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A contribution to the generalized Noether's theorem

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IN THIS paper the following theorem is proved: If the Lagrangian of some physical system expressed in Lagrange's variables is gauge-variant under a certain class of infinitesimal transformation, then the same Lagrangian, expressed in Hamilton's form, is gauge-variant under the same class of infinitesimal transformations in which Lagrange's variables are replaced by Hamilton's canonical variables. The corresponding first integrals, which follow from Noether's generalized theorem, are equal but expressed in terms of Lagrange's and Hamilton's variables, respectively.

W pracy udowodniono następujące twierdzenie: Jeśli funkcja Lagrange'a pewnego układu fizycznego, wyrażona w zmiennych Lagrange'a, nie jest niezmiennicza względem transformacji cechowania dla pewnej klasy transformacji infinitezymalnych, to ta sama funkcja Lagrange'a przedstawiona w postaci hamiltonowskiej ma tę samą własność względem tej samej klasy infinitezymalnych transformacji, w których zmienne Lagrange'a zastąpiono kanonicznymi zmiennymi Hamiltona. Odpowiednie pierwsze całki wynikające z uogólnionego twierdzenia Noether są sobie równe, lecz wyrażają się odpowiednio za pomocą zmiennych Lagrange'a lub Hamiltona.

В работе доказана следующая теорема: если лагранжиан некоторой физической системы, выраженный в лагранжевых координатах, является калибровочно-вариантным по отношению к некоторому классу бесконечно малых преобразований, то этот же лагранжиан, выраженный в гамильтоновой форме, является калибровочно-вариантным по отношению к тому же классу бесконечно малых преобразований, в котором переменные Лагранжа заменяются каноническими переменными Гамильтона. Соответствующие первые интегралы, которые следуют из обобщенной теоремы Нэтер, равны друг другу, но выражены соответственно в терминах лагранжевых и гамильтоновых координат.

1. Introduction

NOETHER's theorem has an important role in classical mechanics, contemporary physics and variational calculus. In fact, in some of the most significant cases, Lagrange's function remains invariant or gauge-variant under certain continuous changes in the physical variables. The existence of these transformations enables first integrals to be found. The proofs of Noether's theorem and Noether's generalized theorem, and applications for obtaining the basic laws of conservation, can be found in [1–7]. In these papers, all considerations are based on examination of Lagrange's function and the corresponding infinitesimal transformations expressed in terms of Lagrange's variables.

The purpose of this note is to show the connection of the invariance and gauge-variance properties of Lagrange's function in Hamilton's form (see for example [8], p. 59), subject to infinitesimal transformations as functions of Hamilton's variables, and the corresponding properties of Lagrange's function under transformations as functions of Lagrange's variables. In this manner, the concept of Killing's generalized equations [5], whose solution yields transformations under which Lagrange's function is invariant or gauge-variant, will be developed.

2. Noether's theorem and Killing's equations in terms of Hamilton's variables

Let $q = \{q^1, \ldots, q^n\}$ be the generalized coordinates which specify the configuration of a holonomic mechanical system with n degrees of freedom at time t, and let $p = \{p_1, \ldots, \ldots, p_n\}$ be the generalized momenta. Let us suppose that the motion of the system is such that it is possible to construct the Lagrangian function L in Hamilton's form:

$$L = p_i \dot{q}^i - H(t, q, p), \tag{2.1}$$

and the corresponding action integral

$$J = \int_{t_0}^{t_1} (p_i \dot{q}^i - H(t, q, p)) dt, \tag{2.2}$$

where: t_0, t_1 is an arbitrary interval of time, and H is the Hamiltonian. Repeated indices are summed.

The objective of these considerations is to find the elementary transformations for which the Lagrangian (2.1) is gauge-variant.

Let us consider a continuous r-parameter transformation of the time, generalized coordinates and generalized momenta in the form:

$$\bar{t} \approx t + \varepsilon^{(s)} \zeta_{(s)}(t, q, p), \tag{2.3}$$

$$\bar{q}^i \approx q^i + \varepsilon^{(s)} \zeta^i_{(s)}(t, q, p), \tag{2.4}$$

$$\bar{p}_i \approx p_i + \varepsilon^{(s)} \nu_{i(s)}(t, q, p), \qquad s = 1, \ldots, r, \tag{2.5}$$

in which the $\varepsilon^{(s)} (s = 1, \ldots, r)$ are the r independent parameters of the transformation, and $\zeta_{(s)}, \zeta^i_{(s)}$, and $\nu_{i(s)}$ are functions of time, generalized coordinates and generalized momenta. Hence, corresponding to (2.3)–(2.5) there exists an infinitesimal transformation of the form:

$$\delta t \approx \varepsilon^{(s)} \zeta_{(s)}; \; \delta q^i \approx \varepsilon^{(s)} \zeta^i_{(s)}; \; \delta p_i \approx \varepsilon^{(s)} \nu_{i(s)}. \tag{2.6}$$

From (2.3) it follows that

$$d\bar{t} \approx dt \left(1 + \varepsilon^{(s)} \frac{d\zeta_{(s)}}{dt}\right), \tag{2.7}$$

and from (2.7) and (2.4), we have:

$$\delta \dot{q}^i \equiv \frac{d\bar{q}^i}{d\bar{t}} - \frac{dq^i}{dt} = \frac{dq^i + \varepsilon^{(s)} d\zeta^i_{(s)}}{dt\left(1 + \varepsilon^{(s)} \frac{d\zeta_{(s)}}{dt}\right)} - \dot{q}^i.$$

By developing this expression in series and retaining only members linear in the small parameters $\varepsilon^{(s)}$, we obtain:

$$\delta \dot{q}^i \approx \varepsilon^{(s)} \left(\frac{d\zeta^i_{(s)}}{dt} - \dot{q}^i \frac{d\zeta_{(s)}}{dt}\right). \tag{2.8}$$

Now, the action integral (2.2) transforms into

$$\bar{J} = \int_{\bar{t}_0}^{\bar{t}_1} [\bar{p}_i \dot{\bar{q}}^i - H(\bar{t}, \bar{q}, \bar{p})] d\bar{t}. \tag{2.9}$$

Further, let us suppose that the Lagrangian function (2.1) is gauge-variant i.e., is "invariant up to an exact differential" in the sense that (see for example [4], p. 73 or [7])

$$[\bar{p}_i\dot{\bar{q}}^i - H(\bar{t}, \bar{q}, \bar{p})]d\bar{t} - [p_i\dot{q}^i - H(t, q, p)]dt = \varepsilon^{(s)}d\Lambda_{(s)}(t, q, p), \tag{2.10}$$

when it is the object of the transformation. Here the $\Lambda_{(s)}$ are known functions of t, q and p.

Combining (2.2), (2.6), (2.7), (2.9), (2.10), changing the domain of integration in (2.9) (see [1], p. 173), developing the term $\bar{p}_i\dot{\bar{q}}^i - H(\bar{t}, \bar{q}, \bar{p})$ in series and retaining only members linear in $\varepsilon^{(s)}$ $(s = 1, \ldots, r)$, we arrive at the following condition:

$$\varepsilon^{(s)}\int_{t_0}^{t_1}\left(p_i\frac{d\zeta^i_{(s)}}{dt} - \frac{\partial H}{\partial t}\zeta_{(s)} - \frac{\partial H}{\partial q^i}\zeta^i_{(s)} - H\frac{d\zeta_{(s)}}{dt} - \frac{d\Lambda_{(s)}}{dt}\right)dt = 0, \tag{2.11}$$

which must be satisfied for mechanical systems with gauge-variant Lagrangians. Here in the sense of Hamilton's mechanics, generalized velocities are eliminated by the relations $\dot{q}^i = \partial H/\partial p_i$.

After simple manipulation, we have from (2.11)

$$\varepsilon^{(s)}\int_{t_0}^{t_1}\left\{\frac{d}{dt}(p_i\zeta^i_{(s)} - H\zeta_{(s)} - \Lambda_{(s)})\right\}dt = \varepsilon^{(s)}\int_{t_0}^{t_1}\left\{\zeta^i_{(s)}\left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right) - \zeta_{(s)}\left(\dot{q}^i\frac{\partial H}{\partial q^i} + \dot{p}_i\frac{\partial H}{\partial p_i}\right)\right\}dt. \tag{2.12}$$

Assuming that the motion of the holonomic mechanical system satisfies Hamilton's equations of motion

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{2.13}$$

we see that the right-hand side of (2.12) vanishes. Since the $\varepsilon^{(s)}$ are, by definition of an r-parameter transformation, linearly independent, we have from (2.12) the following theorem:

Noether's Theorem: If, under the continuous infinitesimal transformation (2.6) the Lagrangian (2.1) is invariant up to an exact differential in the sense of the Eq. (2.10), then the r expressions

$$p_i\zeta^i_{(s)}(t, q, p) - H\zeta_{(s)}(t, q, p) - \Lambda_{(s)}(t, q, p) = \text{const}, \tag{2.14}$$

are constants of the motion. This theorem is a generalization of the existing forms of Noether's theorem {compare (2.14) with [5, 7, 2 and 4, p. 73]}.

Taking into account that $\zeta^i_{(s)}$ and $\zeta_{(s)}$ are functions of t, q^i and p_i and that the $\varepsilon^{(s)}$ are linearly independent, we obtain by means of the first system of Eqs. (2.13) and (2.11) the necessary condition which must be satisfied for the Lagrangian (2.3) to be gauge-variant:

$$\begin{aligned} p_i\left(\frac{\partial\zeta^i_{(s)}}{\partial t} + \frac{\partial\zeta^i_{(s)}}{\partial q^\nu}\frac{\partial H}{\partial p_\nu} + \frac{\partial\zeta^i_{(s)}}{\partial p_\nu}\dot{p}_\nu\right) - \frac{\partial H}{\partial t}\zeta_{(s)} - \frac{\partial H}{\partial q^i}\zeta^i_{(s)} \\ - H\left(\frac{\partial\zeta_{(s)}}{\partial t} + \frac{\partial\zeta_{(s)}}{\partial q^\nu}\frac{\partial H}{\partial p_\nu} + \frac{\partial\zeta_{(s)}}{\partial p_\nu}\dot{p}_\nu\right) = \frac{\partial\Lambda_{(s)}}{\partial t} + \frac{\partial\Lambda_{(s)}}{\partial q^\nu}\frac{\partial H}{\partial p_\nu} + \frac{\partial\Lambda_{(s)}}{\partial p_\nu}\dot{p}_\nu. \end{aligned} \tag{2.15}$$

If by assumption $\zeta_{(s)}$ and $\zeta^i_{(s)}$ do not depend on the $\dot{p}$'s, the condition (2.15) leads to the systems of partial differential equations, obtained by equating terms in corresponding degrees of $\dot{p}$ on the left and right-hand sides of (2.15) (see for example [3, p. 28 and 5]):

$$(2.16)\qquad p_i\frac{\partial\zeta^i_{(s)}}{\partial t}+p_i\frac{\partial H}{\partial p_\nu}\frac{\partial\zeta^i_{(s)}}{\partial q^\nu}-\frac{\partial H}{\partial t}\zeta_{(s)}-\frac{\partial H}{\partial q^i}\zeta^i_{(s)}-H\frac{\partial\zeta_{(s)}}{\partial t}-H\frac{\partial H}{\partial p_\nu}\frac{\partial\zeta_{(s)}}{\partial q^\nu} = \frac{\partial\Lambda_{(s)}}{\partial t}+\frac{\partial\Lambda_{(s)}}{\partial q^\nu}\frac{\partial H}{\partial p_\nu},$$

$$(2.17)\qquad p_i\frac{\partial\zeta^i_{(s)}}{\partial p_\nu}-H\frac{\partial\zeta_{(s)}}{\partial p_\nu}=\frac{\partial\Lambda_{(s)}}{\partial p_\nu},\qquad i,\nu=1,\ldots,n,\ s=1,\ldots,r.$$

These r groups of the $n+1$ partial differential equations, which are linear in the r groups of $n+1$ unknown functions $\zeta_{(s)}$ and $\zeta^i_{(s)}$ $(i = 1, \ldots, n)$, are generalized Killing's equations (see [5]). When these equations, where the functions H and $\Lambda_{(s)}$ are defined, admit a solution in $\zeta_{(s)}$ and $\zeta^i_{(s)}$, then the equations of motion of the holonomic mechanical system (2.13) admit first integrals (2.14).

Here, we may make one important remark. The solution of the Eqs. (2.16) and (2.17) and the first integrals (2.14) do not depend on the quantities $\nu_{i(s)}$, which appear in the transformation laws for the generalized momenta (2.5). At first it may appear that these quantities may take arbitrary values. This supposition can be seen to be erroneous if we recall that in Hamiltonian mechanics the generalized momenta p_i are known functions of t, q^i and $\dot{q}^i$. Hence, $\bar{t}$, $\bar{q}^i$ and $\dot{\bar{q}}^i$ given by (2.3), (2.4) and (2.8) in terms of the functions $\zeta_{(s)}$ and $\zeta^i_{(s)}$, are completely determined and the transformed generalized momenta $\bar{p}_i$ — i.e., the quantities $\nu_{i(s)}$ can be obtained as functions of $\zeta_{(s)}$ and $\zeta^i_{(s)}$. Thus the logic of Hamilton's mechanics is maintained.

3. Noether's theorem and Killing's equations in terms of Lagrange's variables

Here will be given only the main results corresponding to the above problem considered from the point of view of Lagrangian mechanics. This is done because the procedure used in Sec. 2 is also used here.

If, under the continuous infinitesimal r-parameter transformation

$$(3.1)\qquad \delta t\approx\varepsilon^{(s)}\eta_{(s)}(t,q,\dot{q});\qquad \delta q^i\approx\varepsilon^{(s)}\eta^i_{(s)}(t,q,\dot{q}),\qquad s=1,\ldots,r,\ i=1,\ldots,n,$$

the Lagrangian L of the action integral

$$(3.2)\qquad J=\int_{t_0}^{t_1}L(t,q,\dot{q})\,dt,$$

is invariant up to an exact differential in the sense of the equation

$$(3.3)\qquad L(\bar{t},\bar{q},\dot{\bar{q}})\,d\bar{t}=L(t,q,\dot{q})\,dt+\varepsilon^{(s)}d\lambda_{(s)}(t,q,\dot{q}),$$

then there exist first integrals of Lagrange's equation of motion:

$$(3.4)\qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}^i}-\frac{\partial L}{\partial q^i}=0$$

in the form:

$$\eta_{(s)}\left(L-\dot{q}^i\frac{\partial L}{\partial \dot{q}^i}\right)+\eta^i_{(s)}\frac{\partial L}{\partial \dot{q}^i}-\lambda_{(s)} = \text{const}. \tag{3.5}$$

The quantities $\eta_{(s)}$ and $\eta^i_{(s)}$ are solutions to the $n+1$ generalized Killing's equations:

$$\eta_{(s)}\frac{\partial L}{\partial t}+\eta^i_{(s)}\frac{\partial L}{\partial q^i}+\left[\frac{\partial \eta^i_{(s)}}{\partial t}+\frac{\partial \eta^i_{(s)}}{\partial q^\nu}\dot{q}^\nu-\dot{q}^i\frac{\partial \eta_{(s)}}{\partial t}-\frac{\partial \eta_{(s)}}{\partial q^\nu}\dot{q}^i\dot{q}^\nu\right]\frac{\partial L}{\partial \dot{q}^i} + L\left[\frac{\partial \eta_{(s)}}{\partial t}+\frac{\partial \eta_{(s)}}{\partial q^\nu}\dot{q}^\nu\right] = \frac{\partial \lambda_{(s)}}{\partial t}+\frac{\partial \lambda_{(s)}}{\partial q^\nu}\dot{q}^\nu, \tag{3.6}$$

$$\frac{\partial L}{\partial \dot{q}^i}\left[\frac{\partial \eta^i_{(s)}}{\partial \dot{q}^\nu}-\dot{q}^i\frac{\partial \eta_{(s)}}{\partial \dot{q}^\nu}\right]+L\frac{\partial \eta_{(s)}}{\partial \dot{q}^\nu} = \frac{\partial \lambda_{(s)}}{\partial \dot{q}^\nu}, \tag{3.7}$$

$$i, \nu = 1, \ldots n, \; s = 1, \ldots r.$$

4. A connection between Noether's theorem and Killing's equations in terms of Lagrange's and Hamilton's variables

In this section an important relation between Noether's theorem and Killing's equations will be developed in terms of Lagrange's and Hamilton's variables, respectively.

Let us denote the functions $\zeta^i_{(s)}$, $\zeta_{(s)}$ in the transformation laws (2.3), (2.4) and the functions $\Lambda_{(s)}$ (2.10), where the generalized momenta are replaced by generalized velocities using

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{4.1}$$

by $\bar{\zeta}^i_{(s)}$, $\bar{\zeta}_{(s)}$ and $\bar{\Lambda}_{(s)}$

$$\zeta^i_{(s)}(t, q, p) = \bar{\zeta}^i_{(s)}(t, q, \dot{q}), \qquad \zeta_{(s)}(t, q, p) = \bar{\zeta}_{(s)}(t, q, \dot{q}), \tag{4.2}$$

$$\Lambda_{(s)}(t, q, p) = \bar{\Lambda}_{(s)}(t, q, \dot{q}). \tag{4.3}$$

Using (2.1), (2.13), (4.1)–(4.3) and the well known relations (see for example [8], p. 47)

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \qquad \frac{\partial H}{\partial q^\nu} = -\frac{\partial L}{\partial q^\nu}, \tag{4.4}$$

Killing's Eqs. (2.16), (2.17) and the first integrals (2.14) transform into:

$$\bar{\zeta}_{(s)}\frac{\partial L}{\partial t}+\bar{\zeta}^i_{(s)}\frac{\partial L}{\partial q^i}+\left[\frac{\partial \bar{\zeta}^i_{(s)}}{\partial t}+\frac{\partial \bar{\zeta}^i_{(s)}}{\partial q^\nu}\dot{q}^\nu-\dot{q}^i\frac{\partial \bar{\zeta}_{(s)}}{\partial t}-\dot{q}^i\dot{q}^\nu\frac{\partial \bar{\zeta}_{(s)}}{\partial q^\nu}\right] + L\left[\frac{\partial \bar{\zeta}_{(s)}}{\partial t}+\dot{q}^\nu\frac{\partial \bar{\zeta}_{(s)}}{\partial q^\nu}\right] = \frac{\partial \bar{\Lambda}_{(s)}}{\partial t}+\frac{\partial \bar{\Lambda}_{(s)}}{\partial q^\nu}\dot{q}^\nu, \tag{4.5}$$

$$\frac{\partial \dot{q}^j}{\partial p_\nu}X_j = 0, \tag{4.6}$$

$$\bar{\zeta}_{(s)}\left(L-\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)+\bar{\zeta}^i_{(s)}\frac{\partial L}{\partial \dot{q}^i}-\bar{\Lambda}_{(s)} = \text{const}, \tag{4.7}$$

where

$$X_j = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{\partial \bar{\zeta}^i_{(s)}}{\partial \dot{q}^j} - \dot{q}^i \frac{\partial \bar{\zeta}_{(s)}}{\partial \dot{q}^j}\right] + L\frac{\partial \bar{\zeta}_{(s)}}{\partial \dot{q}^j} - \frac{\partial \bar{\Lambda}_{(s)}}{\partial \dot{q}^j}, \tag{4.8}$$

$$i, j, \nu = 1, \dots n, s = 1, \dots r.$$

The system (4.6) is a system of algebraic homogeneous equations. The determinant of this system

$$\Delta = \left|\frac{\partial \dot{q}^j}{\partial p_\nu}\right| \tag{4.9}$$

is also the Jacobian to the reversible transformation $\dot{q} \rightleftarrows p$ and for this reason must be different from zero. Hence, the solution of the system (4.6) is

$$X_j = 0, \tag{4.10}$$

where X_j is given by (4.8).

Thus, Killing's equations and the first integrals expressed in Hamilton's variables transform into the Eqs. (4.5), (4.7), (4.8) and (4.10).

If we suppose that the functions $\bar{\Lambda}_{(s)}$, given by (4.3), are equal to the functions $\lambda_{(s)}$, given by (3.3), then the solutions to the partial differential equations (4.5), (4.8) and (4.10) are same as those obtained from Killing's equations (3.6), (3.7) i.e.,

$$\bar{\zeta}_{(s)} \equiv \eta_{(s)}, \quad \bar{\zeta}^i_{(s)} \equiv \eta^i_{(s)}, \tag{4.11}$$

and for this reason the first integrals (4.7) and (3.5) are of the same form.

Now, we can formulate the following proved theorem: If the Lagrangian $(p_i\dot{q}^i - H)$ of the holonomic mechanical system in terms of the Hamilton's variables (2.1) is invariant up to an exact differential of the functions $\Lambda_{(s)}$ [given by (2.10)] under an r-parameter transformation (2.3), (2.4), then the Lagrangian L in terms of Lagrange's variables (3.2) is invariant up to an exact differential of the same functions $\Lambda_{(s)}$ under the same transformations (2.3), (2.4), where Hamilton's variables are expressed by Lagrange's variables. The corresponding first integrals (4.7) and (3.5), which follow from Noether's theorem, are the same when we use the reversible transformation $\dot{q} \rightleftarrows p$.

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