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On the existence and uniqueness of solutions in the linear theory of mixtures of two elastic solids

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THE EXISTENCE, the uniqueness and the continuous dependence of weak solutions upon the given data for the first, the second and the mixed boundary-value problems in the linear theory of mixtures of two isotropic homogeneous elastic solids are proved. Also given are conditions for the differentiability of weak solutions.

Przeprowadzono dowód istnienia i jednoznaczności oraz ciągłej zależności rozwiązań słabych od wartości brzegowych dla pierwszego, drugiego i mieszanego zagadnienia brzegowego liniowej teorii mieszanin dwóch jednorodnych, izotropowych składników sprężystych. Podano również warunki różniczkowalności rozwiązań słabych.

Доказаны единственность и существование, а также непрерывность слабых решений по отношению к задаваемым значениям краевых условий в первой, второй и смешанной краевой задачах линейной теории смесей двух изотропных однородных упругих материалов. Даны условия дифференцируемости слабых решений.

1. Introduction

IN OUR previous paper [1], we discussed the existence and the uniqueness of weak solutions for the first boundary-value problem in the linearized theory of isotropic mixtures of two incompressible elastic solids. Here we consider the static case of bounded bodies, the material being compressible, isotropic and homogeneous. Following J. NEČAS, I. HLAVÁČEK in [2] and I. HLAVÁČEK, M. HLAVÁČEK in [3], we shall prove the existence and the uniqueness of weak solutions for the first, the second and the mixed boundary-value problems. In order to find conditions for the differentiability of weak solutions, we refer to a regularity theorem after FICHERA [4].

2. Summary of boundary-value problems for elliptic systems

The results presented in this section can be found with complete proofs in [2, 4, 5 and 10]. We present only the definitions and theorems which are needed here and no proof is given.

Let Ω be a bounded region in the three-dimensional Euclidean space referred to the Cartesian coordinates $\mathbf{x} \equiv (x_1, x_2, x_3)$, and let Γ be the boundary of Ω. We suppose that Γ is a Lipshitz boundary—i.e., a) to each point $\mathbf{x}$ there exists an open sphere $S_{\mathbf{x}}$ with centre $\mathbf{x}$ such that $S_{\mathbf{x}} \cap \Gamma$ may be described by means of a Lipshitz function, and b) $S_{\mathbf{x}} \cap \Gamma$ divides $S_{\mathbf{x}}$ into external and internal parts with respect to Ω.

We consider the following boundary-value problem:

$$\mathbf{A}\mathbf{u}+\mathbf{f} = \mathbf{0}, \tag{1}$$

$$\mathbf{u}|_{\Gamma_1} = \hat{\mathbf{u}}, \tag{2}$$

$$\mathbf{B}\mathbf{u}|_{\Gamma_2} = \mathbf{g}, \quad \bar{\Gamma}_1 \cap \Gamma_2 = \phi, \quad \Gamma_1 \cup \Gamma_2 = \Gamma, \tag{3}$$

where

$$\mathbf{A} = \sum_{|i|,|j|\leqslant k} D^i(\mathbf{a}_{ij} D^j), \quad D^i \equiv \frac{\partial^{|i|}}{\partial x_1^{i_1}\partial x_2^{i_2}\partial x_3^{i_3}}, \tag{4}$$

is a matrix of differential operators of order $2k$; $\mathbf{B}$ is a matrix of differential operators of order k; $\mathbf{u}, \mathbf{f}, \mathbf{g}$ are m-dimensional vectors; and $\mathbf{a}_{ij}$ are vector-valued, bounded and measurable functions on Ω.

Let $W_2^k(\Omega)$ be the Sobolev's spaces [9] provided with the scalar product given by

$$(\mathbf{v}, \mathbf{u}) = \sum_{|\alpha|\leqslant k} \int_\Omega D^\alpha \mathbf{v} D^\alpha \mathbf{u} d\Omega, \tag{5}$$

and let $\mathring{W}_2^k(\Omega)$ be the closure of $D(\Omega)$ in $W_2^k(\Omega)$, $D(\Omega)$ being the space of real functions having continuous partial derivatives of all order and compact support in Ω. Let $\mathbf{W}^k(\Omega)$ denote the Cartesian product

$$\mathbf{W}^k(\Omega) = W_2^k(\Omega)\times \dots \times W_2^k(\Omega) \quad (m \text{ times}). \tag{6}$$

We define a bilinear form $A(\mathbf{v}, \mathbf{u})$ on $\mathbf{W}^k(\Omega)\times\mathbf{W}^k(\Omega)$ by

$$A(\mathbf{v}, \mathbf{u}) = \int_\Omega \sum_{|i|,|j|\leqslant k} \mathbf{a}_{ij}(\mathbf{x}) D^i\mathbf{v} D^j\mathbf{u} d\Omega, \quad \mathbf{u}, \mathbf{v} \in \mathbf{W}^k(\Omega), \tag{7}$$

and the functionals

$$\begin{aligned} f(\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot \mathbf{v} d\Omega, \quad \mathbf{f} \in \mathbf{L}_2(\Omega), \mathbf{v} \in \mathbf{W}^k(\Omega),\\ g(\mathbf{v}) &= \int_{\Gamma_2} \mathbf{g}\cdot \mathbf{v} d\Omega, \quad \mathbf{g} \in \mathbf{L}_2(\Gamma_2), \mathbf{v} \in \mathbf{W}^k(\Omega). \end{aligned} \tag{8}$$

In the above, $\mathbf{L}_2(\Omega) = L_2(\Omega)\times \dots \times L_2(\Omega)$ (m times) and $\mathbf{L}_2(\Gamma_2) = L_2(\Gamma_2)\times \dots \times L_2(\Gamma_2)$ (m times) denote the spaces of vector functions square-integrable on Ω and Γ_2, respectively. The theorems of embedding imply that the functionals $\mathbf{f}(\mathbf{v})$, $g(\mathbf{v})$ are continuous on $\mathbf{W}^k(\Omega)$.

DEFINITION 1. [5] *Let* $\mathring{\mathbf{W}}^k(\Omega)$ *be the Cartesian product*

$$\mathring{\mathbf{W}}^k(\Omega) = \mathring{W}_2^k(\Omega)\times \dots \times \mathring{W}_2^k(\Omega) \quad (m \text{ times}), \tag{9}$$

and let $\mathbf{V}$ *be a closed subspace* of $\mathbf{W}^k(\Omega)$, *so that* $\mathring{\mathbf{W}}^k(\Omega) \subset \mathbf{V} \subset \mathbf{W}^k(\Omega)$. *Let* $\hat{\mathbf{u}} \in \mathbf{W}^k(\Omega)$ *be given. By the weak solution of the boundary-value problem, we understand a functon* $\mathbf{u} \in \mathbf{W}^k(\Omega)$ *satisfying the following conditions*:

$$\mathbf{u}-\hat{\mathbf{u}} \in \mathbf{V} \tag{10}$$

and

$$A(\mathbf{v}, \mathbf{u}) = f(\mathbf{v})+g(\mathbf{v}), \tag{11}$$

for each $\mathbf{v} \in \mathbf{V}$.

DEFINITION 2. [5] *Let the operators*

$$N_l \mathbf{v} : \mathbf{W}^k(\Omega) \to \mathbf{L}_2(\Omega), \quad l = 1, 2, \ldots, h, \tag{12}$$

be given by

$$N_l \mathbf{v} = \sum_{s=1}^{m} \sum_{|\alpha| \leqslant k} n_{ls\alpha} D^\alpha v_s, \quad \mathbf{v} = \{v_1, v_2, \ldots, v_m\}, \tag{13}$$

where $n_{ls\alpha}(\mathbf{x})$ a.e real bounded and measurable functions on Ω. We say that the operators N_l $\mathbf{v}$ form a coercive system on $\mathbf{W}^k(\Omega)$ if for each $\mathbf{v} \in \mathbf{W}^k(\Omega)$

$$\sum_{l=1}^{h} ||N_l \mathbf{v}||^2_{\mathbf{L}_2(\Omega)} + ||\mathbf{v}||^2_{\mathbf{L}_2(\Omega)} \geqslant c_1 ||\mathbf{v}||^2_{\mathbf{W}^k(\Omega)}, \quad c_1 > 0 \tag{14}$$

holds, where c_1 is a constant(¹) *which does not depend on $\mathbf{v}$, and where $||\cdot||_{\mathbf{L}_2(\Omega)}$ and $||\cdot||_{\mathbf{W}^k(\Omega)}$ denote the usual norms in $\mathbf{L}_2(\Omega)$ and $\mathbf{W}^k(\Omega)$, respectively.*

THEOREM 1. [2, 5] *Suppose that $n_{ls\alpha}$ are constants for $|\alpha| = k$. Then the system $N_l \mathbf{v}$ is coercive on $\mathbf{W}^k(\Omega)$ if, and only if, the rank of the matrix*

$$N_{ls}\xi \equiv \sum_{|\alpha|=k} n_{ls\alpha}\xi_\alpha \tag{15}$$

equals m for each $\xi \neq 0$, $\xi \in \mathscr{C}_3$, where $\mathscr{C}_3$ denotes the complex three-dimensional space and $\xi_\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$.

Let us denote

$$\mathbf{P} = \left\{ \mathbf{v} \in \mathbf{V}, \sum_{l=1}^{h} ||N_l \mathbf{v}||^2_{\mathbf{L}_2(\Omega)} = 0 \right\} \tag{16}$$

and let $p_i(\mathbf{v})$, $i = 1, 2, \ldots, s$, be continuous linear functionals on $\mathbf{W}^k(\Omega)$ and linearly independent on $\mathbf{P}$. Furthermore, we suppose that

$$\mathbf{p} \in \mathbf{P}, \quad \sum_{i=1}^{s} p_i^2(\mathbf{p}) = 0 \Rightarrow \mathbf{p} = \mathbf{0}. \tag{17}$$

We define

$$\mathbf{V}_p = \left\{ \mathbf{v} \in \mathbf{V}, \sum_{i=1}^{s} p_i^2(\mathbf{v}) = 0 \right\}. \tag{18}$$

THEOREM 2. [3, 5] *If the conditions* (14) *and*

$$A(\mathbf{v}, \mathbf{v}) \geqslant c_2 \sum_{l=1}^{h} ||N_l \mathbf{v}||^2_{\mathbf{L}_2(\Omega)}, \quad \mathbf{v} \in \mathbf{W}^k(\Omega), \tag{19}$$

hold, then there exists a unique weak solution $\mathbf{u} \in \mathbf{W}^k(\Omega)$ of the boundary-value problem, so that

$$\mathbf{u} - \hat{\mathbf{u}} \in \mathbf{V}_p \tag{20}$$

if, and only if,

$$\mathbf{p} \in \mathbf{P} \Rightarrow f(\mathbf{p}) + g(\mathbf{p}) = 0. \tag{21}$$

(¹) Throughout the paper the quantities $c_2, c_3, \ldots$, etc. have the same significance as c_1.

Moreover, the following inequalities hold:

$$\|\mathbf{u}\|_{\mathbf{W}^k(\Omega)} \leqslant c_3[\|\hat{\mathbf{u}}\|_{\mathbf{W}^k(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}_2(\Omega)} + \|\mathbf{g}\|_{\mathbf{L}_2(\Gamma_2)}], \tag{22}$$

and

$$A(\mathbf{v}, \mathbf{v}) \geqslant c_4\|\mathbf{v}\|^2_{\mathbf{W}^k(\Omega)}, \tag{23}$$

for each $\mathbf{v} \in \mathbf{V}_p$.

R e m a r k 1. The inequality (23) is said to be of Korn's type.

Suppose that $\overline{\Gamma} = \Gamma$.

DEFINITION 3. [4] *We say that* Ω *is of the class* C^ν *at* $\mathbf{x}_0 \in \Gamma$ *if there exists a neighbourhood* I *of* $\mathbf{x}_0$, *so that*:

(i) *there exists a homeomorfism of class* C^ν *between the sets* $\overline{J} = \overline{I} \cap \overline{\Omega}$ *and* $\Sigma^+ = \{(y_1, y_2, t), t \geqslant 0, \|y\|^2 + t^2 \leqslant 1\}$, $(\|y\|^2 = y_1^2 + y_2^2)$.

(ii) *the set* $\overline{I} \cap \Gamma$ *is carried by this homeomorfism into the set* $\{(y_1, y_2, t) \in \Sigma^+, t = 0\}$.

DEFINITION 4. [4] Ω *is of the class* C^ν *if it is of the class* C^ν *at each point of* Γ.

Let $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ be the homeomorfism referred to above. Here $\boldsymbol{\xi}$ is a vector whose components are (y_1, y_2, t). Let us denote by $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ the inverse function.

Consider C^k to be the function space of continuous and k-times differentiable real functions, the derivatives being continuous functions, too. We shall use also in the following the space

$$\mathbf{C}^k(\overline{\Omega}) = C^k(\overline{\Omega}) \times \ldots \times C^k(\overline{\Omega}) \quad (m \text{ times}), \tag{24}$$

provided with the usual norm

$$\|\mathbf{u}\|_{\mathbf{C}^s(\overline{\Omega})} = \sum_{i=1}^{m} \sum_{l=0}^{s} \sum_{j_1, \cdots, j_l} \sup_{\overline{\Omega}} |u_{i, j_1 j_2 \cdots j_l}|. \tag{25}$$

Let δ and σ be two real numbers, so that $0 < \delta < \sigma < 1$ and let $\varphi(\boldsymbol{\xi})$ be a function of $\mathbf{C}^{k+1}$ class, given by

$$\varphi(\boldsymbol{\xi}) = \begin{cases} 0 & \text{for} \quad \boldsymbol{\xi} \notin \{\boldsymbol{\xi}; |\boldsymbol{\xi}| \leqslant \sigma\}, \\ 1 & \text{for} \quad \boldsymbol{\xi} \in \{\boldsymbol{\xi}; |\boldsymbol{\xi}| \leqslant \delta\}, \end{cases} \quad |\boldsymbol{\xi}| = (y_1^2 + y_2^2 + t^2)^{1/2}. \tag{26}$$

Furthermore, we introduce the functions:

$$\mathbf{w}(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi})\mathbf{v}[\mathbf{x}(\boldsymbol{\xi})], \quad \mathbf{v} \in \mathbf{V}, \tag{27}$$

$$\mathbf{v}_0(\mathbf{x}) = \begin{cases} \mathbf{w}[\boldsymbol{\xi}(\mathbf{x})], & \mathbf{x} \in J, \\ \quad 0 \quad , & \mathbf{x} \in \Omega - J, \end{cases} \tag{28}$$

$$\mathbf{v}_\mathbf{h}(\mathbf{x}) = \begin{cases} \dfrac{\mathbf{w}[\boldsymbol{\xi}(\mathbf{x}) + \mathbf{h}] - \mathbf{w}[\boldsymbol{\xi}(\mathbf{x})]}{|\mathbf{h}|}, & \mathbf{x} \in J, \\ \quad 0 \quad , & \mathbf{x} \in \Omega - J, \end{cases} \tag{29}$$

where $\mathbf{h} = (h_1, h_2, 0)$ is an arbitrary real vector, satisfying $0 < |\mathbf{h}| < 1-\sigma$, and the function space defined by

$$\mathbf{H} = \{\mathbf{v}[\mathbf{x}(\boldsymbol{\xi})]:\ \mathbf{v} \in \mathbf{V},\ \mathbf{v} \equiv \mathbf{0} \text{ in } \Omega - J\}. \tag{30}$$

THEOREM 3. [4] *We assume that the following conditions are satisfied*:

1) $$\Omega \in C^{p+k},$$

2) $$\mathbf{a}_{ij} \in \mathbf{C}^{p-1}(\bar{\Omega}),$$

3) $$\mathbf{f} \in \mathbf{W}^{p-k}(\Omega)\ (\mathbf{f} \in \mathbf{L}_2(\Omega) \text{ if } p-k < 0),$$

4) *for each* $\mathbf{x} \in \Gamma$, $\mathbf{V}$ *has the properties*

a) $\mathbf{v}_0(\mathbf{x}) \in \mathbf{V}$, $\mathbf{v}_{\mathbf{h}}(\mathbf{x}) \in \mathbf{V}$ *for each* $\mathbf{v} \in \mathbf{V}$,

b) $\mathbf{H} = \overline{\mathbf{H} \cap \mathbf{C}^p(\Sigma^+)}$ (*the closure in* $\mathbf{W}^k(\Sigma^+)$).

Then, the weak solution $\mathbf{u}$ *of the boundary-value problem* (1)–(3) *with homogeneous data on the boundary belongs to* $\mathbf{W}^{p+k}(\Omega)$ *and the following inequality holds*:

$$||\mathbf{u}||_{\mathbf{W}^{p+k}}(\Omega) \leqslant c_5 ||\mathbf{f}||_{\mathbf{W}^{p-k}}(\Omega). \tag{31}$$

R e m a r k 2. If $p+k > 1$ and the conditions 1), 2), 3), 4) are satisfied, then Sobolev's immersion theorem [9] implies that $\mathbf{u} \in \mathbf{C}^{p+k-1}(\bar{\Omega})$ and

$$||\mathbf{u}||_{\mathbf{C}^{p+k-1}}(\Omega) \leqslant c_6 ||\mathbf{f}||_{\mathbf{W}^{p-k}}(\Omega). \tag{32}$$

R e m a r k 3. [4] If $\Gamma_1 = \Gamma$ or $\Gamma_2 = \Gamma$ then $\mathbf{V} = \mathring{\mathbf{W}}^k(\Omega)$ or $\mathbf{W}^k(\Omega)$ respectively, and in these cases the condition 4) is satisfied. If $\Gamma_i \neq \Gamma$ $(i = 1, 2)$, then the weak solution $\mathbf{u} \in \mathbf{C}^{p+k-1}_{(\Omega \cup \Gamma_1 \cup \Gamma_2)}$ (if $p+k > 1$).

THEOREM 4. [10] *Let* Ω *be of the class* C^∞. *Then the mapping*

$$u \to \left\{\frac{\partial^j u}{\partial n^j},\quad j = 0, 1, \ldots, m-1\right\} \tag{33}$$

of $D(\Omega)$ *into* $D(\Gamma) \times \ldots \times D(\Gamma)$ (*m times*) *can be extended to a surjective continuous linear mapping of* $W_2^m(\Omega)$ *into* $\prod_{j=0}^{m-1} W_2^{m-j-\frac{1}{2}}(\Gamma)$ (²). *There exists also a continuous linear mapping*

$$\{g_j, j = 0, 1, \ldots, m-1\} \to g \tag{34}$$

of $\prod_{j=0}^{m-1} W_2^{m-j-\frac{1}{2}}(\Gamma)$ *into* $W_2^m(\Omega)$ *such that*

$$\frac{\partial^j g}{\partial n^j} = g_j, \quad j = 0, 1, \ldots, m-1. \tag{35}$$

(²) The function spaces with fractional indices are those defined in [10].

3. A mixture of two elastic solids. Linearized theory

The linearization of the theory of mixtures developed in [6] and [7] leads to the following constitutive equations for an isotropic body [8]:

$$\sigma_{ij} = \frac{1}{2}\{-\alpha_2+\lambda_1 e_{pp}+\lambda_3 g_{pp}\}\delta_{ij}+\mu_1 e_{ij}+\mu_3 g_{ij}-\lambda_5 h_{[ij]},$$

$$\pi_{ij} = \frac{1}{2}\{\alpha_2+\lambda_4 e_{pp}+\lambda_2 g_{pp}\}\delta_{ij}+\mu_3 e_{ij}+\mu_2 g_{ij}+\lambda_5 h_{[ij]}, \tag{36}$$

$$\pi_i = \frac{\varrho_2}{\varrho}\alpha_2 e_{pp,i}+\frac{\varrho_1\alpha_2}{\varrho}g_{pp,i}.$$

Here σ_{ki} and π_{ki} are components of the partial stresses, π_i the components of the diffusive force, $\alpha_2, \lambda_1, \mu_1, \ldots$ etc., the material constants, and ϱ_1, ϱ_2 the mass-densities of the two solids. We have denoted

$$\varrho = \varrho_1+\varrho_2,$$

$$e_{ij} = \frac{1}{2}(\omega_{i,j}+\omega_{j,i}), \quad g_{ij} = \frac{1}{2}(\eta_{i,j}+\eta_{j,i}), \quad h_{[ij]} = \frac{1}{2}(\omega_{j,i}-\omega_{i,j}+\eta_{i,j}-\eta_{j,i}), \tag{37}$$

where ω_i and η_i are the components of the two displacement vectors.

The equations of static equilibrium are of the form:

$$\sigma_{ki,k}-\pi_i+F_i = 0, \quad \pi_{ki,k}+\pi_i+G_i = 0, \tag{38}$$

where F_i and G_i denote the components of the body forces.

The Lamé equations can be written in the form:

$$M_{rsij}e_{rs,i}+P_{rsij}g_{rs,i}-\frac{1}{2}\lambda_5(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})h_{[rs],i}+F_j = 0,$$

$$P_{rsij}e_{rs,i}+Q_{rsij}g_{rs,i}+\frac{1}{2}\lambda_5(\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr})h_{[rs],i}+G_j = 0, \tag{39}$$

if we denote:

$$M_{rsij} = \left[\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij}+\frac{1}{2}(\mu_1\delta_{ir}\delta_{js}+\mu_1\delta_{is}\delta_{jr})\right],$$

$$P_{rsij} = \left[\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij}+\frac{1}{2}(\mu_3\delta_{ir}\delta_{js}+\mu_3\delta_{is}\delta_{jr})\right], \tag{40}$$

$$Q_{rsij} = \left[\left(\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij}+\frac{1}{2}(\mu_2\delta_{ir}\delta_{js}+\mu_2\delta_{is}\delta_{jr})\right],$$

and take into account the relation (see [8] and [11]):

$$\lambda_3-\lambda_4 = \alpha_2. \tag{41}$$

We prescribe the following boundary conditions which seem to be of practical interest [11]:

$$\omega_i = \eta_i = k_i \quad \text{on } \Gamma_1,$$

$$(\sigma_{ji}+\pi_{ji})n_j = T_i, \quad \omega_i-\eta_i = 0 \quad \text{on } \Gamma_2. \tag{42}$$

Suppose that

$$k_i \in W_2^1(\Omega), \quad T_i \in L_2(\Gamma_2). \tag{43}$$

Let $F_i, G_i \in L_2(\Omega)$. By definition, the partial stresses represent a statically admissible stress field, if $\sigma_{ij}, \pi_{ij} \in W_2^1(\Omega)$, (38) are satisfied in Ω in the sense of $L_2(\Omega)$ and $(42)_2$ are met in the sense of $L_2(\Gamma_2)$. The two displacement fields ω_i, η_i form a geometrically admissible deformation field, if $(42)_1$ is met in the sense of traces [3].

Let σ_{ij}, π_{ij} be a statically admissible stress field and let ω_i, η_i be a geometrically admissible deformation field. Then, we have

$$\begin{gathered} 0 = \int_\Omega \{[(\sigma_{ji}-\pi\delta_{ji})_{,j}+F_i]\omega_i+[(\pi_{ji}+\pi\delta_{ji})_{,j}+G_i]\eta_i\}\,d\Omega, \\ \pi \equiv \frac{\varrho_2}{\varrho}\alpha_2 e_{pp}+\frac{\varrho_1\alpha_2}{\varrho}g_{pp}. \end{gathered} \tag{44}$$

Integrating by parts, we infer

$$\int_\Omega [\sigma_{(ji)}e_{ij}+\pi_{(ji)}g_{ij}+\sigma_{[ij]}h_{[ij]}]d\Omega = \int_\Omega (F_i\omega_i+G_i\eta_i)d\Omega+\int_{\Gamma_1}(\sigma_{ji}+\pi_{ji})k_in_jd\Gamma+\int_{\Gamma_2}T_i\omega_id\Gamma \tag{45}$$

which expresses the principle of virtual work in the linear theory of mixtures of two isotropic solids (see also [3]).

We remark that the global energy of the body is given by [8]

$$E = \int_\Omega W(e_{rs};g_{rs};h_{[rs]})d\Omega = \frac{1}{2}\int_\Omega [\sigma_{(ji)}e_{ij}+\pi_{(ji)}g_{ij}+\sigma_{[ij]}h_{[ij]}]d\Omega, \tag{46}$$

where

$$W(e_{rs};g_{rs};h_{[rs]}) = M_{rsij}e_{rs}e_{ij}+2P_{rsij}e_{rs}g_{ij}+Q_{rsij}g_{rs}g_{ij}-2\lambda_5h_{[ij]}h_{[ij]}, \tag{47}$$

if we take into account the conditions (42).

We assume in the following that the quadratic form W is positive definite, i.e., there exists a constant $c > 0$, so that

$$W(e_{rs};g_{rs};h_{[rs]}) \geqslant c\sum_{r,s=1}^{3}(e_{rs}^2+g_{rs}^2+h_{[rs]}^2). \tag{48}$$

The last inequality involves the following restrictions on the material constants:

$$\begin{gathered} \lambda_1+\frac{2}{3}\mu_1-\frac{\varrho_2}{\varrho}\alpha_2 > 0, \quad \lambda_2+\frac{2}{3}\mu_2+\frac{\varrho_1}{\varrho}\alpha_2 > 0, \\ \left(\lambda_3+\frac{2}{3}\mu_3-\frac{\varrho_1}{\varrho}\alpha_2\right)^2 < \left(\lambda_1+\frac{2}{3}\mu_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_2+\frac{2}{3}\mu_2+\frac{\varrho_1}{\varrho}\alpha_2\right), \\ \mu_1 > 0, \quad \mu_2 > 0, \quad \mu_3^2 < \mu_1\mu_2, \quad \lambda_5 < 0. \end{gathered} \tag{49}$$

We note that these inequalities have been obtained in [11] as conditions for the uniqueness of classical solutions.

4. The existence and uniqueness of weak solutions

We choose $m = 6$, $k = 1$ and denote:

$$\mathbf{u} \equiv \{\omega_1, \omega_2, \omega_3, \eta_1, \eta_2, \eta_3\},$$
$$\mathbf{v} \equiv \{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3\}. \tag{50}$$

We assume

$$\omega_i, \eta_i \in W_2^1(\Omega), \tag{51}$$

so that $\mathbf{W}^1(\Omega)$ is a Hilbert space, provided with the norm:

$$\|\mathbf{u}\|_{\mathbf{W}^1_{(\Omega)}} = \Big[\sum_{i=1}^{3} (\|\omega_i\|^2_{W_2^1(\Omega)} + \|\eta_i\|^2_{W_2^1(\Omega)})\Big]^{1/2}. \tag{52}$$

We take $\mathbf{V}$ to be the subspace of $\mathbf{W}^1(\Omega)$ of all elements $\mathbf{u} \in \mathbf{W}^1(\Omega)$ which satisfy the homogeneous boundary conditions (42) in the sense of traces.

The bilinear form $A(\mathbf{v}, \mathbf{u})$ is given by

$$A(\mathbf{v}, \mathbf{u}) = \int_{\Omega} [M_{rsij}\tilde{e}_{rs}e_{ij} + P_{rsij}(\tilde{g}_{rs}e_{ij} + \tilde{e}_{rs}g_{ij}) + Q_{rsij}\tilde{g}_{rs}g_{ij} - 2\lambda_5 \tilde{h}_{[ij]}h_{[ij]}]d\Omega, \tag{53}$$

where

$$\tilde{e}_{rs} \equiv e_{rs}(\mathbf{v}), \quad \tilde{g}_{rs} \equiv g_{rs}(\mathbf{v}), \quad \tilde{h}_{[ij]} \equiv h_{[ij]}(\mathbf{v}),$$

and the functionals $f(\mathbf{v})$ and $g(\mathbf{v})$ are defined by

$$f(\mathbf{v}) = \int_{\Omega} (F_i\tilde{\omega}_i + G_i\tilde{\eta}_i)d\Omega, \quad \mathbf{v} \in \mathbf{W}^1(\Omega),\ F_i, G_i \in L_2(\Omega), \tag{54}$$

$$g(\mathbf{v}) = \int_{\Gamma_2} T_i\tilde{\omega}_i d\Gamma. \tag{55}$$

The Cauchy-Schwartz's inequality implies the continuity of the functionals on $\mathbf{W}^1(\Omega)$.

The definition of the weak solution can be read off from the Definition 1, (45), (46), (47), (53), (54), (55) as follows.

DEFINITION 5. *By weak solution of the boundary-value problem we understand a function* $\mathbf{u} \in \mathbf{W}^1(\Omega)$ *such that*

$$\mathbf{u} - \hat{\mathbf{u}} \in \mathbf{V}, \quad \hat{\mathbf{u}} = \{k_1, k_2, k_3, k_1, k_2, k_3\}, \tag{56}$$

and

$$\int_{\Omega} [M_{rsij}\tilde{e}_{rs}e_{ij} + P_{rsij}(\tilde{g}_{rs}e_{ij} + \tilde{e}_{rs}g_{ij}) + Q_{rsij}\tilde{g}_{rs}g_{ij} - 2\lambda_5 \tilde{h}_{[ij]}h_{[ij]}]d\Omega = \int_{\Omega} (F_i\tilde{\omega}_i + G_i\tilde{\eta}_i)d\Omega + \int_{\Gamma_2} T_i\tilde{\omega}_i d\Gamma, \tag{57}$$

holds for each $\mathbf{v} \in \mathbf{V}$.

Taking into account Theorem 2, we choose the operators $N_l\mathbf{v}$ to be of the form:

$$N_1\mathbf{v} = \tilde{e}_{11}, \quad N_2\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{1,2}+\tilde{\omega}_{2,1}), \quad N_3\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{1,3}+\tilde{\omega}_{3,1}),$$

$$N_4\mathbf{v} = \tilde{e}_{22}, \quad N_5\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{2,3}+\tilde{\omega}_{3,2}), \quad N_6\mathbf{v} = \tilde{e}_{33}, \quad N_7\mathbf{v} = \tilde{g}_{11},$$

$$(58) \qquad N_8\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\eta}_{1,2}+\tilde{\eta}_{2,1}), \quad N_9\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\eta}_{1,3}+\tilde{\eta}_{3,1}), \quad N_{10}\mathbf{v} = \tilde{g}_{22},$$

$$N_{11}\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\eta}_{2,3}+\tilde{\eta}_{3,2}), \quad N_{12}\mathbf{v} = \tilde{g}_{33}, \quad N_{13}\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{2,1}-\tilde{\omega}_{1,2}+\tilde{\eta}_{1,2}-\tilde{\eta}_{2,1}),$$

$$N_{14}\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{3,1}-\tilde{\omega}_{1,3}+\tilde{\eta}_{1,3}-\tilde{\eta}_{3,1}), \quad N_{15}\mathbf{v} = \frac{1}{\sqrt{2}}(\tilde{\omega}_{3,2}-\tilde{\omega}_{2,3}+\tilde{\eta}_{2,3}-\tilde{\eta}_{3,2}).$$

Because, in our case, the matrix (15) is composed of three diagonal matrices $\xi_i, \mathbf{1}$, $\mathbf{1}$ being the unit matrix, its rank is 6, so that, in view of the Theorem 1, the system of operators (58) is coercive on $\mathbf{W}^1(\Omega)$.

With the above choice, we have

$$(59) \qquad \int_\Omega \sum_{i,j=1}^{3} (\tilde{e}_{ij}^2+\tilde{g}_{ij}^2+\tilde{h}_{[ij]}^2)d\Omega = \sum_{l=1}^{15} ||N_l\mathbf{v}||^1_{\mathbf{L}_2(\Omega)},$$

so that (48) and (59) imply (19).

The set $\mathbf{P}$ is defined by (16), the operators $N_l\mathbf{v}$ being given by (58). Hence, for each $\mathbf{v} \in \mathbf{P}$, we have

$$(60) \qquad \tilde{e}_{ij} = \tilde{g}_{ij} = \tilde{h}_{[ij]} = 0.$$

The above relations imply that

$$(61) \qquad \mathbf{P} = \{\mathbf{v} \in \mathbf{V}; \quad \tilde{\omega}_k = \tilde{\eta}_k = a_k+\varepsilon_{klm}b_l x_m, \quad a_k, b_k = \mathrm{const}\}.$$

The form of the set $\mathbf{P}$ enables us to choose the functionals p_i as in [2]. Thus, for instance, we can take

$$(62) \qquad \begin{aligned} p_i(\mathbf{v}) &= \int_\Omega \tilde{\omega}_i d\Omega, \quad i = 1, 2, 3, \\ p_j(\mathbf{v}) &= \int_\Omega \varepsilon_{(j-3)kl}\tilde{\omega}_{l,k} d\Omega, \quad j = 4, 5, 6. \end{aligned}$$

In [2] it is proved that these functionals satisfy all the requirements of the general theory, summarized in Sec. 1. The subspace $\mathbf{V}_p$ is defined by (18), the functionals $p_i(\mathbf{v})$ being given by (62). Now, we state the following.

THEOREM 5. *Let the condition* (48) *hold. Then, there exists a unique weak solution* $\mathbf{u} \in \mathbf{W}^1(\Omega)$ *of the boundary-value problem, such that*

$$(63) \qquad \mathbf{u}-\dot{\mathbf{u}} \in \mathbf{V}_p,$$

if, and only if,

$$\int_{\Omega} (F_i \tilde{\omega}_i + G_i \tilde{\eta}_i) d\Omega + \int_{\Gamma_2} T_i \tilde{\omega}_i d\Gamma = 0, \tag{64}$$

for each $\mathbf{v} \in \mathbf{P}$. *The weak solution satisfies a relation similar to* (22), *which implies the continuous dependence of the weak solution on data. The following inequality of Korn's type holds*:

$$\int_{\Omega} [M_{rsij} \tilde{e}_{rs} \tilde{e}_{ij} + 2P_{rsij} \tilde{g}_{rs} \tilde{e}_{ij} + Q_{rsij} \tilde{g}_{rs} \tilde{g}_{ij} - 2\lambda_5 \tilde{h}_{[ij]} \tilde{h}_{[ij]}] d\Omega \geqslant c_7 \sum_{i=1}^{3} (||\tilde{\omega}^i||^2_{W_2^1(\Omega)} + ||\tilde{\eta}^i||^2_{W_2^1(\Omega)}), \tag{65}$$

for each $\mathbf{v} \in \mathbf{V}$.

The proof of Theorem 5 follows from the above considerations and from Theorem 2.

Remark 4. In the case of the first boundary-value problem, $\Gamma_1 = \Gamma$ and $\Gamma_2 = \phi$. From the definition of the set $\mathbf{P}$ it results that $\mathbf{P} = \{0\}$. We have also $\mathbf{P} = \{0\}$ in the case of the mixed boundary-value problem [2]. Then the condition (64) is automatically satisfied and (48) assures the existence, the uniqueness and the continuous dependence on data of the weak solutions.

Remark 5. The second boundary-value problem is obtained by putting $\Gamma_1 = \phi$ and $\Gamma_2 = \Gamma$. Then, $\mathbf{P}$ is given by (61), and, from (64), the necessary and sufficient condition of Theorem 5 takes the form:

$$\int_{\Omega} (F_i + G_i) d\Omega + \int_{\Gamma} T_i d\Gamma = 0, \quad \int_{\Omega} \varepsilon_{ijk} x_j (F_k + G_k) d\Omega + \int_{\Gamma} \varepsilon_{ijk} x_j T_k d\Gamma = 0, \tag{66}$$

which expresses the total equilibrium of external forces.

5. The differentiability of weak solutions

Taking into account Theorem 3 and Remarks 2 and 3, we have:

THEOREM 6. *Suppose that* $\Omega \in C^{l+1}$, $\mathbf{F}, \mathbf{G} \in \mathbf{W}^{l-1}(\Omega)$, *l being a natural number and* $\lambda_5 = 0$ (¹). *Then, the weak solutions of the first, the second and the mixed boundary-value problems with homogeneous data on the boundary belong to* $\mathbf{W}^{l+1}(\Omega)$ *and*

$$||\mathbf{u}||_{\mathbf{W}^{l+1}(\Omega)} \leqslant c_8 (||\mathbf{F}||_{\mathbf{W}^{l-1}(\Omega)} + ||\mathbf{G}||_{\mathbf{W}^{l-1}(\Omega)}). \tag{67}$$

If $l \geqslant 2$, *then*

1) *the weak solutions of the first and the second boundary-value problems belong to* $C^l(\bar{\Omega})$, *and*

2) *the weak solution of the mixed boundary-value problem belongs to* $\mathbf{C}^l(\Omega \cup \Gamma_1 \cup \Gamma_2)$.

We can prove now

THEOREM 7. *Let* $\Omega \in C^{l+1}$, $\mathbf{F}, \mathbf{G} \in \mathbf{W}^{l-1}(\Omega)$ *and* $\lambda_5 = 0$. *We assume that by means of a substitution of the form*

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'', \tag{68}$$

(¹) If $\lambda_5 = 0$, then each solid is initially isotropic [8].

where $\mathbf{u}'' \in \mathbf{C}^{l+1}(\bar{\Omega})$ *and*

$$\|\mathbf{u}''\|_{C^{l+1}(\bar{\Omega})} \leqslant c_9\Big(\|\hat{\mathbf{u}}\|_{\mathbf{W}^1(\Omega)} + \sum_{i=1}^{3} \|T_i\|_{\mathbf{L}_2(\Gamma)}\Big), \tag{69}$$

the boundary-value problem (39), (42) *can be reduced to a boundary-value problem with homogeneous data on the boundary. Then, the weak solution* $\mathbf{u}$ *whose existence is assured by Theorem* 5 *belongs to*

1) $\mathbf{C}^l(\bar{\Omega})$ *for the first and the second boundary-value problems satisfying, respectively, the inequalities*

$$\|\mathbf{u}\|_{C^l(\bar{\Omega})} \leqslant c_{10}(\|\mathbf{F}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\mathbf{G}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{u}}\|_{\mathbf{W}^1(\Omega)}), \tag{70}$$

and

$$\|\mathbf{u}\|_{C^l(\bar{\Omega})} \leqslant c_{11}\Big(\|\mathbf{F}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\mathbf{G}\|_{\mathbf{W}^{l-1}(\Omega)} + \sum_{i=1}^{3} (\|T_i\|_{\mathbf{L}_2(\Gamma)}\Big); \tag{71}$$

2) $\mathbf{C}^l(\Omega \cup \Gamma_1 \cup \Gamma_2)$ *for the mixed boundary-value problem.*

P r o o f. Let $\hat{\mathbf{F}}, \hat{\mathbf{G}}$ be the new mass-terms resulting after substitution (68). In view of hypotheses, it is clear that $\hat{\mathbf{F}}, \hat{\mathbf{G}} \in \mathbf{W}^{l-1}(\Omega)$. The Theorem 6 implies that the weak solutions of the first and of the second homogeneous-boundary-value problems belong to $\mathbf{C}^l(\bar{\Omega})$ and

$$\|\mathbf{u}'\|_{C^l(\bar{\Omega})} \leqslant c_8(\|\hat{\mathbf{F}}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{G}}\|_{\mathbf{W}^{l-1}(\Omega)}). \tag{72}$$

Then, the weak solutions of the first and the second nonhomogeneous boundary-value problems, given by (68), belong also to $\mathbf{C}^l(\bar{\Omega})$ and the following estimate holds:

$$\|\mathbf{u}\|_{C^l(\bar{\Omega})} \leqslant \|\mathbf{u}'\|_{C^l(\bar{\Omega})} + \|\mathbf{u}''\|_{C^{l+1}(\bar{\Omega})}). \tag{73}$$

Taking into account the inequalities (69) and (72), the last-given relation leads to

$$\|\mathbf{u}\|_{C^l(\bar{\Omega})} \leqslant c_{12}(\|\hat{\mathbf{F}}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{G}}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{u}}\|_{\mathbf{W}^1(\Omega)}), \tag{74}$$

for the first boundary-value problem, and to

$$\|\mathbf{u}\|_{C^l(\bar{\Omega})} \leqslant c_{12}\Big(\|\hat{\mathbf{F}}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{G}}\|_{\mathbf{W}^{l-1}(\Omega)} + \sum_{i=1}^{3} \|\mathbf{T}_i\|_{\mathbf{L}_2(\Gamma)}\Big), \tag{75}$$

for the second boundary-value problem. Examination of the Eqs. (39) shows that

$$\|\hat{\mathbf{F}}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\hat{\mathbf{G}}\|_{\mathbf{W}^{l-1}(\Omega)} \leqslant c_{13}(\|\mathbf{u}''\|_{C^{l+1}(\bar{\Omega})} + \|\mathbf{F}\|_{\mathbf{W}^{l-1}(\Omega)} + \|\mathbf{G}\|_{\mathbf{W}^{l-1}(\Omega)}). \tag{76}$$

The last-given inequality, together with (72), (74), (75), implies, for the first and the second boundary-value problems, respectively, (70) and (71). Theorem 6 shows that the weak solution of the homogeneous mixed boundary-value problem belongs to $\mathbf{C}^l(\Omega \cup \Gamma_1 \cup \Gamma_2)$. From (68) we see that the weak solution of the non-homogeneous mixed boundary-value problem belongs to the same function space.

R e m a r k 6. If $l \geqslant 2$, then the weak solutions of the first and the second boundary-value problem are also classical solutions.

Remark 7. Suppose $\Omega \in C^{\infty}$ and $\hat{\mathbf{u}} \in \mathbf{W}^{l+\frac{3}{2}}(\Gamma)$. In view of Theorem 4, there exists $\mathbf{u} \in \mathbf{W}^{l+2}(\Omega)$ such that

(77) $$\bar{\mathbf{u}}|_{\Gamma} = \hat{\mathbf{u}},$$

in the sense of traces, and

(78) $$\|\bar{\mathbf{u}}\|_{\mathbf{W}^{l+2}(\Omega)} \leqslant c_{14} \|\hat{\mathbf{u}}\|_{\mathbf{W}^{l+\frac{3}{2}}(\Gamma)}.$$

Sobolev's immersion Theorem [9] implies that $\bar{\mathbf{u}} \in C^{l+1}(\Omega)$ and, from (78), we have

(79) $$\|\bar{\mathbf{u}}\|_{C^{l+1}(\Omega)} \leqslant c_{15} \|\hat{\mathbf{u}}\|_{\mathbf{W}^{l+\frac{3}{2}}(\Omega)}.$$

Hence, we can take, in the case of the first boundary-value problem $\mathbf{u}'' = \bar{\mathbf{u}}$.

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