Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 4, pp. 613–620, Warszawa 1973

An anisotropic linear Cosserat surface and linear shell theory

Z. T. KURLANDZKA (WARSZAWA)

THE AIM of the paper is to show that the state of stress and strain of the middle surface of a thin elastic shell is equivalent to the state of stress and strain of a certain anisotropic, elastic Cosserat surface if external loads are equal. To this end, the basic definitions and equations of the Cosserat theory in the case of the anisotropic Cosserat surface and those of the linear shell theory are taken into consideration. It is shown then that if certain assumptions are made, the equations of the Cosserat surface and the linear shell theory are identical.

W pracy wykazuje się, że stan naprężenia i odkształcenia powierzchni środkowej liniowej powłoki sprężystej jest równoważny przy pewnych założeniach stanowi naprężenia i odkształcenia w anizotropowej powierzchni Cosseratów. Podano definicję i podstawowe równania dla sprężystej powierzchni Cosseratów opierając się na modelu Voigta. Przytoczono podstawowe równania liniowej teorii powłok cienkich a następnie wykazano, że przy pewnych założeniach równania obu teorii są identyczne.

В статье показано, что напряженное и деформированное состояние срединной поверхности линейной упругой оболочки эквивалентны, при некоторых предположениях, напряженному и деформированному состояниям анизотропной поверхности Коссера. Даются определение и основные уравнения, описывающие упругую поверхность Коссера, причем исходной является модель Фойта. Приводятся основные уравнения линейной теории тонких оболочек, а затем дается доказательство, что при некоторых предположениях уравнения обоих теорий совпадают.

1. Introduction

THE PURPOSE of the paper is to obtain the equations of the linear elastic theory of thin shells on the basis of the model of an anisotropic Cosserat surface.

The linear elastic shell theory has been obtained from the model of the classical elastic continuum. Physical and geometrical assumptions used there lead to the equations describing a medium the natural model of which is the Cosserat surface. These equation contain such quantities as stress and couple resultants, displacements and rotations — that is, quantities the physical meaning of which is the same as the meaning of corresponding quantities used in description of the Cosserat medium.

Throughout the paper, the basic definitions and equations of the Cosserat theory in a case of the Cosserat surface will be used as well as those of the linear shell theory. Then the identity of both models is shown, if proper values of the elastic tensors are chosen and certain associations of dynamical and geometrical quantities of the micropolar elasticity and those of the shell theory are assumed. This makes it possible to apply the results obtained in the Cosserat theory to the linear elastic shell theory.

The problem of applying of the Cosserat theory to derivation equilibrium equations (in stresses) of the shell theory has been considered by J. L. ERICKSEN and. C. TRUESDELL [1]. In their paper, a generalized Cosserat medium was considered but without any constitutive assumptions.

A. E. GREEN, P. M. NAGHDI and W. L. WAINWRIGHT considered in their papers [2, 3, 4, 5] a general theory of the Cosserat surface. In paper [4], A. E. GREEN and P. M. NAGHDI considered an isotropic Cosserat surface and indicated the possibility of obtaining certain equations of the Kirchhoff-Love theory.

2. The coordinate system

The considerations presented here will involve the curvilinear coordinate system (x^i) referred to a fixed right-handed Cartesian system (z^i), $(i = 1, 2, 3)$ by

$$(2.1)\qquad x^i = x^i(z^1, z^2, z^3), \quad \det\left[\frac{\partial x^i}{\partial z^i}\right] > 0.$$

Let us consider a surface $x^3 = 0$ and identify x^i as convected coordinates with x^α $(\alpha = 1, 2)$ and x^3 along the normal to the surface.

Throughout the paper, Latin indices will range over the values (1, 2, 3) while the Greek ones will be required to have the values (1, 2).

For the base vectors $\mathbf{g}_i$ of the system (x^i), we have:

$$(2.2)\qquad \mathbf{g}_\alpha \cdot \mathbf{g}_\beta, = g_{\alpha\beta}, \quad \mathbf{g}_\alpha \cdot \mathbf{g}_3 = 0, \quad \mathbf{g}_3 \cdot \mathbf{g}_3 = 1, \quad \mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta^\beta_\alpha$$

The base vectors $\mathbf{g}_i$, when evaluated on the surface $x_3 = 0$, are $\mathbf{g}_\alpha = \mathbf{a}_\alpha$, $\mathbf{g}_3 = \mathbf{a}_3$ and the metric tensor of (x^i) when evaluated on $x_3 = 0$ is:

$$(2.3)\qquad g_{ij}|_{x^3=0} = \begin{bmatrix} [a_{\alpha\beta}] & 0 \\ 0 & 1 \end{bmatrix},$$

where $[a_{\alpha\beta}]$ is the surface metric tensor.

In the coordinate system considered here, the only non-vanishing Christoffel symbols are $\Gamma^\alpha_{\beta\delta}, \Gamma^\alpha_{\beta 3}, \Gamma^3_{\alpha\beta}$, and on the surface $x^3 = 0$ we have:

$$(2.4)\qquad \Gamma^3_{\alpha\beta}|_{x^3=0} = b_{\alpha\beta}, \quad \Gamma^\alpha_{3\beta}|_{x^3=0} = -b^\alpha_\beta,$$

where $b_{\alpha\beta}$ is the second fundamental form of the surface.

The covariant derivative will be designated by a stroke (|). The covariant differentiation with respect to the surface metric will be designated by double strokes (||) and when applied to a surface tensor reads:

$$(2.5)\qquad A^\alpha{}_\beta||_\gamma = A^\alpha{}_{\beta,\gamma} + \bar{\Gamma}^\alpha_{\delta\gamma} A^\delta{}_\beta - \bar{\Gamma}^\delta_{\beta\gamma} A^\alpha{}_\delta,$$

where $\bar{\Gamma}^\alpha_{\delta\gamma} = \Gamma^\alpha_{\delta\gamma}|_{x^3=0}$. Note that the covariant derivative of the surface tensor is:

$$(2.6)\qquad A^\alpha{}_\beta|_\gamma = A^\alpha{}_\beta||_\gamma - b^\alpha_\gamma A^3{}_\beta - b_{\beta\gamma} A^\alpha{}_{.3}.$$

3. An anisotropic Cosserat surface

Analogously as in three-dimensional case, definition of the Cosserat surface is assumed here according to the Voigt model [7]. A surface such that actions between its two parts are defined by the stress vector and couple stress vector will be called the Cosserat surface. A stress vector and a couple stress vector are defined by:

$$\mathbf{p} = \lim_{\Delta l \to 0} \frac{\Delta \mathbf{T}}{\Delta l}, \quad \mathbf{m} = \lim_{\Delta l \to 0} \frac{\Delta \mathbf{M}}{\Delta l}, \tag{3.1}$$

where $\Delta \mathbf{T}$ is a force acting on the line element Δl of the surface and $\Delta \mathbf{M}$ is a couple acting on the element Δl.

Let us consider a surface $x^3 = 0$. Over a curve with unit normal vector $\mathbf{n} = n_\alpha \mathbf{a}^\alpha$, there acts the force vector $\mathbf{p}$. If the stress (physical), vectors acting over each coordinate line are $\mathbf{p}^{(\alpha)}$ (Fig. 1), we have:

$$\mathbf{p} = \sum_\alpha n_\alpha \mathbf{p}^{(\alpha)} (a^{\alpha\alpha})^{1/2} = \mathbf{p}^\alpha n_\alpha \tag{3.2}$$

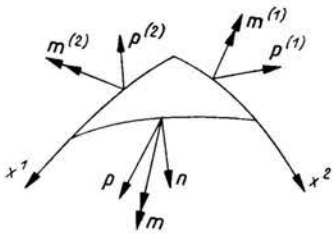

FIG. 1.

Since $\mathbf{p}^\alpha$ transforms as a contravariant surface vector, we can introduce the definition of the stress tensor:

$$\mathbf{p}^\alpha = \sigma^{\alpha i} \mathbf{a}_i, \tag{3.3}$$

where $\sigma^{\alpha i}$ are components of the stress tensor $[\sigma^{\alpha i}]$.

Analogously, for the couple force vector $\mathbf{m}$, we obtain:

$$\mathbf{m} = \sum_\alpha n_\alpha \mathbf{m}^{(\alpha)} (a^{\alpha\alpha})^{1/2} = \mathbf{m}^\alpha n_\alpha, \quad \mathbf{m}^\alpha = \mu^{\alpha i} \mathbf{a}_i, \tag{3.4}$$

where $\mu^{\alpha i}$ are the components of the couple stress tensor.

If the surface element $dS = dx^1 dx^2$ is loaded by an external force $\mathbf{X} dS$ and an external moment $\mathbf{Y} dS$, then the equations of motion are:

$$\begin{gathered} \int_S (\varrho \ddot{\mathbf{u}} - \mathbf{X}) dS - \int_l \mathbf{p}\, dl = 0, \\ \int_S [I \ddot{\boldsymbol{\varphi}} - (\mathbf{r} \times \mathbf{X} + \mathbf{Y})] dS - \int_l (\mathbf{r} \times \mathbf{p} + \mathbf{m}) dl = 0, \end{gathered} \tag{3.5}$$

where $\mathbf{u}(x^\alpha, t) = u_i \mathbf{a}^i$ is a displacement vector, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x^\alpha, t)$ is rotation, ϱ is the mass density, I — the inertia term due to rotation $\boldsymbol{\varphi}$, dots denote differentiation with respect to time, $\mathbf{r} = z^i \mathbf{i}_i$.

Using Stokes theorem to the line integral, from vanishing of the integrands the local equations of motion follow, which in component form are:

$$
\begin{aligned}
&\sigma^{\alpha\beta}||_{\alpha} - b^{\beta}_{\alpha}\sigma^{\alpha 3} + X^{\beta} - \varrho\ddot{u}^{\beta} = 0,\\
&b_{\alpha\beta}\sigma^{\alpha\beta} + \sigma^{\alpha 3}||_{\alpha} + X^{3} - \varrho\ddot{u}^{3} = 0,\\
&\varepsilon_{\alpha\beta}\sigma^{\beta 3} + \mu^{\beta}_{\cdot\alpha}||_{\beta} - b_{\alpha\beta}\mu^{\beta}_{\cdot 3} + Y_{\alpha} - I\ddot{\varphi}_{\alpha} = 0,\\
&\varepsilon_{\alpha\beta}\sigma^{\alpha\beta} + \mu^{\alpha}_{\cdot 3}||_{\alpha} + b_{\alpha\beta}\mu^{\alpha\beta} + Y_{3} - I\ddot{\varphi}_{3} = 0,
\end{aligned}
\tag{3.6}
$$

where $\varepsilon_{\alpha\beta}$ is the surface Ricci tensor.

The equation of balance of energy has the form:

$$
\frac{D}{Dt}\int\limits_{S}\left(\frac{1}{2}\varrho\dot{u}^{2} + I\dot{\varphi}^{2} + U\right)dS = \int\limits_{S}(\mathbf{X}\cdot\dot{\mathbf{u}} + \mathbf{Y}\cdot\dot{\boldsymbol{\varphi}})\,dS + \int\limits_{l}(\mathbf{p}\cdot\dot{\mathbf{u}} + \mathbf{m}\cdot\dot{\boldsymbol{\varphi}})\,dl,
\tag{3.7}
$$

where U is the internal energy (elastic potential).

After applying the Stokes theorem and taking into considerations the equations of motion, we obtain the local form of (3.7), which written in component form is:

$$
\dot{U} = \sigma^{\alpha\beta}(\dot{u}_{\beta|\alpha} - \varepsilon_{\alpha\beta}\dot{\varphi}^{3}) + \sigma^{\alpha 3}(\dot{u}_{3|\alpha} + \varepsilon_{\alpha\beta}\dot{\varphi}^{\beta}) + \mu^{\alpha i}\dot{\varphi}_{i|\alpha}.
\tag{3.8}
$$

From the above equation, the definition of the strain tensors for the Cosserat surface follows:

$$
\gamma_{\alpha\beta} = u_{\beta|\alpha} - \varepsilon_{\alpha\beta}\varphi^{3}, \quad \gamma_{\alpha 3} = u_{3|\alpha} + \varepsilon_{\alpha\beta}\varphi^{\beta}, \quad \varkappa_{\alpha i} = \varphi_{i|\alpha},
\tag{3.9}
$$

and the constitutive equations are

$$
\sigma^{\alpha i} = \frac{\partial U}{\partial \gamma_{\alpha i}}, \quad \mu^{\alpha l} = \frac{\partial U}{\partial \varkappa_{\alpha l}}.
\tag{3.10}
$$

The general form of the constitutive equations for the anisotropic Cosserat surface may be written [10]:

$$
\begin{aligned}
\sigma^{\alpha k} &= C_{1}^{\alpha k\gamma l}\gamma_{\gamma l} + C_{2}^{\alpha k\gamma l}\varkappa_{\gamma l},\\
\mu^{\alpha k} &= C_{2}^{\alpha k\gamma l}\gamma_{\gamma l} + D_{1}^{\alpha k\gamma l}\varkappa_{\gamma l}.
\end{aligned}
\tag{3.11}
$$

From the existence of the elastic potential U, the following symmetries of the elastic tensors C_1, C_2, D_1 result:

$$
C_{1}^{\alpha k\gamma l} = C_{1}^{\gamma l\alpha k}, \quad C_{2}^{\alpha k\gamma l} = C_{2}^{\gamma l\alpha k}, \quad D_{1}^{\alpha k\gamma l} = D_{1}^{\gamma l\alpha k}.
\tag{3.12}
$$

For further considerations, the following form of (3.11) will be assumed:

$$
\begin{aligned}
\sigma^{\alpha\beta} &= C_{1}^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta} + C_{2}^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta},\\
\sigma^{\alpha 3} &= C_{1}^{\alpha 3\gamma 3}\gamma_{\gamma 3},\\
\mu^{\alpha\beta} &= C_{2}^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta} + D_{1}^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta}, \quad \mu^{\alpha 3} = 0.
\end{aligned}
\tag{3.13}
$$

In the case of an isotropic Cosserat surface, the constitutive relations have the form:

$$
\sigma^{\alpha i} = C_{1}^{\alpha i\gamma k}\gamma_{\gamma k}, \quad \mu^{\alpha i} = D_{1}^{\alpha i\gamma k}\varkappa_{\gamma k}
\tag{3.14}
$$

resulting from the isotropy of the elastic potential U, where the coefficients $C_{1}^{\alpha i\gamma k}$, $D_{1}^{\alpha i\gamma k}$ are homogeneous, linear functions of products of components of the metric tensor.

4. Linear, elastic shell theory

On the basis of [6] the fundamental definitions and equations of the linear, elastic shell theory will be given.

The components of the displacement of a point placed at a distance $x^3 = z$ from the middle surface are assumed to be of the form:

$$\tag{4.1} \begin{aligned} \tilde{u}_\alpha(x^i)|_{x^3=z} &= \mathring{u}_\alpha(x^\alpha)+z\beta_\alpha(x^\alpha),\\ \tilde{u}_3(x^i)|_{x^3=z} &= \mathring{u}_3(x^\alpha), \end{aligned}$$

where $\mathring{u}_i(x^\alpha)$ are components of the displacement of the point of the middle surface, $\beta_\alpha(x^\beta)$ is rotation of the normal to the middle surface.

Definitions of the strain tensors are:

$$\tag{4.2} \tilde{\gamma}_{\alpha\beta} = \mathring{u}_\alpha|_\beta, \quad \tilde{\gamma}_{\alpha 3} = \mathring{u}_{3,\alpha}+\beta_\alpha+b^\beta_\alpha\mathring{u}_\beta, \quad \tilde{\varkappa}_{\alpha\beta} = \beta_\alpha||_\beta.$$

Action of the classical elastic stress tensor on the surfaces of the shell element is replaced by the stress and couple resultants acting on the middle surface. Then, the state of stress of the middle surface is characterized by the following stress and couple resultants (Fig. 2):

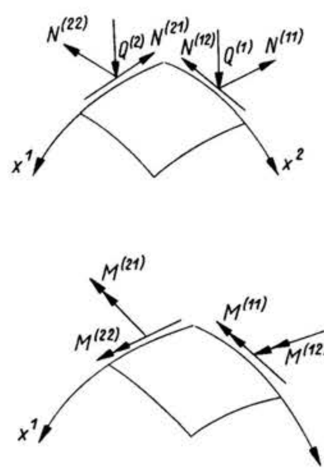

FIG. 2.

$$\tag{4.3} N^{\alpha\beta}(x^\alpha) = \int\limits_{-h/2}^{h/2} \tilde{\sigma}^{\alpha\beta}(x^i)\,dx^3, \quad Q^\alpha(x^\alpha) = \int\limits_{-h/2}^{h/2} \tilde{\sigma}^{\alpha 3}(x^i)\,dx^3,$$

$$M^{\alpha\beta}(x^\alpha) = \int\limits_{-h/2}^{h/2} \tilde{\sigma}^{\alpha\beta}(x^i)x^3\,dx^3,$$

where h is a shell thickness, and $\tilde{\sigma}^{\alpha\beta}$, $\tilde{\sigma}^{\alpha 3}$ are the components of the classical elastic stress tensor.

The constitutive equations for the anisotropic shell, resulting from (4.3) are:

$$N^{\alpha\beta} = A_1^{\alpha\beta\delta\gamma}\tilde{\gamma}_{\gamma\delta}+A_2^{\alpha\beta\delta\gamma}\tilde{\varkappa}_{\gamma\delta}, \quad Q^\lambda = A^{\lambda 3\alpha 3}\tilde{\gamma}_{\alpha 3},$$
$$M^{\alpha\beta} = A_2^{\alpha\beta\delta\gamma}\tilde{\gamma}_{\gamma\delta}+B_1^{\alpha\beta\delta\gamma}\tilde{\varkappa}_{\gamma\delta}. \tag{4.4}$$

The components of the tensors A_1, A_2, B_1 have the properties:

$$A_1^{\alpha\beta\delta\gamma} = A_1^{\delta\gamma\alpha\beta}, \quad A_2^{\alpha\beta\delta\gamma} = A_2^{\delta\gamma\alpha\beta}, \quad B_1^{\alpha\beta\delta\gamma} = B_1^{\delta\gamma\alpha\beta}. \tag{4.5}$$

The equilibrium equations have the form:

$$N^{\alpha\beta}||_\alpha - b_\alpha^\beta Q^\alpha + \tilde{X}^\beta = 0, \quad b_{\alpha\beta}N^{\alpha\beta}+Q^\alpha||_\alpha+\tilde{X}^3 = 0,$$
$$-Q^\lambda + M^{\alpha\lambda}||_\alpha = 0, \quad \varepsilon_{\beta\alpha}(N^{\alpha\beta}-b_\lambda^\alpha M^{\lambda\beta}) = 0, \tag{4.6}$$

where $\tilde{X}^i$ is the component of the external, surface load of the shell. It is assumed in the shell theory that it acts on the middle surface.

A particular case of linear shell theory is the Kirchhoff-Love theory. This results from the general theory if it is assumed that

$$\tilde{\gamma}_{\alpha 3} = 0. \tag{4.7}$$

This assumption leads to the following connections:

$$\beta_\alpha = -\mathring{u}_{3,\alpha} - b_\alpha^\beta \mathring{u}_\beta, \tag{4.8}$$

and the constitutive equation for Q^α is missing.

In the general theory, a number of quantities which are to be obtained from the equilibrium equations, taking into consideration the constitutive relations is five $\mathring{u}_i$, β_α, while the number of equations is six. It has been shown that the last of the equations (4.6) is satisfied identically [9]. This results from certain properties of the elastic tensors of the shell theory.

In the Kirchhoff-Love theory, also five quantities are to be evaluated from the equilibrium equations, $\mathring{u}_i$ and β_α.

The linear shell theory is approximate and applicable to thin shells—that is, to shells of small thickness h as compared with other dimensions of the shell. Usually, as a small parameter is chosen h/R_α, where R_α is the radius of curvature of the middle surface. The form of the coefficients of the constitutive equations depends on the order of approximation and is differently assumed by different authors.

5. The anisotropic Cosserat surface and the shell theory

It will be shown now that the state of strain and stress of the anisotropic Cosserat surface loaded by surface forces **X** is, with some assumptions, equivalent to the state of strain and stress of the middle surface of the linear, elastic shell loaded by surface forces $\tilde{\mathbf{X}} = \mathbf{X}$.

From physical interpretation of the dynamical quantities of the two theories, it results that the stresses and couple stresses of the Cosserat theory are equivalent to the proper stress and couple resultants of the shell theory (Fig. 2, 3). This equivalence is as follows:

$$N^{\alpha\beta} = \sigma^{\alpha\beta}, \quad Q^\alpha = \sigma^{\alpha 3}, \quad M^{\alpha\beta} = \varepsilon^{\beta\lambda}\mu^{\alpha}_{\cdot\cdot\lambda}. \tag{5.1}$$

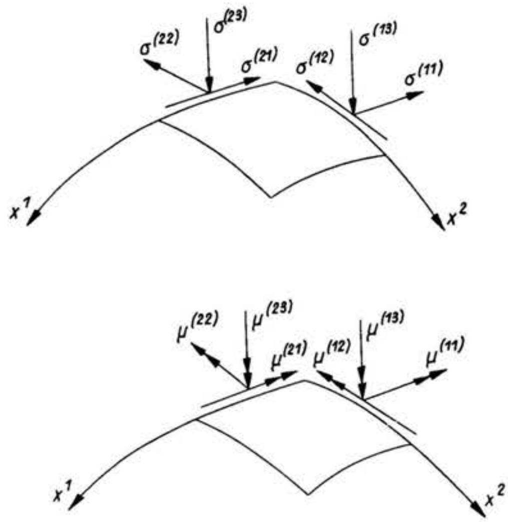

FIG. 3.

The above relations produce identity of a static case of the equations (3.6) and (4.6), if $\mathbf{Y} = 0$ and $\mathbf{X} = \tilde{\mathbf{X}}$.

If it is assumed that the components of the displacement of the Cosserat surface u_i are equivalent to the displacement components $\mathring{u}_i$ of the middle surface of the shell

$$u_i(x^\alpha) = \mathring{u}_i(x^\alpha), \tag{5.2}$$

and that between rotations of the Cosserat theory φ_α and rotations of the shell theory β_α the following correspondence holds

$$\beta_\alpha = \varepsilon_{\alpha\beta}\varphi^\beta, \tag{5.3}$$

then, with the additional assumption $\varphi_3 = 0$ from (3.9) and (4.2), we obtain

$$\tilde{\gamma}_{\alpha\beta} = \gamma_{\beta\alpha}, \quad \tilde{\gamma}_{\alpha3} = \gamma_{\alpha3}, \quad \tilde{\varkappa}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\varkappa_{\beta\cdot}^{\cdot\gamma}. \tag{5.4}$$

It can be observed that (5.1) and (5.4) lead to the identities:

$$N^{\alpha\beta}\tilde{\gamma}_{\beta\alpha} = \sigma^{\alpha\beta}\gamma_{\alpha\beta}, \quad Q^\lambda\tilde{\gamma}_{\lambda3} = \sigma^{\lambda3}\gamma_{\lambda3}, \quad M^{\alpha\beta}\tilde{\varkappa}_{\beta\alpha} = \mu^{\alpha\beta}\varkappa_{\alpha\beta}.$$

It will be shown further that if the values of the elastic tensors C_1, C_2, D_1 in the constitutive equations (3.13), are properly chosen identity of the Eqs. (3.13) and (4.4) can be obtained.

Let us assume the following equalities:

$$\begin{gathered} C_1^{\alpha\beta\gamma\delta} = A_1^{\alpha\beta\gamma\delta}, \quad C_2^{\alpha\beta\delta\gamma} = \varepsilon_{\nu\tau}a^{\tau\gamma}A_2^{\alpha\beta\delta\nu}, \\ D_1^{\alpha\beta\delta\gamma} = \varepsilon_{\xi\lambda}\varepsilon_{\nu\tau}a^{\lambda\beta}a^{\tau\gamma}B_1^{\alpha\xi\delta\nu}, \quad C_1^{\alpha3\gamma3} = A^{\alpha3\gamma3}. \end{gathered} \tag{5.5}$$

Substituting (5.5) into (3.13), and taking into account (5.4), the following connections hold:

$$\begin{aligned} \sigma^{\alpha\beta} &= C_1^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta} + C_2^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta} = A_1^{\alpha\beta\delta\gamma}\tilde{\gamma}_{\gamma\delta} + A_2^{\alpha\beta\gamma\nu}\tilde{\varkappa}_{\nu\gamma} = N^{\alpha\beta}, \\ \sigma^{\alpha3} &= C_1^{\alpha3\gamma3}\tilde{\gamma}_{\gamma3} = A^{\alpha3\gamma3}\tilde{\gamma}_{\gamma3} = Q^\alpha, \\ \mu^{\alpha\beta} &= C_2^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta} + D_1^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta} = (A_1^{\alpha\nu\gamma\delta}\tilde{\gamma}_{\delta\gamma} + B_1^{\alpha\nu\gamma\delta}\tilde{\varkappa}_{\delta\gamma})\varepsilon_{\nu\tau}a^{\tau\beta} = \varepsilon_{\nu\tau}a^{\tau\beta}M^{\alpha\nu}. \end{aligned} \tag{5.6}$$

It is easy to show, taking into account (4.5), that the correspondence (5.5) does not disturb the symmetries (3.12).

It is proved then, that all basic equations of the linear shell theory are equivalent to the corresponding equations of the anisotropic Cosserat surface, assuming $\varphi_3 = 0$. This means that the state of stress and strain of the Cosserat surface and of the middle surface of the shell are equivalent if exterior loads are equal $\mathbf{X} = \tilde{\mathbf{X}}$.

The Kirchhoff-Love theory can be obtained from the Cosserat theory with the above assumptions, analogously as it is obtained in the shell theory — that is, putting $\gamma_{\alpha 3} = 0$.

It can be observed that the constitutive equations of the isotropic shell theory are equivalent to the constitutive equations of the anisotropic Cosserat surface. For example, the constitutive equations obtained by KOITER for isotropic shells [6] include non-vanishing components of the tensor A_2. Taking into consideration (5.5) and (3.14), the tensor A_2 should vanish if the equivalent Cosserat surface has to be isotropic.

6. Conclusions

From above considerations, it results that the natural model applicable in description of the linear theory of thin shells, where the state of stress and strain is determined by the state of stress and strain of the middle surface, is the theory of the Cosserat surface.

The linear shell theory starts from the equations of classical elasticity but by certain manipulations the equations obtained there describe a certain model of an anisotropic Cosserat medium.

References

1. J. L. ERICKSEN, C. TRUESDELL, *Exact theory of stress and strain in rods and shells*, Arch. Rat. Mech Anal., **1**, 295–323, 1958.
2. A. E. GREEN, P. M. NAGHDI, W. L. WAINWRIGHT, *A general theory of a Cosserat surface*, Arch. Rat. Mech. Anal., **20**, 4, 1965.
3. A. E. GREEN, P. M. NAGHDI, *The linear elastic Cosserat surface and shell theory*, Int. J. Sol. Struct., **4**, 209–244, 1968.
4. A. E. GREEN, P. M. NAGHDI, *The Cosserat surface*, IUTAM Symp. Freudenstadt-Stuttgart, 1967 (Springer-Verlag 1968).
5. A. E. GREEN, P. M. NAGHDI, *Non-isothermal theory of rods, plates and shells*, Int. J. Sol. Struct., **6**, 209–244, 1970.
6. P. M. NAGHDI, *Foundations of elastic shell theory*, Prog. Sol. Mech., **4**, 1963.
7. W. NOWACKI, *Theory of non-symmetrical elasticity* [in Polish], Warszawa 1971.
8. W. NOWACKI, *Theorems of non-symmetrical elasticity* (Lectures, Udine 1969).
9. В. В. НОВОЖИЛОВ [V. V. Novozhilov], *Теория тонких оболочек*, Ленинград 1962 [Theory of thin shells, in Russian].
10. R. STOJANOVIČ, *Mechanics of polar continua* (Lectures, Udine 1969).

Received June 6, 1972

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received June 6, 1972