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On the calculation of buckling loads by means of hybrid Ritz equations

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USING the solution of the given problem and its adjoint problem, a functional H^* is defined which is a kind of generalized Hamiltonian. This functional is found to be stationary. Applying a direct method to the variational equation involving H^*, a system of hybrid Ritz equations is obtained which is used for the calculation of the buckling load.

Posługując się rozwiązaniem danego problemu i problemu z nim sprzężonego określono funkcjonał H^*, będący pewnym uogólnieniem hamiltonianu. Stosując do równania wariacyjnego, zawierającego H^* metodę bezpośrednią, otrzymano układ hybrydowych równań Ritza, które użyto następnie do obliczania obciążeń flatterowych.

Пользуясь решениями рассматриваемой и сопряженной с ней задач можно построить функционал H^*, являющийся некоторым обобщением гамильтониана. Из вариационного уравнения, содержащего H^*, получена система гибридных уравнений Ритца, которая затем используется для расчета флаттерных нагрузок.

1. Introduction

CONSIDER the rod shown in Fig. 1 subjected to nonconservative follower forces. The small vibrations of the rod about its equilibrium position are mathematically described by the boundary-eigenvalue problem

$$D[w] = \mu\ddot{w} + \alpha w_{xxxx} + [q(l-x)w_x]_x + qw_x = 0, \tag{1.1}$$

$$w(0, t) = w_x(0, t) = w_{xx}(l, t) = w_{xxx}(l, t) = 0. \tag{1.2}$$

In (1.1), (1.2), $w(x, t)$ is the lateral deflection of the rod, μ its linear mass density, α its flexural rigidity, l its length, and q is a uniformly distributed compressive load.

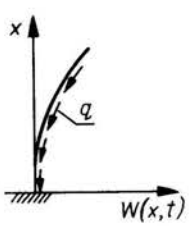

FIG. 1.

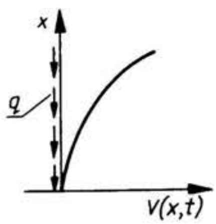

FIG. 2.

Consider also the rod shown in Fig. 2. In this case, the uniformly distributed load q remains in its original line of action regardless of the deformation of the rod. Everything else is the same as in the example. Whether a system like this can be realized physically shall not matter. It has above all been taken into consideration because the boundary-eigenvalue problem

$$(1.3)\qquad D^*[v] = \mu\ddot{v}+\alpha v_{xxxx}+[q(l-x)v_x]_x-qv_x = 0,$$

$$(1.4)\qquad v(0,t) = v_x(0,t) = v_{xx}(l,t) = v_{xxx}(l,t)-\alpha^{-1}qv(l,t) = 0,$$

describing the small vibrations of the system in Fig. 2, is the adjoint problem of (1.1), (1.2).

In the following it will be shown that the combination of these two systems, — the given one (1.1), (1.2), and its adjoint one (1.3), (1.4), — can be used to formulate a variational principle for the calculation of the buckling load q_{cr} of the first system (1.1), (1.2).

The adjointness of the two systems is characterized by the following fact. Let ψ be an admissible function satisfying the boundary conditions (1.2) and ϕ be another admissible function satisfying boundary conditions (1.4). Then

$$(1.5)\qquad \int_{t_1}^{t_2}\int_0^l D[\psi]\phi\,dx\,dt = \int_{t_1}^{t_2}\int_0^l D^*[\phi]\psi\,dx\,dt,$$

if

$$(1.6)\qquad [\dot{\psi}\phi-\psi\dot{\phi}]_{t_1}^{t_2} = 0.$$

2. A variational principle for nonconservative adjoint systems

A generalized Lagrangian L^* may be defined as

$$(2.1)\qquad L^* = \int_0^l \mathscr{L}^*\,dx$$

having the generalized Lagrangian density

$$(2.2)\qquad \mathscr{L}^* = \mu\dot{w}\dot{v}-\alpha w_{xx}v_{xx}+q(l-x)w_xv_x-qw_xv.$$

As has been shown in [1], the generalized Lagrangian (2.1), (2.2) can be used to formulate a generalized system of Lagrange's equations of the second order which yields the differential equations (1.1) and (1.3) as follows:

$$(2.3)\qquad \frac{d}{dt}\frac{\delta L^*}{\delta\dot{v}}-\frac{\delta L^*}{\delta v} = D[w] = 0,$$

$$(2.4)\qquad \frac{d}{dt}\frac{\delta L^*}{\delta\dot{w}}-\frac{\delta L^*}{\delta w} = D^*([v] = 0.$$

In (2.3) and (2.4), variational (functional) derivatives of L^* have been used, for example,

$$(2.5)\qquad \frac{\delta L^*}{\delta\dot{w}} = \frac{\partial\mathscr{L}^*}{\partial\dot{w}},\qquad \frac{\delta L^*}{\delta w} = \frac{\partial\mathscr{L}^*}{\partial w}-\frac{\partial}{\partial x}\frac{\partial\mathscr{L}^*}{\partial w_x}+\frac{\partial^2}{\partial x^2}\frac{\partial\mathscr{L}^*}{\partial w_{xx}}.$$

Since (1.1) holds, the following is an identity:

$$(2.6)\qquad \delta\int_{t_1}^{t_2}\int_0^l D[w]v\,dx\,dt = 0.$$

Carrying out the variation in (2.6),

$$\int_{t_1}^{t_2}\int_0^l \{D[\delta w]v + D[w]\delta v\}\,dx\,dt = 0 \tag{2.7}$$

is obtained. Under the assumption that the variations δw satisfy (1.2), and the variations δv satisfy (1.4), the adjointness condition (1.5) can be applied to (2.7) which yields

$$\int_{t_1}^{t_2}\int_0^l \{D^*[v]\delta w + D[w]\delta v\}\,dx\,dt = 0. \tag{2.8}$$

Using (2.3) and (2.4) in (2.8) results in

$$\int_{t_1}^{t_2}\int_0^l \left\{\left[\frac{d}{dt}\frac{\delta L^*}{\delta \dot{w}} - \frac{\delta L^*}{\delta w}\right]\delta w + \left[\frac{d}{dt}\frac{\delta L^*}{\delta \dot{v}} - \frac{\delta L^*}{\delta v}\right]\delta v\right\}dx\,dt = 0. \tag{2.9}$$

This is obviously equivalent to

$$\delta\int_{t_1}^{t_2} L^* dt = 0. \tag{2.10}$$

Hence a generalized Hamilton's principle, i.e., a variational principle is valid for the two nonconservative adjoint systems under consideration.

Taking (2.1) and (2.2) into account, (2.10) leads after some calculation to

$$\int_0^l \left[\int_{t_1}^{t_2} (\mu\delta\dot{w}\dot{v} + \mu\dot{w}\delta\dot{v})\,dt\right]dx + \delta\int_{t_1}^{t_2}\int_0^l [-\alpha w_{xx}v_{xx} + q(l-x)w_x v_x - q w_x v]\,dx\,dt = 0. \tag{2.11}$$

In (2.11), the first integral on the left side can be changed into

$$\int_0^l \left[\int_{t_1}^{t_2} (\mu\delta\dot{w}\dot{v} + \mu\dot{w}\delta\dot{v})\,dt\right]dx = \int_{t_1}^{t_2}\int_0^l [-\mu\ddot{v}\delta w - \mu\ddot{w}\delta v]\,dx\,dt. \tag{2.12}$$

This is possible by means of integration by parts using the fact that

$$[\delta w]_{t_1}^{t_2} = 0, \quad [\delta v]_{t_1}^{t_2} = 0. \tag{2.13}$$

This is a basic assumption of Hamilton's theory which is not affected by the system being conservative or nonconservative. Therefore it is valid here.

Using (2.12) in (2.11), we have

$$\int_{t_1}^{t_2}\int_0^l [-\mu\ddot{v}\delta w - \mu\ddot{w}\delta v]\,dx\,dt + \delta\int_{t_1}^{t_2}\int_0^l [-\alpha w_{xx}v_{xx} + q(l-x)w_x v_x - q w_x v]\,dx\,dt = 0. \tag{2.14}$$

Assuming solutions of the form

$$w = e^{i\omega t}y(x), \quad v = e^{i\omega t}z(x), \tag{2.15}$$

the variables are separated, and adjointness condition (1.6) is automatically satisfied. Now,

$$\mu\ddot{v}\delta w = -e^{2i\omega t}\mu\omega^2 z\delta y, \tag{2.16}$$

$$\mu\ddot{w}\delta v = -e^{2i\omega t}\mu\omega^2 y\delta z. \tag{2.17}$$

With (2.15)-(2.17) Eq. (2.14) changes into

$$\int_{t_1}^{t_2} e^{2i\omega t} dt\Big\{\int_0^l [\mu\omega^2(y\delta z + z\delta y)]dx + \delta\int_0^l [-\alpha y''z'' + q(l-x)y'z' - qy'z]dx\Big\} = 0. \tag{2.18}$$

This is obviously equal to

$$\int_{t_1}^{t_2} e^{2i\omega t} dt\Big\{\delta\int_0^l [\mu\omega^2 yz - \alpha y''z'' + q(l-x)y'z' - qy'z]dx\Big\} = 0. \tag{2.19}$$

Since (2.19) shall hold for any interval of time $t_1 - t_2$, this can only be true if

$$\delta\int_0^l [\mu\omega^2 yz - \alpha y''z'' + q(l-x)y'z' - qy'z]dx = 0. \tag{2.20}$$

Introducing the functional

$$H^* = \int_0^l \mathscr{H} dx, \tag{2.21}$$

with the density

$$\mathscr{H} = -\mu\omega^2 yz + \alpha y''z'' - q(l-x)y'z' + qy'z, \tag{2.22}$$

(2.20) can be changed into

$$\delta H^* = 0. \tag{2.23}$$

This variational principle will be used in the following as a foundation for calculating the eigenvalues of (1.1), (1.2) and (1.3), (1.4) by means of a direct method.

3. Hybrid Ritz equations

Let

$$y = \sum_i \gamma_i \psi_i, \quad z = \sum_i c_i \phi_i. \tag{3.1}$$

The coordinate functions ψ_i and ϕ_i in the expansions (3.1) are supposed to be admissible functions satisfying the boundary conditions (1.2) and (1.4), respectively.

Using (3.1) in (2.21), (2.22), condition (2.23) can be replaced by

$$\frac{\partial H^*}{\partial c_i} = 0, \quad i = 1, 2, 3, \ldots \tag{3.2}$$

Carrying out this operation,

$$\sum_i \gamma_i \int_0^l [-\mu\omega^2\psi_i\phi_j + \alpha\psi_i''\phi_j'' - q(l-x)\psi_i'\phi_j' + q\psi_i'\phi_j]dx = 0, \quad i, j = 1, 2, 3, \ldots \tag{3.3}$$

is the result. (3.3) is the system of the so-called hybrid Ritz equations.

4. Calculation of the buckling load

The system (3.3) is linear, algebraic, homogeneous, and admits a nontrivial solution for the γ_i only if

(4.1) $$\det(-\omega^2 A_{ij} + B_{ij} - qC_{ij}) = 0,$$

when

(4.2) $$A_{ij} = \mu \int_0^l \psi_i \phi_j dx, \quad B_{ij} = \alpha \int_0^l \psi_i'' \phi_j'' dx,$$
$$C_{ij} = \int_0^l [(l-x)\psi_i' \phi_j' - \psi_i' \phi_j] dx.$$

Restricting ourselves to $i, j = 1, 2$, and using the notations

(4.3) $$G(q) = (B_{11} - qC_{11})(B_{22} - qC_{22}) - (B_{21} - qC_{21})(B_{12} - qC_{12}),$$
$$H(q) = -A_{11}(B_{22} - qC_{22}) - A_{22}(B_{11} - qC_{11}) + A_{12}(B_{21} - qC_{21}) + A_{21}(B_{12} - qC_{12}),$$
$$K(q) = A_{11}A_{22} - A_{21}A_{12},$$

(4.1) can be rewritten as

(4.4) $$\det(-\omega^2 A_{ij} + B_{ij} - qC_{ij}) = F(\omega^2, q) = G(q) + \omega^2 H(q) + \omega^4 K(q) = 0.$$

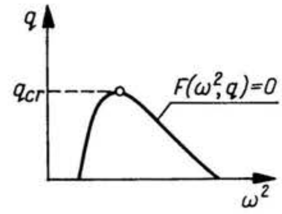

FIG. 3.

The geometric equivalent of equation $F(\omega^2, q) = 0$ is the eigenvalue curve as shown in Fig. 3. The buckling load q_{cr} is obtained from (4.4) and from

(4.5) $$\frac{dq}{d\omega} = -\frac{\partial F/\partial \omega^2}{\partial F/\partial q} = 0, \quad \frac{\partial F}{\partial \omega^2} = 0,$$

respectively.

Using (4.4), the second condition in (4.5) yields

(4.6) $$\frac{\partial F}{\partial \omega^2} = H(q) + 2\omega^2 K(q) = 0.$$

Hence

(4.7) $$\omega^2 = -\frac{H(q)}{2K(q)}.$$

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Using (4.7) in (4.4) yields

$$4G(q)K(q)-H^2(q) = 0. \tag{4.8}$$

The buckling load q_{cr} can be obtained by solving (4.8) for q.

5. Numerical example

The following set of functions

$$\begin{aligned} \psi_1 &= x^2-\frac{2}{3l}x^3+\frac{1}{6l^2}x^4, \\ \psi_2 &= x^2-\frac{1}{l}x^4+\frac{3}{10l^2}x^5, \end{aligned} \tag{5.1}$$

$$\begin{aligned} \phi_1 &= x^2-\frac{5ql^3+24\alpha}{3l(ql^3+12x)}x^3+\frac{3ql^3+6\alpha}{3l^2(ql^3+12\alpha)}x^4 \\ \phi_2 &= x^3-\frac{7ql^3+120\alpha}{4l(ql^3+30\alpha)}x^4+\frac{3ql^3+36\alpha}{4l^2(ql^3+20\alpha)}x^5 \end{aligned} \tag{5.2}$$

has been used for a numerical calculation. The set (5.1) satisfies the boundary conditions

$$\psi_i(0) = \psi_i'(0) = \psi_2''(l) = \psi_i'''(l) = 0, \quad i = 1, 2, \tag{5.3}$$

corresponding to (1.2). The set (5.2) satisfies the boundary conditions

$$\phi_i(0) = \phi_i'(0) = \phi_2''(l) = \phi_i'''(l)-\frac{q}{\alpha}\phi_i(l) = 0, \quad i = 1, 2, \tag{5.4}$$

corresponding to (1.4). Hence the functions in (5.1) and (5.2) are admissible. The rather elementary calculation consists in determining the quantities (4.2) by means of (5.1), (5.2), and then G, H and K according to (4.3).

Finally, (4.8) can be solved for q which yields

$$q_{cr} = 41.51\frac{\alpha}{l^3}. \tag{5.5}$$

The corresponding square of the critical frequency follows from (4.7) as

$$\omega_{cr}^2 = 128.46\frac{\alpha}{\mu l^4}. \tag{5.6}$$

The result (5.5) is in satisfactory agreement with the buckling load

$$q_{cr} = 40.7\frac{\alpha}{l^3}$$

calculated in [2] by means of Galerkin's method.

6. Convergence

Integration by parts applied to (3.3) yields

$$\sum_i \gamma_i \int_0^l \{-\mu\omega^2\psi_i + \alpha\psi_i'^V + [q(l-x)\psi_i']' + q\psi_i'\}\phi_j dx = 0, \tag{6.1}$$

which is the system of hybrid Galerkin's equations. Since (3.3) and (6.1) are completely equivalent under the assumption that ψ_i satisfies all boundary conditions in (1.2) and ϕ_i likewise in (1.4), (6.1) can be used for convergence considerations in place of (3.3). This has already been done in [3]. The result was that the method is convergent.

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Referernces

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