Archives of Mechanics • Archiwum Mechaniki Stosowanej • 26, 2, pp. 251–261, Warszawa 1974

A thermodynamic derivation of non-history dependent constitutive relations for elastic, viscoelastic, fluid, and perfectly plastic bodies

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Constitutive relations based upon a general solution of the Clausius-Duhem inequality are applied to isotropic bodies under the requirements of invariance under superimposed rigid body motions and invariance under orthogonal transformations of the reference state. Under the assumption that the powerless part of the constitutive relations vanishes, it is shown that the theory models elastic, viscoelastic, and fluid behavior in the presence of heat conduction. Constitutive relations for perfect plastic bodies are also obtained under a straigh-tforward relaxation of certain differentiability conditions.

Równania konstytutywne, oparte na rozwiązaniu ogólnym nierówności Clausiusa-Duhema, zastosowano do ciał izotropowych zakładając niezmienniczość względem nałożonych ruchów ciała sztywnego oraz względem ortogonalnych transformacji układu odniesienia. Zakładając znikanie tych członów związków konstytutywnych, które nie są związane z wykonywaniem pracy, stwierdzono, że teoria ta modeluje ciała sprężyste, lepkosprężyste i ciekłe uwzględniając zarazem zjawiska przewodnictwa cieplnego. Otrzymać można także związki konstytutywne dla ciał doskonale plastycznych drogą prostego osłabienia pewnych wymagań dotyczących różniczkowalności.

Определяющие уравнения, основанные на общем решении неравенства Клазиуса-Дюхема, применены для описания изотропных тел, на которые наложены требования инвариантности по отношению к жестким движениям и инвариантности по отношению к ортогональным преобразованиям исходного состояния. Показано, что, в предположении исчезания части определяющего уравнения, теория описывает упругое, вязкоупругое и жидкостное поведение материала при наличии теплопроводности. При некотором ослаблении условий дифференцируемости получаются также определяющие уравнения для идеально пластических материалов.

1. Introduction

One of the basic endeavors of modern work in continuum mechanics is the derivation of constitutive relations for real material bodies from fundamental thermodynamic considerations. The results that have been reported to date in this area stem from two distinct schools of thought concerning non-equilibrium thermodynamics and the forms and interpretations of the second law of thermodynamics. One school has followed the pioneering work of Onsager [1] and has obtained useful and important results for linear phenomenological problems [2, 3, 4]. Extensions of the Onsager theory have also been made, so as to include general non-linear phenomenological problems [5, 6, 7, 8, 9]. This has led to significant increases in both the generality of the theory and the scope of problems which can be handled within the framework of the Onsager theory. In particular, Verhás [7] has used the non-linear Onsager theory to give the first thermodynamic derivation of constitutive relations for a perfect plastic solid of the v. Mises type.

The second school bases their development on the Clausius-Duhem inequality and the principle of equipresence (see [10] for a report on some of the results obtained by this line of investigation). Use of the principle of equipresence, however, leads to significant mathematical complexities in order to obtain results which may serve as a basis for modeling viscoelastic and fluid behavior. On the other hand, we have given [11] a complete solution of the Clausius-Duhem inequality without adherence to the principle of equipresence, and have thereby obtained a very simple thermodynamic theory which is adequate for modeling elastic, viscoelastic, and fluid bodies with heat conduction. The purpose of this paper is to show that this same thermodynamic theory is also capable of modeling perfect plastic behavior. Thus, a unified thermodynamic derivation of the constitutive relations for elastic, viscoelastic, fluid, and plastic bodies is provided.

2. Summary of previous results

The work reported in this paper is a direct extension of a general thermodynamics (primitive thermodynamics) which was obtained in [11]. Since this thermodynamics is based upon a general, rather than a particular, solution of the Clausius-Duhem inequality, it exhibits a number of features which are absent in most treatments given in the current literature. We therefore give a summary of our previous results for the convenience of the reader and for reference in later sections.

The reduced dissipation inequality is obtained in the standard way by using the energy balance equation to eliminate the energy source per unit volume in the Clausius-Duhem inequality. For a one-component nonpolar material body referred to a material description, we have

$$-(\dot{\Psi}+\eta\dot{\theta})+\varrho_0^{-1}T_i^A\partial_A\dot{x}^i+(\varrho_0\theta)^{-1}h^A\partial_A\theta \geqslant 0, \tag{2.1}$$

where

$$\varrho_0 t_i^j = \varrho T_i^A\partial_A x^j, \quad \varrho_0 h^i = \varrho h^A\partial_A x^i \tag{2.2}$$

and all field variables are considered as functions of the reference configuration coordinates, X^A, and the time, t. Here, and throughout, we use the notation $\partial_A \equiv \partial/\partial x^A$, $\partial_i \equiv \partial/\partial x^i$, where x^i are the coordinates of the current configuration. We also assume, in the interest of simplicity, that the reference and the current configurations are referred to Cartesian coordinate covers. The symbolism used in the above equations and throughout this paper for designation of physical quantities follows that of TRUESDELL and TOUPIN [12].

Under the explicit assumption that the Helmholtz free energy density can be written in the form

$$\Psi = \Psi\big(\theta(X^A,t),\, \pi_\alpha(X^A,t),\, \partial_B x^i(X^A,t)\big), \tag{2.3}$$

where $\pi_\alpha(X^A,t)$, $\alpha = 1, \dots, q$, are q additional arguments which may be present as consequences of nonstatic conditions, the reduced dissipation inequality can be written in the equivalent form

$$\mathbf{X}\cdot\mathbf{J} \geqslant 0. \tag{2.4}$$

Here, $\mathbf{X}$ and $\mathbf{J}$ are vectors, in a $(q+13)$-dimensional vector space with inner product $\mathbf{A}\cdot\mathbf{B}$, which are identified according to the scheme

$$\mathbf{X} \equiv \{\dot{\theta},\, \dot{\pi}_\alpha,\, \partial_A\dot{x}^i,\, \partial_A\theta\}, \tag{2.5}$$

(2.6) $$\mathbf{J} \equiv \left\{-\eta - \frac{\partial \Psi}{\partial \theta}, -\frac{\partial \Psi}{\partial \pi_\alpha}, \varrho_0^{-1} T_i^A - \frac{\partial \Psi}{\partial(\partial_A x^i)}, (\varrho_0 \theta)^{-1} h^A\right\},$$

and $\boldsymbol{\omega}$ is a vector, in a $(q+10)$-dimensional vector space, which is identified according to the scheme

(2.7) $$\boldsymbol{\omega} \equiv \{\theta, \pi\ , \partial_A x^i\}.$$

Thus, for instance, (2.3) and (2.7) give $\Psi = \Psi(\boldsymbol{\omega})$. The general solution of the reduced dissipation inequality is thus given by all functions $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ such that

(2.8) $$\mathbf{X} \cdot \mathbf{J}(\mathbf{X}; \boldsymbol{\omega} \geqslant 0.$$

The basis for obtaining the general solution of the inequality (2.8) is the following decomposition theorem which was established in [13]. *Every vector field* $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$, *which is of class* C^1 *in* $\mathbf{X}$ *and of class* C^0 *in* $\boldsymbol{\omega}$, *admits the unique decomposition*

(2.9) $$\mathbf{J}(\mathbf{X}; \boldsymbol{\omega}) = \nabla_X \Phi(\mathbf{X}; \boldsymbol{\omega}) + \mathbf{U}(\mathbf{X}; \boldsymbol{\omega}),$$

where the vector $\mathbf{U}(\mathbf{X}; \boldsymbol{\omega})$ *is such that*

(2.10) $$\mathbf{X} \cdot \mathbf{U}(\mathbf{X}; \boldsymbol{\omega}) \equiv 0.$$

This theorem allows a significant simplification of the reduced dissipation inequality, since (2.9) and (2.10) give $\mathbf{X} \cdot \mathbf{J}(\mathbf{X}; \boldsymbol{\omega}) = \mathbf{X} \cdot \nabla_X \Phi(\mathbf{X}; \boldsymbol{\omega})$; that is, the problem is reduced from that of finding vector functions $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ such that $\mathbf{X} \cdot \mathbf{J}(\mathbf{X}; \boldsymbol{\omega}) \geqslant 0$ to the problem of finding scalar functions $\Phi(\mathbf{X}; \boldsymbol{\omega})$ such that $\mathbf{X} \cdot \nabla_X \Phi(\mathbf{X}; \boldsymbol{\omega}) \geqslant 0$. In fact [13], if $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ is known, then $\Phi(\mathbf{X}; \boldsymbol{\omega})$ is given by

(2.11) $$\Phi(\mathbf{X}; \boldsymbol{\omega}) = \int_0^1 \mathbf{X} \cdot \mathbf{J}(\tau \mathbf{X}; \boldsymbol{\omega}) d\tau.$$

With the aid of the identifications (2.5) through (2.7), it is now a simple matter to show that a $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ which is of class C^1 in $\mathbf{X}$ and of class C^0 in $\boldsymbol{\omega}$ satisfies the reduced dissipation inequality if, and only if,

(2.12) $$\eta = -\frac{\partial \Psi}{\partial \theta} - \frac{\partial \Phi}{\partial \dot{\theta}} - U,$$

(2.13) $$\frac{\partial \Psi}{\partial \pi_\beta} = -\frac{\partial \Phi}{\partial \dot{\pi}_\alpha} - U\ ,$$

(2.14) $$\varrho_0^{-1} T_i^A = \frac{\partial \Psi}{\partial(\partial_A x^i)} + \frac{\partial \Phi}{\partial(\partial_A \dot{x}^i)} + U_i^A,$$

(2.15) $$(\varrho_0 \theta^{-1}) h^A = \frac{\partial \Phi}{\partial(\partial_A \theta)} + U^A,$$

(2.16) $$\dot{\theta} U + \dot{\pi}\ U^\alpha + \partial_A \dot{x}^i U_i^A + \partial_A \theta U^A \equiv 0,$$

(2.17) $$\Phi = \int_0^1 P(\tau X; \omega) \frac{d\tau}{\tau},$$

for some scalar function $P(\mathbf{X}; \boldsymbol{\omega})$ of class C^1 in $\mathbf{X}$ and of class C^0 in $\boldsymbol{\omega}$ such that

$$P(\mathbf{0}; \boldsymbol{\omega}) = 0, \quad P(\mathbf{X}; \boldsymbol{\omega}) \geqslant 0 \tag{2.18}$$

(see Section 3 of [11]). If these conditions are satisfied, the "dissipation" $\mathbf{X}\cdot\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ is given by

$$\mathbf{X}\cdot\mathbf{J}(\mathbf{X}; \boldsymbol{\omega}) = P(\mathbf{X}; \boldsymbol{\omega}). \tag{2.19}$$

The vector

$$\mathbf{U} \equiv \{U, U^{\alpha}, U_i^A, U^A\}, \tag{2.20}$$

whose components occur in the relations (2.12) through (2.15), is the *nondissipative* part of the vector $\mathbf{J} = \nabla_X \Phi + \mathbf{U}$. This follows from the observation that the dissipation obtained from the vector $\mathbf{U}$ is given by $\mathbf{X}\cdot\mathbf{U}$, while (2.16) shows that $\mathbf{X}\cdot\mathbf{U} \equiv 0$. Explicit nonzero forms for the vector $\mathbf{U}$ have been given in [11 and 13]. Clearly, the occurrence of the vector $\mathbf{U}$ provides for a wide range of new possibilities. For instance, there is a choice of $\mathbf{U}$ for which a fluid will flow perpendicular to the gradient of its thermodynamic pressure, while for another choice of $\mathbf{U}$, the balance of energy for a rigid heat conducting body becomes a hyperbolic partial differential equation for the determination of the temperature field whenever $\nabla\theta\cdot\nabla\theta \neq 0$ (see Section 4 of [11]). In view of these results and (2.12), we set two additional restrictions in the analysis given in [11]. A thermodynamics is said to be *simple* if, and only if, $\mathbf{U} = \mathbf{0}$; a thermodynamics is said to be *regular* if, and only if, Φ is independent of the argument $\dot{\theta}$. The following results are then obtained in Section 4 of [11]. A $\mathbf{J}(\mathbf{X}; \boldsymbol{\omega})$ of class C^1 in $\mathbf{X}$ and of class C^0 in $\boldsymbol{\omega}$ gives a simple and regular solution of the reduced dissipation inequality if, and only if,

$$\Psi = \Psi(\theta, \partial_A x^i) = \Psi(\hat{\boldsymbol{\omega}}), \tag{2.21}$$

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \tag{2.22}$$

$$\varrho_0^{-1} T_i^A = \frac{\partial\Psi}{\partial(\partial_A x^i)} + \frac{\partial\Phi}{\partial(\partial_A \dot{x}^i)}, \tag{2.23}$$

$$(\varrho_0\theta)^{-1} h^A = \frac{\partial\Phi}{\partial(\partial_A\theta)}, \tag{2.24}$$

$$\hat{\boldsymbol{\omega}} \equiv \{\theta, \partial_A x^i\}, \tag{2.25}$$

$$\hat{\mathbf{X}} \equiv \{\partial_A \dot{x}^i, \partial_A\theta\}, \tag{2.26}$$

$$\Phi = \int_0^1 P(\tau\hat{\mathbf{X}}; \hat{\boldsymbol{\omega}}) \frac{d\tau}{\tau}, \tag{2.27}$$

for some function $P(\hat{\mathbf{X}}; \hat{\boldsymbol{\omega}})$ of class C^2 in $\mathbf{X}$ and class C^0 in $\boldsymbol{\omega}$ such that

$$P(\mathbf{0}; \hat{\boldsymbol{\omega}}) = 0, \quad P(\hat{\mathbf{X}}; \hat{\boldsymbol{\omega}}) \geqslant 0. \tag{2.28}$$

We also have the reciprocity relations

$$\frac{\partial}{\partial(\partial_B \dot{x}^j)} T_i^A = \frac{\partial}{\partial(\partial_A x^i)} T_j^B, \tag{2.30}$$

$$\frac{\partial}{\partial(\partial_B \theta)} T_i^A = \frac{1}{\theta} \frac{\partial}{\partial(\partial_A \dot{x}^i)} h^B, \tag{2.31}$$

$$\frac{\partial}{\partial(\partial_B \theta)} h^A = \frac{\partial}{\partial(\partial_A \theta)} h^B. \tag{2.32}$$

These relations constitute a non-linear generalization of Onsager's relations, the form of which was first obtained by GYARMATI [5, 6]. We also have the relations

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\varrho_0} T_i^A - \frac{\partial\Phi}{\partial(\partial_A \dot{x}^i)}\right) = -\frac{\partial}{\partial(\partial_A x^i)}\eta, \tag{2.33}$$

$$\frac{\partial}{\partial(\partial_B x^j)}\left(\frac{1}{\varrho_0} T_i^A - \frac{\partial\Phi}{\partial(\partial_A \dot{x}^i)}\right) = \frac{\partial}{\partial(\partial_A x^i)}\left(\frac{1}{\varrho_0} T_j^B - \frac{\partial\Phi}{\partial(\partial_B \dot{x}^j)}\right), \tag{2.34}$$

which are the nonequilibrium form of the Maxwell relations obtained from the cross derivatives of the Helmholtz free energy. The assumption of simplicity reduces (2.12) to the form $\eta = -\frac{\partial\Psi}{\partial\theta} - \frac{\partial\Phi}{\partial\dot{\theta}}$, and hence $\varepsilon = \Psi + \theta\eta = \Psi - \theta\left(\frac{\partial\Psi}{\partial\theta} + \frac{\partial\Phi}{\partial\dot{\theta}}\right)$ is a function of $\hat{\omega}$, $\hat{\mathbf{X}}$ and $\dot{\theta}$ unless $\partial\Phi/\partial\dot{\theta} \equiv 0$. It is for this reason that we assume the thermodynamics to be regular. A simple and regular thermodynamics thus preserves the thermostatic relations $\eta = -\partial\Psi/\partial\theta$, $\varepsilon = \Psi - \theta\,\partial\Psi/\partial\theta$ in the dynamic case.

3. Invariance and symmetry requirements

The results reported in the previous Section were strictly thermodynamic in nature and hence they do not reflect the invariance and symmetry properties of real material bodies. The natural invariance properties of material bodies require that all scalar valued functions of the field variables be invariant under superimposed rigid body motions [14, 15], while symmetry properties pertain to specific properties of the bodies such as isotropy, homogeneity, etc. The consequences of the invariance and symmetry requirements will be obtained in this Section after the constitutive relations are rewritten in terms of a spatial description.

A straightforward substitution of the material constitutive relations (2.23) and (2.24) into (2.2) gives

$$\begin{aligned} t_i^j &= \varrho\partial_A x^i\left(\frac{\partial\Psi}{\partial(\partial_A x^i)} + \frac{\partial\Psi}{\partial(\partial_A \dot{x}^i)}\right), \\ h^i &= \varrho\theta\partial_A x^i \frac{\partial\Phi}{\partial(\partial_A\theta)}. \end{aligned} \tag{3.1}$$

If we define a new function ϕ by

$$\phi(\partial_k \dot{x}^i, \partial_k\theta; \theta, \partial_A x^i) = \Phi(\partial_A x^k \partial_k \dot{x}^i, \partial_A x^k \partial_k \theta; \theta, \partial_A x^i), \tag{3.2}$$

then (3.1) yields the constitutive relations

$$t_i^j = \varrho \partial_A x^j \frac{\partial \Psi}{\partial(\partial_A x^i)} + \varrho \frac{\partial \phi}{\partial(\partial_j \dot{x}^i)}, \tag{3.3}$$

$$h^i = \varrho \theta \frac{\partial \phi}{\partial(\partial_i \theta)}, \tag{3.4}$$

where

$$\begin{aligned} \Psi &= \Psi(\theta, \partial_A x^i), \\ \phi &= \phi(\partial_k \dot{x}^i, \partial_k \theta; \theta, \partial_A x^i), \end{aligned} \tag{3.5}$$

and (2.27) and (2.28) become

$$\phi = \int_0^1 P(\tau \partial_k \dot{x}^i, \tau \partial_k \theta; \theta, \partial_A x^i) \frac{d\tau}{\tau}, \tag{3.6}$$

$$P(\partial_k \dot{x}^i, \partial_k \theta; \theta, \partial_A x^i) \geqslant 0, \quad P(0, 0; \theta, \partial_A x^i) = 0, \tag{3.7}$$

where P is of class C^2 in $(\partial_k \dot{x}^i, \partial_k \theta)$ and of class C^0 in $(\theta, \partial_A x^i)$.

Since all quantities are determined in terms of the scalar valued functions Ψ and ϕ (and P), isotropy and invariance under superimposed rigid body motions obtain if, and only if,

$$\Psi = \Psi(\theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)), \tag{3.8}$$

$$\phi = \phi\big(\mathrm{tr}(\mathbf{d}), \mathrm{tr}(\mathbf{d}^2), \mathrm{tr}(\mathbf{d}^3), \partial_i \theta \partial_j \theta g^{ij}, d^{ij} \partial_i \theta \partial_j \theta, d^i_k d^{kj} \partial_i \theta \partial_j \theta; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)\big), \tag{3.9}$$

where

$$\mathbf{C} = \big((\partial_A x^i \partial_B x^j g_{ij})\big) \tag{3.10}$$

is the strain tensor and

$$2\mathbf{d} = \big((\partial_i v_j + \partial_j v_i)\big) \tag{3.11}$$

is the rate of strain tensor (see [14, 15, 16] for the details). The form of the function ϕ is not completely arbitrary, however, in view of the requirements expressed by (3.6) and (3.7);

$$\phi = \int_0^1 P\big(\tau \mathrm{tr}(\mathbf{d}), \tau^2 \mathrm{tr}(\mathbf{d}^2), \tau^3 \mathrm{tr}(\mathbf{d}^3), \tau^2 \partial_i \theta \partial_j \theta g^{ij}, \tau^3 d^{ij} \partial_i \theta \partial_j \theta, \tau^4 d^i_k d^{kj} \partial_i \theta \partial_j \theta; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)\big) \frac{d\tau}{\tau}, \tag{3.12}$$

$$P \geqslant 0, \quad P|_{\mathbf{d} = \partial_i \theta = 0} = 0. \tag{3.13}$$

If ϕ is taken to be independent of the rate of strain tensor, then (3.3) and (3.4) give the customary constitutive equations for an isotropic, elastic, heat conducting body.

If Ψ depends only on $\mathrm{tr}(\mathbf{C})$, $\mathrm{tr}(\mathbf{C}^2)$ and $\mathrm{tr}(\mathbf{C}^3)$ through the form

$$\det(\partial_A x^i) = \frac{1}{6} \mathrm{tr}(\mathbf{C})^3 - \frac{1}{2} \mathrm{tr}(\mathbf{C}) \mathrm{tr}(\mathbf{C}^2) + \frac{1}{3} \mathrm{tr}(\mathbf{C}^3), \tag{3.14}$$

and

$$\phi = \phi\big(\mathrm{tr}(\mathbf{d}), \mathrm{tr}(\mathbf{d}^2), \mathrm{tr}(\mathbf{d}^3), \partial_i \theta \partial_j \theta g^{ij}; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)\big),$$

then (3.3) and (3.4) give the customary constitutive equations for an isotropic, non-linear, heat conducting fluid. In fact, the contribution to the stress from the rate of strain tensor assumes the form $A\delta_i^j + Bd_i^j + Cd_{ik}d^{kj}$ and the signs of the coefficients A, B, and C derive from the positivity requirement expressed by (3.12) and (3.13). For a linear body, we have $\lambda d_k^k \delta_i^j + 2\mu d_i^j$, and the positivity condition (3.13) gives the standard results $\mu \geqslant 0$, $3\lambda + 2\mu \geqslant 0$.

The general case, wherein the only restriction is the positivity condition (3.13), gives the constitutive equations for isotropic, viscoelastic bodies with heat conduction. In particular, there are possible contributions to the stress from tensors of the form

$$\frac{\partial \phi}{\partial(d^{km}\partial_k\theta\partial_m\theta)}\partial_i\theta\partial_j\theta,$$

$$2\frac{\partial \phi}{\partial(d_m^k d^{mr}\partial_k\theta\partial_r\theta)} d^{ir}\partial_r\theta\partial_j\theta.$$

We thus conclude that simple, regular thermodynamic systems, which satisfy the conditions imposed by invariance under superimposed rigid body motions and isotropy, encompass a wide range of physical properties of material bodies. The one outstanding exception is the lack of a basis for the representation of plastic material behavior.

4. The perfect plastic solid

The general form of the constitutive relations for an isotropic fluid which we obtained in the last Section is

$$t_i^j = \varrho\partial_A x^j \frac{\partial \Psi}{\partial(\partial_A x^i)} + \varrho\frac{\partial \phi}{\partial(\partial_j \dot{x}^i)}, \tag{4.1}$$

where

$$\Psi = \Psi(\theta, \det(\partial_A x^i)), \tag{4.2}$$

$$\phi = \phi\big(\mathrm{tr}(\mathbf{d}), \mathrm{tr}(\mathbf{d}^2), \mathrm{tr}(\mathbf{d}^3), \partial_i\theta\partial_j\theta g^{ij}; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)\big). \tag{4.3}$$

We thus have

$$t_i^j = -p\delta_i^j + \varrho\frac{\partial \phi}{\partial(tr(\mathbf{d}))}\delta_i^j + 2\varrho\frac{\partial \phi}{\partial(\mathrm{tr}(\mathbf{d}^2))}d_i^j + 3\varrho\frac{\partial \phi}{\partial(\mathrm{tr}(\mathbf{d}^3))}d_i^k d_k^j, \tag{4.4}$$

where

$$-p = \det(\partial_A x^i)\frac{\partial \psi}{\partial(\det(\partial_A x^i))} \tag{4.5}$$

is the thermostatic pressure.

The theory given above is based on the assumption that the function P, which occurs in the relations (3.12) and (3.13), is of class C^2 in the rate of strain variables. Thus, the coefficients which occur in (4.4) are continuous with continuous first derivatives as a consequence of (3.3) and (3.12). We will now show that the continuity conditions on the function P, and hence on the function ϕ can be relaxed for *simple* thermodynamic systems.

The vector $\mathbf{U}$ vanishes for a simple thermodynamic system, and hence the vector $\mathbf{J}$ is given by

$$\mathbf{J}(\mathbf{X};\boldsymbol{\omega}) = \nabla_X \Phi. \tag{4.6}$$

The reduced dissipation inequality $\mathbf{X}\cdot\mathbf{J} \geqslant 0$ thus takes the equivalent form

$$\mathbf{X}\cdot\nabla_X \Phi \geqslant 0,$$

so that we must have

$$\mathbf{X}\cdot\nabla_X \Phi = P(\mathbf{X};\boldsymbol{\omega}) \tag{4.7}$$

with

$$P(\mathbf{X};\boldsymbol{\omega}) > 0, \quad P(\mathbf{0};\boldsymbol{\omega}) = 0. \tag{4.8}$$

The important thing is thus the continuity of $\mathbf{X}\cdot\mathbf{J}(\mathbf{X};\boldsymbol{\omega}) = P(\mathbf{X};\boldsymbol{\omega})$, not the continuity of $\mathbf{J}(\mathbf{X};\boldsymbol{\omega})$. In fact, it is clear that whenever the vector $\mathbf{U}$ vanishes, it is sufficient to require $P(\mathbf{X};\boldsymbol{\omega})$ to be continuous such that (4.8) is satisfied. In particular, $P(\mathbf{X};\boldsymbol{\omega})$ need not be differentiable at $\mathbf{X} = \mathbf{0}$. It is thus clear that $\nabla_X\phi = \mathbf{J}(\mathbf{X};\boldsymbol{\omega})$ can have a jump discontinuity at the origin and still satisfy the reduced dissipation inequality. This observation provides us with the added generality which is needed in order to model perfect plastic bodies as we shall now show.

If we introduce the abbreviations

$$\begin{gathered} j = \mathrm{tr}(\mathbf{d}), \quad \alpha = \mathrm{tr}(\mathbf{d}^2), \quad \beta = \mathrm{tr}(\mathbf{d}^3), \\ \mathbf{t} = ((t_i^j)), \quad \mathbf{E} = ((\delta_i^j)), \end{gathered} \tag{4.9}$$

the constitutive Eqs. (4.4) take the equivalent form

$$\mathbf{t} = \left(-p+\varrho\frac{\partial\phi}{\partial j}\right)\mathbf{E}+2\varrho\frac{\partial\phi}{\partial\alpha}\mathbf{d}+3\varrho\frac{\partial\phi}{\partial\beta}\mathbf{d}^2. \tag{4.10}$$

We thus have

$$\mathrm{tr}(\mathbf{t}) = -3p+3\varrho\frac{\partial\phi}{\partial j}+2\varrho j\frac{\partial\phi}{\partial\alpha}+3\varrho\alpha\frac{\partial\phi}{\partial\beta}. \tag{4.11}$$

One of the characteristic properties of a perfect plastic body is that the mean pressure, $-\mathrm{tr}(\mathbf{t})$, is a function of the arguments θ and $\det(\partial_A x^i)$ only. Since p is a function of θ and $\det(\partial_A x^i)$ only, (4.11) shows that $\mathrm{tr}(\mathbf{t})$ will have the required property if, and only if, ϕ is a solution of the partial differential equation

$$3\frac{\partial\phi}{\partial j}+2j\frac{\partial\phi}{\partial\alpha}+3\alpha\frac{\partial\phi}{\partial\beta} = h\big(\theta, \det(\partial_A x^i)\big). \tag{4.12}$$

The general solution of this partial differential equation is given by

$$\phi = \frac{1}{3}jh+\hat{\phi}, \tag{4.13}$$

where $\hat{\phi}$ depends on the arguments j, α, β only through the functions

$$\alpha_0 = \mathrm{tr}({}_0\mathbf{d}^2) = \alpha-\frac{1}{3}j^2, \tag{4.14}$$

$$\beta_0 = \mathrm{tr}({}_0\mathbf{d}^3) = \beta-j\alpha+\frac{2}{9}j^3, \tag{4.15}$$

where

$$(4.16)\qquad {}_0\mathbf{d} = \mathbf{d} - \frac{1}{3} j\mathbf{E}$$

is the deviator of the rate of strain tensor. However, the function ϕ must be nonnegative for all $\mathbf{d}$ as a consequence of (3.12) and (3.13). An inspection of (4.13) shows that this can be the case if, and only if, $h(\theta, \det(\partial_A x^i)) = 0$, in which case we have

$$(4.17)\qquad \phi = \phi(\alpha_0, \beta_0, \partial_i\theta\partial_j\theta g^{ij}; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3))$$

and

$$(4.18)\qquad \mathbf{t} = -p\mathbf{E} + 2\varrho\frac{\partial\phi}{\partial\alpha_0}\,{}_0\mathbf{d} + 3\varrho\frac{\partial\phi}{\partial\beta_0}\left({}_0\mathbf{d}^2 - \frac{1}{3}\alpha_0\mathbf{E}\right),$$

$$(4.19)\qquad \mathrm{tr}({}_0\mathbf{t}^2) = \left\{4\alpha_0\left(\frac{\partial\phi}{\partial\alpha_0}\right)^2 + 12\beta_0\frac{\partial\phi}{\partial\alpha_0}\frac{\partial\phi}{\partial\beta_0} + \frac{2}{3}\alpha_0^2\left(\frac{\partial\phi}{\partial\beta_0}\right)^2\right\}\varrho^2,$$

$$(4.20)\qquad \mathrm{tr}({}_0\mathbf{t}^3) = \left\{8\beta_0\left(\frac{\partial\phi}{\partial\alpha_0}\right)^3 + 6\alpha_0^2\left(\frac{\partial\phi}{\partial\alpha_0}\right)^2\frac{\partial\vartheta}{\partial\beta_0} + 9\alpha_0\beta_0\frac{\partial\phi}{\partial\alpha_0}\left(\frac{\partial\phi}{\partial\beta_0}\right)^2 + \left(9\beta_0^2 - \frac{3}{4}\alpha_0^3\right)\left(\frac{\partial\phi}{\partial\beta_0}\right)^3\right\}\varrho^3,$$

where

$$(4.21)\qquad {}_0\mathbf{t} = \mathbf{t} - \frac{1}{3}\mathrm{tr}(\mathbf{t})\mathbf{E} = \mathbf{t} + p\mathbf{E}$$

is the deviator of the stress tensor.

A second characteristic feature of perfect plastic solids is the existence of a (positive) yield stress, σ, which is independent of the rate of strain tensor (i.e., σ can depend on the arguments $\theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)$) and a yield function $Y(a, b)$ such that

$$(4.22)\qquad Y(\sigma^{-2}\mathrm{tr}({}_0\mathbf{t}^2), \sigma^{-3}\mathrm{tr}({}_0\mathbf{t}^3)) = 0.$$

One of the simplest yield functions is $Y(a, b) = a - K$, where K is a positive function of the arguments $\theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)$. This gives

$$(4.23)\qquad \mathrm{tr}({}_0\mathbf{t}^2) = \sigma^2 K$$

which is the v. Mises yield condition if σ and K are constants [17, 18]. Substituting (4.19) into (4.23), we obtain

$$(4.24)\qquad 4\alpha_0\left(\frac{\partial\phi}{\partial\alpha_0}\right)^2 + 12\beta_0\frac{\partial\phi}{\partial\alpha_0}\frac{\partial\phi}{\partial\beta_0} + \frac{2}{3}\alpha_0^2\left(\frac{\partial\phi}{\partial\beta_0}\right)^2 = \frac{\sigma^2 K}{\varrho^2}$$

which is an equation for the determination of ϕ. One solution of this equation which vanishes with α_0 and β_0 is given by

$$(4.25)\qquad \phi = \frac{\sigma}{\varrho}\sqrt{K\alpha_0}.$$

This solution is admissible in the thermodynamic sense, since

$$\phi = \int_0^1 \frac{\sigma}{\varrho}\sqrt{K\tau^2\alpha_0}\frac{d\tau}{\tau} = \int_0^1 \frac{\sigma}{\varrho}\sqrt{K\mathrm{tr}(\tau^2{}_0\mathbf{d}^2)}\frac{d\tau}{\tau},$$

i.e., $P(\alpha_0, \beta_0; ...) = \frac{\sigma}{\varrho}\sqrt{K\alpha_0}$ and P is real and nonnegative, since K is positive and α_0 is nonnegative. As a function of the rate of strain tensor, however, the function ϕ given by (4.25) is continuous, but not differentiable at $\mathbf{d} = \mathbf{0}$. A substitution of (4.25) into (4.18) now gives

$$\mathbf{t} = -p\mathbf{E}+\sigma\sqrt{\frac{K}{\alpha_0}}\,{}_0\mathbf{d} = -p\mathbf{E}+\sigma\sqrt{K}\frac{{}_0\mathbf{d}}{\sqrt{\mathrm{tr}({}_0\mathbf{d}^2)}}. \tag{4.26}$$

The only other irreversible thermodynamics which gives this result is that obtained by VERHÁS [7] as an application of GYARMATI's principle [6]. However, Verhás' derivation does not allow the coefficients σ and K to depend on the temperature and the invariants of the strain tensor.

There are numerous other yield functions which can be examined from the standpoint of the method given above. A simpler procedure is obtained, however, by simply selecting the dissipation function ϕ by means of the prescription

$$\phi = \frac{1}{\varrho}\gamma\left(\sqrt{\alpha_0}, \sqrt[3]{\beta_0}, \partial_i\theta\partial_j\theta g^{ij}; \theta, \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{C}^2), \mathrm{tr}(\mathbf{C}^3)\right) = \int_0^1 P\left(\tau\sqrt{\alpha_0}, \tau\sqrt[3]{\beta_0}, \tau^2\partial_i\theta\partial_j\theta g^{ij}; ...\right)\frac{d\tau}{\tau}, \tag{4.27}$$

where P is a nonnegative function which vanishes with α_0, β_0 and $\partial_i\theta$. When (4.27) is substituted into (4.18), we obtain

$$\mathbf{t} = -p\mathbf{E}+\frac{\partial\gamma}{\partial\sqrt{\alpha_0}}\frac{{}_0\mathbf{d}}{\sqrt{\alpha_0}}+\frac{\partial\gamma}{\partial\sqrt[3]{\beta_0}}\frac{\left({}_0\mathbf{d}^2-\frac{1}{3}\alpha_0\mathbf{E}\right)}{\sqrt[3]{\beta_0^2}}. \tag{4.28}$$

This general form of the constitutive relations for a perfect plastic solid gives the desired property that the dependence on the rate of strain tensor which is not governed by the function γ is in terms of the two forms

$$\frac{\mathbf{d}_0}{\sqrt{\mathrm{tr}({}_0\mathbf{d}^2)}}, \qquad \frac{{}_0\mathbf{d}^2-\frac{1}{3}\mathrm{tr}({}_0\mathbf{d}^2)\mathbf{E}}{\sqrt[3]{\mathrm{tr}({}_0\mathbf{d}^3)^2}}$$

which are homogeneous of degree zero in the deviator of the rate of strain tensor. In addition, since the arguments of the function γ include the variables θ, $\mathrm{tr}(\mathbf{C})$, $\mathrm{tr}(\mathbf{C}^2)$ and $\mathrm{tr}(\mathbf{C}^3)$ in addition to the argument $\partial_i\theta\partial_j\theta g^{ij}$, a wide class of material behavior can be modeled with the constitutive relations (4.28) obtained from the general thermodynamic theory given in this paper. We also note that the function γ is only required to be continuous in the arguments θ, $\mathrm{tr}(\mathbf{C})$, $\mathrm{tr}(\mathbf{C}^2)$, $\mathrm{tr}(\mathbf{C}^3)$, so that there is no problem with modeling stresses which are not differentiable with respect to these arguments.

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Received June 14, 1973.