Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 5, pp. 719-731, Warszawa 1973

# Ultimate bearing capacity of structural systems with minimal critical sets having joint elements in pairs

W. DZIUBDZIELA, B. KOPOCIŃSKI and Z. KOWAL (WROCŁAW)

THE PAPER discusses two models of rigid-plastic structures in the boundary state: model 1, in which the critical sets form an open chain with joint elements of the adjacent sets — assumptions of this model are satisfied by, among other factors, the continuous beams; model 2, in which the critical sets form a closed chain with joint elements of the adjacent sets. The assumptions of model 2 are satisfied by single-chamber frames. The models are described in the manner used in the measure theory. The distribution of the probability of appearance of ultimate bearing capacity can be determined by means of electronic computers on the basis of four theorems proved in the paper. On the basis of the theorems proved, some estimations, simple in application, from the top of the risk of failure of the system are given. These estimations do not require the application of electronic computers. The work is illustrated by examples.

W pracy rozpatruje się dwa modele konstrukcji sztywno-plastycznych w stanie granicznym: model 1, w którym zbiory krytyczne tworzą łańcuch otwarty o wspólnych elementach sąsiadujących zbiorów (założenia tego modelu spełniają między innymi belki ciągłe) oraz model 2, w którym zbiory krytyczne tworzą łańcuch zamknięty o elementach wspólnych sąsiadujących zbiorów (założenia modelu 2 spełniają ramy jednokomorowe). Modele opisano w terminach teorii miary. Rozkłady prawdopodobieństw wystąpienia nośności granicznej wyznaczać można za pomocą maszyn cyfrowych na podstawie udowodnionych w pracy 4 twierdzeń. W oparciu o udowodnione twierdzenia podano proste w zastosowaniu oszacowania od góry niebezpieczeństwa zniszczenia systemu. Oszacowania te nie wymagają zastosowania maszyn cyfrowych. Pracę zilustrowano przykładami.

В работе рассмотрены две модели жестко-пластических сооружений в предельном состоянии. В первой модели критические множества образуют открытую цепь, соединенную общими элементами соседних множеств (предположениям этой модели удовлетворяют, например, непрерывные балки). Во второй модели критические множества образуют закрытую цепь с общими элементами соседних множеств (предположениям этой модели удовлетворяют однополостные рамы). Модели описаны в терминах теории меры. Распределения вероятностей возникновения предельной нагрузки можно вычислить при помощи ЭЦВМ, исходя из четырех теорем, доказанных в данной работе. Исходя из доказанных теорем даны простые прикладные верхние оценки опасности разрущения системы. Пользование этими оценками не требует применения вычислительных мащин. Работа иллюстрируется примерами приложений.

## 1. Introduction

LET us take into consideration elasto-plastic structures composed of elements in such formation that at the boundary state they can be considered as rigid-plastic structures, and their reliability characteristic determined by means of the kinematically permissible mechanisms of destruction [6]. These systems can be described by application of the minimal critical set of elements (a critical set of elements is a set of elements such that failure of the structure takes place when all the elements of the set are subjected to failure; the critical set is minimal if none of its sub-sets is critical [4, 5]). Let us consider structures the reliability models of which can be described by means of $r$ minimal critical sets,

$A_1, A_2, \ldots, A_r$ of elements having joint elements in pairs. Two models are distinguished in this work: model 1, in which the critical sets form an open chain with joint elements of the adjacent sets, and model 2, in which the critical sets form a closed chain with joint elements of the adjacent sets.

In [4] was discussed the case of a structure with two minimal critical collections having joint elements. Models 1 and 2 are the natural generalization of that case. Problems of objective determination of ultimate bearing capacity have also been dealt with during recent years by the authors of [8, 9, 10, 11, 12].

Our problem is to find the distribution of the ultimate bearing capacity of structures satisfying the assumptions of the models discussed on the basis of the distribution of ultimate bearing capacity elements, estimation of such distribution from the top, and on this basis estimation of ultimate bearing capacity from below.

It is assumed that the distribution $F_j(x)$ of the ultimate bearing capacities of $N_j$ elements in each set is known:

$$\tilde{F}_j(x) = P(N_j < x), \quad j \in A_i, \quad i = 1, 2, \ldots, r, \tag{1.1}$$

together with the means $E(N_j)$ and variances $D^2(N_j)$.

Since the ultimate bearing capacities of elements are independent random variables, therefore the distribution $F_{A_i}(x)$ of the ultimate bearing capacities $N_{A_i}$ of the minimal critical sets of elements $A_i$

$$F_{A_i}(x) = P(N_{A_i} < x), \quad i = 1, 2, \ldots, r \tag{1.2}$$

can be determined on the basis of distributions $F_i(x)$ of the ultimate bearing capacities of elements ([2])

$$F_{A_i}(x) = \prod_{j \in A_i}{}^{*} \tilde{F}_j\left(\frac{x}{a_j}\right), \tag{1.3}$$

where $\Pi^*$ denotes the commutative convolutive product of distributions $\tilde{F}_j(x/a_j)$ for $j \in A_i$, $a_j$ is the weight of the $j^{\text{th}}$ element in set $A_i$.

Parameters of distribution $F_{A_i}(x)$ determined from the Eq. (1.3) are as follows:
the expected value equals

$$E(N_{A_i}) = \sum_{j \in A_i} a_j E(N_j), \tag{1.4}$$

and the variance is equal to:

$$D^2(N_{A_i}) = \sum_{j \in A_i} a_j^2 D^2(N_j). \tag{1.5}$$

## 2. Distribution of the ultimate bearing capacity of a structure constructed according to model 1

A reliable model 1 of the structure is shown in Fig. 1. It can be described in the manner used in the measure theory [3]

$$A_i \cap A_j = 0, \quad |i-j| > 1, \quad A_i \cap A_{i+1} \neq 0, \quad i = 1, 2, \ldots, r-1. \tag{2.1}$$

The sets $B_1, B_2, \ldots, B_{2k-1}, B_{2k}, \ldots, B_{2r-1}$, can be defined in the following manner:

Let the elements with numbers $1, 2, ..., l_1$ belong to the set $B_1 \doteq A_1 \setminus A_2$. Then, for simplicity, the sets of elements are determined by means of the set of their numbers,

$$B_1 = A_1 \setminus A_2 = \{l_0+1, ..., l_1\},$$

and let also

$$B_{2k} = A_k \cap A_{k+1} = \{l_{2k-1}+1, ..., l_{2k}\}, \quad k = 1, 2, ..., r-1,$$
$$B_{2k-1} = A_k \setminus (A_{k-1} \cup A_{k+1}) = \{l_{2k-2}+1, ..., l_{2k-1}\}, \quad k = 2, 3, ..., r-1,$$
$$B_{2r-1} = A_r \setminus A_{r-1} = \{l_{2r-2}+1, ..., k\},$$

where

$$1 \leqslant l_1 < l_2 < ... < l_{2r-1} = k, \quad l_0 = 0.$$

In the measure theory, the product $A_k \cap A_{k+1}$ denotes a set the elements of which belong to the set $A_k$ and also to the set $A_{k+1}$.

Further, use is made of the symbol of the sum $A_k \cup A_{k+1}$ denoting the set elements of which belong to the set $A_k$ or to the set $A_{k+1}$, and the symbol of the difference $A_k \setminus A_{k-1}$, denoting the set elements of which belong to the set $A_k$, and do not belong to the set $A_{k-1}$.

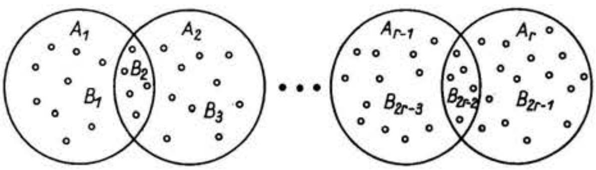


FIG. 1. Model 1 of the reliability of structure.

Figure 2 presents an example of a structure satisfying the assumptions of model 1. This model is satisfied by continuous beams loaded in a typical manner by forces of the same nature as shown in Fig. 2. (The loading arrangement of the system determines the

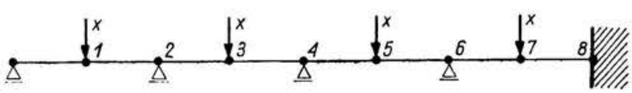


FIG. 2. Example of a structure built on the basis of model 1.

probability — different from zero — of the appearance of individual mechanisms of failure from among the full set).

The ultimate bearing capacity of the minimal critical set of elements is the sum of the ultimate bearing capacities of elements belonging to the set, taken with the corresponding weights [5]. The ultimate bearing capacities $N_i$ can have weights $a_i$ $(i = 1, 2, ..., k)$ in the structural system. Making use of this assumption it will be sufficient to deal with the sequence of the independent random variables:

$$X_i = \sum_{m \in B_i} a_m N_m, \quad i = 1, 2, ..., 2r-1.$$

These variables can be interpreted as the ultimate bearing capacity of the element sets $B_1, B_2, \ldots, B_{2r-1}$.

The ultimate bearing capacity of a structure with $r$ minimal critical sets is a random variable in the form:

$$N_I^{(r)} = \min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-4}+X_{2r-3}+X_{2r-2}, X_{2r-2}+X_{2r-1}). \tag{2.2}$$

The first objective will be to find the distribution function of this random variable

$$F_I^{(r)}(x) = P(N_I^{(r)} < x) = 1-H_{2r-1}(x), \tag{2.3}$$

where

$$H_{2r-1}(x) = P(N_I^{(r)} \geqslant x).$$

### 2.1. Recurrence formulas for $H_{2r-1}(x)$

THEOREM 1. *If $X_1, X_2, \ldots, X_{2r-1}$ $(r = 2, 3, \ldots)$ are the independent random variables with distributions $F_i(x) = P(X_i < x)$, $\bar{F}_i(x) = 1-F_i(x)$ for $i = 1, 2, \ldots, 2r-1$, then*

$$H_{2r-1}(x) = \int_{-\infty}^{\infty} H_{2r-3}(x, x_{2r-2})\bar{F}_{2r-1}(x-x_{2r-2})dF_{2r-2}(x_{2r-2}), \tag{2.4}$$

*where*

$$H_1(x, x_2) = \bar{F}_1(x-x_2), \tag{2.5}$$

$$H_{2r-3}(x, x_{2r-2}) = \int_{-\infty}^{\infty} H_{2r-5}(x, x_{2r-4})\bar{F}_{2r-3}(x-x_{2r-2}-x_{2r-4}) \times \\ \times dF_{2r-4}(x_{2r-4}), \quad r = 3, 4, \ldots \tag{2.6}$$

Proof. Let

$$H_1(x, x_2) = P(X_1 \geqslant x-x_2) = \bar{F}_1(x-x_2),$$

$$H_{2r-3}(x, x_{2r-2}) = P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-6}+X_{2r-5}+X_{2r-4}, \\ X_{2r-4}+X_{2r-3}+X_{2r-2}) \geqslant x].$$

Hence

$$H_3(x) = P[\min(X_1+X_2, X_2+X_3) \geqslant x] = \int_{-\infty}^{\infty} P[\min(X_1+x_2, x_2+X_3) \geqslant x]dF_2(x_2)$$

$$= \int_{-\infty}^{\infty} P(X_1 \geqslant x-x_2)P(X_3 \geqslant x-x_2)dF_2(x_2) = \int_{-\infty}^{\infty} H_1(x, x_2)\bar{F}_3(x-x_2)dF_2(x_2),$$

and then

$$H_{2r-1}(x) = P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-4}+X_{2r-3}+X_{2r-2}, X_{2r-2}+X_{2r-1}) \geqslant x]$$

$$= \int_{-\infty}^{\infty} P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-4}+X_{2r-3}+X_{2r-2}, X_{2r-2}+X_{2r-1}) \geqslant x]\times$$

$$\times dF_{2r-2}(x_{2r-2}) = \int\limits_{-\infty}^{\infty} P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-4}+X_{2r-3}+x_{2r-2}) \geqslant x]\times$$

$$\times P(X_{2r-1} \geqslant x-x_{2r-2})dF_{2r-2}(x_{2r-2}) = \int\limits_{-\infty}^{\infty} H_{2r-3}(x, x_{2r-2})\bar{F}_{2r-1}(x-x_{2r-2})dF_{2r-2}(x_{2r-2}).$$

In this way, the Eq. (2.4) is obtained.

Now, the recurrence equations for $H_{2r-3}(x, x_{2r-2})$, $r = 2, 3, \ldots$ will be found $H_1(x, x_2) = P(X_1 \geqslant x-x_2) = \bar{F}_1(x-x_2)$,

$$H_3(x, x_4) = P[\min(X_1+X_2, X_2+X_3+X_4) \geqslant x]$$

$$= \int\limits_{-\infty}^{\infty} P[\min(X_1+x_2, x_2+X_3+x_4) \geqslant x]dF_2(x_2)$$

$$= \int\limits_{-\infty}^{\infty} P(X_1 \geqslant x-x_2)P(X_3 \geqslant x-x_2-x_4)dF_2(x_2) = \int\limits_{-\infty}^{\infty} H_1(x, x_2)\bar{F}_3(x-x_2-x_4)dF_2(x_2).$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$H_{2r-3}(x, x_{2r-2}) = P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-6}+X_{2r-5}+X_{2r-4},$$

$$X_{2r-4}+X_{2r-3}+X_{2r-2}) \geqslant x] = \int\limits_{-\infty}^{\infty} P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-6}+X_{2r-5}$$

$$+x_{2r-4}, x_{2r-4}+X_{2r-3}+x_{2r-2}) \geqslant x]dF_{2r-4}(x_{2r-4})$$

$$= \int\limits_{-\infty}^{\infty} P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-6}+X_{2r-5}+x_{2r-4} \geqslant x]\times$$

$$\times P(X_{2r-3} \geqslant x-x_{2r-4}-x_{2r-2})dF_{2r-4}(x_{2r-4})$$

$$= \int\limits_{-\infty}^{\infty} H_{2r-5}(x, x_{2r-4})\bar{F}_{2r-3}(x-x_{2r-4}-x_{2r-2})dF_{2r-4}(x_{2r-4}).$$

Theorem 1 has been proved.

THEOREM 2. *If the assumptions of Theorem* 1 *are satisfied, then* $H_1(x, x_2) = \bar{F}_1(x-x_2)$,

$$(2.7)\qquad H_{2r-3}(x, x_{2r-2}) = \int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots\bar{F}_{2r-3}(x-x_{2r-2}-x_{2r-4})\times$$

$$\times dF_2(x_2)dF_4(x_4)\ldots dF_{2r-4}(x_{2r-4}),$$

*where* $r = 3, 4, \ldots$

This theorem will be proved by means of mathematical induction.

Proof. The proof of the theorem for $r = 3$ will be checked. From (2.6), we have

$$H_3(x, x_4) = \int\limits_{-\infty}^{\infty} H_1(x, x_2)\bar{F}_3(x-x_2-x_4)dF_2(x_2) = \int\limits_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)dF_2(x_2),$$

which was to be proved.

Let us assume that Eq. (2.7) is true for $r = k-1$ — i.e., let us assume that

$$H_{2k-5}(x, x_{2k-4}) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots$$

$$\ldots \bar{F}_{2k-5}(x-x_{2k-6}-x_{2k-4})dF_2(x_2)\ldots dF_{2k-6}(x_{2k-6}).$$

Now, the proof of the theorem for $r = k$ will be demonstrated. From (2.6), we have

$$H_{2k-3}(x, x_{2k-2}) = \int_{-\infty}^{\infty} H_{2k-5}(x, x_{2k-4})\bar{F}_{2k-3}(x-x_{2k-4}-x_{2k-2})dF_{2k-4}(x_{2k-4});$$

therefore, making use of the inductive assumptions, we obtain:

$$H_{2k-3}(x, x_{2k-2}) = \int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots \bar{F}_{2k-5}(x-x_{2k-6}x_{2k-4}) \times$$

$$\times dF_2(x_2)\ldots dF_{2k-6}(x_{2k-6})\bar{F}_{2k-3}(x-x_{2k-4}-x_{2k-2})dF_{2k-4}(x_{2k-4})$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots \bar{F}_{2k-3}(x-x_{2k-4}-x_{2k-2})\,dF_2(x_2)\ldots dF_{2k-4}(x_{2k-4}).$$

Theorem 2 has been proved.

THEOREM 3. *If the assumptions of Theorem 1 are satisfied, then*

(2.8)

$$F_1^{(r)}(x) = P(N_1^{(r)} < x) = 1-\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots \bar{F}_{2r-3}(x-x_{2r-2}-x_{2r-4}) \times$$

$$\times \bar{F}_{2r-1}(x-x_{2r-2})dF_2(x_2)dF_4(x_4)\ldots dF_{2r-2}(x_{2r-2}).$$

P r o o f. From the relation (2.3) and Theorem 1, we have:

$$F_1^{(r)}(x) = 1-H_{2r-1}(x) = 1-\int_{-\infty}^{\infty} H_{2r-3}(x, x_{2r-2})\bar{F}_{2r-1}(x-x_{2r-2})dF_{2r-2}(x_{2r-2}).$$

From Theorem 2, we obtain:

$$F_1^{(r)}(x) = 1-\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2-x_4)\ldots \bar{F}_{2r-3}(x-x_{2r-2}-x_{2r-4})dF_2(x_2)\ldots$$

$$\ldots dF_{2r-4}(x_{2r-4})\Big\}\bar{F}_{2r-1}(x-x_{2r-2})dF_{2r-2}(x_{2r-2}) = 1-\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2) \times$$

$$\times \bar{F}_3(x-x_2-x_4)\ldots \bar{F}_{2r-3}(x-x_{2r-2}-x_{2r-4})\bar{F}_{2r-1}(x-x_{2r-2})dF_2(x_2)\ldots dF_{2r-2}(x_{2r-2}).$$

Theorem 3 has been proved.

COROLLARY 1. *Let us consider a structure with two minimal critical sets having joint elements. Making use of Theorem* 3 *for* $r = 2$, *we have*:

$$F_1^{(2)}(x) = 1-\int_{-\infty}^{\infty} \bar{F}_1(x-x_2)\bar{F}_3(x-x_2)dF_2(x_2) = 1-\int_{-\infty}^{\infty}[1-F_1(x-x_2)][1-F_3(x-x_2)] \times$$

$$\times dF_2(x_2) = F_1 * F_2(x)+F_2 * F_3(x) = \int_{-\infty}^{\infty} F_1(x-x_2)F_3(x-x_2)dF_2(x_2).$$

This equation was found in [4].

## 3. Evaluation from below of the ultimate bearing capacity of a structure built according to model 1

Application in practice of the theorems proved in Sec. 2 is troublesome but quite possible by means of electronic computers. In this section, we shall find the estimations from the top of distribution of the ultimate bearing capacity of the structure. Having these estimations at the given level of confidence, the estimation from below of the ultimate bearing capacity of the structure can be found.

In model 1, estimation of the simple form resulting from (2.8) can be accepted

$$F_I^{(r)}(x) \leqslant F_1 * F_2(x) + F_2 * F_3 * F_4(x) + \ldots + F_{2r-4} * F_{2r-3} * F_{2r-2}(x) + F_{2r-2} * F_{2r-1}(x), \tag{3.1}$$

where $*$ is a symbol of the convolution of the two functions.

Accepting the assumption that the random variable $X_i$ has normal distribution with the anticipated value $m_i$ and variance $\sigma_i^2$

$$P(X_i < x) = \Phi\left(\frac{x-m_i}{\sigma_i}\right), \quad i = 1, 2, \ldots, 2r-1,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \exp\left(-\frac{1}{2}u^2\right) du,$$

from (3.1), we obtain

$$F_I^{(r)}(x) \leqslant \sum_{i=1}^{r} \Phi(z_i), \tag{3.2}$$

where

$$z_1 = \frac{x-(m_1+m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}},$$

$$z_k = \frac{x-(m_{2k-2}+m_{2k-1}+m_{2k})}{\sqrt{\sigma_{2k-2}^2+\sigma_{2k-1}^2+\sigma_{2k}^2}}, \quad k = 2, 3, \ldots, r-1,$$

$$z_r = \frac{x-(m_{2r-2}+m_{2r-1})}{\sqrt{\sigma_{2r-2}^2+\sigma_{2r-1}^2}}.$$

The estimation (3.1) can be improved as follows: The distribution of the ultimate bearing capacity can be written in the form:

$$\begin{aligned} F_I^{(r)}(x) &= P[\min(X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-4}+X_{2r-3}+X_{2r-2}, X_{2r-2}+X_{2r-1}) < x] \\ &= P\{\min[\min(X_1+X_2, X_4+X_5+X_6, X_8+X_9+X_{10}, \ldots), \\ &\qquad \min(X_2+X_3+X_4, X_6+X_7+X_8, \ldots)] < x\}. \end{aligned}$$

For any random variables $X$ and $Y$, we have

$$\begin{aligned} P[\min(X, Y) < x] &= P(X < x \quad \text{or} \quad Y < x) = P(X < x) + P(Y < x) \\ &\quad - P(X < x, Y < x) \leqslant P(X < x) + P(Y < x). \end{aligned}$$

Hence

$$F_I^{(r)}(x) \leqslant P[\min(X_1+X_2, X_4+X_5+X_6, \ldots) < x] + P[\min(X_2+X_3+X_4, X_6+X_7+X_8, \ldots) < x].$$

Let us consider two cases.

Case 1. If $r$ is an even number, then

$$(3.3) \quad F_I^{(r)}(x) \leqslant 2 - [1-F_1 * F_2(x)][1-F_4 * F_5 * F_6(x)] \ldots [1-F_{2r-4} * F_{2r-3} * F_{2r-2}(x)] - [1-F_2 * F_3 * F_4(x)][1-F_6 * F_7 * F_8(x)] \ldots [1-F_{2r-2} * F_{2r-1}(x)].$$

Case 2. If $r$ is an odd number, then

$$(3.4) \quad F_I^{(r)}(x) \leqslant 2 - [1-F_1 * F_2(x)][1-F_4 * F_5 * F_6(x)] \ldots [1-F_{2r-2} * F_{2r-1}(x)] - [1-F_2 * F_3 * F_4(x)][1-F_6 * F_7 * F_8(x)] \ldots [1-F_{2r-4} * F_{2r-3} * F_{2r-2}(x)].$$

The Eqs. (3.3) and (3.4) can be expressed by means of distributions of the bearing capacities of sets $A_i$, $i = 1, 2, \ldots, r$. When $r$ is an even number, then

$$(3.5) \quad F_I^{(r)}(x) \leqslant 2 - [1-F_{A_1}(x)][1-F_{A_3}(x)] \ldots [1-F_{A_{r-1}}(x)] - [1-F_{A_2}(x)][1-F_{A_4}(x)] \ldots [1-F_{A_r}(x)].$$

When, however, $r$ is an odd number, then

$$(3.6) \quad F_I^{(r)}(x) \leqslant 2 - [1-F_{A_1}(x)][1-F_{A_3}(x)] \ldots [1-F_{A_r}(x)] - [1-F_{A_2}(x)][1-F_{A_4}(x)] \ldots [1-F_{A_{r-1}}(x)],$$

where $F_{A_r}(x)$ is the distribution of ultimate bearing capacity of the $r^{\text{th}}$ set of elements.

Example 1. A four-span continuous beam as shown in Fig. 2. Boundary moments at dangerous sections of this beam have normal distribution with the expected value $E(M) = 5400$ kGm and with coefficient of variability $\mu = \sigma/E(M) = 0.1$. Spans are $l_1 = 6$ m, $l_2 = l_3 = l_4 = 8$ m. Concentrated load at the middle of each of the spans is the same, and equals $x = 3780$ kG. We want to estimate the probability of beam failure and to estimate the calculated ultimate bearing capacity of the beam at the level of 0.99865. This magnitude will be determined on the basis of estimation from the top of the area of danger.

The random ultimate bearing capacity $N$ of the beam, measured by the transverse loading $x$, will be determined from the relation:

$$N_{A_i} = \min(N_{A_1}, N_{A_2}, \ldots, N_{A_r}),$$

where $N_{A_i}$ denotes the random ultimate bearing capacity of the minimal critical set of elements $A_i$ and equals:

$$N_{A_i} = 4(M_{2i-1} + 0.5M_{2i} + M_{2i+1})/l_i.$$

In the example, there are four consecutive critical sets of elements — 1, 2; 2, 3, 4; 4, 5, 6; 6, 7, 8[1]. The expected ultimate bearing capacity and standard deviations of the

[1] Note. At the change of direction of the individual loadings, $x$ other mechanisms of failure are possible. In the example in Fig. 2, they could be the critical sets of elements 1, 3, 5, 7, 8 or 1, 3, 5, 6. These simple examples will not be dealt with here, since they are solved on the basis of [5]. In the system of loadings assumed by us, the probability of appearance of such destruction mechanisms equals zero (is physically impossible).

individual sets of element are

$$E(N_{A_1}) = 4\,\frac{5400+2700}{6} = 5400\,\text{kG},$$

$$E(N_{A_2}) = 4\,\frac{5400+2\times 2700}{8} = 5400\,\text{kG},$$

$$\sigma_{A_1} = \sqrt{360^2+180^2} = 402\ \text{kG},$$

$$\sigma_{A_2} = \sqrt{270^2+2\times 135^2} = 331\ \text{kG}.$$

The probability of failure $q_i = 1-p_i$ of the individual critical sets will be determined from the Tables [7].

$$1-p_1 = P(N_{A_1} < 3780) = \Phi(z_1) = 0.0^4 2789,$$

$$1-p_2 = P(N_{A} < 3780) = \Phi(z_2) = 0.0^6 4792,$$

where

$$z_1 = \frac{5400-3780}{402} = 4.03, \quad z_2 = \frac{5400-3780}{331} = 4.9.$$

Estimation from the top of the risk of beam failure, on the basis of (3.5), equals:

$$P(N < 3780) = 2-p_1p_2-p_2^2 = 0.0^4 2933.$$

The beam safety is therefore:

$$P(N \geqslant 3780) = 0.99997067.$$

For determination of the utlimate bearing capacity $N_0$ of the beam at the level of confidence of 0.99865, we take the equation found above:

$$p_2(p_1+p_2) = 1.99865,$$

and the second equation (in this particular case) resulting from the condition of equal loading of the beam:

$$5400-N_0 = z_1\cdot 402 = z_2\cdot 331\,;$$

hence, $z_1/z_2 = 0.823$.

The system of equations

$$p_2(p_1+p_2) = 1.99865, \quad z_1 = 0.823 z_2$$

can be solved by, for example, the method of consecutive approximations.

For $z_1 = 3.075$, $z_2 = 3.73$, we shall have:

$$N_0 = 5400-3.075\cdot 402 = 4165\ \text{kG}, \quad P(N_0 < 4165) = 0.998659.$$

## 4. Distribution of the ultimate bearing capacity of a structure built according to model 2

Model 2 is shown in Fig. 3. The minimal sets of elements have the join elements in pairs forming a closed chain with $r$ links.

Let $A_1, A_2, \ldots, A_r$ be the minimal critical sets of the structure with $k$ elements, form-

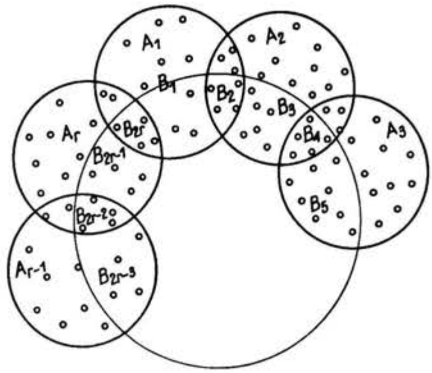


FIG. 3. Model 2 of the reliability of a structure.

ing a closed chain with $r$ links; more accurately, let us assume that these sets satisfy the relations;

$$\text{(4.1)} \qquad \begin{gathered} A_i \cap A_{i+1} \neq 0, \quad i = 1, 2, \ldots, r-1, \quad A_1 \cap A_r \neq 0, \\ A_i \cap A_j = 0 \quad \text{if} \quad 1 < |i-j| < r-1 \end{gathered}$$

(see Fig. 3).

We shall define the sets $B_1, B_2, \ldots, B_{2r}$ as follows:

$$B_{2k} = A_k \cap A_{k+1}, \quad B_{2k-1} = A_k \setminus (A_{k-1} \cup A_{k+1}), \quad k = 2, 3, \ldots, r-1.$$

Analogously to the Eqs. (2.1) and in the subsequent equations not numbered, let

$$\begin{aligned} B_1 &= A_1 \setminus (A_2 \cup A_r) = \{1, 2, \ldots, l_1\}, \\ B_2 &= A_1 \cap A_2 = \{l_1+1, \ldots, l_2\}, \\ B_{2r} &= A_r \cap A_1 = \{l_{2r-1}, \ldots, k)\}, \\ B_{2r-1} &= A_r \setminus (A_1 \cup A_{r-1}) = \{l_{2r-2}+1, \ldots, l_{2r-1}\}. \end{aligned}$$

Examples of structures satisfying the assumptions of model 2 are shown in Fig. 4.

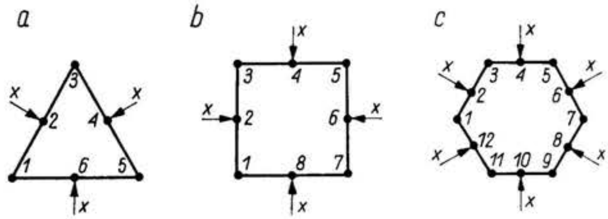


FIG. 4. Example of a structure satisfying the principles of model 2.

Let $X_1, X_2, \ldots, X_{2r}$ denote the ultimate bearing capacities of elements belonging to the sets $B_1, B_2, \ldots, B_{2r}$.

The ultimate bearing capacity of a structure built according to model 2 is a random variable in the form:

$$N_{II}^{(r)} = \min(X_{2r}+X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-2}+X_{2r-1}+X_{2r}). \tag{4.2}$$

THEOREM 4. *If the random variables* $X_1, X_2, \ldots, X_{2r}$, $r = 3, 4, \ldots$ *are independent with distributions* $F_i(x) = P(X_1 < x)$, $\bar{F}_i(x) = 1-F_i(x)$ for $i = 1, 2, \ldots, 2r$, *then*

$$F_{II}^{(r)}(x) = P(N_{II}^{(r)} < x) = 1-\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_2-x_{2r})\bar{F}_3(x-x_2-x_4) \ldots \tag{4.3}$$

$$\ldots \bar{F}_{2r-1}(x-x_{2r-2}-x_{2r})\,dF_2(x_2)\,dF_4(x_4) \ldots dF_{2r}(x_{2r}).$$

P r o o f. From the equation for complete probability, we have

$$F_{II}^{(r)}(x) = P[\min(X_{2r}+X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-2}+X_{2r-1}+X_{2r}) < x]$$

$$= \int_{-\infty}^{\infty} P[\min(X_{2r}+X_1+X_2, X_2+X_3+X_4, \ldots, X_{2r-2}+X_{2r-1}+x_{2r}) < x]\,dF_{2r}(x_{2r}).$$

From Theorem 3, we have:

$$F_{II}^{(r)}(x) = \int_{-\infty}^{\infty} \Big(1-\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_{2r}-x_2)\bar{F}_3(x-x_2-x_4) \ldots \bar{F}_{2r-1}(x-x_{2r-2}-x_{2r}) \times$$

$$\times\, dF_2(x_2)\,dF_4(x_4) \ldots dF_{2r-2}(x_{2r-2})\Big)\,dF_{2r}(x_{2r})$$

$$= 1-\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \bar{F}_1(x-x_{2r}-x_2)\bar{F}_3(x-x_2-x_4) \ldots \bar{F}_{2r-1}(x-x_{2r-2}-x_{2r}) \times$$

$$\times\, dF_2(x_2)\,dF_4(x_4) \ldots dF_{2r}(x_{2r}).$$

Theorem 4 has been proved.

It is easy to check by means of the same methods as in part 3 that the estimate is correct

$$F_{II}^{(r)}(x) \leqslant F_{2r} * F_1 * F_2(x) + F_2 * F_3 * F_4(x) + \ldots + F_{2r-2} * F_{2r-1} * F_{2r}(x). \tag{4.4}$$

In the case of the random variables $X_i$, $i = 1, 2, \ldots, 2r$ with normal distributions, estimation (4.4) can be written in the following form:

$$F_{II}^{(r)}(x) \leqslant \sum_{i=1}^{r} \Phi(z_i), \tag{4.5}$$

where

$$z_i = \frac{x-(m_{2i-2}+m_{2i-1}+m_{2i})}{\sqrt{\sigma_{2i-2}^2+\sigma_{2i-1}^2+\sigma_{2i}^2}}, \quad i = 1, 2, \ldots, r$$

and $m_0 = m_{2r}$, $\sigma_0 = \sigma_{2r}$.

COROLLARY 2. *Let us discuss a structure with three minimal critical sets satisfying the relation* (4.1). *Making use of the Theorem* 4 *for* $r = 3$, *we have*:

$$F_{II}^{(3)}(x) = 1-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{F}_1(x-x_2-x_6)\,\bar{F}_3(x-x_2-x_4)\bar{F}_5(x-x_4-x_6) \times$$

$$\times\, dF_2(x_2)\,dF_4(x_4)\,dF_6(x_6).$$

This equation was found in [1].

Example 2. We have a single-chamber rectangular frame, as shown in Fig. 4b, with identical expected rod sections. Boundary moments in dangerous sections in this frame have normal distribution with the expected value $E(M) = 5400$ kGm and with coefficient of variability 0.1. The height and width of the frame are 8.0 m. Loading is identical and equals to 3780 kG. We want to estimate from the top the probability of frame failure, and to estimate the calculated boundary bearing capacity of the frame at the level of confidence of 0.99865.

The expected ultimate bearing capacity of all four minimal sets of elements measured by transverse loading is identical and equals:

$$E(N_{A_i}) = \frac{4(5400+2\times 0.5\times 5400)}{8} = 5400 \text{ kG}.$$

The standard deviation of the ultimate bearing capacity of all four minimal critical sets of the elements is identical:

$$\sigma_{A_i} = \frac{4}{8}\sqrt{540^2+2\times(0.5\times 540)^2} = 331 \text{ kG}.$$

The probability of destruction of one minimal critical set of elements has been calculated by means of tables of normal distribution [7]: $q_1 = P(N_A < 3780) = 0.0^6\,4792$. Estimation (4.5) of the danger of frame failure is

$$q = P(N < 3780) \leqslant 4\times 0.0^6 4792 = 0.0^5 1917.$$

The ultimate bearing capacity $N_0$ of the frame at the level of $p = 0.99865$ is determined as follows: from (4.5), we have

$$q_1 = q/4 = \frac{1-p}{4} = 0.25\cdot 0.00135 = 0.0003375,$$

argument $z_1 = 3.4$ is taken from tables [7], and as a result we have $N_0 \geqslant 5400-3.4\cdot 331 =$ $= 4275$ kG.

## 5. Conclusions

On the basis of the theorems derived in this work, it is possible to estimate from below, in a simple way, the safety of elasto-plastic continuous beams and single-chamber frames loaded in a typical manner, or to estimate the danger from the top. On the basis of these estimates, it is possible to determine objectively the ultimate bearing capacity of the elasto-plastic structure class discussed at any level of confidence.

## References

1. W. DZIUBDZIELA, B. KOPOCIŃSKI, Z. KOWAL, *Load carrying of a structure with three critical minimum sets containing joint elements* [in Polish], in print.
2. B. W. GNIEDENKO, J. K. BIELAJEV, A. D. SOLOVIOV, *Mathematical methods in the reliability theory* [Polish transl.], WNT, Warsaw 1968.

3. S. HARMAN, J. MIKUSIŃSKI, *Theory of measure and Lebesgue integral* [in Polish], PWN, Warszawa 1957.
4. B. KOPOCIŃSKI, Z. KOWAL, *Load carrying capacity of a structure with two critical minimum sets containing joint elements* [in Polish], Arch. Inżyn. Lądow., **8**, 1, 103–109, 1972.
5. Z. KOWAL, *Load carrying capacity of structures made of elements connected in parallel from the point of view of reliability* [in Polish], Arch. Inżyn. Lądow., **17**, 4, 611–619, 1971.
6. *Theory of plasticity*, ed. by W. OLSZAK, P. PERZYNA, A. SAWCZUK, PWN, Warszawa 1965.
7. *Tables of probability function*, ed. by A. N. LOWAN, National Bureau of Standards, N. York 1941.
8. S. AUGUSTI, A. BARRATA, *Limit analysis of structures with stochastic strength variations*, J. Struct. Mech., 1, 1972.
9. J. R. BENJAMIN, *Probabilistic structural analysis and design*, ASCE, J. Struct. Div., ST 7, July 1968.
10. A. CASTELLANI, *Average value of the safety coefficient in a framed structure corresponding to random distribution of yield stress*, Meccanica **4**, 4, 1969.
11. G. SACCHI, *Upper and lower bounds for the safety coefficients of structures with random resistance*, Meccanica, **6**, 1, 1971.
12. M. TICHY, M. VORLICEK, *Statistical theory of load capacity of concrete structures*, Ph. D. -Thesis [in Czech.], Praha 1966.

TECHNICAL UNIVERSITY OF WROCŁAW.

*Received June* 6, 1972.