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BRIEF NOTES

A note on the constitutive law of plastic flow

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THE PHYSICAL characteristics of the constitutive law of plastic flow are considered and the most general form of the constitutive law having the same physical characteristics is derived. The constitutive law which is obtained is slightly different from the original constitutive law of plastic flow. Using this new constitutive law one can explain the results of the experiment of complex loading performed by M. FEIGEN [1].

1

LET us denote by ε the infinitesimal strain tensor of components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where by u_i $(i = 1, 2, 3)$ we denote the components of the displacement vector, by σ the Cauchy stress tensor of components σ_{ij} $(\sigma_{ij} = \sigma_{ji})$, $i, j = 1, 2, 3$, by θ the dilatation

$$\theta = \varepsilon_{ii},$$

by Θ the spherical part of stress tensor

$$\Theta = \sigma_{ii},$$

by ε' and σ' the deviators

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\theta\delta_{ij},$$

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\Theta\delta_{ij},$$

and by τ the second invariant of the stress deviator

$$\tau^2 = \sigma'_{ij}\sigma'_{ij}.$$

Consider the following constitutive law of plastic flow:

$$\theta = k\Theta,$$

$$(1.1)\qquad d\varepsilon'_{ij} = \frac{1}{2\mu}d\sigma'_{ij} + h(\tau)\sigma'_{ij}d\tau \quad \text{for} \quad \dot{\tau} \geqslant 0,$$

$$d\varepsilon'_{ij} = \frac{1}{2\mu}d\sigma'_{ij} \quad \text{for} \quad \dot{\tau} < 0,$$

where k and μ are positive constants and $h(\tau)$ a continuous function on τ. This constitutive law is used in the books of R. HILL [2] and W. OLSZAK, P. PERZYNA, A. SAWCZUK [3].

Let us denote by $d\mathcal{L}$ the variation of the work done by the stress deviator

$$d\mathcal{L} = \sigma'_{ij} d\varepsilon'_{ij}.$$

We note that the constitutive law (1.1) belongs to the following general class of constitutive laws:

$$\begin{aligned} \theta &= k\Theta, \\ d\varepsilon' &= G^+(\tau)d\sigma' \quad \text{for} \quad \dot{\mathcal{L}} \geqslant 0, \qquad (1.2)\\ d\varepsilon' &= G^-(\tau)d\sigma' \quad \text{for} \quad \dot{\mathcal{L}} < 0, \end{aligned}$$

where $G^{\pm}(\tau)$ are matrix whose components are functions on the second invariant of the stress deviator τ.

We evidence the following physical properties of the material described by the constitutive law (1.1):

1) is isotropic,

2) to a compression (or decompression) behaves like a linear elastic material,

3) if $\dot{\mathcal{L}} \leqslant 0$, to continuous stress rate corresponds continuous strain rate.

2

A short computation shows that the general constitutive equations of the form (1.2) which satisfy the conditions 1)–2) are

$$\begin{aligned} \theta &= k\Theta, \\ d\varepsilon'_{ij} &= d[g_1^+(\tau)\sigma'_{ij}] + g_2^+(\tau)d\sigma'_{ij} \quad \text{for} \quad \dot{\mathcal{L}} \geqslant 0, \qquad (2.1)\\ d\varepsilon'_{ij} &= d[g_1^-(\tau)\sigma'_{ij}] + g_2^-(\tau)d\sigma'_{ij} \quad \text{for} \quad \dot{\mathcal{L}} < 0. \end{aligned}$$

R e m a r k. The equations $(2.1)_2$ and $(2.1)_3$ belong to the class of equations of hypo-elastic materials derived by C. TRUESDELL [4]. A. E. GREEN [5, 6] used the equations of hypo-elasticity for describing the loading $\dot{\mathcal{L}} \geqslant 0$) and the unloading ($\dot{\mathcal{L}} < 0$), and derived the general isotropic form of these equations (for large deformations). Naturally, the Eqs. (2.1) are particular case of those derived by A. E. GREEN [6] (we use the Green's notations):

$$s^m_m = 3h_0 d^m_m,$$

$$\begin{aligned} (2.2) \qquad \tilde{t}^i_j &= h_1^{\pm} f^i_j + 2t^i_k f^k_j + \frac{1}{2} h_4^{\pm}(f^i_k t^k_j + t^i_k f^k_j) + (h_3^{\pm} M + h_7^{\pm} N)\delta^i_j \\ &+ (h_6^{\pm} M + h_{10}^{\pm} N) t^i_j + (h_9^{\pm} M + h_{11}^{\pm} N) t^i_k t^k_j + \frac{1}{2} h_8^{\pm}(f^i_m t^m_n t^n_j + t^i_m t^m_n f^n_j), \end{aligned}$$

where f_{ij} and t_{ij} are the components of the deviators

$$f^i_j = d^i_j - \frac{1}{3} d^r_r \delta^i_j, \qquad t^i_j = s^i_j - \frac{1}{3}\delta^i_j,$$

$d_{ij} = \dfrac{1}{2}(v_{i,j} + v_{j,i})$ and $2\mu s_{ij}$ being the components of the rate deformations and stress tensor, respectively (μ is one of the Lamé constants),

$$\tilde{s}^i_j = \frac{\partial s^i_j}{\partial t} + v^m t^i_{j,m} + v^m{}_{,j} t^i_m - v^i{}_{,m} t^m_j,$$

$h^\pm_1, h^\pm_2, \ldots, h^\pm_{11}$ are dimensionless analytic functions on

$$J = \frac{1}{2} t^i_j t^j_i, \qquad K = \frac{1}{3} t^i_j t^k_j t^k_i,$$
$$M = t^i_j f^j_i, \qquad N = t^i_j t^j_k f^k_i,$$

the sign + corresponds to $M \geqslant 0$ and $-$ to $M < 0$. Let us take $h^\pm_3 = h^\pm_7 = h^\pm_8 = \ldots = h^\pm_{11} = 0$, and $h_0 = \dfrac{1}{3k} =$ const. Then from the condition $t^i_i = 0$ we obtain $h^\pm_4 = -2$.

In rectangular Cartesian coordinates we have, for small deformations:

$$2\mu s_{ij} = \sigma_{ij}, \quad t_{ij} = \sigma'_{ij}, \quad s^m_m = \Theta,$$
$$\tilde{t}^i_j + f^i_k t^k_j - t^i_k f^k_j = \dot{\sigma}_{ij}, \quad \tilde{s}^m_m = \dot{\Theta},$$
$$d_{ij} = \dot{\varepsilon}_{ij}, \quad d^m_m = \dot{\theta}, \quad M = \dot{\mathscr{L}},$$

and the equations (2.2) become

$$\Theta = k^{-1}\theta,$$
$$\dot{\sigma}_{ij} = 2\mu(h^+_1 \dot{\varepsilon}_{ij} + h^+_6 \sigma_{ij}\dot{\mathscr{L}}) \quad \text{for} \quad \dot{\mathscr{L}} \geqslant 0,$$
$$\dot{\sigma}_{ij} = 2\mu(h^-_1 \dot{\varepsilon}_{ij} + h^-_6 \sigma_{ij}\dot{\mathscr{L}}) \quad \text{for} \quad \dot{\mathscr{L}} < 0.$$

Now, if we take

$$2\mu h^\pm_1 = q^\pm_1(\tau), \quad 2\mu h^\pm_6 = q^\pm_2(\tau),$$

where

$$q^\pm_1(\tau) = \frac{1}{g^\pm_1(\tau) + g^\pm_2(\tau)},$$

$$q^\pm_2(\tau) = \frac{{}^\pm_1(\tau)\dfrac{dg^\pm_1}{d\tau}(\tau)}{\tau\left[g^\pm_1(\tau) + \tau\dfrac{dg^\pm_1}{d\tau}(\tau) + g^\pm_2(\tau)\right]},$$

we obtain (2.1).

The relation $(2.1)_2$ and $(2.1)_3$ could be written in the following equivalent form:

(2.3)
$$\dot{\varepsilon}'_{ij} = g^+_1(\tau)\dot{\sigma}'_{ij} + \frac{dg^+_1}{d\tau}(\tau)\sigma'_{ij}\dot{\tau} + g^+_2(\tau)\dot{\sigma}'_{ij} \quad \text{for} \quad \dot{\mathscr{L}} \geqslant 0,$$
$$\dot{\varepsilon}'_{ij} = g^-_1(\tau)\dot{\sigma}'_{ij} + \frac{dg^-_1}{d\tau}(\tau)\sigma'_{ij}\dot{\tau} + g^-_2(\tau)\dot{\sigma}'_{ij} \quad \text{for} \quad \dot{\mathscr{L}} < 0.$$

From (2.2) and (2.3) it follows that $\dot{\tau} = 0$ implies $\dot{\mathscr{L}} = 0$. The condition 3) written at any moment for which $\tau = 0$ leads us to the relation

$$g^+_1(\tau) + g^+_2(\tau) = g^-_1(\tau) + g^-_2(\tau),$$

therefore g_1^-, g_2^-, g_1^+ and g_2^+ could be expressed in the form

$$g_1^-(\tau) = g_1(\tau), \qquad g_2^-(\tau) = g_2(\tau),$$
$$g_1^+(\tau) = g_1(\tau)+f(\tau), \quad g_2^+(\tau) = g_2(\tau)-f(\tau).$$

Let us denote $h(\tau) = \frac{df}{d\tau}(\tau)$. Then $(2.3)_1$ and $(2.3)_2$ become

$$d\varepsilon'_{ij} = d[g_1(\tau)\sigma'_{ij}]+g_2(\tau)d\sigma'_{ij}+h(\tau)\sigma'_{ij}d\tau \quad \text{if} \quad \dot{\mathscr{L}} \geqslant 0, \tag{2.4}$$

$$d\varepsilon'_{ij} = d[g_1(\tau)\sigma'_{ij}]+g_2(\tau)d\sigma'_{ij}, \quad \text{if} \quad \dot{\mathscr{L}} < 0. \tag{2.5}$$

The relations (2.4) and (2.5) depend on three material functions $g_1(\tau)$, $g_2(\tau)$ and $h(\tau)$. For the determination of these functions we can imagine the following experiments:

i) loading according to the law

$$\sigma_{11} = \sigma = \sigma_0 t, \quad \sigma_0 > 0, \quad 0 \leqslant t \leqslant t_0,$$
$$\sigma_{ij} = 0 \quad \text{for} \quad (i,j) \neq (1, 1),$$

ii) unloading according to the law

$$\sigma_{11} = \sigma = \sigma_0 t_0 - \sigma_0(t-t_0), \quad t \geqslant t_0,$$
$$\sigma_{ij} = 0 \quad \text{for} \quad (i,j) \neq (1, 1),$$

iii) complex loading

$$\sigma_{11} = \sigma, \quad \sigma_{12} \neq 0,$$
$$\sigma_{ij} = 0 \quad \text{for} \quad (i,j) \notin (1, 1) \cup (1, 2).$$

The experimental realization of such a kind of experiments is described by M. FEIGEN (op.cit.). Let us denote by dL the variation of total work

$$dL = \sigma_{ij} d\varepsilon_{ij}. \tag{2.6}$$

Obviously, we have

$$dL = d\mathscr{L} + \frac{1}{9}k\Theta d\theta. \tag{2.7}$$

Denote by $\varepsilon'_{11} = \frac{2}{3}e_1(\sigma)$ the value of the component ε'_{11} in the first experiment. Assume that in this experiment the rate of the total work is positive

$$\dot{L} > 0.$$

According to (2.7), we have

$$\dot{\mathscr{L}} = \frac{8}{9}\dot{L} > 0.$$

From (2.4) it follows

i) $$g_1\left(\sqrt{\frac{2}{3}}\sigma\right)+\sqrt{\frac{2}{3}}\sigma\frac{dg_1}{d\tau}\left(\sqrt{\frac{2}{3}}\sigma\right)+g_2\left(\sqrt{\frac{2}{3}}\tau\right)+\sqrt{\frac{2}{3}}h\left(\sqrt{\frac{2}{3}}\sigma\right)\sigma = \frac{de_1}{d\sigma}(\sigma).$$

Let us denote by $\varepsilon'_{11} = \frac{2}{3}e_2(\sigma)$ the value of the components ε'_{11} measured in the second experiment. Assume that in this experiment the rate of the total work is negative

$$L < 0.$$

Then we have also

$$\dot{\mathscr{L}} < 0$$

and from (2.5) we obtain

$$\text{ii)} \qquad g_1\left(\sqrt{\frac{2}{3}}\sigma\right)+\sqrt{\frac{2}{3}}\sigma\frac{dg_1}{d\tau}\left(\sqrt{\frac{2}{3}}\sigma\right)+g_2\left(\sqrt{\frac{2}{3}}\sigma\right) = \frac{de_2}{d\sigma}(\sigma).$$

Let $\frac{d\varepsilon_{12}}{d\sigma_{12}} = G(\sigma)$ be the derivative with respect to σ_{12} of the response ε_{12} measured in the last experiment. The derivative is calculated for $\sigma_{11} = \sigma$ and $\sigma_{12} = 0$. We must have

$$\text{iii)} \qquad g_1\left(\sqrt{\frac{2}{3}}\sigma\right)+g_2\left(\sqrt{\frac{2}{3}}\sigma\right) = G(\sigma).$$

3. Conclusions

From the precedent paragraphs it follows that the general constitutive law of the form (1.2) having the physical characteristic 1)–3) is:

$$\theta = k\Theta,$$

$$(3.1) \qquad \begin{aligned} d\varepsilon'_{ij} &= d[g_1(\tau)\sigma'_{ij}]+g_2(\tau)d\sigma'_{ij}+h(\tau)\sigma'_{ij}d\tau \qquad &&\text{for} \quad \dot{\mathscr{L}} \geqslant 0,\\ d\varepsilon'_{ij} &= d[g_1(\tau)\sigma'_{ij}]+g_2(\tau)d\sigma'_{ij} &&\text{for} \quad \dot{\mathscr{L}} < 0. \end{aligned}$$

We note that the classical law of plastic flow (1.1) is a degenerate form of (3.1). It is known that, using the constitutive law (1.1), in the describing of the complex loading experiment it appears a non-concordance between the experimental data and theoretical predictions, see M. Feigen (op. cit.). For the constitutive law (3.1) this non-concordance is excluded.

If in the second experiment we obtain

$$(3.2) \qquad e_2(\sigma) = \frac{1}{2\mu}\sigma,$$

then at one-dimensional unloading (i.e. when only a component of stress tensor is non-zero and $\dot{\mathscr{L}} < 0$), the material described by (3.1) behaves like a linear elastic material.

We remark that (3.1) and (1.1) have the same plastic part, i.e.,

$$(3.5) \qquad d\varepsilon^p_{ij} = h(\tau)\sigma'_{ij}d\tau;$$

the difference is occuring only in the non-plastic parts.

From $(3.1)_2$ and $(3.1)_3$ it follows

$$\dot{\mathscr{L}} = \dot{\tau}\tau\left[g_1(\tau)+\tau\frac{dg_1}{d\tau}(\tau)+g_2(\tau)+\tau h(\tau)\right] \quad \text{if} \quad \dot{\mathscr{L}} \geqslant 0,$$

(3.6)

$$\dot{\mathscr{L}} = \dot{\tau}\tau\left[g_1(\tau)+\tau\frac{dg_1}{d\tau}(\tau)+g_2(\tau)\right] \quad \text{if} \quad \dot{\mathscr{L}} < 0.$$

If in the first and second experiment, the rate of total work is positive and negative, respectively, then the following relations must be satisfied:

$$g_1(\tau)+\tau\frac{dg_1}{d\tau}(\tau)+g_2(\tau)+\tau h(\tau) > 0,$$

$$g_1(\tau)+\tau\frac{dg_1}{d\tau}(\tau)+g_2(\tau) > 0.$$

From (3.6) it follows that the loading and unloading criteria ($\dot{\mathscr{L}} \geqslant 0, \dot{\mathscr{L}} < 0$) could be replaced by ($\dot{\tau} \geqslant 0, \dot{\tau} < 0$).

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