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Acceleration wave and progressive wave in non-linear elastic material

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CONSIDERATION is given to propagation of an acceleration wave in elastic materials subject to large strains. The condition of propagation of such a wave is constructed and then, by introducing the notion of an acoustic radius, a general solution of the equations of motion is derived. The progressive wave is then discussed, its phase and group velocities being determined. It is demonstrated that the velocity of propagation is approximately equal to the geometric mean of the phase and group velocities.

Rozważa się propagację fali przyspieszenia w materiale sprężystym, poddanym dużym odkształceniom. Buduje się warunek propagacji takiej fali, a następnie po wprowadzeniu pojęcia promienia akustycznego wyznacza ogólne rozwiązanie równań ruchu. Z kolei rozważa się falę postępującą i wyznacza prędkość fazową i grupową. Pokazuje się, że w przybliżeniu prędkość propagacji jest średnią geometryczną prędkości fazowej i grupowej.

Рассмотрено распространение волны ускорения в упругом материале, подвергнутом конечным деформациям. Получено условие распространения этой волны, а затем, на основе введенного понятия акустического луча, определено общее решение уравнений движения. Далее, исследуется прогрессивная волна, для которой определены фазовая и групповая скорости. Показано, что скорость распространения равна в приближении средней геометрической величине фазовой и групповой скоростей.

THE PRESENT paper is aimed at developing the simplest possible theory of waves in a non-linear elastic material. That is why we shall confine considerations to small amplitudes, which will enable us to apply the linearized equations of motion. A number of results concerning large amplitudes may be found in various papers published in recent years (cf. [1] and the references cited there), but the corresponding equations are very complicated. In particular, the equations governing the amplitude variations (analogous to the Eqs. (3.17)) are extremely complex, and relations corresponding to those presented in Sec. 4 of this paper have not been derived at all in the case of large amplitudes.

It should be stressed that the majority of the general considerations given in this paper (except those presented in Sec. 4) may be found in various books and papers dealing with the theory of differential equations; however, they are rather dispersed and generally unknown. Thus it seems useful to collect them, to apply them to non-linear elasticity and to present the results in a concise form.

1. Equations of non-linear elasticity

Let $\{X^\alpha\}$ and $\{x^i\}$ denote two, generally curvilinear coordinate systems. The body in a natural configuration B_R is referred to the system $\{X^\alpha\}$, and the body in actual configuration B is referred to the system $\{x^i\}$. Coordinates of a typical material point in the respective configurations B_R and B are X^α and x^i.

Let us consider the motion

$$x^i = \xi^i(X^\alpha, t). \tag{1.1}$$

Denote by $T_{Ri}{}^\alpha$ the Piola-Kirchhoff stress tensor. If σ is the stored energy (elastic potential), and $x^i_\alpha = \partial x^i/\partial X^\alpha$ — the strain gradient, ϱ_R denoting the mass density in the natural state B_R, then holds true the relation (cf. [2]):

$$T_{Ri} = \varrho_R \frac{\partial\sigma}{\partial x^i{}_\alpha}, \quad \sigma = \sigma(x^i{}_\alpha, X^\beta). \tag{1.2}$$

The tensor inverse to $X^i{}_\alpha$ is denoted by $X^\alpha{}_i$,

$$x^i{}_\alpha X^\alpha{}_k = \delta^i{}_k, \quad X^\alpha{}_k x^k{}_\beta = \delta^\alpha{}_\beta.$$

The equations of motion have the form:

$$T_{Ri}{}^\alpha||_\alpha = \varrho_R \ddot{x}^i, \tag{1.3}$$

where double vertical lines denote the total covariant differentiation

$$\cdot||_\alpha = \cdot|_\alpha + \cdot|_r x^r, \tag{1.4}$$

and a single vertical line corresponds to the partial covariant differentiation (cf. the formula $d/dX = \partial/\partial X + (\partial/\partial x)(dx/dX)$; a dot denotes the material time derivative.

Let us consider another motion

$$x^{*i} = \xi^i(X^\alpha, t) + u^i(X^\alpha, x^k, t), \tag{1.5}$$

which differs only slightly from the motion (1.1). Vector u^i is the vector of additional displacement. If the Eqs. (1.3) are satisfied, then the disturbed motion equations (1.5) are (cf. [3])

$$(A_i{}^\alpha{}_k{}^\beta u^k||_\beta)||_\alpha = \varrho_R \ddot{u}_i, \tag{1.6}$$

where

$$A_i{}^\alpha{}_k{}^\beta = \varrho_R \frac{\partial^2\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta}. \tag{1.7}$$

Let us denote

$$J = \varepsilon_{ikm}\varepsilon^{\alpha\beta\gamma} x^i{}_\alpha x^k{}_\beta x^m{}_\gamma, \tag{1.8}$$

where ε_{ikm} and $\varepsilon^{\alpha\beta\gamma}$ are the Ricci tensors. If both the coordinate systems $\{x^i\}$ and $\{X^\alpha\}$ are Cartesian, then $J = \det x^i$. Since J is the measure of the ratio of material volumes in B and B_R, the relation

$$\varrho = \frac{1}{J}\varrho_R \text{ holds true}. \tag{1.9}$$

Let us introduce the tensor $B_i{}^r{}_k{}^s$ defined by one of two equivalent formulae:

$$B_i{}^r{}_k{}^s = J^{-1} A_i{}^\alpha{}_k{}^\beta x^r{}_\alpha x^s{}_\beta, \tag{1.10}$$

$$A_i{}^\alpha{}_k{}^\beta = J B_i{}^r{}_k{}^s X^\alpha{}_r X^\beta{}_s. \tag{1.11}$$

The relation (1.11) is now substituted into the linearized equation of motion (1.6). Taking into account the Eq. (1.9) and the identity

$$(JX^\alpha{}_r)|| = 0, \tag{1.12}$$

we obtain a different form of the linearized equation of motion, namely the equation:

$$(B_i{}^r{}_k{}^s u^k||_s)||_r = \varrho \ddot{u}_i. \tag{1.13}$$

Without any loss of generality, it will be assumed in what follows that the displacement u^i and tensor $B_i{}^r{}_k{}^s$ are functions of the only variables x^i and t (variables X^α are eliminated by means of the Eq. (1.1)). Thus the total covariant differentiation in the Eq. (1.13) is reduced to the usual covariant differentiation. The Eqs. (1.13) are then reduced to:

$$\mathscr{L}_{ir} u^r = B_i{}^r{}_k{}^s u^k|_{rs} + B_i{}^r{}_k{}^s|_r u^k|_s - \varrho \ddot{u}_i = 0. \tag{1.14}$$

These equations will be subject to further analysis. They describe the dynamics of small deviations from the fundamental motion (1.1). A particular case of the Eqs. (1.14) is represented by the Lamé equations which correspond to the case in which the fundamental motion does not exist. On comparing the Eqs. (1.14) with the Lame equations, it is found that in the classical elasticity theory the functions $B_i{}^r{}_k{}^s$ are equal to

$$\begin{gathered} B_1{}^1{}_1{}^1 = \lambda+2\mu, \quad B_1{}^2{}_1{}^2 = B_1{}^3{}_1{}^3 = \mu, \quad B_{12}^{(12)} = B_{13}^{(13)} = \frac{1}{2}(\lambda+\mu), \\ B_1{}^1{}_2{}^2 = B_1{}^1{}_3{}^3 = \lambda, \quad B_1{}^2{}_2{}^1 = B_1{}^3{}_3{}^1 = \mu. \end{gathered} \tag{1.15}$$

The functions $B_2{}^r{}_k{}^s$ and $B_3{}^r{}_k{}^s$ result from cyclic interchange of indices.

2. Surface of discontinuity

Let $\mathscr{S}$ be a time-dependent surface described by one of the relations

$$t = \psi(x^r), \tag{2.1}$$

$$x^i = \pi^i(M^K, t), \quad K = 1, 2. \tag{2.2}$$

where M^1, M^2 parametrize the surface $\mathscr{S}$. The relations (2.1) and (2.2) are not independent, since the Eq. (2.1) may be obtained from the Eq. (2.2) by elimination of the parameters M^1, M^2. The unit vector orthogonal to $\mathscr{S}$ is denoted by n_i:

$$n_i = \frac{\psi_{,i}}{\sqrt{\psi_{,r}\psi_{,s}g^{rs}}}. \tag{2.3}$$

Here, and throughout the paper, a comma denotes the partial differentiation. The vector $\pi^i{}_{,K} = \partial\pi^i/\partial M^K$ is tangent to $\mathscr{S}$, and hence its scalar product with the vector n_i vanishes,

$$n_i \pi^i{}_{,K} = 0. \tag{2.4}$$

Substituting the Eqs. (2.2) into (2.1), and differentiating in time t, we obtain the relation:

$$\psi_{,r}\pi^r{}_{,t} = 1. \tag{2.5}$$

Using in turn the Eq. (2.3), we have

$$U \overset{\text{df}}{=} \frac{1}{\sqrt{\psi_{,r}\psi_{,}{}^r}} = n_r \pi^r{}_{,t}, \quad U = \frac{1}{\psi_{,r} n^r}. \tag{2.6}$$

U is now the velocity of surface $\mathscr{S}$ in the direction of a vector normal to $\mathscr{S}$. That velocity will be termed the velocity of propagation of the surface $\mathscr{S}$.

Let H be an arbitrary function of variables x^i and t, $H = H(x^i, t)$. On each side of the surface $\mathscr{S}$, the magnitude H may be represented as a function of M^K and t,

$$\begin{aligned} H &= H_F(M^K, t) \quad \text{on } \mathscr{S}_F, \\ H &= H_B(M^K, t) \quad \text{on } \mathscr{S}_B. \end{aligned} \tag{2.7}$$

The function H and its derivatives $H_{,r}$, $H_{,t}$ are, in general, discontinuous on $\mathscr{S}$. Obviously, we may write the relations

$$\begin{aligned} \frac{dH_F}{dM^K} &= (H_{,i})_F \pi^i{}_{,K}, \\ \frac{dH_F}{dt} &= (H_{,t})_F + (H_{,i})_F \pi^i{}_{,t}. \end{aligned} \tag{2.8}$$

The magnitude dH_F/dt represents the time rate of change at the point of $\mathscr{S}$ with coordinates $M^K = \text{const}$. Similar relations hold true on the side $\mathscr{S}_B$. Denoting the jump by double brackets

$$[\![\cdot]\!] = (\cdot)_F - (\cdot)_B,$$

we have then:

$$[\![H]\!]_{,K} = [\![H_{,i}]\!] \pi^i{}_{,K}, \tag{2.9}$$

$$[\![H]\!]_{,t} = [\![H_{,t}]\!] + [\![H_{,i}]\!] \pi^i{}_{,t}. \tag{2.10}$$

Let us now consider the particular case in which H is continuous over $\mathscr{S}$, and only the derivatives of H suffer certain discontinuities. Inserting $[\![H]\!] = 0$ into the Eq. (2.9) and making use of the Eq. (2.4) yields:

$$[\![H_{,i}]\!] = A n_i, \tag{2.11}$$

A being an indeterminate parametr. Substituting now the Eq. (2.11) into (2.10) and taking into account the Eq. (2.6), we obtain

$$[\![H_{,t}]\!] = -AU. \tag{2.12}$$

The acceleration wave, or the wave of weak discontinuity, is the name attributed to all the phenomena occuring at such a discontinuity surface that u^i, $u^i{}_{,k}$ and $u^i{}_{,t}$ remain continuous. The surface $\mathscr{S}$ itself is called the wave front; it separates the disturbed region from the undisturbed region. Assuming in the Eqs. (2.11), (2.12) consecutively $H = u^i{}_{,k}$ and $H = u^i{}_{,t}$, and taking into account the symmetry of derivatives $u^i{}_{,km} = u^i{}_{,mk}$, $u^i{}_{,kt} = u^i{}_{,tk}$, we obtain:

$$\begin{aligned} [\![u^i{}_{,rs}]\!] &= a^i n_r n_s, \\ [\![u^i{}_{,rt}]\!] &= -a^i U n_r, \\ [\![u^i{}_{,tt}]\!] &= a^i U^2. \end{aligned} \tag{2.13}$$

Here a^i is an indeterminate vector. It determines the magnitudes of jumps of the second derivatives of the displacement vector and is called the amplitude. The covariant derivatives and the material time derivative are obtained from the partial derivatives by adding the terms involving only the first derivatives of the vector u^i. For an acceleration wave, the first-order derivatives are — according to the definition — continuous on $\mathscr{S}$, and hence the conclusion follows that the Eqs. (2.13) also hold true for the covariant and material time derivatives. Finally, we obtain

$$\begin{aligned} [\![u^i|_{rs}]\!] &= a^i n_r n_s, \\ [\![\dot{u}_i|_r]\!] &= -a^i U n_r, \\ [\![\ddot{u}_i]\!] &= a^i U^2. \end{aligned} \tag{2.14}$$

3. Propagation condition and the equation of the acceleration wave amplitude

Let us now pass to the derivation of the condition of propagation of the acceleration wave. Since the magnitudes $B_i{}^r{}_k{}^s$, $B_i{}^r{}_k{}^s|_r$ and ϱ are independent of u^i, they must be continuous on $\mathscr{S}$; but $u^i|_s$ are also continuous, and in accordance with the Eq. (1.14), we have:

$$B_i{}^r{}_k{}^s[\![u^k|_{rs}]\!] = \varrho[\![\ddot{u}_i]\!]. \tag{3.1}$$

Substituting here the compatibility conditions (2.14), we obtain the condition of propagation of the acceleration wave:

$$(Q_{ik} - \varrho U^2 g_{ik})a^k = 0, \tag{3.2}$$

where

$$Q_{ik} = Q_{ki} = B_i{}^r{}_k{}^s n_r n_s, \tag{3.3}$$

is the acoustic tensor. By means of the Eqs. (2.3) and (2.6), that condition may also be written in another, equivalent form

$$(B_i{}^r{}_k{}^s \psi_r \psi_s - \varrho g_{ik})a^k = 0, \quad \psi_r = \psi_{,r}. \tag{3.4}$$

From the Eq. (3.2) it follows that a^k is the eigenvector, and the product ϱU^2 — the eigenvalue of the acoustic tensor Q_{ik}. This is a symmetric tensor, therefore there always exist three orthogonal admissible amplitudes $\overset{(1)k}{a}$, $\overset{(2)k}{a}$, $\overset{(3)k}{a}$, and three corresponding real products ϱU^2. If the products happen to be positive, then the real velocities $\overset{1}{U}$, $\overset{2}{U}$, $\overset{3}{U}$ exist, and the wave can be propagated. It is easily verified that for the tensor $B_i{}^r{}_k{}^s$ as given by the Eq. (1.15) the product ϱU^2 is positive, since $\lambda + 2\mu > 0$ and $\mu > 0$. If $a^k || n_k$, then the wave is longitudinal, and if $a^k \perp n_k$, the wave is transversal. A typical wave is neither longitudinal nor transversal.

According to the propagation condition (3.2), the tensor $Q_{ik} - \varrho U^2 g_{ik}$ is singular. By means of the Eqs. (2.3) and (2.6) we obtain the equation

$$\det(B_i{}^r{}_k{}^s \psi_r \psi_s - \varrho g_{ik}) = 0. \tag{3.5}$$

It is a non-linear equation for the function $\psi(x^i)$ determining the wave front motion.

The condition of propagation (3.2) determines the direction of the amplitude but not the amplitude itself. Let us now pass to constructing the equation governing the changes of amplitude. From now on, a^k will denote an arbitrary, fixed vector satisfying the condition (3.2). The real, actual amplitude which is collinear with a^k will be denoted by another symbol. Displacement $u^k(x^r, t)$ is represented in the following form (cf. e.g. [5]):

$$u^k(x^r, t) = \sum_{\nu=0}^{\infty} S_{\nu+2}(\varphi) g_\nu^k(x^r, t), \tag{3.6}$$

where

$$S_\nu = \frac{1}{\nu!}\left[\frac{1}{2}(|\varphi|+\varphi)\right]^\nu, \quad \nu = 1, 2, \ldots, \tag{3.7}$$

$$\varphi = \psi - t, \tag{3.8}$$

and $g_\nu^k(x^r, t)$ are functions of the class C^2. The following identities are easily derived:

$$\frac{dS_\nu}{d\varphi} = S_{\nu-1},$$

$$S_0 \overset{\text{df}}{=} \frac{dS_1}{d\varphi} = \eta(\varphi) = \begin{cases} 1 & \text{for} \quad \varphi > 0, \\ 0 & \text{for} \quad \varphi < 0, \end{cases} \tag{3.9}$$

$$S_0 S_0 = S_0, \quad S_0 S_\nu = S_\nu, \; S_\mu S_\nu = \frac{(\mu+\nu)!}{\mu!\,\nu!} S_{\mu+\nu}.$$

S_0 is hence the Heaviside function, and all functions S_ν, $\nu \geqslant 1$ are continuous. The summation in the Eq. (3.6) starts at S_2 to ensure the continuity of displacement u^k and of the derivatives $u^k{}_{,r}$, $u^k{}_{,t}$. Let us confine our considerations to the case of stationary, fixed initial deformation. Differentiation of the expression (3.6) and the relations (3.9) yield

$$u^k|_r = S_1 \varphi_r g_0^k + \sum_{\nu=0}^{\infty} S_{\nu+2}(g_\nu{}^k|_r + \varphi_r g_{\nu+1}^k),$$

$$\begin{aligned} u^k|_{rs} = {} & S_0 \varphi_r \varphi_s g_0^k + S_1(\varphi_r|_s g_0^k + \varphi_r g_0^k|_s + \varphi_s g_0^k|_r + \varphi_r \varphi_s g_0^k) \\ & + \sum_{\nu=0}^{\infty} S_{\nu+2}(g_\nu^k|_{rs} + \varphi_r g_{\nu+1}^k|_s + \varphi_s g_{\nu+1}^k|_j + \varphi_r|_s g_{\nu+1}^k + \varphi_r \varphi_s g_{\nu+2}^k), \end{aligned} \tag{3.10}$$

$$\ddot{u}^k = S_0 g_0^k + S_1(-2\dot{g}_0^k + g_1^k) + \sum_{\nu=0}^{\infty} S_{\nu+2}(\ddot{g}_\nu^k - 2\dot{g}_{\nu+1}^k + \ddot{g}_{\nu+2}^k).$$

Function g_0^k denotes the magnitude of the jump of second derivatives of the displacement vector u^k.

Let us substitute the above expression into the Eqs. (1.14), and group the terms involving S_ν. We obtain the equation:

$$\mathscr{L}_{ir} u^r = S_0 B_0 + S_1 B_1 + \sum_{\nu=0}^{\infty} S_{\nu+2} B_{\nu+2} = 0, \tag{3.11}$$

in which

$$B_0 = (B_i{}^r{}_k{}^s \varphi_r \varphi_s - \varrho g_{ik}) g_0^k = 0, \tag{3.12}$$

$$\begin{aligned} B_1 = {} & (B_i{}^r{}_k{}^s \varphi_r \varphi_s - \varrho g_{ik}) g_1^k + [B_i{}^r{}_k{}^s(\varphi_r g_0^k|_s + \varphi_s g_0^k|_r) \\ & + 2\varrho \dot{g}_0^k + (B_i{}^r{}_k{}^s \varphi_r|_s + B_i{}^r{}_k{}^s|_r \varphi_s) g_0^k] = 0, \end{aligned} \tag{3.13}$$

$$B_{\nu+2} = (B_i{}^r{}_k{}^s \varphi_r \varphi_s - \varrho g_{ik}) g^k_{\nu+2} + B_i{}^r{}_k{}^s (\varphi_r g^k_{\nu+1}|_s + \varphi_s g^k_{\nu+1}|_r) + 2\varrho \dot{g}^k_{\nu+1} + (B_i{}^r{}_k{}^s \varphi_r|_s + B_i{}^r{}_k{}^s|_r \varphi_s) g^k_{\nu+1} + \mathscr{K}_{ir} g^r_\nu = 0. \tag{3.14}$$

The function $S_0, S_1, S_2, \ldots$ are linearly independent and thus each of their coefficients B has to vanish. Consequently, the signs of equality and zero were added at the right-hand sides of the relations (3.12)–(3.14). In the Eq. (3.12), the expression in parenthesis is identical with that in the propagation condition (3.3). It follows (under the assumption that the Eq. (3.4) has no double roots) that the equation

$$g^k_0 = \varkappa_0 a^k \tag{3.15}$$

holds, where $\varkappa_0$ is a scalar multiplier. It should be stressed that a^k is assumed to be an arbitrary, fixed solution of the Eq. (3.2).

Let us now multiply the Eq. (3.13) by a^i. Pursuant to the Eq. (3.3), the first term equals zero and, after substitution of the Eq. (3.15), the equation is reduced to the form:

$$a^i a^k [B_i{}^r{}_k{}^s (\varphi_r \varkappa_{0,s} + \varphi_s \varkappa_{0,r}) + 2\varrho \dot{\varkappa}_0 g_{ik}] + \varkappa_0 a^i [B_i{}^r{}_k{}^s (\varphi_r a^k|_s + \varphi_s a^k|_r) + 2\varrho g_{ik} \dot{a}^k + (B_i{}^r{}_k{}^s \varphi_r|_s + B_i{}^r{}_k{}^s|_r \varphi_s) a^k] = 0. \tag{3.16}$$

This is a partial differential equation for the function $\varkappa_0$. Let $x^i = x^i(\lambda)$, $t = t(\lambda)$ denote a curve in the four-dimensional space $\{x^i\} \times t$ determined by the differential relations

$$\frac{dx^s}{d\lambda} = a^i a^k (B_i{}^r{}_k{}^s + B_i{}^r{}_k{}^s) \varphi_r, \qquad \frac{dt}{d\lambda} = 2\varrho a^i a^k g_{ik}. \tag{3.17}$$

Let us make the assumption that the parameter λ is so selected that at the instant $t = 0$ also $\lambda = 0$. According to the Eq. (3.17), we have

$$\frac{d\varkappa_0}{d\lambda} = \frac{\partial \varkappa_0}{\partial x^s} \frac{dx^s}{d\lambda} + \frac{\partial \varkappa_0}{\partial t} \frac{dt}{d\lambda} = a^i a^k B_i{}^r{}_k{}^s (\varphi_r \varkappa_{0,s} + \varphi_s \varkappa_{0,r}) + 2\varrho a^i a^k g_{ik} \dot{\varkappa}_0. \tag{3.18}$$

The first term in the Eq. (3.16) is then equal to $d\varkappa_0/d\lambda$. On the curve $\{\lambda\}$, the coefficient at $\varkappa_0$ is in this equation a function of λ only. This function is denoted by $P(\lambda)$. The Eq. (3.16) is now reduced to the ordinary differential equation:

$$\frac{d\varkappa_0}{d\lambda} + \varkappa_0 P(\lambda) = 0, \tag{3.19}$$

with the solution

$$\varkappa_0 = C_0 \exp\left(-\int_0^\lambda P(\lambda) d\lambda\right). \tag{3.20}$$

Here, C_0 denotes a constant of integration.

Let the curve $\{r\}$ be a projection of the curve $\{\lambda\}$ upon the three-dimensional space. The curve $\{r\}$ is determined by the relations $(3.17)_1$. From the Eq. (3.20), it follows that if at one point of the curve $\{r\}$ $\varkappa_0 = 0$ (or $\varkappa_0 \neq 0$), then at any other point of that curve

$\varkappa_0 = 0$ (or $\varkappa_0 \neq 0$). Therefore, the curve $\{r\}$ is the acoustic radius, [1]. The Eq. $(3.17)_1$ is closely connected with the acoustic tensor Q_{ik}, since from the Eq. (3.2), we obtain

$$\frac{dx^s}{d\lambda} = \frac{1}{U} a^i a^k \frac{\partial Q_{ik}}{\partial n_s}. \tag{3.21}$$

Let us now return to the Eq. (3.13). The expression in brackets is already known, so we are able to determine g_1^k. The expression in parenthesis being a singular tensor, the vector g_i^k may be represented in the form

$$g_1^k = \varkappa_1 a^k + k_1^k, \tag{3.22}$$

where

$$a^i k_1^r g_{ir} = 0. \tag{3.23}$$

In compliance with the Eq. (3.3), only the vector ${k_1}^k$ enters the Eq. (3.13). This equation does not lead to contradiction and enables ${k_1}^k$ to be determined. To determine the parameter $\varkappa_1$, let us consider the Eq. (3.14) with $\nu = 0$. Multiplying it by a^i, g_2^k is eliminated. In the resulting equation, the expressions (3.22) are substituted to yield the differential equation for the parameter $\varkappa_1$

$$\begin{aligned} a^i a^k [B_i{}^r{}_k{}^s(\varphi_r \varkappa_{1,s} + \varphi_s \varkappa_{1,r}) + 2\varrho \dot{\varkappa}_1 g_{ik}] + \varkappa_1 a^i [B_i{}^r{}_k{}^s(\varphi_r a^k|_s + \varphi_s a^k|_r) + 2\varrho g_{ik} \dot{a}^k \\ + (B_i{}^r{}_k{}^s|_r \varphi_s|_r + B_i{}^r{}_k{}^s|_r \varphi_s) a^k] = -a^i [B_i{}^r{}_k{}^s(\varphi_r k_1^k|_s + \varphi_s k_1^k|_r) + 2\varrho \dot{k}_1^k \\ + (B_i{}^r{}_k{}^s \varphi_s|_s + B_i{}^r{}_k{}^s|_r \varphi_s) k_1^k + \mathscr{L}_{ir} g_0^r]. \end{aligned} \tag{3.24}$$

The left-hand side is exactly the same as in the Eq. (3.16), provided that $\varkappa_0$ is replaced by $\varkappa_1$. Therefore, the entire expression may be replaced by $d\varkappa_1/d\lambda + \varkappa_1 P(\lambda)$. On the curve $\{\lambda\}$, the right-hand side of the Eq. (3.24) is a function of λ. Denoting this function by $K_1(\lambda)$, we obtain:

$$\frac{d\varkappa_1}{d\lambda} + \varkappa_1 P(\lambda) = K_1(\lambda). \tag{3.25}$$

It follows that the solution of the Eq. (3.24) is:

$$\varkappa_1 = C_1 \exp\Big(-\int_0^\lambda P(\lambda) d\lambda\Big) + D_1(\lambda). \tag{3.26}$$

Here, D_1 is the particular integral of the Eq. (3.25). Proceeding in a similar manner with the Eq. (3.14), for $\nu = 1, 2, 3, \ldots$ we obtain for each $\nu > 1$

$$g_\nu^k = \varkappa_\nu a^k + k_\nu^k, \tag{3.27}$$

$$\varkappa_\nu = C_\nu \exp\Big(-\int_0^\lambda P(\lambda) d\lambda\Big) + D_\nu(\lambda). \tag{3.28}$$

The functions $k_\nu(\lambda)$ and $D_\nu(\lambda)$ are known if the parameters $\varkappa_\mu$ for $\mu < \nu$ are known.

The unit vector in the direction of the acoustic radius $\{r\}$ is denoted by r^k. It is collinear with the vector $dx^k/d\lambda$ given by the Eq. (3.21). The velocity at which the discontinuity surface $\mathscr{S}$ propagates along the radius $\{r\}$ is the radial velocity. The relation

$$U_r r^k n_k = U \tag{3.29}$$

holds. Using the conditions (2.6) and (3.29), we obtain

$$U_r = \frac{1}{r^k \psi_k}. \qquad U_r \geqslant U. \tag{3.30}$$

4. Progressive wave

The solution derived enables us to construct a different solution which has no discontinuity at the surface $\varphi = 0$. Let us in the relation (3.6) replace the functions $S_\nu(\varphi)$, defined by (3.7), by arbitrary functions $T_\nu(\varphi)$ satisfying the relation

$$\frac{dT_\nu}{d\varphi} = T_{\nu-1}, \quad \nu = 0, 1, 2, \ldots, \tag{4.1}$$

and let us construct the series:

$$u^k(x^r, t) = \sum_{\nu=0}^{\infty} T_{\nu+2}(\varphi) g_\nu^k(x^r, t). \tag{4.2}$$

The series, if it is convergent, represents the solution of the Eq. (1.14). For the displacement u^k in the form (4.2), the Eq. (1.14) assumes the form (3.11) with functions S_ν replaced by T_ν. All the coefficients B_ν are zero and hence $\mathscr{L}_{ir} u^r = 0$.

In particular, we may assume

$$T_{\nu+2} = \frac{1}{(i\omega)^\nu} e^{i\omega\varphi}, \quad \omega = \text{const}, \quad i = \sqrt{-1}, \tag{4.3}$$

and then

$$u^k(x^r, t) = e^{i\omega\varphi}\left(g_0^k + \frac{1}{i\omega} g_1^k + \frac{1}{(i\omega)^2} g_2^k + \ldots\right). \tag{4.4}$$

The solution (4.4) is called the progressive wave.

Since our considerations are confined to the case in which the function ξ^i in (1.1) does not depend on the time, then the functions $B_i{}^r{}_k{}^s$ depend solely on x^m, in accordance with the Eqs. (1.7) and (1.10). Consequently, from the considerations presented in Sec. 3 it follows that the functions $\varkappa_\nu$, g_ν^k are time-independent, $g_\nu^k = g_\nu^k(x^m)$. By using the definition (3.8), the solution (4.4) is reduced to

$$u^k(x^r, t) = e^{-i\omega t} e^{i\omega\psi} \sum_{\nu=0}^{\infty} \frac{1}{(i\omega)^\nu} g_\nu^k(x^r), \tag{4.5}$$

and represents a product of a function of time and a function of place. The solution (4.5) is closely connected with the surface of discontinuity. It should be stressed that separation of the variables in the Eq. (1.14) does not directly lead to the solution (4.5).

In order to write the Eq. (4.4) in a real form, let us first observe that, in the situation described, the solution may also be represented by:

$$u^k(x^r, t) = e^{(-i\omega)\varphi}\left(g_0^k + \frac{1}{(-i\omega)} g_1^k + \frac{1}{(-i\omega)^2} g_2^k + \ \ldots\right) \tag{4.6}$$

Summing both sides of the Eqs. (4.4) and (4.6), we obtain the real solution

$$(4.7)\qquad u^k = \left(g_0^k - \frac{1}{\omega^2} g_2^k + \frac{1}{\omega^4} g_4^k - \frac{1}{\omega^6} g_6^k + \dots\right)\cos\omega\varphi + \left(\frac{1}{\omega} g_1^k - \frac{1}{\omega^3} g_3^k + \frac{1}{\omega^5} g_5^k - \frac{1}{\omega^7} g_7^k + \dots\right)\sin\omega\varphi.$$

Pursuant to the Eq. (3.27), the displacement u may be written in the following form:

$$(4.8)\qquad u^k = a^k\left[\left(\varkappa_3 - \frac{1}{\omega^2}\varkappa_2 + \frac{1}{\omega^4}\varkappa_4 - \frac{1}{\omega^6}\varkappa_6 + \dots\right)\cos\omega\varphi + \left(\frac{1}{\omega}\varkappa_1 - \frac{1}{\omega^3}\varkappa_3 + \frac{1}{\omega^5}\varkappa_5 - \frac{1}{\omega^7}\varkappa_7 + \dots\right)\sin\omega\varphi\right] + R^k, \qquad R^k \perp a^k.$$

Denoting

$$(4.9)\qquad \begin{aligned} M &= \varkappa_0 - \frac{1}{\omega^2}\varkappa_2 + \frac{1}{\omega^4}\varkappa_4 - \frac{1}{\omega^6}\varkappa_6 + \dots \\ N &= \frac{1}{\omega}\varkappa_1 - \frac{1}{\omega^3}\varkappa_3 + \frac{1}{\omega^5}\varkappa_5 - \frac{1}{\omega^7}\varkappa_7 + \dots \\ \alpha &= \operatorname{arc\,tg}\frac{N}{M}, \end{aligned}$$

we obtain

$$(4.10)\qquad u^k = a^k\sqrt{M^2+N^2}\cos(\omega\varphi - \alpha) + R^k, \qquad R^k \perp a^k.$$

The expression $\omega\varphi - \alpha = -\omega t + \omega\psi - \alpha$ is called the phase. The point of space at which the phase is constant form, a certain surface $\mathscr{S}_f$ which is moving in time. The surface $\mathscr{S}_f$ moves, in general, in a different manner than the discontinuity surface $\mathscr{S}$. Various velocities may be attributed to the surface $\mathscr{S}_f$, such as velocity in the direction of its normal (velocity of propagation of $\mathscr{S}_f$), velocity in the direction of the normal n, and the velocity in the direction of the acoustic radius $\{r\}$; the last named is called the phase velocity. By means of the Eq. (4.10), the equation of the constant phase surface is

$$(4.11)\qquad -\omega t + \omega\psi - \alpha = \text{const}.$$

When written in a differential form

$$(4.12)\qquad -\omega\, dt + \left(\omega\psi_k - \frac{\partial\alpha}{\partial x^k}\right)U_f r^k dt = 0,$$

the expression for the phase velocity may be written as:

$$(4.13)\qquad U_f = \frac{\omega}{\omega\psi_k r^k - \dfrac{\partial\alpha}{\partial x^k} r^k}.$$

From the Eq. (3.30) it follows that the product $\psi_k n^k$ is equal to $1/U$. Thus we finally obtain:

$$(4.14)\qquad U_f = U_r \frac{1}{1 - U_r r^k \dfrac{1}{\omega}\dfrac{\partial\alpha}{\partial x^k}}.$$

The vector R^k is a function of the variables x^k and t and has the form of a trigonometric function $\cos(\omega\varphi+k^k)$. The R^k vector cannot be taken into account in the evaluation of the phase velocity, since for $k = 1, 2, 3$ three different phase velocities are obtained, different also from U_f. Let us, however, observe that for large ω the vector R^k is small in comparison with the first term of (4.10), since each of the components of R^k is divided by ω^n, $n \geqslant 1$ [it is known from the Eq. (3.15) that $k_0^k = 0$). The first term of (4.10) represents then the principal part of the displacement.

Usually, we have to deal not with a single wave (4.10) but with a system of waves with frequencies from within a certain interval $\omega_1 < \omega < \omega_2$ and with amplitudes forming a continuous function of the frequency ω. The displacement has then the form

$$u^k = a^k(x^r)\int_{\omega_1}^{\omega_2} P(\omega)\cos(\omega\varphi-\alpha)\,d\omega+\hat{R}^k, \tag{4.15}$$

where a^k, in compliance with the analysis of the preceding section, is independent of ω.

The case in which ω_2 is close to ω_1 is of special interest. The Eq. (4.15) may then be considered as a superposition of two waves with identical amplitudes, and with frequencies $\omega+\Delta\omega$ and $\omega-\Delta\omega$, $\Delta\omega \ll \omega$.

$$u^k = a^k\sqrt{M^2+N^2}\left\{\cos\left[(\omega+\Delta\omega)\varphi-\left(\alpha+\frac{\partial\alpha}{\partial\omega}\Delta\omega\right)\right]\right.$$
$$\left.+\cos\left[(\omega-\Delta\omega)\varphi-\left(\alpha-\frac{\partial\alpha}{\partial\omega}\Delta\omega\right]\right\}+\hat{\hat{R}}^k, \qquad \alpha_w = \frac{\partial\alpha}{\partial\omega}, \tag{4.16}$$

whence it follows

$$u^k = \left[2a^k\sqrt{M^2+N^2}\cos\left(\varphi-\frac{\partial\alpha}{\partial\omega}\right)\Delta\omega\right]\cos(\omega\varphi-\alpha)+\hat{\hat{R}}^k. \tag{4.17}$$

The motion represents a wave $\cos(\omega\varphi-\alpha)$ with an amplitude (expression in brackets) varying in time and space as $\cos(\varphi-\alpha_\omega)\Delta\omega$; thus we are dealing with groups of waves which move as the surfaces described by the equation

$$\left(\varphi-\frac{\partial\alpha}{\partial\omega}\right)\Delta\omega = \left(-t+\psi-\frac{\partial\alpha}{\partial\omega}\right)\Delta\omega = \text{const}. \tag{4.18}$$

The propagation velocity of these surfaces measured in the direction of r^k is the group velocity U_g. According to (4.18), we have

$$-dt+\left(\psi_k-\frac{\partial^2\alpha}{\partial\omega\partial x^k}\right)U_g r^k dt = 0, \tag{4.19}$$

and

$$U_g = U_r\frac{1}{1+U_r r^k\dfrac{\partial^2\alpha}{\partial\omega\partial x^k}}. \tag{4.20}$$

Let us concentrate upon the first two approximations. For sufficiently large frequencies ω, all terms in the Eq. (4.7) may be disregarded except $g_0^k\cos\omega\varphi$; that leads to the first approximation. Then $N = \alpha = 0$ [cf. Eq. (4.9)] and from the Eqs. (4.14) and (4.20), we obtain:

$$U_f = U_g = U_r. \tag{4.21}$$

In deriving the second approximation, all terms of orders higher than $1/\omega^2$ are disregarded:

$$\begin{gathered} M = \varkappa_0 - \frac{1}{\omega^2}\varkappa_2, \quad N = \frac{1}{\omega}\varkappa_1, \\ \alpha = \operatorname{arc\,tg}\frac{1}{\omega}\frac{\varkappa_1}{\varkappa_0 - \varkappa_2/\omega^2}. \end{gathered} \tag{4.22}$$

Determining the derivatives $\partial\alpha/\partial x^k$, $\partial^2\alpha/\partial x^k\partial\omega$, expanding into series and disregarding the terms of orders higher than $1/\omega^3$, we obtain

$$\frac{1}{\omega}\frac{\partial\alpha}{\partial x^k} = \frac{1}{\omega^2}\left(\frac{\varkappa_0}{\varkappa_1}\right)_{,\varkappa}, \qquad \frac{\partial^2\alpha}{\partial x^k\partial\omega} = -\frac{1}{\omega^2}\left(\frac{\varkappa_0}{\varkappa_1}\right)_{,k}.$$

This result together, with the Eqs. (4.14) and (4.20), yields the phase and group velocities U_f and U_g

$$\begin{aligned} U_f &= U_r\left[1 - \frac{U_r}{\omega^2}\left(\frac{\varkappa_1}{\varkappa_0}\right)_{,k} r^k\right]^{-1} \approx U_r\left[1 + \frac{U_r}{\omega^2}\left(\frac{\varkappa_0}{\varkappa_1}\right)_{,k} r^k\right], \\ U_g &= U_r\left[1 + \frac{U_r}{\omega^2}\left(\frac{\varkappa_0}{\varkappa_1}\right)_{,k} r^k\right]^{-1} \approx U_r\left[1 - \frac{U_r}{\omega^2}\left(\frac{\varkappa_1}{\varkappa_0}\right)_{k,} r^k\right]. \end{aligned} \tag{4.23}$$

The velocities evidently satisfy the relation

$$U_f U_g = U_r^2. \tag{4.24}$$

In the order of approximation assumed, the radial velocity represents the geometric mean of the phase and group velocities. It should be stressed that, in general, $U_f > U_r$ and $U_g < U_r$. There exist, however, waves for which $U_f < U_r$ and $U_g > U_r$, cf. e.g. [6].

A forced displacement on a certain surface $\mathscr{S}_0$ which has the form of vibrations sinusoidal in time is called a signal

$$u^k(\mathscr{S}_0, t) = a^k(\mathscr{S}_0)\cos\omega_0 t, \quad \begin{cases} t < 0, \\ 0 < t < t_1, \\ t_1 < t. \end{cases} \tag{4.25}$$

Here, ω_0 and t_1 are fixed. At the point x^k lying not on $\mathscr{S}_0$, the signal is received in the form of vibrations of various frequencies ω from the interval $0 < \omega < \infty$. The vibrations start at a certaint instant $t_p(x^k)$, but are very weak at the beginning. The main portion of the signal arrives in x^k at the instant $t_s(x^k)$. The instant t_p is determined by the propagation velocity U, while the instant t_s — by the signal velocity U_s. In general, the signal velocity is equal neither to U_f nor to U_g. The signal velocity was, in the simplest case, analyzed by Sommerfeld in the BRILLOUIN monograph [6]. The corresponding Eq. (1.14) derived in this paper has to author's knowledge not so far been analyzed.

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