_X f)(P_0).$$

Introduce the function

$$\varphi(t) = f(X, t) - f(X_0, t), \qquad t \in [t_0, t_1]$$

and apply the mean value theorem; then

$$\varphi(t_1) - \varphi(t_0) = (t_1 - t_0)\{\partial_t f(X, \tilde{t}) - \partial_t f(X_0, \tilde{t})\}\ \tilde{t} \in [t_0, t_1];$$

for X sufficiently close to X_0 we can write

$$\varphi(t_1) - \varphi(t_0) = (t_1 - t_0)(X - X_0)\{\partial_X^-(\partial_t f)(X_0, \tilde{t}) + 0_{\tilde{t}}(X - X_0)\}$$
$$= (t_1 - t_0)(X - X_0)\{\partial_X^-(\partial_t f)(X_0, t_0) + 0(t - t_0) + 0_{\tilde{t}}(X - X_0)\}.$$

Then

$$\frac{f(X, t_1) - f(X_0, t_1)}{X - X_0} - \frac{f(X, t_0) - f(X_0, t_1)}{X - X_0}$$
$$= (t_1 - t_0)\{\partial_X^-(\partial_t f)(X_0, t_0) + 0(\tilde{t} - t_0) + 0_{\tilde{t}}(X - X_0)\}.$$

For $X \to X_0$ we get

$$\partial_X f(X_0, t_1) - \partial_X f(X_0, t_0) = (t_1 - t_0)\{\partial_X^-(\partial_t f)(X_0, t_0) + 0(\tilde{t} - t_0)\}$$

and therefore

$$\partial_t^+(\partial_X f)(X_0, t_0) = \partial_X^-(\partial_t f)(X_0, t_0).$$

P r o o f o f L e m m a 3.1. Since $\partial_t^{\pm}(\partial_X f)$ and $\partial_X^{\pm}(\partial_t f)$ are R^2-regulated functions on D, the set of points in which there are at least two discontinuity directions is at most countable (see Corollary 2.2). Let this set be denoted by $\{P_n\}_{n \in N}$. Consider now $P_0 \in D$, a unit vector e_1 in P_0 and $P(s)$ a right tangent curve to e_1 in P_0. Then there is a sequence of points $P_k \to P_0$, $P_k = P(s_k)$ with the property that through any P_k passes only one discontinuity direction. According to Lemma 3.2 and Proposition 2.4, we obtain

$$(\partial_X^{\pm}(\partial_t f))_{\pi_2}^{+}(P_0) = (\partial_t^{\pm}(\partial_X f))_{\pi_2}^{+}(P_0).$$

Here the order of $\pm$ signs is not important (is indifferent).

3.2. Dynamical jump conditions

First, we shall derive dynamical jump conditions corresponding to the shock waves.

DEFINITION 3.1. *A motion $x = x(X, t)$, $(X, t) \in \bar{D} = [a, b] \times [t_0, t_1]$ of a material body $\mathscr{B}$ will be called a motion with first-order discontinuities if*

$$x \in C_T^{01}(\bar{D}) = C^0(\bar{D}) \cap \mathsf{R}_T^1(\bar{D}).$$

Involving (2.9), (1.1) can be written as

(3.3) $$V^{\pm} = \partial_t^{\pm} x(X,t), \quad F^{\pm} = \partial_X^{\pm} x(X,t), \quad \gamma^{\pm} = F^{\pm}-1.$$

We put down the following hypotheses:

(i) x is a motion with first-order discontinuities.

(ii) The stress $T = T(X,t)$ is a totally regulated function on $\bar{D}$.

(iii) The following equality

$$I(a_1, b_1, t) = \int_{a_1}^{b_1} V^+(X,t)dX = \int_{a_1}^{b_1} (V^+)^-_{\pi_2}(X,t)dX = \int_{a_1}^{b_1} (V^+)^+_{\pi_2}(X,t)dX$$

holds for any fixed t, any fixed basis e_1, e_2 and any interval $(a_1, b_1) \subset (a, b)$ such that T is continuous with respect to x at a_1 and b_1.

From Corollary 2.5 and Proposition 2.4, we also have

(iii) $$I(a_1, b_1, t) = \int_{a_1}^{b_1} V^-(X,t)dX = \int_{a_1}^{b_1} (V^-)^-_{\pi_2}(X,t)dX = \int_{a_1}^{b_1} (V^-)^+_{\pi_2}(X,t)dX.$$

If the directions i and j are not directions of discontinuity for $V^{\pm}$, then the condition (iii) is automatically satisfied.

Note that the hypotheses (i)–(iii) are only sufficient conditions for results we derive here.

In conditions (i) and (ii), the momentum of material body $\mathcal{B}$ which, in the reference configuration occupied the segment $[a_1, b_1]$ and had mass density $\varrho_0 =$ const, is at time t, by definition

(3.4) $$H(a_1, b_1, t) = \int_{a_1}^{b_1} V^+(X,t)\varrho_0 dX.$$

Let $P_0 \in D$ and e_1 be a unit vector. Consider two tangent curves to e_1 at P_0 on each side of e_1, of equations

$$C_1: \begin{cases} b_1 = X_0 + \lambda \operatorname{cotan}\theta + 0_1(\lambda), \\ t = t_0 + \lambda; \end{cases}$$

$$C_2: \begin{cases} a_1 = X_0 + \lambda \operatorname{cotan}\theta - 0_1(\lambda), \\ t = t_0 + \lambda, \end{cases}$$

where $0_1(\lambda)/\lambda \to 0$ for $\lambda \to 0, 0_1(\lambda) > 0$ for $\lambda \in (0, \lambda_0]$ and θ is the angle between e_1 and X-axis. The straight line through P_0 along e_1 has the equation

$$X(t) = X_0 + (t-t_0)\operatorname{cotan}\theta.$$

Now, consider those t for which $X(t) \in (a_1, b_1)$. Then, for $\Delta t > 0$, $\Delta t \to 0$, taking into account the conditions (i)–(iii), the following formula can be derived

(3.5) $$\partial_t^+ H(a_1, b_1, t) = \int_{a_1}^{x(t)} \varrho_0 \partial^+_{\pi_2^{11}} V^+(X,t)dX + \int_{x(t)}^{b_1} \varrho_0 \partial^-_{\pi_2^{11}} V^+(X,t)dX$$
$$+\varrho_0 \operatorname{cotan}\theta\{(V^+)^+_{\pi_2}(X(t),t) - (V^+)^-_{\pi_2^1}(X(t),t)\} = T^+_{\pi_1}(b_1,t) - T^-_{\pi_1^1}(a_1,t),$$

where $\pi_2 = (P, e_1, e_2)$, $\pi_2^1 = (P, i, j)$, $\pi_2^{11} = (P, j, -i)$, $\pi_1 = (P, i)$, $\pi_1^1 = (P, -i)$. Equation (3.5) represents, in this case, the law of balance of momentum for zero body forces.

We now state the following

THEOREM 3.2. *In hypotheses* (i)–(iii), *the following dynamical jump relation*

$$\varrho_0\{(V)^+_{\pi_2}(X_0, t_0)-(V)_{\pi_2}(X_0, t_0)\}\cot\theta = T^-_{\pi_2}(X_0, t_0)-T^+_{\pi_2}(X_0, t_0) \tag{3.6}$$

or

$$[T]_{\pi_2}(P_0)+\varrho_0 \cot\theta[V]_{\pi_2}(P_0) = 0 \tag{3.6'}$$

holds for any $P_0 \in D$ *and any* $\pi_2(P_0, e_1, e_2)$, $e_1 = \cos\theta i+\sin\theta j$. *From now on the sign* $\pm$ *will be omitted when it is indifferent, i.e., we shall write* $(f^\pm)^\varkappa_{\pi_2}(P_0) = (f)^\varkappa_{\pi_2}(P_0)$.

Proof. For $\lambda \to 0$ we have $t \to t_0$, $X(t) \to X_0$, and the theorem follows according to Lemma 2.3.

Now we shall derive dynamical jump condition corresponding to the acceleration waves.

DEFINITION 3.2. We say that the motion x of the body $\mathscr{B}$ has second-order discontinuities if $x \in C^1(\bar{D})$ and $V, F \in C^{01}(\bar{D})$.

We now assume that

(i) x has second-order discontinuities on $\bar{D}$,

(ii) The stress $T = T(X, t)$ is from $C^{01}(\bar{D})$.

The law of balance of momentum yields

$$\int\limits_{a_1}^{b_1} \varrho_0 \partial_t^+ V(X, t)dX = T(b_1, t)-T(a_1, t). \tag{3.7}$$

THEOREM 3.3. *If the hypotheses* (i)–(ii) *hold, then the following dynamical relations*

$$\varrho_0(\partial_t V)^\varkappa_{\pi_2}(P_0) = (\partial_X T)^\varkappa_{\pi_2}(P_0), \quad \varkappa = - \text{ or } + \tag{3.8}$$

hold, too, for any $P_0 \in D$ *and any* $\pi_2 = (P_0, e_1, e_2)$. *The relations* (3.8) *can be written also as*

$$\varrho_0[\partial_t V]_{\pi_2}(P_0) = [\partial_X T]_{\pi_2}(P_0). \tag{3.9}$$

Proof. Let C be a right tangent curve to e_1 at P_0, of equation $b_1 = X_0+\cot\theta(t-t_0)+ +0(t-t_0)$. Denote $X(t) = X_0+\cot\theta(t-t_0)$. Then substituting in (3.7), $a_1 = X(t)$ and dividing by $b_1(t)-a_1(t)$, we get

$$\frac{1}{b_1(t)-a_1(t)}\int\limits_{a_1(t)}^{b_1(t)} \varrho\{\partial_t^+ V(X, t)-\partial_t^+ V(b_1, t)\}dX+\varrho\partial_t^+ V(b_1, t) = \partial_X^- T(b_1, t)+\Omega(t-t_0).$$

Take a sequence $t_n \to t_0$, and using hypotheses (i)–(ii), we obtain

$$\varrho_0(\partial_t V)^-_{\pi_2}(P_0) = (\partial_X T)^-_{\pi_2}(P_0).$$

4. On the solution of a differential equation for the rate-type materials

We shall consider a simple and bounded domain D in γ, T plane, i.e., a domain for which $(\gamma_1, T), (\gamma_2, T) \in D$ imply $(\lambda\gamma_1+(1-\lambda)\gamma_2, T) \in D$ and $(\gamma, T_1), (\gamma, T_2) \in D$ imply $(\gamma, \lambda T_1+(1-\lambda)T_2) \in D$ for any $\lambda \in [0, 1]$. We shall be concerned with functions $\gamma, T:[t_0, t_1] \to R$, $\gamma, T \in R^1[t_0, t_1]$.

Usually we shall take $t_0 = 0$; moreover, we shall assume that γ (respective T) is defined on $[0, t_1]$, so that for any t, $\gamma(t) = \gamma(t-0) = \gamma^-(t)$ or $\gamma(t) = \gamma(t+0) = \gamma^+(t)$. We shall denote the set of functions from $\mathsf{R}^1[0, t_1]$ with this property by $\mathsf{R}^1_*[0, t_1]$.

Suppose $\varphi, \psi: \overline{D} \to R$ and consider the equation

$$\dot{T} = \varphi(\gamma, T)\dot{\gamma} + \psi(\gamma, T), \quad \begin{matrix}\gamma(0) = \gamma_0, \\ T(0) = T_0\end{matrix} \quad (\gamma_0, T_0) \in D \tag{4.1}$$

for functions γ, T of class C^1. Hence, for the Eq. (4.1), the natural class to look for the solutions in some meaning is C^1. Due to Denjoy's theorem, we may modify Eq. (4.1) as to look for solutions in R^1 or in C^{01}. Thus we write

$$\begin{aligned} \dot{T}_d &= \varphi(\gamma^+, T^+)\dot{\gamma}_d + \psi(\gamma^+, T^+), \qquad T^+(0) = T_0, \quad \gamma^+(0) = \gamma_0, \\ \dot{T}_s &= \varphi(\gamma^-, T^-)\gamma_s^- + \psi(\gamma^-, T^-), \qquad (\gamma_0, T_0) \in D. \end{aligned} \tag{4.2}$$

The meaning of $\overline{T}_d$, $\overline{T}_s$, etc., is given by Definition 2.3. In this way, when $\gamma, T \in C^1$, then (4.2) reduces to (4.1).

We shall try to find a kind of map which will associate to each $\gamma \in \mathsf{R}^1_*[0, t_1]$ (or $\gamma \in C^1_{[0,t_1]}$ or $\gamma \in C^{01}_{[0,t_1]}$) a function T, defined on some subinterval $[0, \omega) \subset [0, t_1]$ (ω being the largest possible $t \in (0, t_1]$), of the same class as γ on this subinterval. It will be given some necessary conditions of existence, uniqueness, continuity and differentiability with respect to uniform convergence topology of this map when γ moves in $\mathsf{R}^1_*[0, t_1]$ (or $C^1_{[0,t_1]}$, $C^{01}_{[0,t_1]}$).

DEFINITION 4.1. *A pair of functions* $(\gamma(t), T(t)) \in D$ *for* $t \in [0, \omega_\gamma)$ *is called a solution of class* R^1_* *for the initial value problem* (4.2) *if for any* $\gamma \in \mathsf{R}^1_*[0, t_1]$ *with* $\gamma^+(0) = \gamma_0$ *there is an* $\omega_\gamma \in (0, t_1]$ *and an* R^1_* *function* $T: [0, \omega_\gamma) \to R$, *such that the pair* $(\gamma(t), T(t))$ *verifies* (4.2) *for all* $t \in [0, \omega_\gamma)$ *and for a given* $\gamma \in \mathsf{R}^1_*[0, t_1]$, ω_γ *is the largest* t *with this property.*

In order to find a solution of problem (4.1), we shall apply the so-called Lagrange's method of variation of parameters. Take the problem

$$\dot{T} = \varphi(\gamma, T)\dot{\gamma}, \quad \gamma(0) = \gamma_0, \quad T(0) = T_0, \quad (\gamma_0, T_0) \in D. \tag{4.3}$$

If φ is "good enough" and $\gamma \in C^1[0, t_1]$, $\dot{\gamma}(t) \neq 0$ on $[0, t_1]$, then we can find a "good enough" solution in large (see HARTMAN [8], Chaps. II, III, V; CODDINGTON & LEVINSTON [9], Chaps. I, II) for any $(\gamma_0, T_0) \in D$:

$$T = f(\gamma, \gamma_0, T_0), \quad \gamma \in (\omega_-, \omega_+), \quad \omega_\pm = \omega_\pm(\gamma_0, T_0) \tag{4.4}$$

of the following problem

$$\frac{dT}{d\gamma} = \varphi(\gamma, T), \quad T(\gamma_0) = T_0, \quad (\gamma_0, T_0) \in D. \tag{4.5}$$

The solution (4.4) has the following properties

$$\begin{cases} \dfrac{\partial f}{\partial \gamma} = \varphi(\gamma, f), \quad \dfrac{\partial f}{\partial T_0} > 0, \\ (\gamma, f(\gamma, \gamma_0, T_0)) \in D, \quad \gamma \in (\omega_-, \omega_+), \\ \left(\omega_\pm, \lim\limits_{\gamma \to \omega_\pm} f(\gamma, \gamma_0, T_0)\right) \in \partial D. \end{cases} \tag{4.6}$$

Now it is evident that we can omit the restriction $\dot{\gamma}(t) \neq 0$; moreover, we let γ be a function in $\mathsf{R}^1_*[0, t_1]$ and f from (4.4) will give a solution for the problem

$$\begin{aligned} \dot{T}_d &= \varphi(\gamma^+, T^+)\dot{\gamma}_d, \quad T^+(0) = T_0, \quad \gamma^+(0) = \gamma_0, \\ \dot{T}_s &= \varphi(\gamma^-, T^-)\dot{\gamma}_s, \quad (\gamma_0, T_0) \in D, \end{aligned} \tag{4.7}$$

if we choose $\omega \in (0, t_1]$, so that $(\gamma(t), f(\gamma(t), \gamma_0, T_0)) \in D$ for $t \in [0, \omega)$. In fact, we have to choose ω so that $\gamma(t) \in (\omega_-, \omega_+)$ for $t \in [0, \omega)$.

We can prove the following

THEOREM 4.1. *Let $\varphi: \overline{D} \to R$, φ and $\partial\varphi/\partial T$ be continuous on $\overline{D}$. Suppose D is a bounded simple domain of $\gamma \circ T$ plane. Let $\gamma \in \mathsf{R}^1_*[0, t_1]$, $T_0 \in R$ and $\gamma^+(0) = \gamma_0$ be such that $(\gamma_0, T_0) \in D$. We also assume that there exists a continuous function k depending on γ, $k:[0, t_1] \to R$, $k(t) > 0$, such that for any solution $T \in \mathsf{R}^1_*$ of the problem* (4.7) *for $\gamma \in \mathsf{R}^1_*[0, t_1]$ fixed, we have*

$$[T] = k[\gamma], \tag{4.8}$$

where $[T](t) = T(t+0) - T(t-0)$. Then the problem (4.7) *has a unique solution.*

P r o o f. We have to prove the uniqueness only. We suppose there exist $\omega_1 > 0$, $\omega_2 > 0$ and $T_1, T_2 \in \mathsf{R}^1_*$ such that $(\gamma(t), T_1(t)) \in D$ for $t \in [0, \omega_1)$ and $(\gamma(t), T_2(t)) \in D$ for $t \in [0, \omega_2)$ are solutions of problem (4.7). Then, for $t \in [0, \omega)$, $\omega = \min(\omega_1, \omega_2)$

$$\dot{T}_{1d} - \dot{T}_{2d} = (\varphi(\gamma^+, T_1^+) - \varphi(\gamma^+, T_2^+))\dot{\gamma}_d = \frac{\partial\varphi}{\partial T}(\gamma^+, \tilde{T})(T_1^+ - T_2^+)\dot{\gamma}_d.$$

Write $T = T_1 - T_2$. From (4.8) it follows that T is continuous. Hence, as $\dot{\gamma}_d$ is bounded on $[0, t]$ and $\partial\varphi/\partial T$ is bounded on D, we have

$$|\dot{T}_d| \leqslant M|T|, \quad T(0) = 0. \tag{4.9}$$

According to Lemma 2.1, $T(t) \equiv 0$ for $t \in [0, \omega)$, so $T_1(t) = T_2(t)$ for $t \in [0, \omega)$. If $\omega = \omega_1$, then for $t \in [\omega_1, \omega_2)$, $T_2(t)$ can be considered as an extension of $T_1(t)$, that contradicts the maximality of ω_1 and therefore $\omega_1 = \omega_2$.

If we look for a solution in C^{01} or in C^1, then the condition (4.8) is automatically satisfied, γ and T being continuous functions.

Now, we shall use Lagrange's method to obtain a solution for the problem (4.2). We choose an arbitrary, but fixed $\gamma \in \mathsf{R}^1_*[0, t_1]$ with $\gamma^+(0) = \gamma_0$. Instead of T_0, we put an unknown function of t, $\tau(t)$ and determine this function, so that (4.2) is verified. We have

$$\dot{T}_d = \frac{\partial f}{\partial \gamma}(\gamma^+, \gamma_0, \tau)\dot{\gamma}_d + \frac{\partial f}{\partial \tau}(\gamma^+, \gamma_0, \tau)\dot{\tau}_d = \varphi(\gamma^+, f(\gamma^+, \gamma_0, \tau))\dot{\gamma}_d + \psi(\gamma^+, f(\gamma^+, \gamma_0, \tau)).$$

Taking into account (4.6), we can write

$$\dot{\tau}_d = \mu(\gamma^+, \tau), \quad \tau(0) = T_0, \tag{4.10}$$

or

$$\tau(t) = T_0 + \int_0^t \mu(\gamma(s), \tau(s))\,ds, \tag{4.11}$$

where

$$\mu(\gamma, \tau) = \frac{\psi(\gamma, f(\gamma, \gamma_0, \tau))}{\frac{\partial f}{\partial \tau}(\gamma, \gamma_0, \tau)}. \tag{4.12}$$

Due to the equivalence between (4.10) (if we add the equation for $\dot{\tau}_s$) and (4.11), we can solve the problem (4.2) using functions $\tau \in C^{01}$ for $\gamma \in \mathrm{R}^0_*$ or $\tau \in C^1$ for $\gamma \in C^{01}$, i.e., τ is "roughly speaking", of a class "better" than γ (and T). τ will be called the history parameter.

The existence and uniqueness of the problem (4.10) (or (4.11)) for a fixed $\gamma \in \mathrm{R}^0_*[0, t_1]$, are assumed in very weak hypotheses concerning the function μ. We shall deal with continuity and differentiability of the solution when γ moves in $\mathrm{R}^0_*[0, t_1]$.

In fact, the following theorem can be proved.

THEOREM 4.2. *Suppose the following conditions hold*:

(i) D_0 *is a simple bounded domain*,

(ii) $\mu: \bar{D}_0 \to R$ *is a continuously differentiable function and* $\partial\mu/\partial T$, $\partial\mu/\partial\gamma$ *are bounded on* D_0,

(iii) $\gamma_0, T_0, \mathring{\gamma}_n, \mathring{T}_n$ *are real numbers with properties*, $(\gamma_0, T_0), (\mathring{\gamma}_n, \mathring{T}_n) \in D_0$, $\lim_{n\to\infty} \mathring{\gamma}_n = \gamma_0$, $\lim_{n\to\infty} \mathring{T}_n = T_0$,

(iv) $\gamma_n, \gamma \in \mathrm{R}^0_*[0, t_1]$, $\gamma_n^+(0) = \mathring{\gamma}_n$, $\gamma^+(0) = \gamma_0$ and $\gamma_n \to \gamma$, *when* $n \to \infty$ *in the norm* $\|\gamma\| = \max_{t\in[0, t_1]} |\gamma(t)|$.

Then: (I) *for each problem*

$$\tau(t) = T_0 + \int_0^t \mu(\gamma(s), \tau(s))\, ds,$$

$$\tau_n(t) = \mathring{T}_n + \int_0^t \mu(\gamma_n(s), \tau_n(s))\, ds, \quad n = 1, 2, \ldots,$$

there exist the maximal intervals $[0, \omega) \subset [0, t_1]$, $[0, \omega_n) \subset [0, t_1]$ $(\omega, \omega_n \in (0, t_1])$ *and* $\tau(t), \tau_n(t)$ *uniquely determined functions from* C^{01}, *so that* $(\gamma(t), \tau(t)) \in D_0$ *for* $t \in [0, \omega)$, $(\gamma_n(t), \tau_n(t)) \in D_0$ *for* $t \in [0, \omega_n)$ *and* $(\gamma(\omega-0), \tau(\omega-0)) \in \bar{D}_0$, $(\gamma_n(\omega_n-0), \tau_n(\omega_n-0)) \in \bar{D}_0$ *and if* $\omega < t_1$ *and* $\omega_n < t_1$, *then* $(\gamma(\omega+0), \tau(\omega-0)) \notin D_0$, $(\gamma_n(\omega_n+0), \tau_n(\omega_n-0)) \in D_0$. *When* $n \to \infty$, *we have*

$$\omega \leqslant \liminf_{n\to\infty} \omega_n, \quad \tau(t) = \lim_{n\to\infty} \tau_n(t), \quad t \in [0, \omega).$$

(II) *The map* $\gamma(\cdot) \to \tau(\cdot)$ *is continuous* (FRÉCHET) *differentiable mapping with respect to uniform convergence.* Gateaux

P r o o f. The proof of the first part of this theorem is similar to classical proofs concerning the continuous dependence of the solution upon initial data (see, for example, HARTMAN [8], Chaps. II, III, V, and CODDINGTON & LEVINSON [9], Chaps. I, II). We shall give a proof for the last part of this theorem.

Let $\gamma, l \in \mathsf{R}^0_*[0, t_1]$ (one-side continuous function from $\mathsf{R}^0_{[0,t_1]}$) such that (γ_0, T_0), $(\gamma_0+\lambda l_0, T_0) \in D_0$ $\big(\gamma_0 = \gamma^+(0), l_0 = l^+(0)\big)$ for a sufficiently small λ. Then there are $\omega, \omega_\lambda \in (0, t_1]$ $(\omega \leqslant \liminf_{\lambda\to 0} \omega_\lambda)$ and $\tau(t)$, $\tau_\lambda(t)$ such that

$$(4.12)\qquad \tau_\lambda(t)-\tau(t) = \int_0^t [\mu(\gamma+\lambda l, \tau_\lambda)-\mu(\gamma, \tau_\lambda)+\mu(\gamma, \tau_\lambda)-\mu(\gamma, \tau)]ds$$

$$= \int_0^t \left[\lambda \frac{\partial\mu}{\partial\gamma}(\gamma+\lambda\theta_1 l, \tau_\lambda)l+ \frac{\partial\mu}{\partial\tau}(\gamma, \theta_2(\tau_\lambda-\tau))(\tau_\lambda-\tau)\right]ds$$

for $t \in [0, \tilde\omega)$, with $\tilde\omega = \min(\omega, \omega_\lambda)$, $\theta_1 = \theta_1(s)$, $\theta_2 = \theta_2(s)$ and $0 \leqslant \theta_1, \theta_2 \leqslant 1$.

We introduce the notations

$$V_\lambda(t) = \frac{\tau_\lambda(t)-\tau(t)}{\lambda}, \quad C = \max_{(\gamma,\tau)\in D_0}\left|\frac{\partial\mu}{\partial\gamma}(\gamma, \tau)\right|, \quad P = \max_{(\gamma,\tau)\in D_0}\left|\frac{\partial\mu}{\partial\tau}(\gamma, \tau)\right|$$

and we show that $|V_\lambda(t)| \leqslant M < \infty$ for any $|\lambda| \leqslant a$, and $t \in [0, \tilde\omega)$. Indeed, from (4.12) we get

$$|V_\lambda(t)| \leqslant C\|l\|\tilde\omega+P\int_0^t |V_\lambda(s)|ds$$

and applying Gronwall's lemma

$$(4.13)\qquad |V_\lambda(t)| \leqslant C\|l\|\tilde\omega\exp(\tilde\omega P), \quad t \in [0, \tilde\omega).$$

Relation (4.12) can be written also as

$$(4.14)\qquad V_\lambda(t)-\int_0^t \left[\frac{\partial\mu}{\partial\gamma}(\gamma, \tau)l(s)+\frac{\partial\mu}{\partial\tau}(\gamma, \tau)V_\lambda(s)\right]ds = \int_0^t \left[\frac{\partial\mu}{\partial\gamma}(\gamma+\lambda\theta_1 l, \tau_\lambda)\right.$$

$$\left.-\frac{\partial\mu}{\partial\gamma}(\gamma, \tau)\right]l(s)ds+\int_0^t \left[\frac{\partial\mu}{\partial\tau}(\gamma, \tau+\lambda\theta_2 V_\lambda)-\frac{\partial\mu}{\partial\tau}(\gamma, \tau)\right]V_\lambda(s)ds.$$

Consider now the initial value problem

$$V(t)-\int_0^t \left[\frac{\partial\mu}{\partial\gamma}(\gamma, \tau)l+\frac{\partial\mu}{\partial\tau}(\gamma, \tau)V\right]ds = 0.$$

This problem possesses a unique continuous solution $V \in C^{01}[0, \omega)$ for $\gamma, l \in \mathsf{R}^*_0[0, t_1]$ and $\tau \in C^{01}[0, \omega)$, given by

$$(4.15)\qquad V(t) = \exp\left(\int_0^t \frac{\partial\mu}{\partial\tau}(\gamma, \tau)ds\right)\left(\int_0^t \exp\left(-\int_0^s \frac{\partial\mu}{\partial\tau}(\gamma, \tau)ds_1\right)\frac{\partial\mu}{\partial\gamma}(\gamma, \tau)l\,ds\right).$$

As τ_λ is continuous with respect to $\gamma+\lambda l$ and $\partial\mu/\partial\gamma$, $\partial\mu/\partial\tau$ are continuous, according to (4.13) we get from (4.14), for $\lambda \to 0$

$$\lim_{\lambda\to 0} V_\lambda(t) = V(t), \quad t \in [0, \omega).$$

Based on the first part of this theorem, if we introduce the functional $\mathcal{F}$ which associates with each function $\gamma \in R^0_*[0, t_1]$ and each number T_0 $\big((\gamma_0, T_0) \in D_0\big)$ a real valued function τ defined on some interval $[0, \omega_\gamma)$

$$\tau(t) = \mathcal{F}[\gamma(\cdot), t, T_0], \quad t \in [0, \omega_\gamma), \tag{4.16}$$

and taking into account the hypothesis (ii) of the theorem and formula (4.15), we can say $\mathcal{F}$ is a continuously differentiable mapping and write

$$\begin{aligned}(4.17)\quad \delta\mathcal{F}[\gamma(\cdot), t, T_0 | l(\cdot)] &= V(t)\\ &= \exp\left(\int_0^t \frac{\partial\mu}{\partial\tau}(\gamma, \tau) ds\right)\left(\int_0^t \exp\left(-\int_0^s \frac{\partial\mu}{\partial\tau}(\gamma, \tau) ds_1\right)\frac{\partial\mu}{\partial\gamma}(\gamma, \tau) l ds\right).\end{aligned}$$

Now, we can return to problem (4.2) and prove the following.

THEOREM 4.3. *For the problem* (4.2), *we suppose the following conditions to be satisfied*:

(i) *D is a plane bounded simple domain,*

(ii) $\varphi, \psi: \overline{D} \to R$ *are continuous on* $\overline{D}$ *and* $\partial\varphi/\partial\gamma$, $\partial\varphi/\partial T$, $\partial^2\varphi/\partial T^2$, $\partial\psi/\partial\gamma$, $\partial\psi/\partial T$ *exist and are continuous and bounded on D, and*

(iii) *the condition* (4.8) *holds.*

Then for any $\gamma \in R^1_*[0, t_1]$, $\gamma^+(0) = \gamma_0$.

(I) *There are* $\omega \in (0, t_1]$ *and* $T: [0, \omega) \to R$ *such that* $\gamma(t), T(t)$, *where* $T(t)$ *is given by*

$$\begin{aligned} T(t) &= f(\gamma(t), \gamma_0, \tau(t)), \quad t \in [0, \omega),\\ \tau(t) &= \mathcal{F}[\gamma(\cdot), t, T_0] \end{aligned} \tag{4.18}$$

is the unique solution (in the sense of Definition 4.1) *of problem* (4.2), *f and* $\mathcal{F}$ *being determined by* (4.4) *and* (4.16), *respectively.*

(II) *The map* $\gamma \to T$ *defined by* (4.18) *is a continuously differentiable mapping* (*for any* $t \in [0, \omega)$) *in the sense of uniform convergence topology.*

P r o o f. Let $\gamma \in R^1_*[0, t_1]$ and $T_0 \in R$, so that $\gamma^+(0) = \gamma_0$ and $(\gamma_0, T_0) \in D$. Now, we can follow the procedure which we have followed when we derived the results under the formulas (4.3)–(4.7) and (4.10)–(4.12).

We intend to apply the Theorem 4.2 to obtain a solution for the problem (4.2). For that we need to make precise the domain of definition for $\mu(\gamma, \tau)$ defined by (4.12) for a fixed γ_0. $f(\gamma, \gamma_0, \tau)$ is defined for any τ with $(\gamma_0, \tau) \in D$ and $\omega_-(\gamma_0, \tau) < \gamma < \omega_+(\gamma_0, \tau)$, and establishes a one-to-one correspondence between τ and T, for a fixed γ_0 and γ.

Denote by τ_+ and τ_- the two values of τ for which the segment $(\gamma_0 \tau)$ crosses ∂D; $(\gamma_0, \tau) \in D$ for $\tau \in (\tau_-, \tau_+)$. Consider the domains $D_n \subset D$ with $\overline{D}_n \subset D_{n+1}$ and $\bigcup_1^\infty D_n = D$ (for example, $D_n = \left\{(\gamma, T)/(\gamma, T) \in D, |\gamma| < n,\ |T| < n, \text{dist}((\gamma, T), \partial D) > \frac{1}{n}\right\}$ see, for instance, HARTMAN [8], Chap. II) and denote by τ_{n-}, τ_{n+} the intersections of (γ_0, τ) with D_n. Then dist $\big((\gamma_0, \tau_{n\pm}), \partial D\big) \geqslant 1/n$ and $(\gamma_0, \tau) \in D$ for $\tau \in [\tau_{n-}, \tau_{n+}]$. The properties of

$\omega_\pm(\gamma_0, \tau)$ imply that for any $\varepsilon > 0$ and any $\bar{\tau} \in [\tau_{n-}, \tau_{n+}]$ there exist the neighbourhoods $V'(\bar{\tau})$, $V''(\bar{\tau})$ such that

$$\omega_+(\gamma_0, \bar{\tau})+\varepsilon < \omega_+(\gamma_0, \tau), \quad \tau \in V'(\bar{\tau}),$$
$$\omega_-(\gamma_0, \tau) < \omega_-(\gamma_0, \bar{\tau})+\varepsilon, \quad \tau \in V''(\bar{\tau}).$$

Take $\varepsilon = 1/n$ and $V_n(\bar{\tau}) = V'_n(\bar{\tau}) \bigcap V''_n(\bar{\tau})$; then we can write

$$\omega_-[\gamma_0, \tau)-\frac{1}{n} < \omega_-(\gamma_0, \bar{\tau}) < \gamma_0 < \omega_+(\gamma_0, \bar{\tau}) < \omega_+(\gamma_0, \tau)-\frac{1}{n}$$

for $\tau \in V_n(\bar{\tau})$. Now we may find a finite covering of $[\tau_{n-}, \tau_{n+}]$, say $V_n(\tau_1), \ldots, V_n(\tau_m)$. Choose $\omega'_-(\gamma_0) = \max\limits_{i=1,\ldots,m} \omega_-(\gamma_0, \tau_i)$ and $\omega'_+(\gamma_0) = \min\limits_{i=1,\ldots,m} \omega_+(\gamma_0, \tau_i)$. Then $\omega'_-(\gamma_0) < \gamma_0 <$ $< \omega'_+(\gamma_0)$. Denote

$$\Delta(\gamma_0, T_0) = \{(\gamma, \tau)/\gamma \in (\omega'_-(\gamma_0), \omega'_+(\gamma_0)), \tau \in (\tau_{n-}, \tau_{n+})\},$$

then $(\gamma, T) \in D$, where $T = f(\gamma, \gamma_0, \tau)$ for any $(\gamma, \tau) \in \Delta(\gamma_0, T_0)$.

According to Theorem 4.2 for $\mu: \Delta(\gamma_0, T_0) \to R$ and $\gamma \in \mathsf{R}^1_*[0, t_1] \subset \mathsf{R}^0_*[0, t_1]$ with $\gamma^+(0) = \gamma_0$, there is an $\omega_1 \in (0, t_1]$ such that

(4.19) $$\tau(t) = \mathscr{F}[\gamma(\cdot), t, T_0], \quad t \in [0, \omega_1)$$

is a solution of the problem (4.11) and

(4.20) $$T(t) = f(\gamma(t), \gamma_0, \tau(t)), \quad t \in [0, \omega_1)$$

is a solution of the problem (4.2).

If $\omega_1 = t_1$, the existence is proved. Assume $\omega_1 < t_1$. We have $(\gamma(\omega_1-0), \tau(\omega_1-0) \in$ $\in \bar{\Delta}(\gamma_0, T_0)$ and $(\gamma(\omega_1+0), \tau(\omega_1-0)) \notin \Delta(\gamma_0, T_0)$. There are two alternatives: either $\gamma(\omega_1-0), T(\omega_1-0) \in \bar{D}$ and $\gamma(\omega_1+0)$ does not belong to the segment $(\omega_-(\gamma_0, \tau(\omega_1-0)), \omega_+(\gamma_0, \tau(\omega_1-0)))$ and hence $\omega_1 = \omega$ (this concludes the proof), or $(\gamma(\omega_1+0), T(\omega_1+0)) \in D$ (we have used here the fact that τ is continuous). If $(\gamma(\omega_1+0), T(\omega_1+0)) \in D_n$, we choose this point instead of (γ_0, T_0) and building another $\Delta(\gamma(\omega_1+0), T(\omega_1+0))$, we can apply the same procedure as before and we get an $\omega_1^1 > \omega_1$. If we are still in D_n, then applying successively this procedure, we obtain either $\omega_1^1 = t_1$ or $(\gamma(\omega_1^1+0), T(\omega_1^1+0)) \notin D_n$, but it is a point from D. Then there is an $n_1 > n$ such that $(\gamma(\omega_1^1+0), T(\omega_1^1+0)) \in D_{n_1}$, etc.

Finally, we find an $\omega \leqslant t_1$, so that $(\gamma(t), T(t)) \in D$ $t \in [0, \omega)$ and if $\omega < t_1$, $(\gamma(\omega-0), T(\omega-0)) \in \bar{D}$ and $\gamma(\omega+0)$ does not belong to the segment $(\omega_-(\gamma_0, \tau(\omega-0)),$ $\omega_+(\gamma_0, \tau(\omega-0)))$.

To prove the uniqueness of the solution, we suppose there are two solutions of the problem (4.2), $(\gamma(t), T_1(t)) \in D$ for $t \in [0, \omega^1)$ and $(\gamma(t), T_2(t)) \in D$ for $t \in [0, \omega^2)$, with $\gamma^+(0) = \gamma_0$, $T_1^+(0) = T_2^+(0) = T_0$.

Then, as D is a simple domain relative to axes, φ and ψ are smooth bounded functions on D and $\dot{\gamma}_d$ is bounded on $[0, t_1]$, we can write

$$\dot{T}_{1d}-\dot{T}_{2d} = \left[\frac{\partial\varphi}{\partial\gamma}(\gamma^+, T^*)\dot{\gamma}_d+\frac{\partial\psi}{\partial\gamma}(\gamma^+, T^{**})\right](T_1^+-T_2^+)$$

and, therefore

$$|\dot{T}_{1d}-\dot{T}_{2d}| \leqslant M|T_1^+-T_2^+|, \quad T_1^+(0)-T_2^+(0) = 0, \quad t \in [0, \omega)$$

with $\omega = \min(\omega^1, \omega^2)$. Now, taking into account the assumption (iii) and Lemma 2.1, we obtain $T_1(t) = T_2(t)$ and $\omega^1 = \omega^2$.

The continuity and differentiability of the map $\gamma \to f(\gamma, \gamma_0, \tau)$ are consequences of the properties of φ and ψ; these imply $\partial f/\partial\gamma$, $\partial f/\partial\tau$ are continuous. On the other hand, the continuity and differentiability of $\tau(t) = \mathscr{F}[\gamma(\cdot), t, T_0]$ are given by Theorem 4.2. The differential of f can be written as

$$\text{(4.21)} \quad \delta f(\gamma(t), \gamma_0, \tau(t)|l(t)) = \frac{\partial f}{\partial\gamma}(\gamma(t), \gamma_0, \tau(t))\,l(t)+\frac{\partial f}{\partial\tau}(\gamma(t), \gamma_0, \tau(t)) \times$$
$$\times \exp\left(\int_0^t \frac{\partial\mu}{\partial\tau}(\gamma(s), \tau(s))\,ds\right)\left[\int_0^t \exp\left(-\int_0^s \frac{\partial\mu}{\partial\tau}(\gamma(s_1), \tau(s_1))\,ds_1\right) \times\right.$$
$$\left.\times \frac{\partial\mu}{\partial\gamma}(\gamma(s), \tau(s))\,l(s)\,ds\right],$$

where $\mu(\gamma, \tau)$ is given by (4.12). The theorem is completely proved.

As a consequence of Theorems 4.3 and 4.2, we have, for $\gamma \in \mathsf{R}_*^1[0, t_1]$, $\tau(t)$ of class C^{01} for $t \in [0, \omega)$ and therefore the discontinuities of T at time t are given only by discontinuities of γ at the same time t. The discontinuities of the derivative of T are also depending on τ. If we suppose $\gamma \in C^{01}_{[0,t_1]}$, then τ is of class C^1, hence the discontinuity of the derivative of T at $t \in [0, \omega)$ depends only on the discontinuity of the derivative of γ at t, the discontinuities of γ's derivative on the interval $[0, t)$ having no influence.

If we consider that $T(t)$ is determined as a function of two variables (not necessary as a solution of Eq. (4.2)), $\gamma(t)$, and some history parameter $\tau(t)$ which is a functional of $\gamma(\cdot)$ up to time t, then from the above remarks it appears as natural to suppose some smoothness properties on the functional relation and not on the past history of γ.

5. Discontinuous motions in elastic and rate-type materials

5.1. Elastic materials

5.1.1. *Motions with second-order discontinuities*. Suppose that the motion x is a motion with second-order discontinuities (see Definition 3.2) and the stress T is given by formula (1.3), $T = g(\gamma)$. Then Theorem 3.3 can be applied (the hypothesis (ii) being automatically satisfied) and yields

$$\text{(5.1)} \quad \varrho_0[\partial_t V]_{\pi_2}(P_0) = \frac{dg(\gamma)}{d\gamma}[\partial_X \gamma]_{\pi_2}(P_0).$$

Using Theorem 3.1 for f equal to V and γ, respectively, and denoting

$$\text{(5.2)} \quad C = \mathrm{ctg}\,\theta$$

we get

$$[\partial_t V]_{\pi_2}(P_0)+C[\partial_X V]_{\pi_2}(P_0) = 0, \tag{5.3}$$

$$[\partial_t \gamma]_{\pi_2}(P_0)+C[\partial_X \gamma]_{\pi_2}(P_0) = 0. \tag{5.4}$$

Lemma 3.1 implies the additional relation

$$[\partial_X V]_{\pi_2}(P_0) = [\partial_t \gamma]_{\pi_2}(P_0). \tag{5.5}$$

The relations (5.1)–(5.5) yield

$$\left(\varrho_0 C^2 - \frac{dg(\gamma)}{d\gamma}\right)[\partial_X \gamma]_{\pi_2}(P_0) = 0. \tag{5.6}$$

If $[\partial_X \gamma]_{\pi_2}(P_0) \neq 0$, we have

$$\varrho_0 C^2 = \frac{dg(\gamma)}{d\gamma} \tag{5.7}$$

and (5.7) will admit real solutions only if $dg/d\gamma \geqslant 0$.

Relation (5.7) shows that through any $P_0 \in D$ there are at most two discontinuity directions. Now, according to Corollaries 2.1–2.3, the following theorem is true.

THEOREM 5.1. *For an elastic material in a motion with second-order discontinuities, the number of discontinuity directions through any point is at most two, and the set of points at which there are effectively two discontinuity directions, is at most countable.*

5.1.2. *Motions with first-order discontinuities.* Suppose now that the motion x of an elastic material is a motion with first-order discontinuities in the sense of Definition 3.1. The hypothesis that the material is elastic involves condition (ii) from Theorem 3.2. Applying Theorems 3.1, 3.2, we get

$$\begin{gathered} [V]_{\pi_2}(P_0)+C[\gamma]_{\pi_2}(P_0) = 0, \\ g(\gamma^+_{\pi_2}(P_0))-g(\gamma^-_{\pi_2}(P_0))+\varrho_0 C(V^+_{\pi_2}(P_0)-V^-_{\pi_2}(P_0)) = 0. \end{gathered} \tag{5.8}$$

From (5.8) we have

$$\varrho_0 C(\gamma^+_{\pi_2}(P_0)-\gamma^-_{\pi_2}(P_0)) = g(\gamma^+_{\pi_2}(P_0))-g(\gamma^-_{\pi_2}(P_0)). \tag{5.9}$$

Therefore, if $[\gamma]_{\pi_2}(P_0) \neq 0$, we obtain

$$\varrho_0 C^2 = \frac{g(\gamma^+_{\pi_2}(P_0))-g(\gamma^-_{\pi_2}(P_0))}{\gamma^+_{\pi_2}(P_0)-\gamma^-_{\pi_2}(P_0)}. \tag{5.10}$$

Obviously, (5.10) will have real solutions for C only if g is an increasing function.

Relation (5.9) or (5.10) may be interpreted as follows: in a point $P_0 \in D$, two given numbers γ^+ and γ^- can be one-side limits of the strain at P_0 along the unit vector e_1, if the slope of e_1 is given by (5.10); on the other hand, if e_1 is known as a discontinuity unit vector and one of one-side limits is also known, then the value of the second one is given by (5.10).

For motions with first-order discontinuities, there are no other limitations relative to the number of discontinuity points or directions than general ones for R^2-regulated functions.

5.2. Rate-type materials

Let D be a simple bounded domain in the plane $\gamma \circ T$ and suppose $(0, 0) \in D$. Let $\varphi, \psi: D \to R$, having the properties from Theorem 4.3. A quasi-linear rate-type material for smooth strain and stress histories is described by the Eq. (1.4).

DEFINITION 5.1. *One says that a quasi-linear rate-type material admits a natural rest configuration, if* $\gamma(t) \equiv 0$ *for* $t \in (t_1, t_2)$ *and* $T(t_1) = 0$ *implies* $T(t) \equiv 0$ *for* $t \in (t_1, t_2)$ *for any* $t_1 < t_2$, $t_1, t_2 \in R$.

PROPOSITION 5.1. *If* $\psi(\gamma, T)$ *is Lipschitzean in* T *on* D (in the conditions of Theorem 4.3, ψ verifies this requirement) *and* $\psi(0, 0) = 0$, *then the rate-type material described by* the Eq. (1.4) *admits a natural rest configuration.*

The fact that a material possesses a natural rest configuration can be expressed in other words as follows: if the material is not deformed and has no initial stresses, then stresses cannot appear in it.

Suppose that the rate-type material described by (1.4) possesses a natural rest configuration; i.e., $\psi(0, 0) = 0$ and for $t < 0$ this material was in the natural rest configuration. Then the set of initial conditions (γ_0, T_0) from the problem (4.7) verifies the following relation

$$T_0 = f(\gamma_0, 0, 0) = f_0(\gamma_0), \gamma_0 \in \big(\omega_-(0, 0), \omega_+(0, 0)\tag{5.11}$$

(see Theorems 4.1, 4.3). It means that the initial states lie on a curve, which is solution of the problem $dT/d\gamma = \varphi(\gamma, T)$ $T(0) = 0$.

As we shall see further, the rate-type material has to have the property that $\varphi(\gamma, T) \geqslant 0$; therefore the function $f_0(\gamma)$ is an increasing function. The curve $(\gamma, f_0(\gamma)) \in D$, $\gamma \in (\omega_-, \omega_+)$ can be called instantaneous response curve from the rest configuration.

From this discussion it follows that for a rate-type material with a natural rest configuration, the initial stress cannot be given arbitrarily, but is determined by initial strain. In other words, a rate-type constitutive equation of the form (1.4), in the condition of Theorem 4.3, and $\psi(0, 0) = 0$ describes a single material rather than a family of materials.

5.2.1. *Motions with second-order discontinuities.* Suppose that the motion x of a rate-type material is a motion with second-order discontinuities on the domain Δ from the $X \circ t$ plane. Then the strain $\gamma \in C^{01}(\Delta)$ and by Theorem 4.3 the stress T belongs to the same class $C^{01}(\Delta)$. Of course the initial conditions $\gamma_0(X)$ and $T_0(X)$ have to be of class C^{01} with respect to X. By Theorem 4.2, the history parameter τ has smooth derivatives with respect to t and is at least of class $C^{01}(\Delta)$ with respect to X, t jointly.

Applying Theorem 3.1 to τ, we have

$$[\partial_X \tau]_{\pi_2}(P_0)\cos\theta + [\partial_t \tau]_{\pi_2}(P_0)\sin\theta = 0,\tag{5.12}$$

and for $\theta \neq \pm\pi/2$, we obtain

$$(\partial_X \tau)^+_{\pi_2}(P_0) = (\partial_X \tau)^-_{\pi_2}(P_0) = (\partial_X \tau)^+_{\pi_1}(P_0), \quad \pi_2 = (P_0, e_1, e_2),\tag{5.13}$$

$$\pi_1 = (P_0, e_1);$$

it means that except the t direction, the function τ has no other discontinuity directions.

Taking into account formulas (4.18) (Theorem 4.3), we can write (3.8) (Theorem 3.3) as

$$(5.14)\quad \varrho_0(\partial_t V)^{\varkappa}_{\pi_2}(P_0) = \frac{\partial f}{\partial \gamma}(\gamma, \gamma_0, \tau)(\partial_X \gamma)^{\varkappa}_{\pi_2}(P_0) + \frac{\partial f}{\partial \gamma_0}(\gamma, \gamma_0, \tau)(\partial_X \gamma_0)^{\varkappa_1}(X, 0) + \frac{\partial f}{\partial \tau}(\gamma, \gamma_0, \tau)(\partial_X \tau)^{+}_{\pi_1}(P_0).$$

Here $\varkappa = +$ or $-$, depending on the tangent curve to the unit vector $e_1 = \cos\theta i + \sin\theta j$ from $\pi_2 = (P_0, e_1, e_2)$, plus for the left tangent curve and minus for the right tangent curve. $\varkappa_1 = +$ for $\theta \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$ and $\varkappa_1 = -$ otherwise. We have supposed $\theta \neq \pm \frac{\pi}{2}$ and we have applied (5.13). Using (4.16) and (4.17), we obtain

$$(5.15)\quad (\partial_X \tau)^{+}_{\pi_1}(P_0) = \big(\delta \mathscr{F}[\gamma(X, \cdot), t, T_0 | \partial^{\ddagger}_X \gamma(X, \cdot)]\big)^{+}_{\pi_1}(P_0) + \frac{\partial \mathscr{F}}{\partial T_0}[\gamma(X, \cdot), t, T_0]\,(\partial_X T_0)^{\varkappa_1}(X, 0).$$

Here, the expression of $\delta\mathscr{F}[\ldots | \partial^{\ddagger}_X \gamma(X, \cdot)]$ is given by (4.17), where we put $l(\cdot) = \partial^{\ddagger}_X \gamma(X, \cdot)$, $\varkappa_1$ has the same meaning as in (5.14).

Now, from (5.14), we obtain

$$(5.16)\quad \varrho_0[\partial_t V]_{\pi_2}(P_0) = \frac{\partial f}{\partial \gamma}(\gamma, \gamma_0, \tau)\,[\partial_X \gamma](P_0),$$

and following the same way as for elastic materials, we get

$$(5.17)\quad \varrho_0 C^2 = \frac{\partial f}{\partial \gamma}(\gamma, \gamma_0, \tau) = \varphi(\gamma, f(\gamma, \gamma_0, \tau)).$$

In obtaining the last equality the formula (4.6) has been involved.

Therefore, there are three possible discontinuous directions for a motion with second-order discontinuities in a rate-type material; the two directions given by (5.17), which are depending on the whole history of strain and the direction $X = \text{const}$. Similar conclusions as those from Theorem 5.1 hold here as well. Also, from (5.14) we can see that as a result of integration of constitutive equation, the equation of motion contains some additional terms but the characteristics of the system remain the same.

5.2.2. *Motions with first-order discontinuities*. In the condition of Theorem 4.3, taking into account Proposition 2.5, the results for motions with first-order discontinuities in rate-type materials are similar to the corresponding results for elastic materials. The main difference consists in the fact that slope of discontinuity directions are depending in this case on the history parameter τ as well.

We shall discuss now the meaning of the condition (iii)–(4.8) involved in the Theorem 4.3, in order to obtain uniqueness in the case when motions with first-order discontinuities are considered.

Suppose that the motion $x: \overline{\Delta} \to R$ satisfies the supplementary assumption:

(iii′) The set of points $P \in \Delta$ with property

$$\partial^{+}_t x(P) \neq \partial^{-}_t x(P) \quad \text{or} \quad \partial^{+}_X x(P) \neq \partial^{-}_X x(P)$$

is formed from isolated arcs of smooth curves, whose slope is not parallel to the axes; that is, if C is an arc of smooth curve for which $\partial_t^+ x(P_0) \neq \partial_t^- x(P_0)$, $P_0 \in C$, then there is an open disc $d(P_0)$ with the center in P_0, divided in two parts by C such that $\partial_t^+ x(P) = \partial_t^- x(P)$ for $P \in d(P_0) - C$. (See also [11]).
As an immediate consequence of hypothesis (iii') and Theorem 3.1 is the following: if $\partial_t x$ is discontinuous when crossing C, then the same is true for $\partial_X x$. If we denote by $\delta(P_0)$ the tangent straight line to C in P_0, then

$$\partial_t^+ x(P_0) = (\partial_t x)^{*}_{\pi_1^2}(P_0) = (\partial_t x)^{*}_{\pi_2^+}(P_0),$$
$$\partial_t^- x(P_0) = (\partial_t x)^{*}_{\pi_2^{11}}(P_0) = (\partial_t x)^{*}_{\pi_2^-}(P_0),$$

where

$$\pi_2^1 = (P_0, j, -i), \quad \pi_2^{11} = (P_0, -j, i), \quad \pi_2^+ = (P_0, e_1^+, e_2^+), \quad \pi_2^- = (P_0, e_1^-, e_2^-)$$

and e_1^+ is a unit vector at P_0 lying on the same side of $\delta(P_0)$ as j; similarly for e_1^-.

For a fixed X and an interval $[t_0, t_1]$ such that $(X, t) \in \bar{\Delta}$ for $t \in [t_0, t_1]$, there is a finite number of discontinuity curves which cut the segment $(X, t), t \in [t_0, t_1]$, say C_i, $1 \leqslant i \leqslant N$. Denote by $P_i = (X, t_i)$ the intersection points of C_i with the segment and by c_i the slope of C_i. Then there is a function $k:[t_0, t_1] \to R_+$ with $k(t_i) = \varrho c_i^2$. From Theorems 3.1, 3.2 and the Eq. (5.2), we get

$$[\gamma](X, t_i) = \varrho_0 c_i^2 [T](X, t_i),$$

or using the function k, for any $t \in [t_0, t_1]$

$$[\gamma](X, t) = k(t)[T](X, t).$$

This means that the condition (iii) holds when condition (iii') holds.

References

1. C. TRUESDELL and W. NOLL, *The non-linear field theories of mechanics*, Handbuch der Physik, Band III/3, Springer, 1965.
2. B. D. COLEMAN, M. E. GURTIN, and I. HERRERA, R. Arch. Rat. Mech. Anal., **19**, 1–18, 1965.
3. N. CRISTESCU, *Dynamic plasticity*, North Holland, Amsterdam 1967.
4. M. NICOLESCU, *Mathematical analysis* [in Romanian], Vol. II, Endituro Tehnica, Bucuresti 1958.
5. J. DIEUDONNÉ, *Foundations of modern analysis*, Academic Press, 1969.
6. J. HADAMARD, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris 1903.
7. C. TRUESDELL and R. A. TOUPIN, *The classical field theories*, Handbuch der Physik, Band III/1, Springer, 1960.
8. P. HARTMAN, *Ordinary differential equations*, John Wiley & Sons, 1964.
9. E. A. CODDINGTON and N. LEVINSON, *Theory of ordinary differential equations*, McGraw-Hill, 1955.
10. I. SULICIU, *Motions with discontinuities in solids—non-linear models* [in Romanian], Studii si Cercetari Mat., **25**, 1, 53–170, 1973.
11. B. D. COLEMAN, J. M. GREENBERG, and M. E. GURTIN, Arch. Rat. Mech. Anal., **22**, 333–354, 1966.

INSTITUTE OF MATHEMATICS,
BUCHAREST.

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