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Riemann invariants for nonhomogeneous systems of first-order partial quasi-linear differential equations — algebraic aspects. Examples from gasdynamics

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IN THIS paper, systems of partial differential equations of the form (1.1) are considered from the point of view of integral elements defined by the Eqs. (1.2). In particular the connections between the structure of the set of integral elements and the possibility of a construction of special classes of solutions are studied. These classes consist of what are called simple waves, simple states and solutions describing interactions among them. We deal with them in Chapter I. A classification of the set of all integral elements is introduced. It is a generalization of that given in [7]. A couple of theorems useful for this classification are given in the Chapter III. The final part of the work contains analysis of nonhomogeneous gasdynamic equations from the point of view of the method described above.

Praca niniejsza dotyczy układów równań różniczkowych cząstkowych postaci (1.1), które rozpatrywane są z punktu widzenia elementów całkowych zdefiniowanych przez równania (1.2). W szczególności rozważa się związki między strukturą elementów całkowych i możliwością konstrukcji pewnych specjalnych klas rozwiązań. Klasy te składają się z tzw. fal prostych, stanów prostych oraz rozwiązań opisujących ich wzajemne oddziaływania. Zajmiemy się nimi w rozdziale I. Następnie wprowadza się klasyfikację wszystkich elementów całkowych, która stanowi uogólnienie klasyfikacji zaproponowanej w pracy [7]. W ostatnim rozdziale przedstawiono kilka twierdzeń użytecznych z punktu widzenia tej klasyfikacji. Druga część pracy zawiera analizę niejednorodnych równań gazodynamiki z punktu widzenia omawianej metody.

В работе рассматриваются квазилинейные уравнения вида (1.1.) с точки зрения интегральных элементов определённых уравнениями (1.2). Анализируются связи между структурой интегральных элементов и возможностью конструкции некоторых специальных классов решений, которые состоят из так называемых простых волн, простых состояний и решений, описывающих взаимодействия между ними. Затем, вводится классификация всех интегральных элементов, обобщающая классификацию, предложенную в работе [7]. Несколько теорем полезных для этой классификации представлено в последней главе. Вторая часть работы содержит анализ неоднородных уравнений газодинамики с точки зрения описанного метода.

I. Introduction

1. Integral elements

Let us consider systems of first order partial differential equations which, according to the summation convention, may be written as follows:

$$a_j^{s\nu}(u^1, \ldots, u^l)u^j_{,x^\nu} = b^s(u^1, \ldots, u^l), \tag{1.1}$$

where

$s = 1, \ldots, m$ is the number of equations,
$\nu = 1, \ldots, n$ is the number of independent variables,
$j = 1, \ldots, l$ is the number of unknown functions.

The system (1.1) is nonhomogeneous with coefficients dependent on the unknown functions (even when $m \geqslant l$). The space $\mathscr{R}^n$ of independent variables $x = (x^1, \ldots, x^n)$ will be denoted by E and called physical space $E \subset \mathscr{R}^n$. The space $\mathscr{R}^l$ of dependent variables $u = (u^1, \ldots, u^l)$ is denoted by $\mathscr{H}$ and is called the hodograph space $\mathscr{H} \subset \mathscr{R}^l$. At each point (x_0, u_0) of the Cartesian product $E \times \mathscr{H}$, we define the hyperplane $\mathscr{L}(x_0, u_0)$ in the linear space $\mathscr{R}^{n\times l}$ consisting of all matrices (integral elements) $\{L^j_\nu\}$ satisfying the following algebraic equations:

$$a^{s\nu}_j(u^1, \ldots, u^l)L^j_\nu = b^s(u^1, \ldots, u^l), \tag{1.2}$$

where $\max\operatorname{rank}\|L^j_\nu\| = \min(n, l)$. If $\underset{N}{L}$ is any solution of the system (1.2), then:

$$\mathscr{L} = \mathscr{K} + \underset{N}{L}, \tag{1.3}$$

where $\mathscr{K} = \{K \in \mathscr{R}^{n\times l}\colon a^{s\nu}_j K^j_\nu = 0\}$ is the vector space of solutions of the homogeneous system (1.2). The dimension of the space $\mathscr{K}(x_0, u_0)$ of the homogeneous integral elements is given by

$$\dim\mathscr{K}(x_0, u_0) = n\cdot l - m(x_0, u_0), \tag{1.4}$$

where m is the number of independent Eqs. (1.2) or the number of linearly independent matrices $a^s = \{a^{s\nu}_j(u_0)\}$.

By the definitions given above, for each $L_1, \ldots, L_p \in \mathscr{L}$, their linear combination $\mu^1 L_1 + \ldots + \mu^p L_p$ belongs to $\mathscr{L}$, provided that

$$\sum_{s=1}^{P} \mu^s = 1. \tag{1.5}$$

If there exists at least one solution of the nonhomogeneous system (1.2), then

$$\dim\mathscr{K}(x_0, u_0) = \dim \mathscr{L}(x_0, u_0). \tag{1.6}$$

2. Simple elements

An element $L \in \mathscr{L}(x_0, u_0)$ is called simple (or decomposable) if there exists $\lambda \in \mathscr{R}^n$ and $\gamma \in \mathscr{R}^l$ such that L may be written in the form:

$$L^j_\nu = \gamma^\nu \lambda_j \tag{2.1}$$

— i.e.,

$$\operatorname{rank}\|L^j_\nu(u_0, x_0)\| = 1. \tag{2.2}$$

It is convenient to consider λ as an element of E^*. Here E^* denotes the space of linear forms: $E^* \in \lambda\colon E \to \mathscr{R}$, or in other words, if $x \in E$ is a contravariant vector, then $\lambda \in E^*$ is a covariant one. In this terminology, L is an element of the tensor product space $\mathscr{H} \otimes E^*$ of the form:

$$L = \gamma \otimes \lambda \in \mathscr{H} \otimes E^*. \tag{2.3}$$

Simple elements of a homogeneous system are denoted by $\gamma \otimes \lambda$ and of a nonhomogeneous system by $\underset{N}{\gamma} \otimes \underset{N}{\lambda}$. Homogeneous elements are connected directly with the existence of characteristic vectors. Namely:

STATEMENT 1. If $\gamma \otimes \lambda$ is a simple element of a homogeneous system, then λ is a characteristic vector.

Indeed, $a_j^{s\nu}\gamma^j\lambda_\nu = 0$ implies $\operatorname{rank}||a_j^{s\nu}\lambda_\nu|| < l$, or if $l = m$ then $\det||a_j^{s\nu}\lambda_\nu|| = 0$.

DEFINITION 1. *If $\gamma \otimes \lambda$ is a simple element, then γ will be called the characteristic victor in hodograph space $\mathscr{H}$ and λ will be called a characteristic covector in the dual E^* of the physical space.*

Now we introduce the notion of simple waves and simple states. (These notions will provide us with a tool for an extraction of simple integral elements from the set of all integral elements). Let the mapping $u\colon D \to \mathscr{H}$, $D \subset E$ be a solution of the system (1.1). This solution is called a simple wave for a homogeneous system (or a simple state in the case of a nonhomogeneous system) if the tangent mapping([1]) du, which is a linear mapping $E \to \mathscr{H}$ defined by

$$TE \ni (x_0, X^\nu) \xrightarrow{du} \big(u(x_0), u^j_{,x^\nu}(x_0)\, X^\nu\big) \in T\mathscr{H}, \tag{2.4}$$

is a simple element at each point $x_0 \in D$. In other words, the derived mapping (tangent mapping du) of a simple wave, is a simple element.

THEOREM 1. *The hodograph of a simple wave (or a simple state) $u(D)$ for homogeneous (nonhomogeneous) systems is given by the curve in the hodograph space $\mathscr{H}$, such that at each point of this curve the vector γ is tangent to it.*

Proof. The tangent mapping

$$du^j(x) = \gamma^j(x)\,\lambda_\nu(x)\,dx^\nu \tag{2.5}$$

is of rank one, hence the image of the mapping $u\colon E \to \mathscr{H}$ is a curve in the hodograph space $\mathscr{H}$. Let this curve be determined by $u = u(R)$, then $u(x)$ may be represented by $u\big(R(x)\big)$. Hence

$$du = u_{,R}\big(R(x)\big)\,dR(x). \tag{2.6}$$

Thus

$$u_{,R}(x) \approx \gamma(x) \quad \text{and} \quad dR(x) \approx \lambda(x). \tag{2.7}$$

The solution $u(x)$ is constant on the $(n-1)$-dimensional hyperplane perpendicular to the field $\lambda(x)$ satisfying:

$$\lambda_\nu(x)\,dx^\nu = 0. \tag{2.8}$$

Such a surface exists if the Frobenius condition is satisfied

$$\lambda \wedge d\lambda = 0. \tag{2.9}$$

By the definition of integral elements we have:

STATEMENT 2. The mapping $E \supset D \xrightarrow{u} \mathscr{H}$ is a solution iff

$$du \in \mathscr{L}. \tag{2.10}$$

([1]) Denoted also by $T_{x_0}u$.

We have in our case the isomorphisms $T_{x_0}E \approx E$, $T_u\mathscr{H} \approx \mathscr{H}$; therefore we can regard λ as a vector from E^* and γ as a vector from $\mathscr{H}$.

Thus if $\pi \subset \mathscr{L}$, then we may seek solutions such that $du \in \pi$. For example, if we have a family of integral elements depending on any parameters $\xi^1, \ldots, \xi^l$: $L(u, x, \xi^1, \ldots, \xi^l) \in \mathscr{L}(u, x)$, then the solutions

$$du = L(u, x, \xi^1, \ldots, \xi^l) \tag{2.11}$$

exist iff the integrability conditions:

$$0 = d(du) = dL \text{ modulo } (2.11), \tag{2.12}$$

are satisfied. This imposes certain conditions on a class of elements $L(u, x, \xi^1, \ldots, \xi^l)$. We shall consider these conditions in what follows. In particular, we can choose:

$$L(u, x, \xi^1, \ldots, \xi^l, \mu^1, \ldots, \mu^p) = \xi^1 \gamma_1 \otimes \lambda^1 + \ldots + \xi^r \gamma_r \otimes \lambda^r + \mu^1 \underset{N}{\gamma_1} \otimes \underset{N}{\lambda^1} + \ldots + \mu^p \underset{N}{\gamma_p} \otimes \underset{N}{\lambda^p}, \tag{2.13}$$

where $\sum_{s=1}^{p} \mu^s = 1$ and $\gamma_q \otimes \lambda^q$ are simple elements of a homogeneous system and $\underset{N}{\gamma_s} \otimes \underset{N}{\lambda^s}$ are simple elements of a nonhomogeneous system.

The physical meanings of these two sets of elements $\gamma \otimes \lambda$ and $\underset{N}{\gamma} \otimes \underset{N}{\lambda}$ are different. While the homogeneous elements are usually connected with certain waves, which may propagate in the medium, the nonhomogeneous elements lead to certain special solutions which will be called simple states and which, in general, may be not attributed to waves([2]). But we may seek solutions of the form (2.13), where the tangent mapping du is the sum of homogeneous and nonhomogeneous elements. Correct choice of the element of the form (2.13) leaves considerable freedom and compels us to study the structure of its components as well as a solution, in which the integral conditions are satisfied. The physical sense of solutions of this type may be regarded as an interaction of waves with medium in a certain state.

3. Simple waves and simple states

It has been shown in [1–4, 7–9] that simple elements for homogeneous systems of the form (1.1) (i.e., such that $b^s = 0$) are connected with a certain rich family of solutions of what are called simple waves([3]). Let us consider a curve $\Gamma: u = f(R)$ in the hodograph space $\mathscr{H}^l$, where R is a parameter. Let us assume Γ is such that the tangent vector:

$$\frac{\partial}{\partial R} f(R) = \gamma\big(f(R)\big) \tag{3.1}$$

([2]) In the literature [6, 10], solutions u of rank $u = 1$ (i.e., rank $\|u^j_{,x^\nu}\| = 1$ are actually called simple waves. We use the word "wave" for solutions which are interpreted as physical waves. E.g. elliptic nonhomogeneous systems also have solutions of rank one, but we cannot call them "waves". That is why we have chosen to call solutions such that $du = \underset{N}{\gamma} \otimes \underset{N}{\lambda}$ "simple states".

([3]) In those papers are considered systems with coefficients independent of x, and only such systems are considered in this section.

is the characteristic vector. Then there exists a field of the characteristic covector $\lambda(u)$ dual to $\gamma\big(f(R)\big)$, defined on the curve $\Gamma: \lambda = \lambda\big(f(R)\big)$ (4).

THEOREM 2. *If the curve $\Gamma \subset \mathscr{H}$ satisfies* (3.1) *and if φ (.) is any differentiable function with one variable, then the function $u = u(x)$ given by*:

$$u = f(R),$$
$$(3.2) \qquad R = \varphi\big(\lambda_\nu\big(f(R)\big)x^\nu\big)$$

is a solution of the system: $a_J^{s\nu}(u)u^J_{,\nu} = 0$.

This solution is called simple wave. Each curve Γ satisfying (3.1) is called characteristic curve in the hodograph space $\mathscr{H}$. Theorem 2 holds that if a mapping $E \xrightarrow{u} \mathscr{H}$ is a simple wave, then the image of u is a characteristic curve in $\mathscr{H}$. The parameter R is called Riemann's invariant.

The form of solution (3.2) suggests that the covector λ should be treated as an analogue of the wave vector $(\omega, \bar{k})$, which determines the velocity and direction of the propagation of the wave. By contrast with the case of linear equations, here (ω, k) depends also on the value of the solution; therefore the profile of the wave is changed during propagation. It is due to the form of the expression (3.2). The solution (3.2) is constant on $(n-1)$-dimensional hyperplanes perpendicular to λ. By differentation of

$$R = \varphi\big(\lambda_\nu(R)x^\nu\big),$$

we obtain:

$$(3.3) \qquad R_{,\nu} = \frac{\dot{\varphi}}{1-\dot{\varphi}\lambda_\mu(R)_{,R}x^\mu}\,\lambda_\nu(R), \quad \mu = 1, \ldots, n.$$

It follows that on hypersurface $\mathfrak{M}$, which is given by the two relations:

$$R = \varphi\big(\lambda_\nu(R)x^\nu\big),$$
$$(3.4) \qquad \dot{\varphi}(\lambda_\nu(R)x^\nu)\,\lambda_\mu(R)_{,R}\,x^\mu = 1,$$

the gradient of the function R becomes infinite and this situation is called the gradient catastrophe. Our solution does not make sense on the hypersurface $\mathfrak{M}$. In this case, certain discontinuities can arise — e.g., shock waves. It was mentioned above that the function $R(x)$ determined by (3.2) is constant on hyperplanes orthogonal to the covector λ. (For each of these hyperplanes there is determined a certain value of the parameter R). Hence in general (except for a few cases — e.g., if planes are parallel) there exists a developable surface, which is an envelope of this family of planes. This surface is exactly the place of gradient catastrophe.

It is easy to check that, in the case of simple wave, *du* is a simple element. In other words, simple waves are just solutions of the system:

$$(3.5) \qquad du = \xi\gamma(u) \otimes \lambda(u),$$

(4) It may happen that there exists more than one characteristic covector λ for given γ. Then the set of simple waves is richer [8, 9].

where $\gamma(u) \otimes \lambda(u)$ is the field of simple elements over the space $\mathscr{H} \otimes E^*$. This system always has solutions. Indeed, if $u = f(R)$ is a solution of the system $du/dR = \gamma(u)$ of ordinary equations, then the relations:

$$
\begin{aligned} u &= f(R), \\ R &= \varphi\big(\lambda_\nu(f(R))\, x^\nu\big) \end{aligned} \tag{3.6}
$$

represent a simple wave.

Following analogy with simple wave, we introduce the notions of simple state. A mapping $u(x)$ is called a simple state iff

$$
du = \underset{N}{\gamma}(u) \otimes \underset{N}{\lambda}(u), \quad \underset{N}{\gamma} = \underset{N}{\lambda}(u), \quad \underset{N}{\lambda} = \underset{N}{\lambda}(u). \tag{3.7}
$$

By contrast with the case of simple waves for homogeneous systems, the formula (3.7) of du has no free parameter ξ, and the integral conditions are not automatically satisfied as in (3.5).

By exterior differentiation (3.7), we obtain:

$$
d\underset{N}{\gamma} \wedge \underset{N}{\lambda} + \underset{N}{\lambda}\, d\underset{N}{\lambda} = 0, \tag{3.8}
$$

where

$$
\left.\begin{aligned} d\underset{N}{\gamma} &= \underset{N}{\gamma}_{,\underset{N}{\gamma}} \underset{N}{\lambda} \\ d\underset{N}{\lambda} &= \underset{N}{\lambda}_{,\underset{N}{\gamma}} \underset{N}{\lambda} \end{aligned}\right\} \text{modulo (3.7)}.
$$

From the Eq. (3.8), we obtain [5]:

$$
\underset{N}{\lambda} \wedge \underset{N}{\lambda}_{,\underset{N}{\gamma}} = 0 \text{ modulo (3.7)}
$$

$$
\text{modulo (3.7)}
$$

because
$$
d\underset{N}{\gamma} \wedge \underset{N}{\lambda} = \underset{N}{\gamma}_{,\underset{N}{\gamma}} \underset{N}{\lambda} \wedge \underset{N}{\lambda} \equiv 0.
$$

From this we see that the system (3.7) has a solution iff: $\underset{N}{\lambda}_{,\underset{N}{\gamma}} \wedge \underset{N}{\lambda} = 0$ — i.e.,

$$
\underset{N}{\lambda}_{,\underset{N}{\gamma}} \approx \underset{N}{\lambda}. \tag{3.9}
$$

This means that the direction of covector $\underset{N}{\lambda}$ does not change in the direction $\underset{N}{\gamma}$. The image of simple state is also a curve tangent to $\underset{N}{\gamma}$. Let this image be given by $u = f(R_0)$. Then the condition (3.9) becomes

$$
\underset{N}{\lambda} \wedge \underset{N}{\lambda}_{,R_0} = 0, \text{ where } \lambda = \lambda\big(f(R_0)\big)
$$

or

$$
\underset{N}{\lambda}_{,R_0} \approx \underset{N}{\lambda}.
$$

This means that the direction of $\underset{N}{\lambda}$ does not depend on R_0; hence it is constant in the physical space E. Thus solution is constant on hyperplanes which are disjoint — i.e., there is no

[5] We denote $\lambda_{,\gamma} = \lambda_{,u^i}\gamma^i$.

gradient catastrophe. By so choosing the length of $\underset{N}{\lambda}$ that $\underset{N}{\lambda}_{,R_0} = 0$, we may represent our simple state in the form:

$$u = f(R_0), \quad R_0 = \underset{N}{\lambda}_\nu x^\nu. \tag{3.10}$$

In the case of nonhomogeneous systems, simple waves attributed to homogeneous elements are not solutions of the (nonhomogeneous) systems we have started. We may seek slightly more general solutions, which would correspond to an interaction of simple wave with simple state and which would be "good solutions":

$$du = \xi\gamma(u)\otimes\lambda(u) + \underset{N}{\gamma}(u)\otimes\underset{N}{\lambda}(u). \tag{3.11}$$

As in the case of simple state, the existence of solutions of (3.11) needs certain conditions, called involutivity conditions. Namely, closing (3.11) (by exterior differentiation), we obtain:

$$\gamma\otimes d\xi\wedge\lambda + \xi d\gamma\wedge\lambda + \xi\gamma\otimes d\lambda + d\underset{N}{\gamma}\wedge\underset{N}{\lambda} + \underset{N}{\gamma}\otimes d\underset{N}{\lambda} = 0. \tag{3.12}$$

Let Φ be the set of $(l-2)$ covectors r in the space $\mathscr{H}^*$, such that

$$\langle r, \gamma\rangle = 0 \quad \text{and} \quad \langle r, \underset{N}{\gamma}\rangle = 0. \tag{3.13}$$

The scalar multiplication of the Eq. (3.2) with the vector r yields:

$$\xi\langle r, d\gamma\rangle\wedge\lambda + \langle r, d\underset{N}{\gamma}\rangle\wedge\underset{N}{\lambda} = 0, \quad (r\in\Phi), \tag{3.14}$$

where by (3.11) we have:

$$\left.\begin{aligned} d\gamma &= \gamma_{,u^i}du^i = \xi\gamma_{,\gamma}\lambda + \gamma_{,\underset{N}{\gamma}}\underset{N}{\lambda} \\ d\underset{N}{\gamma} &= \underset{N}{\gamma}_{,u^i}du^i = \xi\underset{N}{\gamma}_{,\gamma}\lambda + \underset{N}{\gamma}_{,\underset{N}{\gamma}}\underset{N}{\lambda} \end{aligned}\right\} \quad \text{modulo (3.11).}$$

Hence,

$$\xi\langle r, \gamma_{,\underset{N}{\gamma}}\rangle\underset{N}{\lambda}\wedge\lambda + \xi\langle r, \underset{N}{\gamma}_{,\gamma}\rangle\lambda\wedge\underset{N}{\lambda} = 0. \tag{3.15}$$

But ξ being an undetermined parameter, we require the coefficients of powers of ξ to be zero. Hence,

$$(\langle r, \gamma_{,\underset{N}{\gamma}}\rangle - \langle r, \underset{N}{\gamma}_{,\gamma}\rangle)\lambda\wedge\underset{N}{\lambda} = 0. \tag{3.16}$$

If we assume $\lambda\wedge\underset{N}{\lambda}\neq 0$, we obtain

$$\langle r, (\gamma_{,\underset{N}{\gamma}} - \underset{N}{\gamma}_{,\gamma})\rangle = 0.$$

But the expression in brackets is the commutator of the fields $\gamma, \underset{N}{\gamma}$; hence we have

$$\langle r, [\gamma, \underset{N}{\gamma}]\rangle = 0, \tag{3.17}$$

where $[\gamma, \underset{N}{\gamma}]$ denotes the commutator of the fields $\gamma, \underset{N}{\gamma}$. It follows from the form of vectors r, that the Eq. (3.17) is equivalent to the following condition:

$$[\gamma, \underset{N}{\gamma}]\in\{\gamma, \underset{N}{\gamma}\} = \text{linear space spanned by } \gamma, \underset{N}{\gamma}. \tag{3.18}$$

This means that the Frobenius theorem is satisfied; hence there exist surfaces tangent to vector $\gamma, \underset{N}{\gamma}$. Let covectors $\omega, \underset{N}{\omega} \in \mathscr{H}^*$ be such that:

$$\langle \omega, \gamma \rangle = 1, \quad \langle \omega, \underset{N}{\gamma} \rangle = 0. \tag{3.19}$$

and

$$\langle \underset{N}{\omega}, \gamma \rangle = 0, \langle \underset{N}{\omega}, \underset{N}{\gamma} \rangle = 1. \tag{3.20}$$

By multiplication of the Eq. (3.12) by ω and $\underset{N}{\omega}$ respectively, we obtain:

$$d\xi \wedge \lambda + \xi \langle \omega, d\gamma \rangle \wedge \lambda + \xi d\lambda + \langle \omega, d\underset{N}{\gamma} \rangle \wedge \underset{N}{\lambda} = 0, \tag{3.21}$$

$$\xi \langle \underset{N}{\omega}, d\gamma \rangle \wedge \lambda + \langle \underset{N}{\omega}, d\underset{N}{\gamma} \rangle \wedge \underset{N}{\lambda} + d\underset{N}{\lambda} = 0, \tag{3.22}$$

where, using the Eq. (3.11), we have:

$$\left.\begin{aligned} d\lambda &= du^i \wedge \lambda_{,u^i} = \xi \lambda \wedge \lambda_{,\gamma} + \underset{N}{\lambda} \wedge \lambda_{,\underset{N}{\gamma}} && (3.23) \\ d\underset{N}{\lambda} &= du^i \wedge \underset{N}{\lambda}_{,u^i} = \xi \lambda \wedge \underset{N}{\lambda}_{,\gamma} + \underset{N}{\lambda} \wedge \underset{N}{\lambda}_{,\underset{N}{\gamma}} && (3.24) \end{aligned}\right\} \quad \text{modulo (3.11).}$$

Substituting the form (3.23), (3.24) and (3.14) into (3.21), (3.22), we obtain

$$d\xi \wedge \lambda + \xi \langle \omega, \gamma_{,\underset{N}{\gamma}} \rangle \underset{N}{\lambda} \wedge \lambda + \xi(\xi \lambda \wedge \lambda_{,\gamma} + \underset{N}{\lambda} \wedge \lambda_{,\underset{N}{\gamma}}) + \langle \omega, \xi \underset{N}{\gamma}_{,\gamma} \rangle \lambda \wedge \underset{N}{\lambda} = 0, \tag{3.25}$$

$$\xi \langle \underset{N}{\omega}, \gamma_{,\underset{N}{\gamma}} \rangle \underset{N}{\lambda} \wedge \lambda + \xi \langle \underset{N}{\omega}, \underset{N}{\gamma}_{,\gamma} \rangle \lambda \wedge \underset{N}{\lambda} + (\xi \lambda \wedge \underset{N}{\lambda}_{,\gamma} + \underset{N}{\lambda} \wedge \underset{N}{\lambda}_{,\underset{N}{\gamma}}) = 0. \tag{3.26}$$

By means of exterior multiplication of (3.25) by λ and using Cartan's lemma, we obtain

$$\underset{N}{\lambda} \wedge \lambda \wedge \lambda_{,\underset{N}{\gamma}} = 0. \tag{3.27}$$

But coefficients of appropriate powers of ξ in the Eq. (3.26) are assumed to be zero; hence by the Frobenius theorem and because of the form of the covector $\underset{N}{\omega}$, we have the following conditions:

$$\underset{N}{\lambda} \wedge \underset{N}{\lambda}_{,\underset{N}{\gamma}} = 0, \tag{3.28}$$

$$\lambda \wedge (\underset{N}{\lambda}_{,\gamma} + \langle \underset{N}{\omega}, [\gamma, \underset{N}{\gamma}] \rangle \underset{N}{\lambda}) = 0. \tag{3.29}$$

The conditions (3.18) and (3.27)–(3.29), called involutivity conditions, ensure the existence of solutions of the system we have started with (3.11). They ensure (this will be shown in another paper) that the set of solutions of the system (3.11) depends on one arbitrary function with one variable. The physical interpretation of this fact is that the profile of a simple wave connected with the element $\gamma \otimes \lambda$ may be chosen in any manner, but the profile of a simple state connected with the element $\underset{N}{\gamma} \otimes \underset{N}{\lambda}$ is somewhat determined. The solution describes a certain (non-linear) superposition of a simple wave and simple state.

II. Classification of systems of quasi-linear first-order differential equations

The classification [6] introduced in [7] seems to be useful in the construction of special classes of solutions mentioned here. The idea of this classification is to distinguish the following subspaces in the space of homogeneous integral elements $\mathscr{K}$.

4. The space $\mathbf{Q}_1$

By Q_1 we denote the linear space generated by all simple elements belonging to $\mathscr{K}(x_0, u_0)$ — i.e.,

$$Q_1 = \{\gamma_k \otimes \lambda^k\}, \tag{4.1}$$

where $\{\ \}$ denotes linear subspace generated on elements $\gamma_k \otimes \lambda^k$. Obviously, the inclusion

$$Q_1 \subset \mathscr{K} \tag{4.2}$$

holds.

5. The space $\mathbf{Q}_m$

We define Q_m to be the vector space generated by the set

$$\{q(x_0, u_0) \in \mathscr{K} : \langle a^s, q \rangle = 0 \text{ and } \operatorname{rank}\|q(x_0, u_0)\| \leqslant m\}. \tag{5.1}$$

Thus $Q_m(x_0, u_0)$ is the linear space generated by integral homogeneous elements of rank at most m.

Obviously we have

$$\{0\} \subset Q_1 \subset Q_2 \subset \ldots \subset Q_m = \mathscr{K}. \tag{5.2}$$

6. A theorem for hyperbolic system

Now we show the role of simple integral elements in the theory of first-order hyperbolic systems of differential equations. Let us consider systems of the form:

$$a_j^{s\nu} u^j_{,x^\nu} = 0. \tag{6.1}$$

We consider the following polynomial (called a characteristic polynomial) of variable $\xi \in \mathscr{R}$. Namely, let: $\eta, \vartheta \in E^*$, then

$$w(\xi) = a_j^{s\nu}(\xi\eta_\nu + \vartheta_\nu). \tag{6.2}$$

Obviously, if for $\xi^\circ \in \mathscr{R}$ we have $w(\xi^\circ) = 0$, then $\lambda = \xi^\circ \eta + \vartheta$ is a characteristic covector. Thus there exist, dual to it, characteristic vectors $\underset{0,\alpha}{\gamma}$, where $\alpha = 1, \ldots, r_0$; r_0 — is the multiplicity of the root ξ°.

[6] This classification, and also the entire study in this Chapter deal with systems with coefficients dependent on (x, u) only.

DEFINITION 2. *We say that the system* (6.1) *at the point* (x_0, u_0) *is hyperbolic in the direction* $\sigma \in E$ *iff for each* $\bigwedge_{0 \neq \vartheta \in E^*}$ *such that*

$$\langle \vartheta, \sigma \rangle = 0 \tag{6.3}$$

and η *such that*

$$\langle \eta, \sigma \rangle \neq 0 \tag{6.4}$$

the characteristic polynomial (6.2) *has:*

1. $k \leqslant l$ *real roots* $\xi^1 \leqslant \ldots \leqslant \xi^k$ *multiplicities of which do not depend of the choice of* ϑ.

2. *The characteristic vectors:* $\underset{1,1}{\gamma}, \underset{1,2}{\gamma}, \ldots, \underset{1,r_1}{\gamma}, \underset{2,1}{\gamma}, \ldots \ldots \underset{k,r_k}{\gamma}$ *corresponding to* $\lambda^1, \ldots, \lambda^k$ *generate the hodograph space* $(\mathscr{H})$([7]).

DEFINITION 3. *The system is strongly hyperbolic in the direction* $\sigma \in E$ *for each* ϑ *satisfying* (6.3) *iff its characteristic polynomial* (6.2) *has exactly* l *different real roots.*

When $k = l$, *and roots* ξ^i *are different, then vectors* $\underset{p}{\gamma}$, $p = 1, \ldots, l$ *which are associated with all eigenvalues generate the whole hodograph space* $\mathscr{H}^l$ — *i.e.*,

$$\{\gamma_{p, p=1,\ldots,l}\} = \mathscr{H}^l. \tag{6.5}$$

DEFINITION 4. *The system is hyperbolic (resp. strongly hyperbolic) iff there exists* $\sigma \in E$ *such that the system is hyperbolic (resp. strongly hyperbolic) in the direction* σ. *There is a connection between* Q_1 *and hyperbolicity, because of:*

THEOREM 3 [7]. *If the system* (6.1) *is hyperbolic, then all its integral elements may be written as a sum of simple elements — i.e.*,

$$\mathscr{K}(x_0, u_0) = Q_1(x_0, u_0). \tag{6.6}$$

It follows that the entire space of integral elements is generated by simple elements — i.e., every integral element is a linear combination of q — simple elements:

$$\mathscr{K} = \gamma_1 \otimes \lambda^1 + \ldots + \gamma_q \otimes \lambda^q,$$

where $q \leqslant n \cdot l - m$.

The systems for which $\mathscr{K}(x_0, u_0) = Q_1(x_0, u_0)$ will be called Q_1-systems.

III. Classification of nonhomogeneous systems

Having introduced nonhomogeneous elements, let us extend this classification to nonhomogeneous elements. Following the former procedure, let us define the following hyperplanes in the hyperplane $\mathscr{L}$.

7. Hyperplane $\mathscr{L}_1$

Hyperplane $\mathscr{L}_1$ is the plane which contains all the elements L_1 of the form:

$$L_1 = \underset{N}{\gamma} \otimes \underset{N}{\lambda}, \tag{7.1}$$

where $\underset{N}{\gamma} \in \mathscr{R}^l$, $\underset{N}{\lambda} \in \mathscr{R}^{n*}$

([7]) The vectors γ introduced here are called usually the right-side characteristic vectors. Also introduced, corresponding to λ, may be the left-side characteristic vectors $\varkappa = (\varkappa_1, \ldots, \varkappa_l)$ defined by the relation $\varkappa_s a_j^{sv} \lambda_v = 0$. The right-side vectors appearing in the definition 2 may be replaced by left-side characteristic vectors. The two definitions so obtained are equivalent.

and [8]

$$\langle a^s, \underset{N}{\gamma} \otimes \underset{N}{\lambda} \rangle = b^s$$

— i.e., all the elements of $\mathscr{L}_1$ are of the form:

(7.2) $$\mathscr{L}_1 = \sum_{S=1}^{P} \mu^s \underset{N}{\gamma_s} \otimes \underset{N}{\lambda^s},$$

where $\sum_{S=1}^{P} \mu^s = 1$ and $\underset{N}{\gamma_1} \otimes \underset{N}{\lambda^1}, \ldots, \underset{N}{\gamma_p} \otimes \underset{N}{\lambda^p}$ are linearly independent simple nonhomogeneous elements, which generate $\mathscr{L}_1$. Of course,

(7.3) $$\mathscr{L}_1 \subset \mathscr{L}.$$

The systems for which $\mathscr{L}(x_0, u_0) = \mathscr{L}_1(x_0, u_0)$ will be called $\mathscr{L}_1$-systems.

8. Hyperplane $\mathscr{L}_k$

We continue this procedure. Let us denote $\mathscr{L}_k(x_0, u_0)$ the linear subspace generated by all $L \in \mathscr{L}$, such that

(8.1) $$\operatorname{rank} \| L(x_0, u_0) \| \leqslant k.$$

Obviously, we have:

(8.2) $$\mathscr{L}_1 \subset \mathscr{L}_2 \subset \ldots \mathscr{L}_k = \mathscr{L}.$$

For $k = 1$, we have a hyperspace generated by simple elements. The dimensions of the appropriate hyperplanes are closely allied to the richness of the sets of elements with given properties. Let

(8.3) $$\varrho_k = \dim \mathscr{L}_k - \dim \mathscr{L}_{k-1}.$$

Multi-index $\varrho = \{\varrho_1, \ldots, \varrho_k\}$, which is a function of a point, is called the index of classification for system (1.1). (Remark: if $\mathscr{L}_k$ is an empty set, then we define dim $\mathscr{L}_k = -1$). If $\varrho_1 = -1$, then the system (1.1) has no solution built from simple elements (there are no simple states). If $\varrho_1 \neq -1$, then we may seek solutions of the system (1.1) built from simple elements (i.e., $\operatorname{rank} \| L^j_\nu \| = 1$) and in some cases we may obtain solutions which are the interactions of simple waves and simple states.

The study of the structure of elements of hyperplane $\mathscr{L}_1(x_0, u_0)$ enables us to find physical properties of solutions which are simple states or superpositions of a simple state with simple wave.

9. Theorems on type $\mathscr{L}_1$ system

Now, we shall demonstrate several theorems exhibiting the structure of $\mathscr{L}$. They enable us to decide whether a given system is of type $\mathscr{L}_1(x_0, u_0)$, (i.e. $\mathscr{L}(x_0, u_0) = \mathscr{L}_1(x_0, u_0)$ or not. Let us consider a system of the form:

(9.1) $$u_{,x_0} + A u_{,x_1} = b, \quad \text{where} \quad A = \left(A^r_l(x, u)\right) \in (\mathscr{R}^2 \times \mathscr{R}^l).$$

[8] We denote $\langle a^s, \gamma \otimes \lambda \rangle \equiv a^{s\nu}_j \gamma^j \lambda_\nu$.

It follows from the form of the Eq. (9.1) that the covector (1,0) is noncharacteristic. Let us consider any fixed point $(x_0, u_0) \in E \times \mathscr{H}$. The set of noncharacteristic covectors is open in E^* for fixed (x_0, u_0). (This fact is a consequence of the Darboux property applied to the function $\psi(\lambda) = \det||a_J^{s\nu}\lambda_\nu||$). If for some λ $\det||a_J^{s\nu}\lambda_\nu|| \neq 0$, then there exists a neighbourhood of λ such that in this neighbourhood we have $\det||a_J^{s\nu}\tilde{\lambda}_\nu|| \neq 0$ — i.e., all vectors in this neighbourhood are noncharacteristic. Without loss of generality, we may assume that also the covector (0, 1) is noncharacteristic, since we may obtain it by linear transformation of independent variables. Consequently on the remark above, the set of noncharacteristic vectors is open; hence, there exists $\varepsilon > 0$ such that for $|\lambda_0| < \varepsilon$ all covectors $(\lambda_0, 1)$ are noncharacteristic. But we also assumed that the covector (0, 1) is noncharacteristic; hence the matrix A in the Eq. (9.1) has an inverse. Hence the equation for simple elements is of the form:

$$(I\lambda_0 + A)\gamma = b. \tag{9.2}$$

THEOREM 4. *Let us consider the system* (9.1). *If the vector b does not belong to any invariant* (9) *space* $N \subset \mathscr{H}(N \neq \mathscr{H})$ *of the matrix A, then we have*

$$\mathscr{L}_1(x_0, u_0) = \mathscr{L}(x_0, u_0). \tag{9.3}$$

P r o o f. We may assume that the covector (0, 1) is noncharacteristic, and then:

$$\gamma = (I\lambda_0 + A)^{-1} b \quad \text{for} \quad |\lambda_0| < \varepsilon, \tag{9.4}$$

where $\varepsilon > 0$. Since $\gamma = \gamma(\lambda_0)$ is an analytic function in the neighbourhood of zero for $\lambda_0 \in (-\varepsilon, +\varepsilon) = I_\varepsilon$, we may write (9.4) as a von Neumann series for small values of λ_0, such that $\lambda_0 < ||A||$. Thus we have:

$$(I\lambda_0 + A)^{-1} = A^{-1}(1 - \lambda_0 A^{-1} + \lambda_0^2 A^{-2} - \lambda_0^3 A^{-3} + \ldots + (-\lambda_0)^n A^{-n} + \ldots). \tag{9.5}$$

The tensor product $\gamma(\lambda_0) \otimes \lambda(\lambda_0)$ is also an analytic function; hence:

$$\begin{aligned} \gamma(\lambda_0) \otimes \lambda(\lambda_0) &= A^{-1}(1 - \lambda_0 A^{-1} + \ldots + (-\lambda_0)^n A^{-n} + \ldots) b \otimes \big((0,1) + (1,0)\lambda_0\big) = \\ &= A^{-1} b \otimes (0,1) + \sum_{n=1}^{\infty} (-\lambda_0)^n \big(A^{-n} b \otimes (0,1) - A^{n+1} b \otimes (1,0)\big). \end{aligned} \tag{9.6}$$

By way of proof, we need only remark that first $l+1$ coefficients of different powers of λ_0 are linearly independent in the space $\mathscr{H} \otimes E^*$. These elements are:

$$\{(A^{-1}b) \otimes (0,1)\}, \{A^{-2} b \otimes (0,1) - A^{-1} b \otimes (1,0)\}, \ldots, \{A^{-l-1} b \otimes (0,1) - A^{-l} b \otimes (0,1)\}. \tag{9.7}$$

Let us denote $A^{-n} b = b_n$. Since b belongs to none of invariant spaces of the matrix A, then vectors $b_1, b_2, \ldots, b_l$ are linearly independents and generate the whole space $\mathscr{H}$ and $b_{l+1} = \alpha^i b_i$, $i = 1, \ldots, l$. Let us denote: $e_0 = (1, 0), e_1 = (0, 1)$. For (9.6) we have:

$$(b_1 \otimes e_1), (b_2 \otimes e_1 - b_1 \otimes e_0), \ldots, (b_{l+1} \otimes e_1 - b_l \otimes e_0).$$

Now we show that these elements are linearly independent. Let us consider any linear combination

$$c_1(b_1 \otimes e_1) + c_2(b_2 \otimes e_1 - b_1 \otimes e_0) + \ldots + c_{l+1}(b_{l+1} \otimes e_1 - b_l \otimes e_0) = 0. \tag{9.8}$$

(9) This means that $x \in N \Rightarrow Ax \in N$.

From this it follows that

$$\sum_{i=1}^{l+1} c_i b_i = 0 \quad \text{and} \quad \sum_{k=1}^{l} c_{k+1} b_k = 0.$$

But $b_1, \ldots, b_l$ are linearly independent; hence, $c_2, c_3, \ldots, c_{l+1} = 0$, and hence also $c_1 = 0$. Therefore the vectors in (9.8) are independent. Thus the dimension of the space generated by simple elements of the form (9.5) is equal to:

$$\dim\{\gamma(\lambda_0)\otimes\lambda(\lambda_0)\}_{\lambda_0\in I_e} \geqslant l+1. \tag{9.9}$$

Since there are $l+1$ linearly independent elements in (9.7), the dimension of the hyperplane generated by $\mathscr{L}_1(x_0, u_0)$ is at least l.

In addition, the dimension of the linear subspace $\mathscr{L}(x_0, u_0)$ generated by all the integral elements of the system (9.1) is $\dim\{\mathscr{L}(x_0, u_0)\} = l$. Hence: $\mathscr{L}_1(x_0, u_0) = \mathscr{L}(x_0, u_0)$. Q.E.D.

Let N_b be the smallest subspace invariant [10] under the matrix A and containing vector b. This means that N_b is spanned by vectors $b, Ab, A^2b, \ldots; N_b = \{b, Ab, A^2b, \ldots\} = \{A^ib\}_{i=0,1,\ldots}$ [11].

By H^{Hyp} we shall denote a subspace generated by the eigenvectors of the matrix A. We shall consider only the real eigenvectors and eigenvalues. Obviously, H^{Hyp} is an invariant subspace of the matrix A. We obtain the following theorem:

THEOREM 5. *If the eigenvalues of the matrix A are distinct* (*provided they exist*), *then*

$$\dim\mathscr{L}_1 = \dim H^{\mathrm{Hyp}} + \dim N_b - \dim(H^{\mathrm{Hyp}} \cap N_b). \tag{9.10}$$

P r o o f. Applying an appropriate transformation we may assume that the equations determining simple elements are of the form:

$$\left[I\lambda_0 + \begin{bmatrix} \mu_1 & & & \\ & \ddots & 0 & \\ & & \mu_p & \\ & 0 & & B \\ & & & & A_N \end{bmatrix} \right] \begin{bmatrix} \gamma^1 \\ \vdots \\ \gamma^p \\ \gamma_B \\ \gamma_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ b_N \end{bmatrix} \tag{9.11}$$

where $\mu_1, \ldots, \mu_p$ are eigenvalues of the matrix A, B — is a matrix of dimension $(l-p+\dim N)\times(l-p+\dim N)$ and A_N is a matrix $\dim N\times\dim N$. The matrix B has no eigenvector. It is easy to see that: Simple elements $\gamma(\lambda_0)\otimes\lambda(\lambda_0)$ for $\lambda_0 \neq \mu_i$, $i = 1, \ldots, p$ generate a hyperplane of dimension equal to $\dim N_b$ (this follows from Theorem 4) or a linear subspace of dimension equal to $\dim N_b+1$. The vectors $\gamma(\lambda_0)$ are of the form $\gamma^1 = 0, \ldots \ldots, \ldots, \gamma^p = 0$, $\gamma_B = (\gamma^{p+1}, \ldots, \gamma^r) = 0$, where $r = l-p+\dim N$. Now if we set $\lambda_0 = -\mu_i$ we shall obtain the following solution for $\gamma(-\mu_i)$: $\gamma^1 = 0, \ldots, \gamma^{i-1} = 0$, γ^i is undetermined, $\gamma^{i+1} = 0, \ldots, \gamma^p = 0$ and $\gamma_B = 0$. Setting $i = 1, \ldots, p$, we shall obtain further p independent elements, where $p = \dim H^{\mathrm{Hyp}} - \dim(H^{\mathrm{Hyp}} \cap N_b)$. Thus, $\dim \mathscr{L}_1 = \dim H^{\mathrm{Hyp}} + \dim N_b - \dim(H^{\mathrm{Hyp}} \cap N_b)$. Q.E.D.

([10]) N is an invariant subspace of the matrix A, if $N \supset A(N)$ — i.e., $\bigwedge_{x\in N} Ax \in N$.

([11]) As there are at the most l linear independent vectors we may take l successive ones.

From this theorem, we obtain as a corollary:

THEOREM 6. *If the system* (9.1) *is strongly hyperbolic, then* $\mathscr{L}_1(x_0, u_0) = \mathscr{L}(x_0, u_0)$.

P r o o f. If the system is hyperbolic, then $H^{\mathrm{Hyp}} = \mathscr{H}$. From the assumption that our system is strongly hyperbolic it follows that the matrix A has exactly $l = \dim \mathscr{H}$ distinct eigenvalues. Applying the formula (9.10) we have then:

$$\dim \mathscr{L}_1 = \dim \mathscr{H}. \tag{9.12}$$

But also $\dim \mathscr{L} = \dim \mathscr{H}$ and $\mathscr{L}_1 \subset \mathscr{L}$. Hence,

$$\mathscr{L}_1(x_0, u_0) = \mathscr{L}(x_0, z_0) \quad \text{Q.E.D.}$$

COROLLARIES. *Suppose the system* (9.1) *satisfies the assumptions of Theorem* 5. *Since* $\dim Q_1 = \dim H^{\mathrm{Hyp}}$, *we may write* (9.10) *in the following form*:

$$\dim \mathscr{L}_1 = \dim Q_1 + \dim N_b - \dim(Q_1 \cap N_b). \tag{9.13}$$

Hence,

1. *If* $N_b \subset H^{\mathrm{Hyp}}$, *then* $\dim \mathscr{L}_1 = \dim Q_1$.

2. *If* $N_b \not\subset H^{\mathrm{Hyp}}$, *then* $\dim \mathscr{L}_1 > \dim Q_1$. *In fact* $\dim N_b \geqslant \dim(H^{\mathrm{Hyp}} \cap N_b)$.

From this it follows that

$$\dim \mathscr{L}_1 \geqslant \dim Q_1. \tag{9.14}$$

3. *If the system is elliptic — i.e.*, $\dim H^{\mathrm{Hyp}} = \dim Q_1 = 0$, *then*

$$\dim \mathscr{L}_1 = \dim N_b. \tag{9.15}$$

R e m a r k. The inequality (9.14) may not hold if a root of the matrix A has multiplicity greater than one.

E x a m p l e. To give an example for $\dim \mathscr{L}_1 < \dim Q_1$, let us consider an equation of the following form:

$$\mu^j_{,t} + \mu^j_{,x} = b^j, \quad j = 1, \ldots, l, \quad \text{when} \quad A = I,\ b^j = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The equation for the nonhomogeneous simple elements takes the form: $(I\lambda_0 + I)\gamma = b$.

Hence, for $\lambda_0 \neq -1$ the only solution is $\gamma = \begin{bmatrix} \frac{1}{\lambda_0 + 1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and for $\lambda_0 = -1$ there is no solution. But for $\lambda_0 \neq -1$, we have only two linearly independent solutions — e.g.,

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes (0, 1)$ and $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes (1, 0)$. They correspond to $\lambda_0 = 0$ and $\lambda_0 = \infty$. Thus $\dim \mathscr{L}_1 = 1$. But obviously $\dim Q_1 = l$ — i.e., $\dim \mathscr{L}_1 < \dim Q_1$.

Let the matrix A have eigenvalues μ_i with multiplicities $k(i)$. Also, let the number of linearly independent eigenvectors associated with an eigenvalue μ_i be equal to its multiplicity $k(i)$ for $i = 1, \ldots, r$. By H_i we shall denote the space corresponding to the eigenvalue μ_i of the matrix A. By virtue of above assumption, H_i consists only of the eigenvectors corresponding to the eigenvalues μ_i. We define a function:

$$\chi(H_i) = \begin{cases} 0 & \text{when} \quad H_i \cap N_b = 0, \\ 1 & \text{when} \quad H_i \cap N_b \neq 0. \end{cases} \tag{9.16}$$

We shall now formulate a lemma which will be helpful in the proof of the theorem.

LEMMA 1. *The invariant subspace N_b has at the most one common direction with each of the spaces H_i* — i.e.,

$$\dim(H_i \cap N_b) \leqslant 1. \tag{9.17}$$

It is essential in the proof of this lemma that the subspace N_b is generated by the vector b (i.e., $N_b = \{b, Ab, A^2b, \ldots\}$).

We obtain the theorem:

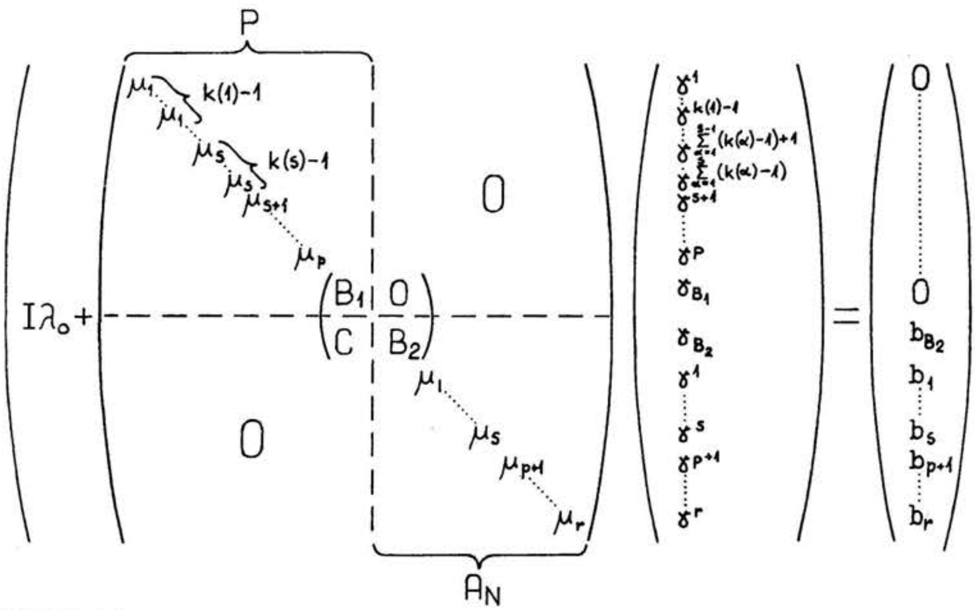
THEOREM 7. *Let us assume that for the system* (9.1) *the matrix A has at the point* (x_0, u_0) *r distinct eigenvalues $\mu_1, \ldots, \mu_r$ with multiplicities $k(1), \ldots, k(r)$. Let us assume moreover, that the number of linearly independent vectors associated with the eigenvalue μ_i is equal to its multiplicity $k(i)$ for $i = 1, \ldots, r$. Then we have*:

$$\dim \mathscr{L}_1 = \dim N_b + \sum_{i=1}^{r} k(i)\big(1-\chi(H_i)\big) \tag{9.18}$$

or [12]

$$\dim \mathscr{L}_1 = \dim N_b + \dim H^{\mathrm{Hyp}} - \dim(H^{\mathrm{Hyp}} \cap N_b) - \sum_{i=1}^{r} \big(k(i)-1\big)\chi(H_i). \tag{9.19}$$

P r o o f. It follows from the Lemma 1 that in a suitable system associated with the eigenvectors, the system of equations for nonhomogeneous simple elements can be expressed in such a form that in the section A_N each eigenvalue occurs at most once:



[12] These expressions are equivalent.

where all the components $b_B, b_1, \ldots, b_r$ are different from zero, A_N — a matrix of the dimension $\dim N \times \dim N$, $B = \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$ — a matrix which has no eigenvector ([13]), B_1 — a matrix of the dimension $\left(l - \left(\sum_{i=1}^{s} (k(i)-1) + p - s + \dim N\right)\right) \times \left(l - \left(\sum_{i=1}^{s} (k(i)-1) + p - s + \dim N\right)\right)$. Some of the eigenvalues appearing in the section A_N may also appear in the section P. They may be so ordered that only $\mu_1, \ldots, \mu_s$ appear in A_N. The eigenvalues $\mu_{s+1}, \ldots, \mu_p$ do not apear in A_N.

We notice that the simple elements $\gamma(\lambda_0) \otimes \lambda(\lambda_0)$ exist for $\lambda_0 \neq -\mu_i$, $i = 1, \ldots, r$, and they span a hyperplane of the dimension equal to $\dim N_b$ (from Theorem 4) or a linear subspace of the dimension equal to $\dim N_b + 1$. The vectors $\gamma(\lambda_0)$ are of the form $(0, 0, \ldots, \ldots, 0, \tilde{\gamma}_N)$ — i.e., the part corresponding to the section P vanishes. If $\lambda_0 = -\mu_i$, and μ_i appears in A_N then the system has no solution for γ. If μ_i does not appear in A_N, then the solution exists and it has the form:

$$\gamma = (\underbrace{0, \ldots, 0}_{k(1)}, \ldots, 0, \underbrace{\alpha_1, \ldots, \alpha_{k(i)}}_{k(i)}, 0, \ldots, 0, \tilde{\gamma}_N), \text{ where } \alpha_1, \ldots, \alpha_{k(i)}$$

are arbitrary components of the vector γ. Thus the dimension of the hyperplane $\mathscr{L}_1$ is equal to $\dim \mathscr{L}_1 = \dim N_b +$ the sum of multiplicities of μ_i not appearing in A_N:

$$\dim \mathscr{L}_1 = \dim N_b + \sum_{i=1}^{r} k(i)(1 - \chi(H_i)).$$

But

$$\sum_{i=1}^{r} k(i) = \dim H^{\mathrm{Hyp}} \quad \text{and} \quad \dim(N_b \cap H^{\mathrm{Hyp}}) = \sum_{i=1}^{r} \chi(H_i),$$

hence,

$$\dim \mathscr{L}_1 = \dim H^{\mathrm{Hyp}} + \dim N_b - \dim(H^{\mathrm{Hyp}} \cap N_b) - \sum_{i=1}^{r} (k(i)-1)\chi(H_i). \qquad \text{Q.E.D.}$$

COROLLARIES. 1. *If the system is hyperbolic then* $\dim N_b = \dim(N_b \cap H)$. *As* $H = H^{\mathrm{Hyp}}$, *or equivalently* $\dim N_b = \sum_{i=1}^{r} \chi(H_i)$, *thus*

$$\dim \mathscr{L}_1 = \sum_{i=1}^{r} k(i) - \sum_{i=1}^{r} k(i)\chi(H_i) + \sum_{i=1}^{r} \chi(H_i);$$

hence:

$$\dim \mathscr{L}_1 = \dim H - \sum_{i=1}^{r} (k(i)-1)\chi(H_i) \quad \textit{for a hyperbolic system;} \tag{9.20}$$

([13]) If N_b is an invariant subspace of the linear mapping $B\colon E \to E$ (having no real eigenvector), and if $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ is a basis in E such that $e_1, \ldots, e_k$ spans the subspace N_b, then the matrix B of the mapping takes on the form: $B = \begin{pmatrix} B_1 & 0 \\ C & B_2 \end{pmatrix}$, where the matrix B of the dimension $(n-k) \times (n-k)$ and matrix B_2 of the dimension $k \times k$ have no real eigenvectors.

of course, we have:

$$\dim \mathcal{L}_1 \leqslant \dim Q_1 = \dim \mathcal{K} \quad \textit{for a hyperbolic system.} \tag{9.21}$$

The inequality appears if the matrix A has roots with multiplicity greater than one.

2. *If* $N_b \subset H^{\mathrm{Hyp}}$, *then* $\dim \mathcal{L}_1 \leqslant \dim Q_1$.

3. *If* $N_b \cap H^{\mathrm{Hyp}} = \{0\}$, *then* $\dim \mathcal{L}_1 > \dim Q_1$.

Proof. We have $H_i \cap N_b = \{0\}$, therefore $\chi(H_i) = 0$. Consequently, $\dim \mathcal{L}_1 = \dim N_b + \dim H^{\mathrm{Hyp}}$. Moreover, $\dim Q_1 = \dim H^{\mathrm{Hyp}}$; hence,

$$\dim \mathcal{L}_1 = \dim N_b + \dim Q_1. \tag{9.22}$$

4. *If the system is elliptic — i.e.,* $\dim H^{\mathrm{Hyp}} = \dim Q_1 = 0$, *then*

$$\dim \mathcal{L}_1 = \dim N_b. \tag{9.23}$$

We may also consider a more general system:

$$\mu_{,x_0} + A^{\alpha}\mu_{,x_\alpha} = b, \quad \alpha = 1, \ldots, n. \tag{9.24}$$

In such a case it is necessary to consider in some places a matrix $A^{\alpha}\lambda_{\alpha}$, $\alpha = 1, \ldots, n$ instead of matrix A. Suppose $\mu_i = \mu_i(\bar{\lambda})$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{R}^n$ are different real analytic functions defined on $\mathcal{R}^n$. Assume, moreover, the number of linearly indepedent eigenvectors corresponding to μ_i to be equal to $k(i)$, $i = 1, \ldots, r$. Then our theorem may be generalized as follows:

$$\dim \mathcal{L}_1 = n\Big\{\dim N_b + \sum_{i=1}^{r} k(i)\big(1-\chi(H_1)\big)\Big\}; \tag{9.25}$$

we may write also

$$\dim \mathcal{L}_1 = \dim Q_1 + n\Big\{\dim N_b - \dim(H^{\mathrm{Hyp}} \cap N_b) - \sum_{i=1}^{r} (k(i)-1)\chi(H_i)\Big\}. \tag{9.26}$$

This expression is a generalization of the formula (9.10), where N_b and H^{Hyp} are computed for any fixed direction $\bar{\lambda} \in \mathcal{R}^n$ such that its roots $\mu_i(\bar{\lambda})$ (with multiplicities $k(i)$) are different ([14]).

In particular, the assumptions are satisfied if for any $\bar{\lambda} \in \mathcal{R}^n$ we have $\mu_i(\bar{\lambda}) \neq \mu_j(\bar{\lambda})$ for $i \neq j$, the multiplicities $k(i)$ are independent of λ, and the number of corresponding linearly independent eigenvectors is equal to $k(i)$, $i = 1, \ldots, r$.

10. Examples — nonhomogeneous equations of gasdynamics

Now we shall consider some examples to illustrate the theoretical considerations above. We have chosen the case of nonhomogeneous equations of gasdynamics; it is possible to apply them to geophysical fluid dynamics.

Let us consider the classical equations of hydrodynamics which describe the motion of fluid medium when the gravitational force and Coriolis force occur. We study only the equations of flow of the one-component nonviscous fluid. Under these assumptions our

([14]) For certain $\bar{\lambda}$, $\mu_i(\bar{\lambda}) = \mu_j(\bar{\lambda})$ is permissible.

equations are of the type (9.1). The form of these equations of hydrodynamics in a non-inertial system is:

$$\varrho\left\{\frac{\partial\overline{v}}{\partial t}+(\overline{v}\nu)\overline{v}\right\}+\nabla p = \varrho\overline{g}-2\varrho\overline{\omega}\times\overline{v},$$

(10.1)

$$\frac{\partial\varrho}{\partial t}+\operatorname{div}(\varrho\overline{v}) = 0, \qquad \frac{d}{dt}\left(\frac{P}{\varrho^{\varkappa}}\right) = 0.$$

Here we treat the physical space $E \subset \mathscr{R}^4$ as the classical space-time, each of its points having coordinates $(t, \overline{x})$, and the space of unknown functions (i.e., the hodograph space) $\mathscr{H} \subset \mathscr{R}^5$ has the coordinates $(\varrho, p, \overline{v})$. Let us denote by $\lambda = (\lambda_0, \overline{\lambda})$, where $\overline{\lambda} \in \mathscr{R}^3$, the vectors which belong to E^* and by $\gamma = (\gamma_\varrho, \gamma_p, \overline{\gamma})$, where $\overline{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$, elements of the space $T\mathscr{H}$, where $\varrho \to \gamma_\varrho, p \to \gamma_p, v \to \overline{\gamma}$. Algebraic equations which determine simple elements for the equations (10.1) are of the form:

$$\begin{aligned} \varrho\delta\overline{\gamma}+\gamma_p\overline{\lambda} &= \varrho(\overline{g}-\overline{\Omega}\times\overline{v}),\\ \delta\gamma_\varrho+\varrho\overline{\gamma}\overline{\lambda} &= 0, \qquad (10.2)\\ \delta(\varrho\gamma_p-\varkappa p\gamma_\varrho) &= 0, \end{aligned}$$

where we use the following notation: $\overline{\Omega} = 2\overline{\omega}$ and

(10.3) $$\delta|\overline{\lambda}| = \lambda_0+\overline{v}\cdot\overline{\lambda}.$$

The physical sense of the function δ is the following: it describes the velocity of the propagation of disturbance relative to the fluid. We may consider the Eqs. (10.2) as a system of linear nonhomogeneous algebraic equations. By the Kronecker-Capelli theorem, there exists a solution γ different from zero iff one of the following cases holds:

(10.4)

$$\begin{aligned}
&1.\ \delta_{E_{N_1}} = \delta = 0 \quad \text{and} \quad \gamma_p = 0,\ \overline{g}-\overline{\Omega}\times\overline{v} = 0,\\
&2.\ \delta_{E_{N_2}} = \delta = 0 \quad \text{and} \quad \gamma_p \neq 0,\\
&3.\ \delta_{A_N} = \delta = \varepsilon\sqrt{\frac{xp}{\varrho}} \ \text{and} \quad \overline{\lambda}\cdot(\overline{g}-\overline{\Omega}\times\overline{v}) = 0,\ \varepsilon = \pm 1;\\
&4.\ \delta_{H_N} = \delta \neq \begin{cases} 0 \\ \varepsilon\sqrt{\dfrac{\varkappa p}{\varrho}} \end{cases},\ \varepsilon = \pm 1.
\end{aligned}$$

The Eqs. (10.4.1) and (10.4.2) determine (by (10.3)) entropic elements, which correlate to simple entropic states denoted by E_{N_1}, E_{N_2}, respectively. The Eq. (10.4.3) determines acoustic elements, which also correlate to simple acoustic states, denoted by A_N. The Eq. (10.4.4) determines hydrodynamic elements, denoted by H_N. It follows from the definition (10.3) above that the velocity of entropic state E_N relative to the fluid is equal to zero—i.e., this state moves with the fluid. The velocity of propagation of acoustic state A_N relative to the medium is equal to sound velocity: $\sqrt{dp/d\varrho} = \sqrt{\varkappa p/\varrho}$. The hydrodynamic state H_N may move with any velocity, except entropic velocity $\delta_{E_N} = 0$ and acoustic $\delta_{A_N} = \varepsilon\sqrt{\varkappa p/\varrho}$. The conditions (10.4) above hold that cones of simple nonhomogeneous elements are determined only in that part of the hodograph space where the vectors belong to spaces tangent to these submanifolds (10.4).

It follows from the analysis of the homogeneous system (10.2) that if there exists a non zero solution on the vector $\gamma = (\gamma_\varrho, \gamma_p, \overline{\gamma})$, then the characteristic determinant of this system must be equal to zero — i.e., the following condition must hold:

$$\delta^3|\overline{\lambda}|^3\left(\delta^2|\overline{\lambda}|^2-\frac{\varkappa p}{\varrho}\,\overline{\lambda}^2\right) = 0. \tag{10.5}$$

We have solutions of two kinds, namely:

$$\begin{aligned} &1.\ \delta_E = \delta = 0, && \text{entropic velocity,}\\ &2.\ \delta_A = \delta = \varepsilon\sqrt{\frac{\varkappa p}{\varrho}} && \text{acoustic velocity.}\end{aligned} \tag{10.6}$$

The Eq. (10.6.1) determines entropic elements, which correlate to simple entropic waves E, and (10.6.2) determines acoustic elements, which correlate to simple acoustic waves A. We shall give later the physical interpretation of these velocities and their simple states.

The Eqs. (10.1) are of the Cauchy-Kowalewska form, and constitute a system of hyperbolic equations, the dimension of which is $\dim Q_1 = 15$. By (10.6), these equations have three eigenvalues of multiplicity of root $\delta = 0$ egual to three. By the corollary to Theorem 6, the dimension of tensor space generated by simple nonhomogeneous elements is

$$\dim \mathscr{L}_1 = \dim Q_1-(n-1)\sum_{i=1}^{r}(k(i)-1)\chi(H_i).$$

The dimension of invariant subspace N_b is $\dim N_b = 3$.

Since $\dim(H_i \cap N_b) \neq 0$, it must occur that $\chi(H_i) = 1$; hence we have:

$$\dim \mathscr{L}_1 = 9. \tag{10.7}$$

Thus the system we consider has linear elements, which are not a linear combination of simple elements. In other words, simple nonhomogeneous elements do not generate the whole space of integral elements.

Simple states

1. Simple entropic states E_{N_1} and E_{N_2}

The condition (10.4.1) applied to the Eqs. (10.2) gives the following system of algebraic equations, which determines entropic elements E_{N_1}:

$$\lambda_0 = -\overline{v}\cdot\overline{\lambda}, \quad \gamma_p = 0, \quad \overline{\gamma}\cdot\overline{\lambda} = 0, \quad \overline{g}-\overline{\Omega}\times\overline{v} = 0. \tag{10.8}$$

The solution of this system gives simple nonhomogeneous elements E_{N_1} of the form:

$$\gamma = (\gamma_\varrho, 0, \overline{\alpha}\times\overline{\lambda}), \quad \lambda = (-\overline{v}\overline{\lambda}, \overline{\lambda}) \quad \overline{g} = \overline{\Omega}\times\overline{v}, \text{ where } \overline{\alpha} \text{ is arbitrary vector.} \tag{10.9}$$

If we require the conditions of integrability (3.9) for (10.9), we have the simple state E_{N_1} of the form:

$$\varrho = \varrho(s), \quad P = P_0, \quad \overline{v} = \beta(s)\overline{\Omega}-c_1\overline{A}, \quad \overline{\Omega}\cdot\overline{A} = 0, \quad \overline{g}\cdot\overline{\Omega} = 0, \tag{10.10}$$

where $|\overline{A}| = 1$, $\beta(s)$ — arbitrary function.

The function $s = S(t, \bar{x})$ should be treated according to the formula (3.6); this means that the four-vector $\nabla S(t, \bar{x})$ is equal to λ; then $s = c_1 t + \bar{A}\cdot\bar{x}$.

This solution (when the condition $c_1 = 0$ is satisfied) describes the gas in the state of equilibrium in the field of gravitation and Coriolis force.

Applying the condition (10.4.2) to the Eq. (10.2), we obtain the equations which describe the entropic elements E_{N_2}:

$$(10.11) \qquad \gamma_p \bar{\lambda} = \varrho(\bar{g} - \bar{\Omega}\times\bar{v}), \quad \bar{\gamma}\cdot\bar{\lambda} = 0, \quad \gamma_p \neq 0, \quad \lambda_0 = -\bar{v}\bar{\lambda}.$$

Thus the simple nonhomogeneous entropic elements E_{N_2} are of the form

$$(10.12) \qquad \gamma = \big(\gamma_\varrho, \varrho, \bar{\alpha}\times(\bar{g} - \bar{\Omega}\times\bar{v})\big), \quad \lambda = (-\bar{v}\bar{g}, \bar{g} - \bar{\Omega}\times\bar{v}),$$

where $\bar{\alpha}$ is arbitrary vector.

If we require the conditions (3.9) for (10.12), we obtain the simple state E_{N_2} of the form:

$$(10.13) \quad \varrho = \dot{p}(s), \quad p = p(s), \quad \bar{g}\cdot\bar{\Omega} = 0, \quad \bar{A} = (1 + \bar{\alpha}\cdot\bar{\Omega})\bar{g}, \quad \bar{v} = \bar{\alpha}\times\bar{g},$$

$p(s)$ is arbitrary function, where the function $s = S(t, \bar{x})$ is defined by $s = \bar{x}\cdot\bar{g}$.

The solution above describes the gas which moves with no acceleration and friction in the direction perpendicular to $\bar{g}$. The flows of this kind occurring along parallel rectilinear izobars are a subject of consideration in geophysics, where they are called geostrophic wind.

2. *Simple acoustic state A_N*

Application of the condition (10.4.3) to the Eqs. (10.2) yields to the following system of algebraic equations, which determines the acoustic elements A_N

$$(10.14) \qquad \begin{aligned} &\varepsilon\varrho\sqrt{\frac{\varkappa p}{\varrho}}\,|\bar{\lambda}|\bar{\gamma} + \gamma_p\bar{\lambda} = \varrho(\bar{g} - \bar{\Omega}\times\bar{v}),\\ &\varepsilon\sqrt{\frac{\varkappa p}{\varrho}}\,|\bar{\lambda}|\gamma_\varrho + \varrho\bar{\gamma}\cdot\bar{\lambda} = 0, \quad \gamma_p = \frac{\varkappa P}{\varrho}\gamma_\varrho. \end{aligned}$$

The solution of this system is given by simple nonhomogeneous elements A_N of the form:

$$(10.15) \qquad \begin{aligned} &\gamma = \left(\gamma_\varrho, \frac{\varkappa p}{\varrho}\gamma_\varrho, \frac{1}{\varepsilon\varrho\sqrt{\dfrac{\varkappa P}{\varrho}}\,|\bar{\lambda}|}\left(\varrho(\bar{g} - \bar{\Omega}\times\bar{v}) - \frac{\varkappa p}{\varrho}\gamma_\varrho\bar{\lambda}\right)\right),\\ &\lambda = \left(\varepsilon\sqrt{\frac{\varkappa p}{\varrho}}\,|\bar{\lambda}| - \bar{v}\bar{\lambda}, \bar{\lambda}\right), \quad (\bar{g} - \bar{\Omega}\times\bar{v})\cdot\bar{\lambda} = 0. \end{aligned}$$

Applying the conditions (3.9) to (10.15), we obtain the simple acoustic state A_N of the form:

$$(10.16) \quad \varrho = \varrho_0, \quad p = c\varrho_0^\varkappa, \quad \bar{v} = \bar{\beta}(s)\times\bar{A} + (\varepsilon\sqrt{c\varkappa}\varrho_0^{\frac{\varkappa+1}{2}} - c_1)\bar{A}, \quad \bar{\Omega} = c_2\bar{A},$$

$$\bar{g}\cdot\bar{A} = 0, \quad \bar{\beta}\cdot\bar{A} = , \quad |\bar{A}| = 1,$$

where: $\varrho_0, c, c_1, c_2, c_3$ denote arbitrary constants and conditions which determine coordinates of the vector $\bar{\beta}(s)$ — namely:

$$1.\quad \bar{g}\cdot\bar{\beta}-\frac{c_2}{2}\bar{\beta}^2 = c_3,$$

$$2.\quad \bar{\beta}^2_{,s} = \frac{2\bar{g}\cdot\bar{\beta}\times\bar{A}}{\varepsilon\sqrt{c\varkappa}\,\varrho_0^{\frac{\varkappa-1}{2}}},$$

where the function $s = S(t, \bar{x})$ is given by $s = c_1 t+\bar{A}\bar{x}$.

This equation describes such a gas in the field of gravitation which is accelerated by the Coriolis force.

3. *The simple hydrodynamic state H_N*

The solution of the system (10.2) under the condition (10.4.4) leads to the following simple nonhomogeneous elements:

$$\gamma = \left(\frac{-\varrho(\bar{g}-\Omega\times\bar{v})\cdot\bar{\lambda}}{\delta^2-\frac{\varkappa p}{\varrho}\bar{\lambda}^2}, \frac{-\varkappa p(\bar{g}-\bar{\Omega}\times\bar{v})\cdot\bar{\lambda}}{\delta^2-\frac{\varkappa p}{\varrho}\bar{\lambda}^2}, \frac{1}{\varrho\delta}\left[\varrho(\bar{g}-\bar{\Omega}\times\bar{v})+\frac{\varkappa p(\bar{g}-\bar{\Omega}\times\bar{v})\cdot\bar{\lambda}}{\delta^2-\frac{\varkappa p}{\varrho}\bar{\lambda}^2}\bar{\lambda}\right]\right)$$

$$\lambda = (\delta-\bar{v}\cdot\bar{\lambda}, \bar{\lambda}), \quad \text{where} \quad \delta \neq \begin{Bmatrix}0\\ \varepsilon\sqrt{\frac{\varkappa p}{\varrho}}\end{Bmatrix}. \tag{10.17}$$

If we require the conditions of involutivity, (3.9) to (10.17), we obtain the simple state H_N of the form:

$$\varrho = \varrho(s), \quad p = c\varrho^{\varkappa}, \quad \bar{v} = \bar{\beta}\times\bar{A}+\varphi(s)\bar{A} \tag{10.18}$$

and the conditions:

$$\frac{d\varrho}{ds} = \frac{-\varrho(\bar{g}\cdot\bar{A}+\bar{\beta}\cdot\bar{\Omega})}{(c_1+\varphi)^2-c\varkappa\varrho^{\varkappa-1}},$$

$$\frac{d\varphi}{ds} = \frac{1}{c_1+\varphi}(\bar{g}\cdot\bar{A}+\bar{\beta}\cdot\bar{\Omega}-c\varkappa\varrho^{\varkappa-2}\varrho_{,s}),$$

$$\frac{d}{ds}(\bar{\beta}\times\bar{A}) = \frac{1}{c_1+\varphi}(\bar{g}-\bar{g}\cdot\bar{A}\;\bar{A}-\bar{\beta}\;\bar{\Omega}\cdot\bar{A}-\varphi\bar{\Omega}\times\bar{A}).$$

Thus the problem is now to solve the system of four nonlinear usual equations on functions $\varrho, \varphi, \bar{\beta}$.

Now we shall consider the special case of the Eqs. (10.1) when the Coriolis force does not occur (i.e., when $\bar{\Omega} = 0$). Simple elements for this case are now:

1. Simple nonhomogeneous entropic elements E_N:

$$\gamma_{E_.} = (\gamma_\varrho, \varrho, \bar{\alpha}\times g), \quad \lambda_{E_N} = (-\bar{v}\bar{g}, \bar{g}). \tag{10.19}$$

2. Simple nonhomogeneous acoustic element A_N

$$\gamma_{A_N} = \left(\gamma_\varrho, \frac{\varkappa p}{\varrho}\gamma_\varrho, \frac{1}{\varepsilon\varrho\sqrt{\frac{\varkappa p}{\varrho}}|\bar{\alpha}\times\bar{g}|}\left[\varrho\bar{g} - \frac{\varkappa p}{\varrho}\gamma_\varrho\bar{\alpha}\times\bar{g}\right]\right), \tag{10.20}$$

$$\lambda_{A_N} = \left(\varepsilon\sqrt{\frac{\varkappa p}{\varrho}}|\bar{\alpha}\times\bar{g}| - \bar{v}\cdot\bar{\alpha}\times\bar{g},\ \bar{\alpha}\times\bar{g}\right).$$

3. Simple nonhomogeneous hydrodynamic element H_N

$$\gamma_{H_N} = \left(\frac{-\varrho\bar{g}\cdot\bar{\lambda}}{\delta^2 - \frac{\varkappa p}{\varrho}\bar{\lambda}^2}, \frac{-\varkappa p\bar{g}\cdot\bar{\lambda}}{\delta^2 - \frac{\varkappa p}{\varrho}\bar{\lambda}^2}, \frac{1}{\varrho\delta}\left[\varrho\bar{g} + \frac{\varkappa p\bar{g}\cdot\bar{\lambda}}{\delta^2 - \frac{\varkappa p}{\varrho}\bar{\lambda}^2}\bar{\lambda}\right]\right), \tag{10.21}$$

$$\lambda_{H_N} = (\delta - \bar{v}\bar{\lambda}, \bar{\lambda}),$$

where $\delta \neq \begin{cases} 0 \\ \varepsilon\sqrt{\frac{\varkappa p}{\varrho}} \end{cases}.$

Simple homogeneous elements are of the same form as before (10.6). The dimension of the tensor space generated by simple nonhomogeneous elements is also $\dim \mathscr{L}_1 = 9$, since nonhomogeneity occurs in the the Euler's equation, exactly as before.

Simple states

1. Simple entropic state E_N

Using the Eqs. (10.19), we obtain the state E_N of the form:

$$\varrho = \dot{p}(s), \quad p = p(s), \quad \bar{v} = \bar{\alpha}\times\bar{g} - c_1\frac{\bar{g}}{\bar{g}^2}, \tag{10.22}$$

where the function $s = S(t, \bar{x})$ is given by $s = c_1 t + \bar{g}\bar{x}$. This solution describes the gas in the state of equilibrium in the field of gravitation when flows in the direction perpendicular to $\bar{g}$ occur. Especially for the stationary case (i.e., when $c_1 = 0$) and when we put $\bar{\alpha} = 0$ (we can always do it, since the vector $\bar{\alpha}$ forces no change either of other physical values or of the function s), the solution we obtain may be interpreted as the description of the atmosphere in the state of statical equilibrium.

2. Simple acoustic state A_N

The Eqs. (10.20) yield to the state A_N of the form:

$$\varrho = \varrho_0, \quad p = c\varrho_0^\varkappa, \quad \bar{g}\cdot\bar{A} = 0, \quad \bar{v} = [\varepsilon(c\varkappa\varrho_0^{\varkappa-1})^{-1/2}s + c_2]\bar{g} + \bar{v}_0, \tag{10.23}$$

where $\bar{v}\cdot\bar{A} = \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} - c_1$, $|\bar{A}| = 1$. The function $s = S(t, \bar{x})$ is given here by $s = c_1 t + \bar{x}\bar{g}$.

This solution describes free fall of the gas in the field of gravitation; this is seen when we choose such a system of coordinates, that $\bar{v}_0 = 0$:

$$\varrho = \varrho_0, \quad p = c\varrho_0^\varkappa, \quad \bar{g}\cdot\bar{A} = 0, \quad \bar{v} = \bar{g}t. \tag{10.24}$$

Thus these solutions are not interesting from the physical point of view.

3. Simple hydrodynamic state H_N

For a study of the behaviour of H_N, we must distinguish two cases:

1. The direction of propagation of the state $\bar{A}$ is parallel to the direction of the field $\bar{g}$ (i.e., $\bar{g}||\bar{A}$); then we have:

$$(10.25)\qquad \varrho = \varrho(s), \quad p = c\varrho^{\varkappa}, \quad \bar{g} = c_2\bar{A}, \quad \bar{v} = (\delta-c_1)\bar{A}+\bar{v}_0, \quad |\bar{A}| = 1,$$

and the conditions:

$$\frac{1}{2}\frac{d\delta^2}{d\varrho} = \frac{-(c_2-c\varkappa\varrho^{\varkappa-2})(\delta^2-c\varkappa\varrho^{\varkappa-1})}{c_2\varrho},$$

$$0 < \delta < \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} \quad \text{or} \quad \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} < \delta,$$

where the function $s = S(t, \bar{x})$ is given by $s = c_1t+\bar{x}\bar{g}$.

2. The direction of propagation of $\bar{A}$ is not perpendicular to the direction of $\bar{g}$(i.e., $\bar{g}\cdot\bar{A} \neq 0$); then we have:

$$(10.26)\qquad \varrho = \varrho(s), \quad p = c\varrho^{\varkappa}, \quad \bar{v} = \alpha(s)\bar{g}+(\delta-c_1)\bar{A}+\bar{v}_0,$$

where $\dot{\alpha}(s) = -\dot{\delta}(s)/c\varkappa\varrho^{\varkappa-2}$ and the conditions:

$$\frac{d\varrho}{d\delta} = \frac{\delta\bar{g}\cdot\bar{A}}{c\varkappa\varrho^{\varkappa-3}(\delta^2-c\varkappa\varrho^{\varkappa-1})},$$

$$0 < \delta < \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} \quad \text{or} \quad \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} < \delta,$$

where the function $s = S(t, \bar{x})$ is given by $s = c_1t+\bar{g}\bar{x}$.

In both these cases, we obtain a dependence of the function δ on ϱ. The condition $0 < \delta < \varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}}$ permits the state to move relative to the medium only with infrasound velocities, and the condition $\varepsilon\sqrt{c\varkappa}\,\varrho^{\frac{\varkappa-1}{2}} < \delta$ permits the state to move only with supersonic velocities. In both cases, when $\varkappa = 2$ we may give an analytic formula which defines the function $\delta = \delta(\varrho)$. Let us illustrate this with an example when $\varkappa = 2$ for the second case. The solutions now are of the form:

$$(10.27)\qquad \varrho = \varrho(s), \quad p = c\varrho^{\varkappa}, \quad \bar{v} = \frac{1}{2c}(\delta+c_3)\bar{g}+(\delta-c_1)\bar{A}+\bar{v}_0, \quad \bar{g}\cdot\bar{A} \neq 0,$$

where

$$\bar{v}_0\cdot\bar{A} = 0, \quad \delta = \varepsilon\sqrt{c_4\varrho^{\frac{4c}{\bar{g}\cdot\bar{A}}}+\frac{8c}{4c-\bar{g}\cdot\bar{A}}}$$

and the conditions:

$$0 < \delta < \varepsilon\sqrt{2c}\,\varrho^{1/2} \quad \text{or} \quad \varepsilon\sqrt{2c}\,\varrho^{1/2} < \delta,$$

where the function $s = S(t, \bar{x})$ is given by $s = c_1t+\bar{g}\bar{x}$.

Interaction of simple waves with simple states

Interaction of simple waves with simple states may be illustrated by Table 1,

Table 1.

Simple wave \ Simple state	E_N	A_N	H_N
E	+	+	—
A	—	+	$\varkappa > 0$? $\varkappa = 1$ —

where + denotes that interaction occurs, – denotes that interaction does not occur, ? denotes the case which is not determined.

Studies of interaction of entropic state with entropic wave, have led us to the existence of Riemann invariants. They give the following solutions:

$$\varrho = \dot{p}\,(r), \quad p = p(r), \quad v = \left(\beta, \frac{c_3 - c_4\beta}{c_2}, -c_1/\bar{g}^2\right). \tag{10.28}$$

The equations on parameters s, r are of the form:

$$r = c_1 t + \bar{g}\cdot\bar{x}, \quad \nabla s || (-c_3 + c_1\alpha, c_4, c_2, \alpha),$$

where α, β — arbitrary function of parameters s, r.

This solution describes the gas in the state of equilibrium in the field of gravitation in which occurs the outflow of flux of the gas in the direction perpendicular to the field $\bar{g}$. If we assume that $\alpha = \alpha(s)$ and $c_1 = 0$, $c_3 = 0$, then

$$s = \varphi(c_4 x + c_2 y + \alpha z), \quad r = |\bar{g}|z. \tag{10.29}$$

We have then the stationary case. The state interacts with the wave, since the wave does not influence the state and, conversely, they interact independently [9].

The interaction of entropic waves or acoustic waves with acoustic state describes free fall of fluid in the field of gravitation in which the given wave propagates.

The considerations above lead to the following results: the simple state may be interpreted physically as a one-dimensional solution constant on the planes parallel one to another (since the direction $\underset{N}{\lambda}$ does not depend on parameter R_0), which may move with constant velocity in the physical space E^3. As was illustrated above, these simple states may serve also to search for more general solutions, which may be interpreted as interactions of waves with medium in certain determined states.

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Symbols

E	physical space,
$\mathscr{H}$	hodograph space,
$\mathscr{K}$	vector space of solutions of the homogeneous system,
$\mathscr{L}$	hyperplane of solutions of the nonhomogeneous system,
Q	linear subspace,
N	invariant subspace,
H^{Hyp}	subspace generated by the eigenvectors of matrix A,
H_i	space corresponding to the eigenvalue μ_i of matrix A,
R, s, r	Riemann invariants,
μ_i	eigenvalues,
$k(r)$	multiplicities of the root,
u	coordinates of $\mathscr{H}$,
λ	characterisitic covector from E,
$\underset{N}{\lambda}$	noncharacteristic covector from E,
γ	characteristic vector from $\mathscr{H}$,
$\underset{N}{\gamma}$	noncharacteristic vector from $\mathscr{H}$,
$x = (t, \bar{x})$	coordinates of E,
$\delta = (\lambda_0 + \bar{v}\cdot\bar{\lambda})$	velocity of wave and state regard to a moving media,
ϱ	density of fluid,
p	pressure of fluid,
$\bar{v}$	velocity of fluid,
$\bar{g}$	gravitation field,
$\varkappa$	adiabetic exponent,
$\bar{A}$	direction of propagation of a state,
$\bar{\Omega}$	angular velocity of fluid.

References

1. M. BURNAT, *Theory of simple waves for non-linear systems of partial differential equations of first-order and applications to gas dynamics*, Arch. Mech. Stos., **18**, 4, 1966.
2. M. BURNAT, *The method of solutions of hyperbolic systems by means of combining simple waves*. In "Fluid Dynamics Transactions", **3**, 1967.
3. M. BURNAT, *Riemann invariants*. In "Fluid Dynamics Transactions", **4**, 1969.
4. M. BURNAT, *The method of Riemann invariants for multidimensional noneliptic systems*. Bull. Acad. Polon. Sci., Série. Sci. Techn., **17**, 11—12, 1969.
5. R. COURANT, *Partial differential equations*. In R. Courant, D. Hilbert "Methods of mathematical physics", **2**, New York 1962.
6. R. COURANT, H. D. FRIEDRICHS, *Supersonic flow and shock waves*, New York 1948.
7. Z. PERADZYŃSKI, *On algebraic aspects of the generalized Riemann invariants method*, Bull. Acad. Polon. Sci., Série Techn., **18**, 4, 1970.
8. Z. PERADZYŃSKI, *Nonlinear plane k-waves and Riemann invariants*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 9, 1971.
9. Z. PERADZYŃSKI, *Riemann invariants for the nonplanar k-waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 10, 1971.
10. B. L. ROZDESTVENSSII, N. V. JANENKO, *Sistemy kvazilinejnych uravnenii i ich priloženija k gazovoj dinamike*, Moskva 1968.
11. G. F. B. RIEMANN, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandl. Königl. Ges. Wiss. Göttingen, 8, 1869.

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