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# Similarity theory of turbulence in a plasma inhomogeneity

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THE INHOMOGENEOUS density and electric field in a plasma cause an instability, which develops into turbulence. The turbulent spectrum is divided into three subranges. The mean density gradient of the plasma supplies an energy to the turbulent motions, and constitutes a source in the production subrange; the non-linear behavior maintains a cascade transfer of energy across the spectrum in the inertia subrange and, finally, the molecular diffusion forms a sink of energy in the dissipation subrange of large wave numbers. A similarity theory is developed, based upon a mixing-length for plasmas and dimensional considerations. It predicts the spectral laws $k^{-3}$, $k^{-1}$ and $k^{-5}$ in the respective production, inertia and dissipation subranges of the density spectrum, and a spectral law $k^{-3}$ in all subranges of the electric field spectrum. Those results of the similarity theory agree with a more complicated analytic theory of repeated cascade.

Niejednorodność gęstości i pola elektrycznego w plazmie prowadzi do niestateczności oraz turbulencji. Widmo turbulencji podzielić można na trzy zakresy. Średni gradient gęstości plazmy dostarcza ruchom turbulentnym energii i stanowi źródło w zakresie produkcji; nieliniowość zjawiska utrzymuje kaskadowy charakter przekazywania energii w zakresie inercyjnym, a dyfuzja molekularna tworzy pochłaniacz energii w zakresie dysypacyjnym dużych liczb falowych. Rozwinięto teorię podobieństwa opartą na "długości mieszania" dla plazmy oraz na rozważaniach wymiarowych. Zgodnie z tą teorią należy oczekiwać zależności widmowych typu $k^{-3}$, $k^{-1}$ i $k^{-5}$ w odpowiednich zakresach: produkcyjnym, inercyjnym i dysypacyjnym widma gęstości, jak również zależności typu $k^{-3}$ we wszystkich zakresach widma pola elektrycznego. Te wyniki teorii podobieństwa są zgodne z wnioskami bardziej złożonej teorii kaskady wielokrotnej.

Неоднородности плотности и электрического поля в плазме приводят к неустойчивости и к турбулентности. Спектр турбулентности можно разделить на три диапазона. Средний градиент плотности плазмы подводит турбулентным движениям энергию и составляет источник в рабочем диапазоне, нелинейность явления поддерживает каскадный характер передачи энергии в инертном диапазоне, а молекулярная диффузия образует поглотитель энергии в диссипативном диапазоне больших волновых чисел. Создана теория подобия, которая опирается на "длине смешения" для плазмы, а также на размерных рассуждениях. Согласно этой теории следует ожидать спектральных зависимостей типа $k^{-3}$, $k^{-1}$ и $k^{-5}$ в соответствующих рабочим, инертным и диссипативным диапазонах спектра плотности, как тоже зависимости типа $k^{-3}$ во всех диапазонах спектра электрического поля. Эти результаты теории совпадают со следствиями более сложной теории многократного каскада.

## 1. Introduction

The spectrum $F(k)$ of a hydrodynamic turbulence in the wave number $k$ space has been first treated by a similarity method of KOLMOGOROFF [1]. Since the governing parameter in the inertia subrange is the rate of dissipation $\varepsilon$, the Kolmogoroff law

$$F = \text{const}\, \varepsilon^{2/3} k^{-5/3} \tag{1.1}$$

directly follows from a simple dimensional consideration, without recourse to hydrodynamical equations. The dissipation subrange adds the molecular viscosity $\nu$ as a second parameter. The determination of the relevant combination of the two parameters $\varepsilon$ and $\nu$ requires an analysis of the physical processes in the hydrodynamic equation. To this end, and with the use of the Navier-Stokes equation of motion, HEISENBERG [2] considered the balance between a molecular dissipation function and an energy transfer function, which are proportional to the molecular viscosity $\nu$ and an eddy viscosity $\eta'$, respectively, with a sum $\nu+\eta'$. The flow of energy from the inertia into the dissipation subranges is governed by the ratio $\varepsilon/\nu+\eta'$, or better by its derivative, which produces a compound parameter $\varepsilon/\nu^2$. A subsequent dimensional analysis leads to the Heisenberg law

$$F = \text{const}\,(\varepsilon/\nu^2)^2 k^{-7}. \tag{1.2}$$

By an analogous argument on the introduction of a gradient $\omega_s \equiv |\nabla \mathbf{U}|$ of mean wind $\mathbf{U}$, the similarity method leads Tchen [3, 4] to the spectral law

$$F = \text{const}\,(\varepsilon/\omega_s)k^{-1} \tag{1.3}$$

in the production subrange, which precedes the inertia law (1.1) and the dissipation law (1.2) in the wave number space.

In order to diminish the degree of ambiguity in the above dimensional theories, analytical attempts have been suggested, among which we only mention the repeated-cascade theory. It considers a velocity fluctuation as consisting of a series of ranks, representing many degrees of randomness. The picture is born from the concept of many interacting scales in turbulence, as suggested by BOUSSINESQ [5] and von WEISZÄCKER [6]. TCHEN [7] described the various ranks by coupled equations of motion of different ranks, and calculated the eddy viscosity by means of a Langevin equation of turbulence. The method achieves the closure of the double hierarchy [8] of correlations and of relaxation memories. Not only is it able to reproduce the spectral laws (1.1)–(1.3), but also to determine their numerical coefficients.

In view of the more complicated processes governing a plasma turbulence, the similarity method does not seem to show a clear headway at the first glance. Therefore Tchen resorted to extending the repeated cascade theory [9]. After having clarified the mathematical and physical foundations of the governing transport processes by the repeated-cascade theory, we may ask whether we are now in a better position of developing a similarity theory which provides a simpler physical tool of studying plasma turbulence. For this purpose we must recognize that the crucial difficulty remains in finding a mixing-length which can characterize a shear eddy viscosity of plasma turbulence. Therefore we shall elaborate this concept and its foundation in Sec. 4, after a brief discussion of the dynamical equations in Sec. 2 and of its transport processes in Sec. 3. Only then will that mixing-length $l'$, which serves as a self-consistent scale of eddy mixing, be able to transform the dynamical equations of plasmas into equations of spectral balance. Since $l'$ is found to be a very simple variable in Sec. 4, the similarity method can then proceed in a simple fashion to solve the equations of the spectral balance in Sec .5.

## 2. Turbulent transport processes

A plasma inhomogeneity in a constant magnetic field $\mathbf{B}$ has a mean density $N$ and a mean electric field $\mathbf{E}_0$ which necessarily are both non-uniform. A density fluctuation $n$ and a field fluctuation $\mathbf{E}$ are superposed to those mean background quantities. In representing the fields, we introduce a mean velocity $\mathbf{U}$ and a velocity fluctuation $\mathbf{u}$, which are $\mathbf{U} = c\mathbf{E}_0/B$, and $\mathbf{u} = c\mathbf{E}/B$, where $c$ is the speed of light. Since the fluctuations are most evident in the plane transversal to the magnetic field, we write the equations governing those fluctuations in that transversal plane, as follows:

$$\frac{\partial n}{\partial t}+\nabla\cdot n(\mathbf{U}+\mathbf{u})\times\mathbf{e}_B = -(\mathbf{u}\times\mathbf{e}_B)\cdot\nabla N+D\nabla^2 n,$$
(2.1)
$$\nabla\cdot n(\mathbf{U}+\mathbf{u}) = -\mathbf{u}\cdot\nabla N+\lambda\nabla^2 n.$$

The productions are represented by $-\mathbf{u}\cdot\nabla N$ and $-(\mathbf{u}\times\mathbf{e}_B)\cdot\nabla N$, the non-linear cascade transfers are represented by $\nabla\cdot(n\mathbf{u})$ and $\nabla\cdot(n\mathbf{u}\times\mathbf{e}_B)$, and the collisions are represented by $D\nabla^2 n$ and $\lambda\nabla^2 n$ as proportional to the molecular diffusivities $D$ and $\lambda$.

Upon multiplying the dynamical Eqs. (2.1) by $n$, we can derive the equations of spectral balance. Therefore, the transport processes, as represented in the system of Eqs. (2.1), essentially determine the various transport functions governing the spectral balance.

In the terminology of a single turbulent cascade, a density or a field spectrum can be divided into a portion with wave numbers up to $k$, which serves as a macroscopic background, and a portion with wave numbers greater than $k$, where the more random fluctuations operate. Those two portions are denoted by the superscripts $(\ldots)^0$ and $(\ldots)'$, respectively. Thus, we can write

(2.2) $$\mathbf{u} = \mathbf{u}^0+\mathbf{u}', \quad n = n^0+n'.$$

The ensemble averages $\langle\ldots\rangle^0$, $\langle\ldots\rangle'$ of macroscopic and random ranks serve to screen between the macroscopic and random components. A quantity having no superscript covers the whole spectrum.

In the above cascade representation, we can introduce a vorticity function of density,

$(2.3)_1$ $$J^0 \equiv \langle(\nabla n^0)^2\rangle^0,$$

with $J\equiv\langle(\nabla n)^2\rangle$, and a mean density vorticity

$(2.3)_2$ $$\bar{J} = (\nabla N)^2,$$

$J^0$ and $\bar{J}$ play the role of backgrounds.

The fluctuations of the eddies produce an eddy viscosity, $\eta = \eta^0+\eta'$, of macroscopic and random components, $\eta^0$ and $\eta'$, which form the transport properties of the mean background vorticity $\bar{J}$ and of the macroscopic background vorticity $J^0$, respectively.

In terms of $\eta^0$, $\eta'$, $\bar{J}$, $J^0$, and on the basis of the transport processes enumerated in Eqs. (2.1) and (2.3), we can write the production function, the transfer function and the dissipation function, governing the transport across a density or a field spectrum, in the sequence of cascade of increasing wave numbers:

$$-\eta^0\bar{J}, \quad \eta'J^0, \quad (D \text{ or } \lambda)J^0.$$

## 3. Spectral balance

It is important to remark that, in view of the orthogonality of the field components governing Eqs. (2.1), and of the presence of the shear in the mean field, the normal eddy viscosity $\eta'$ will govern the equation of balance of the density spectrum, while the shear eddy viscosity $\eta'_{21}$ will govern the equation of balance of the field spectrum, as follows:

$$\eta^0\bar{J}-\eta'J^0-DJ^0 = \eta\bar{J}-DJ,$$
(3.1)
$$-\eta^0_{21}\bar{J}+\eta'_{21}J^0-\lambda J^0 = -\eta_{21}\bar{J}-\lambda J.$$

They originated from the dynamical Eqs. (2.1). The right-hand members are written by the reason of conservation at large wave numbers, including $k = \infty$, and are valid for a plasma turbulence in statistical equilibrium. The system of Eqs. (3.1) can be simplified into the following form:

(3.2)
$$\eta'(\bar{J}+J^0)+DJ^0 = DJ,$$
$$-\eta'_{21}(\bar{J}+J^0)+\lambda J^0 = \lambda J.$$

While the discussion of the eddy viscosity tensor is referred to Sec. 4, the system of Eqs. (3.2) are seen to determine the velocity spectrum $F(k)$ and the density spectrum $G(k)$ in principle. It is to be remarked that the total areas under the spectra are

(3.3)
$$\frac{1}{2}\langle \mathbf{u}^2\rangle = \int_0^\infty dkF(k), \quad \frac{1}{2}\langle n^2\rangle = \int_0^\infty dkG(k).$$

## 4. Plasma mixing-length as a basis of similarity theory

In analogy with the Prandtl formula for the eddy viscosity in a shear flow [10], we write the velocity fluctuation $u'_2$, due to a mean shear $\partial U_2/\partial x_1 \equiv \omega_s$, in the form

(4.1)
$$u'_2 = -l'_1\frac{\partial U_2}{\partial x_1}.$$

Since the mean velocity has a gradient in the $x_1$ direction, the length scale, responsible for the eddy mixing in the same direction, is $l'_1$, called a mixing-length. When the velocity fluctuation (4. 1) is substituted into the shear eddy viscosity

(4.2)
$$\eta'_{21} = \langle u'_2 l'_1\rangle',$$

we find

(4.3)
$$\eta'_{21} = -\langle l'^2_1\rangle'\omega_s,$$

where $\langle l'^2_1\rangle'$ becomes a dispersion, or variance of the eddy displacements.

When we define the normal eddy viscosity $\eta'$ by

(4.4)
$$\eta' \equiv \eta'_{11} = \Big\langle\int_0^\infty d\tau u'_1(0)u'_1(\tau)\Big\rangle' = \langle u'^2\omega'^{-1}\rangle',$$

we can write the dispersion as

(4.5)
$$\langle l'^2_1\rangle' = 2\langle\eta'\omega'^{-1}\rangle',$$

where $\omega'$ is a relaxation frequency, or $\omega'^{-1}$ is a duration of the correlation of velocities in Eq. (4.4). By interpreting $\omega'^{-1}$ as a duration of dispersion, the formula (4.5) is the standard formula for the dispersion by eddy movements.

If we assume that the relaxation process, which brings a transport property in equilibrium, is mainly due to the eddy motions rather than the molecular motions, we can write

(4.6)
$$\omega' = k^2\eta',$$

instead of the molecular relaxation frequency $k^2\nu$. Under those circumstances, and upon substituting for $\omega'$ from (4.6), we can reduce (4.4) and (4.5) to

$$\eta' = [\langle k^{-2}u'^2\rangle']^{1/2}, \tag{4.7}$$

and

$$l'^2 \equiv \langle l_1'^2\rangle' = 2k^{-2}. \tag{4.8}$$

Hence, we have identified the plasma mixing-length $l'$ as proportional to $k^{-1}$, which is the scale dividing the macroscopic and the random ranks in the cascade decomposition (2.2). Such a mixing-length, which is independent of the spectrum, greatly simplifies the solutions of the system of Eqs. (3.2).

## 5. Solutions by the similarity method

We divide the universal range of the spectra into the production, inertia and dissipation subranges. Except for the inertial subranges which are governed by the transfer functions alone, other subranges are governed by a pair of neighbor transport functions. Thus, upon substituting (4.3) into (3.2), we can write the following equations governing the three subranges:

(a) *production subrange*

$$\eta'(\bar{J}+J^0) = DJ, \tag{5.1$_1$}$$

$$\omega_s l'^2(\bar{J}+J^0) = \lambda J; \tag{5.1$_2$}$$

(b) *inertia subrange*

$$\eta' J^0 = DJ, \tag{5.2$_1$}$$

$$\omega_s l'^2 J^0 = \lambda J \equiv \varepsilon_\varphi; \tag{5.2$_2$}$$

(c) *dissipation subrange*

$$(\eta'+D)J^0 = DJ, \tag{5.3$_1$}$$

$$(\omega_s l'^2+\lambda)J^0 = \lambda J. \tag{5.3$_2$}$$

The system of Eqs. (5.1)–(5.3) are the particular forms of the general system of Eqs. (3.2), and will be used here to find the solutions $F(k)$ and $G(k)$ by a similarity method.

### A. Field spectrum $F(k)$

The field spectrum $F(k)$, covering the three subranges (a), (b), and (c), can be obtained by eliminating $\bar{J}+J^0$ from the Eqs. (5.1)–(5.3), giving

$$\eta' = \Omega l'^2, \tag{5.4}$$

with

$$\Omega \equiv D\omega_s/\lambda. \tag{5.5}$$

A subsequent substitution of (4.7) into (5.4) yields

$$u' = \Omega l', \tag{5.6}$$

and, hence,

$$F = \Omega^2 k^{-3}. \tag{5.7}$$

**B. Density spectrum $G(k)$**

(a) *Production subrange.* The density spectrum in the production subrange is governed by Eq. $(5.1)_2$, rewritten in the form

$$l'^2(\bar{J}+J^0) = \lambda J/\omega_s, \tag{5.8}$$

the differential of which can more adequately describe the flow of density transport from $\bar{J}$ to $\bar{J}^0$, without the interference of the right-hand side of (5.8). Therefore, the parameter $\bar{J}$ determines the spectrum $G$ to be

$$G = \bar{J}^\alpha k^{-\beta}. \tag{5.9}$$

Since $G$ and $\bar{J}$ have the dimensions

$$G \sim n^2k^{-1}, \quad \bar{J} \sim n^2k^2, \tag{5.10}$$

respectively, we can rewrite (5.9) as an identity, in the form

$$n^2k^{-1} = (n^2k^2)^\alpha k^{-\beta}, \tag{5.11}$$

requiring $\alpha = 1$, $\beta = 3$, and hence reducing (5.9) to

$$G = \text{const}\,\bar{J}k^{-3}. \tag{5.12}$$

(b) *Inertia subrange.* The density spectrum is governed by equation $(5.2)_2$, having a parameter $\varepsilon_\varphi/\omega_s$ of dimension

$$\varepsilon_\varphi/\omega_s = n^2. \tag{5.13}$$

When we write

$$G = (\varepsilon_\varphi/\omega_s)^\alpha k^{-\beta}, \tag{5.14}$$

and note that $G$ has the dimension $n^2k^{-1}$, we obtain, by identification, $\alpha = 1$, $\beta = 1$, reducing (5.14) to

$$G = \text{const}(\varepsilon_\varphi/\omega_s)k^{-1}. \tag{5.15}$$

(c) *Dissipation subrange.* The spectrum is determined by equation $(5.3)_2$ rewritten in the form

$$J^0 = \frac{J}{1+(k_s/k)^2}, \quad k_s \equiv \omega_s/\lambda. \tag{5.16}$$

The transition from the inertia to the dissipation subranges can be best described by the differential form of (5.16), which is

$$\frac{dJ^0}{dk} = \frac{2Jk_s^2}{k^3[1+(k/k_s)^2]^2} \cong 2Jk_s^2k^{-3}, \tag{5.17}$$

for $k \gg k_s$. Hence, the governing parameter is $Jk_s^2$, which permits to write

$$G \sim (Jk_s^2)^\alpha k^{-\beta}. \tag{5.18}$$

Since $G$ has a dimension given by (5.10), and $Jk_s^2$ has the dimension $n^2k^4$, we can rewrite (5.18) dimensionally, in the form,

$$n^2k^{-1} = (n^2k^4)^\alpha k^{-\beta},$$

identifying $\alpha = 1$, $\beta = 5$, and reducing (5.18) to

$$G = \mathrm{const}\, Jk_s^2 k^{-5}, \tag{5.19}$$

The transition between the inertia and dissipation subranges is characterized by the critical wave number $k_s$, defined by (5.16).

## 6. Conclusions

The similarity method, which had proved to be a simple and useful tool of analysing hydrodynamic turbulence, had always shied away from plasma turbulence, in view of the difficulty of composing the many parameters in plasmas. Even when the relevant sets of parameters were well established the difficulty remained in structuring the shear eddy viscosity, in addition to the normal eddy viscosity, and in determining the mixing-length of plasmas which serves as a basis of the similarity method. The resolution of those two difficulties is approached here from investigating the physical processes, as represented in the dynamical equations of a plasma inhomogeneity, and from modeling the relaxation of an eddy transport property. When the above basis of similarity is found, the equations of spectral balance can easily be solved by a dimensional analysis. We find the laws $k^{-3}$, $k^{-1}$ and $k^{-5}$ in the production, inertia and dissipation subranges of the density spectrum, and a power law $k^{-3}$ in all subranges of the field spectrum. Those results agree with a more complicated analytic theory, based upon the repeated-cascade. Although the similarity method can find a physical basis and hence can proceed to analyse the plasma turbulence in an elementary manner, it cannot compete with the repeated cascade method, which clearly models the closure problem, and determines all the numerical coefficients.

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