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Thermomechanical coupling in materials with memory(*)

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A PROPOSAL for thermodynamic theory of a material with memory, the constitutive functionals of which depend on the summed history of the temperature gradient, the history of the deformation gradient and the history of the temperature is given. The theory is constructed when the weakest possible assumptions for the constitutive functionals are employed. It is supposed that their domain is a subset of a linear topological Hausdorff space of histories. A chain rule property for the free energy functional and consequences of the second law of thermodynamics are demonstrated, together with some properties for the free energy and heat flux in a state of equilibrium.

Przedstawiono termodynamiczną teorię materiału z pamięcią, którego funkcjonały konstytutywne zależą od zsumowanej historii gradientu temperatury, historii gradientu deformacji i historii temperatury. Teorię zbudowano przy możliwie najsłabszych założeniach dotyczących funkcjonałów konstytutywnych. Przyjęto, że ich dziedzina jest podzbiorem liniowej topologicznej przestrzeni Hausdorffa. Pokazano prawo różniczkowania złożonego dla energii swobodnej oraz konsekwencje drugiego prawa termodynamiki. Wykazano pewne własności dla energii swobodnej i strumienia ciepła w stanie równowagi.

Представлена термодинамическая теория материала с памятью, определяющие функционалы которого зависят от просуммированной истории градиента температуры, истории градиента деформации и истории температуры. Теория построена при возможно самых слабых предположениях, касающихся определяющих функционалов. Принимается, что область их определения является подмножеством линейного топологического пространства Хаусдорфа. Указан закон сложного дифференцирования для свободной энергии, а также следствия второго начала термодинамики. Указаны некоторые свойства свободной энергии и потока тепла в состоянии равновесия.

1. Introduction

A MAIN problem in the thermodynamics of materials with memory is that of defining the restrictions which the second law of thermodynamics imposes on constitutive functionals — i.e. on functions describing the response of a material. For a simple material which examplifies material with memory, this problem was formulated and solved in 1964 by B. D. COLEMAN [1–2]. His investigations were based on the strong principle of fading memory, later however, in 1970, in a joint paper with D. R. OWEN [7] he showed that a thermodynamic theory of simple materials could be developed without endowing the domain of the constitutive functionals with the structure of normed space.

The basic theme of the present paper is the thermodynamic theory of a material with memory, the constitutive functionals of which depend on the summed history of the temperature gradient, the history of the deformation gradient and the history of the temperature. The summed history of the temperature gradient as one of the independent variables enables description of thermal disturbances propagating with finite speeds.

The theory presented in this paper is constructed when the weakest possible assumptions for the constitutive functionals are employed. It is supposed that their domain is

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a subset of a linear topological Hausdorff space of histories. The conception of the convergence is introduced into the domain by Moore-Smith sequences. The general assumptions concerning functionals enable the existence of a time-derivative of the free energy to be proved. It is shown that this derivative obeys a chain rule similar to such occurring in the theories of Coleman and Owen.

It is noticed that the linear topological space containing the domain of functionals need not be normable or even metrizable.

On the basis of the chain rule property, it is shown that the second law has the following implications for constitutive functionals: 1) the functionals for the stress, entropy and heat flux are completely determined by the functional for the free energy; 2) a part of the derivative of the free energy obeys an inequality called "the dissipation inequality". If a description of a state is given by a constant history of the deformation gradient, the temperature and the temperature gradient, the dissipation inequality turns into the usual heat conduction inequality.

In the final part of the paper are proved some properties of the free energy and heat flux in a state of equilibrium, such as are obtained in the case of a material with fading memory.

The main aim of these investigations is to formulate a thermodynamic theory of material with memory within the framework of which plastic materials may be described. The smoothness assumptions of theories based on the principle of fading memory are too strong to enable plastic materials to be considered.

2. Definitions of state and process. Choice of a method of preparation

Let us consider a body B with particles X and a fixed reference configuration $\varkappa$. We assume that this body can deform and conduct heat. We identifly each of the particles X of B with the place ζ it occupies in $\varkappa$. We introduce the deformation gradient (1) $F(X, t) = F(\varkappa(X), t) \equiv \frac{\partial}{\partial \zeta}\gamma(\zeta, t)$, where γ is the motion, with $x = \gamma(\zeta, t)$ as the place at time t of the particle $\varkappa^{-1}(\zeta)$. To describe thermal effects, we introduce the absolute temperature $\vartheta(X, t) > 0$ and the temperature gradient $g(X, t) \equiv \frac{\partial}{\partial x}\vartheta(\gamma^{-1}(x, t), t)$.

For fixed time t and particle X, we can define the history F^t of the deformation gradient, the history ϑ^t of the temperature and the history g^t of the temperature gradient by

$$(2.1) \qquad F^t(X, s) \equiv F(X, t-s), \quad \vartheta^t(X, s) \equiv \vartheta(X, t-s), \quad g^t(X, s) \equiv g(X, t-s), \quad s \in [0, \infty].$$

We further define the summed history $\bar{g}^t$ of the temperature gradient(2):

$$(2.2) \qquad \bar{g}^t(X, s) \equiv \int_0^s g^t(X, \lambda)\, d\lambda.$$

(1) We assume that det $F \neq 0$.

(2) For the existence of this integral we assume that $g^t(X, \cdot)$ is a measurable function on $[0, \infty)$.

If f is a function of t, we denote by $\dot{f}$ the left-hand derivative of f: $\dot{f}(t) \equiv \lim_{\sigma \to 0} \frac{1}{\sigma}\{f(t)-$ $-f(t-\sigma)\}$.

The following definitions and two constitutive postulates form the logical structure of the thermodynamic theory developed.

This structure is based on the general structure of the thermodynamic local theory of material proposed by P. PERZYNA in 1971(3).

DEFINITION 1. The ordered pair $(F(X,t), \vartheta(X,t)) \equiv \Lambda(X,t)$ is called the actual deformation-temperature configuration of a particle X of a body B at time t.

DEFINITION 2. A local thermodynamic process (t_p, t_k) at particle X of a body B is a collection of functions given for every $t \in (t_p, t_k)$

$$\mathscr{P}_X = \{\Lambda(X,t), g(X,t), \pi(X,t)\}, \tag{2.3}$$

which satisfies the thermodynamic inequality

$$-\dot{\psi}+\mathrm{tr}(T\dot{F}^T)-\eta\dot{\vartheta}-\frac{1}{\varrho\vartheta}q\cdot g \geqslant 0, \tag{2.4}$$

at all times at which the derivatives $\dot{F}$, $\dot{\vartheta}$ and $\dot{\psi}$ exist, where $\pi(X,t) = \{\psi(X,t), T(X,t), \eta(X,t), q(X,t)\}$ represents specific free energy per unit mass $\psi(X,t)$, the first Piola-Kirchhoff stress tensor $T(X,t)$, the specific entropy $\eta(X,t)$ and the heat flux vector $q(X,t)$ per unit surface in the actual configuration γ.

It is implicitly assumed that the body forces and the rate of heat supply are determined by the requirement that the process obeys the laws of balance of momentum and energy. A symbol ϱ denotes the mass density in the actual configuration.

DEFINITION 3. A thermo-mechanical state of a particle X in time t is a collection of values which take the functions $\mathscr{P}_X$ for particular time $t \in (t_p, t_k)$.

DEFINITION 4. A description of a thermo-mechanical state of a particle X in the time t consists of the actual deformation-temperature configuration $\Lambda(X,t)$, the temperature gradient $g(X,t)$ and of the method of preparation of this configuration.

To determine the actual thermo-mechanical state of a particle during an irreversible thermodynamic procces, it does not suffice to have the actual deformation-temperature configuration of a particle X but we additionally need the method of preparation of this configuration.

A method of preparation of the configuration is a primitive concept in our theory. To give the rule of interpretation for this concept (and to be in agreement with requirements which are formulated for each physical theory(4)) we give a few examples of the method of preparation. The history of the terms appearing in the actual thermo-mechanical configuration of the particle X can form the method of preparation. The thermodynamic theory of a simple material can be developed when this method of preparation is assumed(5). Another example of the method of preparation is by introducing internal parameters

(3) In the present theory a definition of the actual configuration different from that introduced by PERZYNA [16] is assumed.

(4) Cf. GILES [9] and PERZYNA [16].

(5) Cf. COLEMAN [1-2], COLEMAN and OWEN [7]. See also [11].

and initial-value problems for the differential equations. The theory of rheological materials with internal structural changes can be constructed by suitable interpretations of the parameters(6).

In the present theory, we choose a different method of preparation.

Postulate K1. The method of preparation of the actual deformation-temperature configuration is the history $\Lambda^t(X, t)$, $s \in (0, \infty)$ and the summed history of the temperature gradient $\bar{g}^t(X, s)$, $s \in [0, \infty)$.

To specify the material structure in a body B, we shall introduce.

Postulate K2. The thermo-mechanical principle of determinism for the material is expressed by the functional relations

$$\pi(X, t) = \mathscr{R}(\Lambda^t(X, \cdot), \bar{g}^t(X, \cdot)), \tag{2.5}$$

where $\mathscr{R} = \{\mathfrak{p}, \mathfrak{s}, \mathfrak{h}, \mathfrak{q}\}$ represents the constitutive functionals for the free energy $\mathfrak{p}$, stress $\mathfrak{s}$, entropy $\mathfrak{h}$ and heat flux $\mathfrak{q}$.

Definition 5. A local thermodynamic process described by $\mathscr{P}_X$ is said to be admissible in B if it is compatible with Postulate K2 at each particle X.

If the constitutive functionals in (2.5) are chosen arbitrarily, it cannot be expected that the thermodynamic inequality (2.4) will hold. Indeed, the present paper will be concerned mainly with the problem of finding the restrictions which the thermodynamic inequality (2.4) places on constitutive functionals. We shall treat the problem for a broad class of materials, defined by postulates of regularity for $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{h}$, $\mathfrak{q}$ and their domain.

For future considerations, it will be useful to introduce two linear (vector) spaces:
$V_{10} \equiv \{\Gamma : \Gamma = (L, \lambda)$, L — a tensor of order two, λ — a real number$\}$,
$V_3 \equiv \{k : k$ — a vector of three-dimensional Euclidean space$\}$.

In the space V_{10} rules are defined by

$$\alpha\Gamma_1 + \beta\Gamma_2 = (\alpha L_1 + \beta L_2, \alpha\lambda_1 + \beta\lambda_2),$$

$$\Gamma_1 \cdot \Gamma_2 = \mathrm{tr}(L_1 L_2^T) + \lambda_1 \lambda_2, \quad |\Gamma|_{10} = (\Gamma \cdot \Gamma)^{\frac{1}{2}}$$

for $\Gamma_1 = (L_1, \lambda_1) \in V_{10}$, $\Gamma_2 = (L_2 . \lambda_2) \in V_{10}$, α, β — real numbers.

In the space V_3 addition, scalar multiplication, inner product and norm are defined pointwise, as usual.

Introduce a cone $V_{10}^+ \subset V_{10}$ by

$$V_{10}^+ \equiv \{\Gamma \in V_{10} : \Gamma = (F, \vartheta),\ F \text{ — an invertible tensor, } \vartheta > 0\}.$$

In a local thermodynamic process, for each time $t \in (t_p, t_k)$, the total history up to t of deformation gradient and temperature, is a function(7) $(\Lambda^t = (F^t, \vartheta^t))$ mapping $[0, \infty)$ into V_{10}^+. The summed history of the temperature gradient is a function
$\bar{g}: [0, \infty) \to V_3$ such that there exists a measurable function

$$g: [0, \infty) \to V_3 \text{ such that } \bar{g}(s) = \int_0^s g(\lambda)\, d\lambda \text{ for } s \in [0, \infty).$$

(6) See Perzyna [16], where was developed a thermodynamic theory of this material, taking into account highergradients of the deformation and temperature. Cf. also Perzyna [14–15].

(7) We assume that the elements of the set V_{10}^+ are the same as pairs of deformation gradient-temperature.

Let us put [8]

(2.6) $$\Sigma \equiv \left(T, -\eta, \frac{1}{\varrho\vartheta} q\right), \quad \hat{\alpha} \equiv (\dot{F}, \dot{\vartheta}, -g);$$

then (2.4) can be written

(2.7) $$-\dot{\psi}+\Sigma\cdot\hat{\alpha} \geqslant 0, \quad \text{where} \quad \Sigma\cdot\hat{\alpha} = \operatorname{tr}(T\dot{F}^T)-\eta\dot{\vartheta}-\frac{1}{\varrho\vartheta} q\cdot g.$$

Let us introduce the new functionals

(2.8) $$\mathfrak{S} \equiv (\mathfrak{s}, -\mathfrak{h}), \quad \mathfrak{n} \equiv \left(\mathfrak{S}, \frac{1}{\varrho\vartheta}\mathfrak{q}\right).$$

Then we can write (2.5) in the form[9]

(2.9) $$\begin{aligned} \Sigma(t) &= \mathfrak{n}(\Lambda^t, \bar{g}^t),\\ \psi(t) &= \mathfrak{p}(\Lambda^t, \bar{g}^t). \end{aligned}$$

3. Properties of constitutive functionals and their domain

We suppose that there are given two topological linear (vector) Hausdorff spaces $\mathfrak{B}_0$ and $\mathfrak{B}_1$. The space $\mathfrak{B}_0$ is formed from functions mapping $[0, \infty)$ into V_{10} and $\mathfrak{B}_1$ is formed from functions mapping $[0, \infty)$ into V_3. In both spaces, addition and scalar multiplication are defined naturally.

We take the domain of definition of constitutive functionals in (2.9) to be the subset $\mathscr{D} \equiv \mathscr{E}\times\mathfrak{B}_1$ with the topology induced on $\mathscr{D}$ by $\mathfrak{B} \equiv \mathfrak{B}_0\times\mathfrak{B}_1$. Here $\mathscr{E}$ is the set of functions from $[0, \infty)$ into V_{10}^+ belonging to $\mathfrak{B}_0$.

For our considerations we need to introduce a fundamental definition concerning the functions vanishing rapidly with δ. This definition is different from that introduced by COLEMAN and OWEN [7].

DEFINITION 6. Let $a > 0$. A one-parameter family of pairs of functions (Φ_δ, h_δ), where Φ_δ: $[0, \infty) \to V_{10}$, h_δ: $[0, \infty) \to V_3$ for $\delta \in (0, a)$ is a regular family of functions vanishing rapidly with δ if:

a) $\bigwedge_{\delta\in(0,a)} \Phi_\delta, h_\delta$ are continuous functions on $[0, \infty)$ and furthermore Φ_δ is a piecewise continuously differentiable functions on $[0, \infty)$;

b) $$\bigvee_{K>0}\bigvee_{M>0}\bigvee_{N>0}\bigwedge_{\delta\in(0,a)}\bigwedge_{s\in[0,\infty)} |\Phi_\delta(s)|_{10} \leqslant K\chi_{[0,\delta]}(s)\,\delta \wedge \left|\frac{d}{ds}\Phi_\delta(s)\right|_{10} \leqslant N\chi_{[0,\delta]}(s) \wedge |h(s)|_3 \leqslant M\chi_{[0,\delta]}(s)\,\delta.$$

Here $\chi_{[0,\delta]}$ is the characteristic function of the interval $[0, \delta]$ and $\frac{d}{ds}\Phi_\delta$ is the right-hand derivative of Φ_δ:

$$\frac{d}{ds}\Phi_\delta(s) \equiv \lim_{\sigma\to0^+}\frac{1}{\sigma}\{\Phi_\delta(s+\sigma)-\Phi_\delta(s)\}.$$

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[8] Σ and $\hat{\alpha}$ are elements of $V_{10}\times V_3$.

[9] To simplify notations, we shall omit consequently a particle X in all formulae.

Given a number $\sigma \geqslant 0$ and functions $(\Psi, \bar{g}) \in \mathfrak{B}$, we may define new functions:

$$(3.1)\qquad L_\sigma \Psi(s) \equiv \begin{cases} \Psi(\sigma) \text{ for } s \in [0, \sigma], \\ \Psi(s) \text{ for } s \in (\sigma, \infty); \end{cases} \quad l_\sigma \bar{g}(s) \equiv \begin{cases} \bar{g}(\sigma) \text{ for } s \in [0, \sigma], \\ \bar{g}(s) \text{ for } s \in (\sigma, \infty); \end{cases}$$

$$S_\sigma \Psi(s) \equiv \Psi(s+\sigma) \text{ for } s \in [0, \infty);\ \mathfrak{z}_\sigma \bar{g}(s) \equiv \int_0^s g(\lambda+\sigma) d\lambda \text{ for } s \in [0, \infty).$$

The pairs (L_σ, l_σ) and $(S_\sigma, \mathfrak{z}_\sigma)$ are for fixed σ operators (transformations) of space $\mathfrak{B}$. They are helpful is formulating the following postulates. These postulates are similar in a part to those introduced by Coleman and Owen [7].

Postulate P1. The constitutive functionals $\mathfrak{p}$ and $\mathfrak{n}$ are continuous functions on $\mathscr{D}$.

Postulate P2. If (Φ_δ, h_δ) is a regular family of functions vanishing rapidly with δ, then (Φ_δ, h_δ) is a Moore-Smith convergent sequence ([10]) in $\mathfrak{B}$ on $(0, a)$ with the limit $0^\dagger \in \mathfrak{B}$, where $0^\dagger: [0, \infty) \to 0 \in V_{10} \times V_3$.

Postulate P3. There exists a subset $\hat{\mathscr{D}} \subset \mathscr{D}$, which is dense in $\mathscr{D}$. Furthermore, if $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$ then Ψ is continuously differentiable and g is continuous ([11]) both on some interval $[0, \beta], \beta > 0$..

Postulate P4. If $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$ and (Φ_δ, h_δ) is a regular family of functions vanishing rapidly with δ, then for sufficiently small δ, the pair $(\Psi+\Phi_\delta, \bar{g}+h_\delta)$ is in $\hat{\mathscr{D}}$ and there exist $\mathsf{D}_\Gamma \mathfrak{p}: \hat{\mathscr{D}} \to V_{10}$ and $\mathsf{D}_g \mathfrak{p}: \hat{\mathscr{D}} \to V_3$ such that ([12])

$$(3.2)\qquad \mathfrak{p}(\Psi+\Phi_\delta, \bar{g}+h_\delta) = \mathfrak{p}(\Psi, \bar{g}) + \mathsf{D}_\Gamma \mathfrak{p}(\Psi, \bar{g}) \cdot \Phi_\delta(0) + \mathsf{D}_g \mathfrak{p}(\Psi, \bar{g}) \cdot h_\delta(0) + o(\delta).$$

Postulate P5. If $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$, then $(L_\sigma^r \Psi, l_\sigma \bar{g}) \in \mathscr{D}$ and $(S_\sigma \Psi, \mathfrak{z}_\sigma \bar{g}) \in \mathscr{D}$ for all $\sigma \geqslant 0$, and the limit ([13])

$$(3.3)\qquad \mathsf{d}_\Gamma \mathfrak{p}(\Psi, \bar{g}) + \mathsf{d}_g \mathfrak{p}(\Psi, \bar{g}) \equiv \lim_{\sigma \to 0^+} \frac{1}{\sigma} \{\mathfrak{p}(L_\sigma \Psi, l_\sigma \bar{g}) - \mathfrak{p}(S_\sigma \Psi, \mathfrak{z}_\sigma \bar{g})\}$$

exists.

Postulate P6. The functionals $\mathsf{D}_\Gamma \mathfrak{p}$, $\mathsf{D}_g \mathfrak{p}$, $\mathsf{d}_\Gamma \mathfrak{p}$ and $\mathsf{d}_g \mathfrak{p}$, when regarded as functions on $\hat{\mathscr{D}}$ are smooth in the following sense. Let $\mathfrak{f}$ stand for $\mathsf{D}_\Gamma \mathfrak{p}$, $\mathsf{D}_g \mathfrak{p}$, $\mathsf{d}_\Gamma \mathfrak{p}$ or $\mathsf{d}_g \mathfrak{p}$. If $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$, then for each regular family of functions (Φ_δ, h_δ) vanishing rapidly with δ,

$$(3.4)\qquad \lim_{\delta \to 0^+} \mathfrak{f}(\Psi+\Phi_\delta, \bar{g}+h_\delta) = \mathfrak{f}(\Psi, \bar{g}).$$

The functionals appearing in Postulate P6 (as in P4 and P5) have a certain type of interpretation. Roughly speaking $\mathsf{D}_\Gamma \mathfrak{p}(\Psi, \bar{g})$ and $\mathsf{D}_g \mathfrak{p}(\Psi, \bar{g})$ measure the rate of change $\mathfrak{p}$ in consequence of the present rates of change of Ψ and $\bar{g}$ — i.e., in $s = 0$, and $\mathsf{d}_\Gamma \mathfrak{p}(\Psi, \bar{g})$ and $\mathsf{d}_g \mathfrak{p}(\Psi, \bar{g})$ measure the rate at which $\mathfrak{p}$ would be changed if it is possible to ignore the present rates of change of Ψ and $\bar{g}$.

([10]) Cf. Engekling [8].

([11]) Bear in mind that g is linked with $\bar{g}$ by (2.2).

([12]) The sign $o(\delta)$ denotes a term which satisfies the condition $\lim_{\delta \to 0^+} \frac{1}{\delta} o(\delta) = 0$.

([13]) The notation of this limit means that: if $L_\sigma \Psi = S_\sigma \Psi$ for $\sigma \in [0, \infty)$, then $\mathsf{d}_\Gamma \mathfrak{p}(\Psi, \bar{g}) \equiv 0$, and if $l_\sigma \bar{g} = \mathfrak{z}_\sigma \bar{g}$ for $\sigma \in [0, \infty)$, then $\mathsf{d}_g \mathfrak{p}(\Psi, \bar{g}) \equiv 0$.

We wish to find an expression for the derivative of the free energy.

In an admissible thermodynamic process, the left-hand derivative $\dot{\psi}(t)$ is defined by

$$(3.5)\qquad \dot{\psi}(t) \equiv \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\psi(t)-\psi(t-\sigma)\}.$$

If it exists, is given by the formulae

$$(3.6)\qquad \dot{\psi}(t) = \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t,\bar{g}^t)-\mathfrak{p}(\Lambda^{t-\sigma},\bar{g}^{t-\sigma})\} = \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t,\bar{g}^t)-\mathfrak{p}(S_\sigma\Lambda^t,\mathfrak{z}_\sigma\bar{g}^t)\},$$

where was used (3.1), for $\sigma, s \geqslant 0$,

$$\Lambda^{t-\sigma}(s) = \Lambda(t-\sigma-s) = \Lambda^t(s+\sigma) = S_\sigma\Lambda^t(s),$$

$$\bar{g}^{t-\sigma}(s) = \int_0^s g^{t-\sigma}(\lambda)d\lambda = \int_0^s g^t(\sigma+\lambda)d\lambda = \mathfrak{z}_\sigma\bar{g}^t(s).$$

The following theorem demonstrates a property of chain rule for ψ.

THEOREM 1. *If an admissible local thermodynamic process is such that* $(\Lambda^t,\bar{g}^t)\in\hat{\mathscr{D}}$ *for* $t\in(t_p,t_k)$, *then for that process* $\dot{\psi}(t)$ *exists and obeys the formula*:

$$(3.7)\qquad \dot{\psi}(t) = \mathsf{D}_\Gamma\mathfrak{p}(\Lambda^t,\bar{g}^t)\cdot\dot{\Lambda}(t)-\mathsf{D}_g\mathfrak{p}(\Lambda^t,\bar{g}^t)\cdot g(t)+\mathsf{d}_\Gamma\mathfrak{p}(\Lambda^t,\bar{g}^t)+\mathsf{d}_g\mathfrak{p}(\Lambda^t,\bar{g}^t).$$

P r o o f. We can write:

$$(3.8)\qquad \frac{1}{\sigma}\{\psi(t)-\psi(t-\sigma)\} = \frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t,\bar{g}^t)-\mathfrak{p}(S_\sigma\Lambda^t,\mathfrak{z}_\sigma\bar{g}^t)\}$$
$$= \frac{1}{\sigma}\{\mathfrak{p}(L_\sigma\Lambda^t,l_\sigma\bar{g}^t)-\mathfrak{p}(S_\sigma\Lambda^t,\mathfrak{z}_\sigma\bar{g}^t)\}+\{\mathfrak{p}(\Lambda^t,\bar{g}^t)-\mathfrak{p}(L_\sigma\Lambda^t,l_\sigma\bar{g}^t)\}.$$

By Postulate P5, we have

$$(3.9)\qquad \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(L_\sigma\Lambda^t,l_\sigma\bar{g}^t)-\mathfrak{p}(S_\sigma\Lambda^t,\mathfrak{z}_\sigma\bar{g}^t)\} = \mathsf{d}_\Gamma\mathfrak{p}(\Lambda^t,\bar{g}^t)+\mathsf{d}_g\mathfrak{p}(\Lambda^t,\bar{g}^t).$$

Let us define ([14])

$$(3.10)\qquad \Phi_\sigma(s) \equiv L_\sigma\Lambda^t(s)-\Lambda^t(s) = (\Lambda^t(\sigma)-\Lambda^t(s))\chi_{[0,\sigma]}(s).$$

Since, by Postulate P3, Λ^t is continuously differentiable on the interval $[0,\beta]$ with $\beta>0$, for each σ in this interval,

$$|\Lambda^t(\sigma)-\Lambda^t(s)|_{10} \leqslant \int_s^\sigma \left|\frac{d}{d\lambda}\Lambda^t(\lambda)\right|_{10} d\lambda \leqslant \sigma K, \quad \text{for} \quad s\in[0,\sigma),$$

where $K \equiv \max_{\lambda\in[0,\beta]}\left|\frac{d}{d\lambda}\Lambda^t(\lambda)\right|_{10} < \infty$.

Hence, we have

$$|\Phi_\sigma(s)|_{10} \leqslant K\chi_{[0,\sigma]}(s)\sigma.$$

By the definition of Φ_σ and K

$$\left|\frac{d}{ds}\Phi_\sigma(s)\right|_{10} \leqslant K\chi_{[0,\sigma]}(s).$$

([14]) A part of the proof concerning of Φ_σ is the same as in [7].

If we put

$$(3.11)\qquad h_\sigma(s) \equiv l_\sigma \bar{g}^t(s) - \bar{g}^t(s) = (\bar{g}^t(\sigma) - \bar{g}^t(s))\chi_{[0,\sigma]}(s),$$

then

$$h_\sigma(s) = \Big(\int_0^\sigma g^t(\lambda)\,d\lambda - \int_0^s g^t(\lambda)\,d\lambda\Big)\chi_{[0,\sigma]}(s) = \int_s^\sigma g^t(\lambda)\,d\lambda\,\chi_{[0,\sigma]}(s).$$

For each $\sigma \in [0, \beta]$, by Postulate P3, we have

$$|h_\sigma(s)|_3 \leqslant \int_s^\sigma |g^t(\lambda)|_3\,d\lambda \leqslant M\chi_{[0,\sigma]}(s)\,\sigma,$$

where $M \equiv \max_{\lambda\in[0,\beta]} |g^t(\lambda)|_3 < \infty$.

Thus (Φ_σ, h_σ), with Φ_σ given by (3.10) and h_σ by (3.11), is a regular family of functions vanishing rapidly with σ, and by Postulate P4

$$(3.12)\qquad \frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t, \bar{g}^t) - \mathfrak{p}(L_\sigma\Lambda^t, l_\sigma\bar{g}^t)\} = -\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t + \Phi_\sigma, \bar{g}^t + h_\sigma) - \mathfrak{p}(\Lambda^t, \bar{g}^t)\}$$

$$= \mathsf{D}_\Gamma\mathfrak{p}(\Lambda^t, \bar{g}^t)\cdot\left(\frac{\Lambda^t(0) - \Lambda^t(\sigma)}{\sigma}\right) - \mathsf{D}_g\mathfrak{p}(\Lambda^t, \bar{g}^t)\cdot\frac{1}{\sigma}\int_0^\sigma g^t(\lambda)\,d\lambda - \frac{1}{\sigma}o(\sigma).$$

The function Λ^t is differentiable on $[0, \beta]$, hence

$$\lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\Lambda^t(0) - \Lambda^t(\sigma)\} = \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\Lambda(t) - \Lambda(t-\sigma)\} = \dot{\Lambda}(t).$$

The function g is continuous on $[0, \beta]$ and

$$\lim_{\sigma\to 0^+}\frac{1}{\sigma}\int_0^\sigma g^t(\lambda)\,d\lambda = g^t(0) = g(t).$$

Thus, in the limit, the expression (3.12) yields

$$(3.13)\qquad \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^t, \bar{g}^t) - \mathfrak{p}(L_\sigma\Lambda^t, l_\sigma\bar{g}^t)\} = \mathsf{D}_\Gamma\mathfrak{p}(\Lambda^t, \bar{g}^t)\cdot\dot{\Lambda}(t) - \mathsf{D}_g\mathfrak{p}(\Lambda^t, \bar{g}^t)\cdot g(t).$$

It follows immediately from (3.13), (3.9) and (3.8) that $\dot{\psi}(t)$ exists and obeys (3.7).

4. Consequences of the thermodynamic postulate

Let us take a particle X of material for a time $t \in (t_p, t_k)$. Let us choose arbitrarily $(\Gamma^t, \bar{g}^t) \in \hat{\mathscr{D}}$. We may be certain that there exist several admissible processes corresponding to this choice. But, for each of these processes, in the state in the time t the actual deformation-temperature configuration $\Lambda(X, t)$ is the value of $\Gamma^t(s)$ in $s = 0$, and π is given by the constitutive relations (2.5). It should be borne in mind that in this state the inequality (2.4) ought to hold.

Let $\mathscr{P}_X$ be an admissible local process with $(\Lambda^t, \bar{g}^t) \in \hat{\mathscr{D}}$ for $t \in (t_p, t_k)$. It follows from Theorem 1 that $\dot{\psi}(t)$ exists in the process and is given by (3.7). The thermodynamic inequality (2.4) should hold in every state t from the interval (t_p, t_k). We may use (2.6), (2.9) and (3.7) to write (2.7) in the form ([15])

$$[\mathfrak{n}(\Lambda^t, \bar{g}^t) - \mathsf{D}\mathfrak{p}(\Lambda^t, \bar{g}^t)] \cdot \hat{\alpha} - \mathsf{d}_\Gamma \mathfrak{p}(\Lambda^t, \bar{g}^t) - \mathsf{d}_g \mathfrak{p}(\Lambda^t, \bar{g}^t) \geqslant 0. \tag{4.1}$$

Let us form the basic theorem:

THEOREM 2. It *follows from the thermodynamic inequality and Postulates* P1–P6 *that*:

1. *the functional* $\mathfrak{n}$ *is completely determined by the functional* $\mathfrak{p}$ *through the relation*

$$\mathfrak{n}(\Lambda^t, \bar{g}^t) = \mathsf{D}\mathfrak{p}(\Lambda^t, \bar{g}^t) \quad \text{for} \quad (\Lambda^t, \bar{g}^t) \in \hat{\mathscr{D}}; \tag{4.2}$$

2. *the functionals* $\mathsf{d}_\Gamma \mathfrak{p}$ *and* $\mathsf{d}_g \mathfrak{p}$ *obey the following inequality, called the dissipation inequality* ([16]),

$$\mathsf{d}_\Gamma \mathfrak{p}(\Lambda^t, \bar{g}^t) + \mathsf{d}_g \mathfrak{p}(\Lambda^t, \bar{g}^t) \leqslant 0. \tag{4.3}$$

Proof. Let us define functions ([17])

$$\Phi_\sigma(s) \equiv \frac{s-\delta}{\delta} s\Gamma \chi_{[0,\delta]}(s) \tag{4.4}$$

for an arbitrary vector $\Gamma \in V_{10}$ and each $\delta \in (0, a), a > 0$. They are continuous and piecewise smooth functions of s for each δ fulfilling the following bounds:

$$|\Phi_\delta(s)|_{10} \leqslant |\Gamma|_{10} \chi_{[0,\delta]}(s)\,\delta,$$

$$\left|\frac{d}{ds}\Phi_\delta(s)\right|_{10} = \left|\frac{2s-\delta}{\delta}\Gamma\chi_{[0,\delta]}(s)\right|_{10} \leqslant |\Gamma|_{10}\chi_{[0,\delta]}(s).$$

Let us take an arbitrary vector $k \in V_3$ and for each $\delta \in (0, a)$ define

$$h_\delta(s) \equiv \int_0^s k d\lambda \chi_{[0,\delta]}(s). \tag{4.5}$$

For each δ in an interval $(0, a)$, we have

$$|h_\delta(s)|_3 \leqslant |k|_3 \chi_{[0,\delta]}(s)\,\delta.$$

Thus the one-parameter family of functions (Φ_δ, h_δ) is a regular family of functions vanishing rapidly with δ. Since $(\Lambda^t, \bar{g}^t)$ is in $\hat{\mathscr{D}}$, it follows from Postulate P4 that, for sufficiently small δ, the pair $(\Delta_\delta, d_\delta)$ is in $\hat{\mathscr{D}}$, where

$$(\Delta_\delta, d_\delta) \equiv (\Lambda^t + \Phi_\delta, \bar{g}^t + h_\delta). \tag{4.6}$$

Thus for small δ, the inequality (4.1) must hold with $(\Lambda^t, \bar{g}^t)$ replaced by $(\Delta_\delta, d_\delta)$,

$$[\mathfrak{n}(\Delta_\delta, d_\delta) - \mathsf{D}\mathfrak{p}(\Delta_\delta, d_\delta)] \cdot (\dot{\Lambda}(t) + \Gamma, -g(t) - k) - \mathsf{d}_\Gamma \mathfrak{p}(\Delta_\delta, d_\delta) - \mathsf{d}_g \mathfrak{p}(\Delta_\delta, d_\delta) \geqslant 0, \tag{4.7}$$

where was used

$$\dot{\Delta}_\delta(0) \equiv \lim_{\sigma \to 0^+} \frac{1}{\sigma}\{\Delta_\delta(0) - \Delta_\delta(\sigma)\} = \dot{\Lambda}(t) + \Gamma.$$

([15]) It was assumed $(\mathsf{D}_\Gamma \mathfrak{p}, \mathsf{D}_g \mathfrak{p}) \equiv \mathsf{D}\mathfrak{p}$.

([16]) Cf. GURTIN and PIPKIN [10].

([17]) Cf. COLEMAN and OWEN [7].

Since (Φ_δ, h_δ) forms a regular family of functions vanishing rapidly with δ, it follows from (4.6) and Postulates P1, P2 and P6 that, if we let δ go to zero in (4.7), we obtain

$$(4.8)\qquad [\mathfrak{n}(\Lambda^t, \bar{g}^t) - \mathsf{D}\mathfrak{p}(\Lambda^t, \bar{g}^t)]\cdot(\dot{\Lambda}(t)+\Gamma,\ -g(t)-k) - \mathsf{d}_\Gamma \mathfrak{p}(\Lambda^t, \bar{g}^t) - \mathsf{d}_g \mathfrak{p}(\Lambda^t, \bar{g}^t) \geqslant 0.$$

Clearly, (4.8) can hold for all $(\Gamma, k) \in V_{10}\times V_3$ only if the coefficient of $(\dot{\Lambda}(t)+\Gamma,\ -g(t)-k)$ vanishes, and the sum of the remaining terms is not negative. Thus (4.2) and (4.3) are true. Let us note that the functionals $\mathsf{D}_\Gamma\mathfrak{p}$ and $\mathsf{D}_g\mathfrak{p}$ were defined only for histories $(\Lambda^t, \bar{g}^t)$ belonging to $\hat{\mathscr{D}}$. Hence the relation (4.2) is here meaningful only for functions in $\hat{\mathscr{D}}$. However, since $\mathfrak{n}$ is assumed continuous on $\mathscr{D}$ and $\hat{\mathscr{D}}$ is dense in $\mathscr{D}$, $\mathfrak{n}$ is determined by its restriction to $\hat{\mathscr{D}}$. Thus $\mathfrak{n}$ is completely determined by the functional $\mathfrak{p}$, through (4.2), although $\mathsf{D}\mathfrak{p}(\Lambda^t, \bar{g}^t)$ in (4.2) has been defined only for $(\Lambda^t, \bar{g}^t) \in \hat{\mathscr{D}}$.

In the present theory, an interesting fact is that, in contrast to the results obtained by Coleman [1] and Coleman and Owen [7] and in agreement with the recent works of Gurtin and Pipkin [10] and Mc Carthy [12], the heat flux is determined by the functional for the free energy. There is the property of the constitutive relation under consideration. Let us consider a characteristic state of a material, namely a state which is described by a constant history of the deformation gradient, the temperature and the temperature gradient. For this state, we prove two theorems.

Theorem 3. *If a description of a state is given by a constant history* $(\Lambda^\dagger, g_0 i) \in \hat{\mathscr{D}}$, *where* $g_0 \equiv g(t) = g^t(s)$ *for all* $s \in [0, \infty)$, *and* $i(s) = s$ *and* $\Lambda^\dagger(s) = \Lambda(t)$ for all $s \in [0, \infty)$, *then*

$$(4.9)\qquad \mathsf{d}_g\mathfrak{p}(\Lambda^\dagger, g_0 i) = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, g_0 i)\cdot g_0.$$

Proof. If $\Lambda^t = \Lambda^\dagger$ and $\bar{g}^t(s) = g_0 i(s) = g_0 s = \int\limits_0^s g_0\, d\lambda$, then $L_\sigma\Lambda^\dagger = \Lambda^\dagger$, $S_\sigma\Lambda^\dagger = \Lambda^\dagger$ for $\sigma \geqslant 0$, and

$$\mathfrak{l}_\sigma(g_0 s) = \begin{cases} g_0\sigma & \text{for}\quad s\in[0,\sigma],\\ g_0 s & \text{for}\quad s\in[\sigma,\infty);\end{cases}\qquad \mathfrak{z}_\sigma(g_0 s) = \int\limits_0^s g_0\, d\lambda = g_0 s.$$

If we put

$$(4.10)\qquad h_\sigma(s) \equiv \mathfrak{l}_\sigma(g_0 s) - g_0 s = g_0 i(\sigma - s)\chi_{[0,\sigma]}(s)\quad \text{for}\quad \sigma\in(0,a),\ a>0,$$

then it is easy to prove that pairs $(0^\dagger, h_\sigma)$ form a regular family of functions vanishing rapidly with σ, and by Postulate P4, for sufficiently small σ, we have

$$\mathfrak{p}(\Lambda^\dagger, \mathfrak{l}_\sigma(g_0 i)) - \mathfrak{p}(\Lambda^\dagger, g_0 i) = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, g_0 i)\cdot(g_0\sigma) + o(\sigma).$$

Hence

$$(4.11)\qquad \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^\dagger, \mathfrak{l}_\sigma(g_0 i)) - \mathfrak{p}(\Lambda^\dagger, g_0 i)\} = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, g_0 i)\cdot g_0.$$

On the other hand, Postulate P5 yields:

$$(4.12)\qquad \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(L_\sigma\Lambda^\dagger, \mathfrak{l}_\sigma(g_0 i)) - \mathfrak{p}(S_\sigma\Lambda^\dagger, \mathfrak{z}_\sigma(g_0 i))\} = \lim_{\sigma\to 0^+}\frac{1}{\sigma}\{\mathfrak{p}(\Lambda^\dagger, \mathfrak{l}_\sigma(g_0 i)) - \mathfrak{p}(\Lambda^\dagger, g_0 i)\} = \mathsf{d}_g\mathfrak{p}(\Lambda^\dagger, g_0 i).$$

It follows immediately from (4.11) and (4.12) that the equality (4.9) holds.

A consequence of Theorem 3 is

THEOREM 4. *If the assumptions of Theorem 3 are satisfied, then the dissipation inequality* (4.3) *becomes the usual heat conduction inequality*:

$$-\frac{1}{\varrho\vartheta}q\cdot g_0 \geqslant 0. \tag{4.13}$$

P r o o f. In our case, Theorem 2 and (4.12) yield

$$\mathsf{d}_g\mathfrak{p}(\Lambda^\dagger, g_0 i) \leqslant 0 \quad \text{and} \quad \frac{1}{\varrho\vartheta}q = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, g_0 i). \tag{4.14}$$

Comparison of these results with Theorem 3 gives

$$\frac{1}{\varrho\vartheta}q\cdot g_0 = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, g_0 i)\cdot g_0 = \mathsf{d}_g\mathfrak{p}(\Lambda^\dagger, g_0) \leqslant 0.$$

5. State of equilibrium and memory restrictions

In this general topological structure, we cannot prove such properties for the free energy in a state of an equilibrium as have been obtained in the case of a material with fading memory. To do this, we should provide our structure with a relaxation property ([18]). This property will be given in Postulate A1.

Now, we introduce a definition of a state of equilibrium of a material.

DEFINITION 7. We say that the deformation-temperature configuration Λ corresponds to a state of equilibrium if the description of this state is given by $(\Lambda^\dagger, 0^\dagger)$, where $\Lambda^\dagger(s) = \Lambda$ and $0^\dagger(s) = 0$ for all $s \in [0, \infty)$.

Note that for a state of equilibrium $L_\sigma\Lambda^\dagger = \Lambda^\dagger$, $S_\sigma\Lambda = \Lambda^\dagger$, $l_\sigma 0^\dagger = 0^\dagger$, $\mathfrak{z}_\sigma 0^\dagger = 0^\dagger$ for $\sigma \geqslant 0$. If we substitute this functions into (3.3), we arrive at the proof of the theorem.

THEOREM 5. *In a state of equilibrium the dissipation* $\mathsf{d}_\Gamma\mathfrak{p}+\mathsf{d}_g\mathfrak{p}$ *vanishes.*

Given a number $\sigma \geqslant 0$ and functions $(\Psi, \bar{g}) \in \mathfrak{B}$, we may define new functions ([19]).

$$(5.1)\qquad \begin{aligned} C^{(\sigma)}\Psi(s) &\equiv \begin{cases} \Psi(0) & \text{for} \quad s \in [0, \sigma], \\ \Psi(s-\sigma) & \text{for} \quad s \in (\sigma, \infty); \end{cases} \\ c^{(\sigma)}\bar{g}(s) &\equiv \begin{cases} 0 & \text{for} \quad s \in [0, \sigma], \\ \bar{g}(s-\sigma) & \text{for} \quad s \in (\sigma, \infty). \end{cases} \end{aligned}$$

The function $c^{(\sigma)}\bar{g}$ we shall call the homothermal continuation of $\bar{g}$ by the amount σ. The operators $(C^{(\sigma)}, c^{(\sigma)})$ introduced will be helpful in formulating the relaxation property.

For further investigations, it is necessary to define a set

$$\mathscr{G}_0 \equiv \Big\{\bar{g} \in \mathfrak{B}_1: \quad \bar{g}(s) = \int_0^s g(\lambda)d\lambda \wedge g(0) = 0\Big\}. \tag{5.2}$$

Elements of the set $\mathscr{G}_0$ we denote by $\hat{g}$.

([18]) COLEMAN and MIZEL [4–6] introduced a relaxation property for the domain of functionals in the Banach space. COLEMAN and OWEN [7] assumed a relaxation property for the topological space.

([19]) $C^{(\sigma)}\Psi$ is called the static continuation of Ψ by the amount σ. Cf. [7].

We add to the six Postulates P1–P6 the following

POSTULATE A1. If $(\Psi, \hat{g}) \in \hat{\mathscr{D}}$, then [20] $(\Psi(0)^\dagger, 0^\dagger) \in \hat{\mathscr{D}}$ and $(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g}) \in \hat{\mathscr{D}}$ for all $\sigma \in [0, \infty)$, and $(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g})$ is a Moore-Smith convergent sequence in $\mathfrak{B}$ on $[0, \infty)$ with the limit $(\Psi(0)^\dagger, 0^\dagger)$. Furthermore, for each $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$, the function $(C^{(\cdot)}\Psi, c^{(\cdot)}\bar{g})$: $[0, \infty) \to \mathfrak{B}$ is continuous.

POSTULATE A2. The set $\mathscr{G}_0$ is dense in $\hat{\mathscr{D}}_1$ [21].

Since the constitutive functionals are continuous, it follows from Postulate A1 that we may define new functions, called equilibrium functions,

$$
\begin{aligned}
&\lim_{\sigma\to\infty} \mathfrak{p}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g}) = \mathfrak{p}(\Psi(0)^\dagger, 0^\dagger) \equiv \mathsf{p}^\#(\Psi(0)),\\
(5.3)\qquad &\lim_{\sigma\to\infty} \mathfrak{S}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g}) = \mathfrak{S}(\Psi(0)^\dagger, 0^\dagger) \equiv \mathsf{S}^\#(\Psi(0)),\\
&\lim_{\sigma\to\infty} \mathfrak{h}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g}) = \mathfrak{h}(\Psi(0)^\dagger, 0^\dagger) \equiv \mathsf{q}^\#(\Psi(0)).
\end{aligned}
$$

Let us note that $\mathsf{p}^\#$, $\mathsf{S}^\#$ and $\mathsf{q}^\#$ are defined on a set

$$(5.4)\qquad \mathscr{A} \equiv \{\Gamma \in V_{10}:\quad (\Gamma^\dagger, 0^\dagger) \in \mathscr{D}\} \subset V_{10}^+.$$

Let $(\Lambda^t, \bar{g}^t) \equiv (\Psi, \hat{g}) \in \hat{\mathscr{D}}$. Then histories $(\Lambda^{t+\sigma}, \bar{g}^{t+\sigma}) = (C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g})$, for $\sigma \geqslant 0$, define a process. For this process $\dot{\Lambda}(t+\sigma) = 0$ and $g(t+\sigma) = 0$ for $\sigma > 0$. Thus Theorem 1, the dissipation inequality (4.3) and Postulate A1 yield

$$(5.5)\qquad \dot{\psi}(t+\sigma) = \mathsf{d}_\Gamma \mathfrak{p}(\Lambda^{t+\sigma}, \bar{g}^{t+\sigma}) + \mathsf{d}_g \mathfrak{p}(\Lambda^{t+\sigma}, \bar{g}^{t+\sigma}) \leqslant 0 \quad \text{for} \quad \sigma > 0.$$

That is, for all $\sigma > 0$, the left-hand derivative $\mathfrak{p}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g})$ with respect to σ exists, and is not positive. It is clear from Postulate A1 and P1 that $\mathfrak{p}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g})$ is a continuous function of σ for all $\sigma \geqslant 0$, and it follows, by (5.5), that $\mathfrak{p}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g})$ is a non-increasing function of σ for all $\sigma \geqslant 0$ i.e.:

$$(5.6)\qquad \mathfrak{p}(\Psi, \hat{g}) \geqslant \mathfrak{p}(C^{(\sigma)}\Psi, c^{(\sigma)}\hat{g}) \quad \text{for} \quad \sigma \geqslant 0.$$

In view of $(5.3)_1$, we have:

$$(5.7)\qquad \mathfrak{p}(\Psi, \hat{g}) \geqslant \mathsf{p}^\#(\Psi(0)).$$

The last inequality is true for each $(\Psi, \hat{g}) \in \hat{\mathscr{D}}$. It follows by Postulate A2 that the set $\{(\Psi, \hat{g}) \in \hat{\mathscr{D}}:\ \hat{g} \in \mathscr{G}_0\}$ is dense in $\hat{\mathscr{D}}$. Hence the inequality (5.7) is true for each element from $\hat{\mathscr{D}}$ and we obtain

THEOREM 6. *If $(\Psi, \bar{g}) \in \hat{\mathscr{D}}$, then*

$$(5.8)\qquad \mathfrak{p}(\Psi, \bar{g}) \geqslant \mathsf{p}^\#(\Psi(0)).$$

This theorem expresses the well known the extremum property for the free energy for a broad class of materials, including materials with fading memory, and for a rigid conductor has been obtained by GURTIN and PIPKIN [10] [22].

Now, we prove the following

---

[20] $\Psi(0)^\dagger$ denotes a constant history with the value $\Psi(0)$.

[21] $\hat{\mathscr{D}}_1$ denotes the image of $\hat{\mathscr{D}}$ under the projection of $\mathfrak{B}$ on $\mathfrak{B}_1$.

[22] Cf. COLEMAN [1], COLEMAN and GURTIN [3], COLEMAN and OWEN [7].

Remark. If [23] $\Lambda \in \mathscr{A}$, then for each $\Omega \in V_{10}$ there exists an interval $(0, b)$, $b > 0$ such that $(\Lambda + \delta\Omega) \in \mathscr{A}$ for $\delta \in (0, b)$.

Proof. Let $\Omega \in V_{10}$ and $\Lambda \in \mathscr{A}$. For every $\delta > 0$ put

$$\Lambda[\delta, \Omega](s) \equiv \begin{cases} \Lambda + (\delta - s)\Omega & \text{for} \quad s \in [0, \delta], \\ \Lambda & \text{for} \quad s \in (\delta, \infty); \end{cases} \tag{5.9}$$

$$\Phi_\delta^{(\Omega)} \equiv \Lambda[\delta, \Omega] - \Lambda^\dagger.$$

It is clear that since δ varies an interval $(0, a)$, $a > 0$, the functions $(\Phi_\delta^{(\Omega)}, 0^\dagger)$ form a regular family of functions vanishing rapidly with δ. Therefore, since $(\Lambda^\dagger, 0^\dagger) \in \hat{\mathscr{D}}$, it is a consequence of Postulate P4 that $(\Lambda[\delta, \Omega], 0^\dagger) \in \hat{\mathscr{D}}$ for all δ in some non-empty interval $(0, b)$. Hence, by Postulate A1, $(C^{(\sigma)}\Lambda[\delta, \Omega], 0^\dagger) \in \hat{\mathscr{D}}$, for each $\delta \in (0, b)$ and $\sigma \geqslant 0$, and also

$$\lim_{\sigma \to \infty} (C^{(\sigma)}\Lambda[\delta, \Omega], 0^\dagger) = (\Lambda[\delta, \Omega](0)^\dagger, 0^\dagger) = \big((\Lambda + \delta\Omega)^\dagger, 0^\dagger\big) \in \hat{\mathscr{D}}. \tag{5.10}$$

The following assumption of differentiability for $\mathsf{p}^\#$ on $\mathscr{A}$ seems natural.

POSTULATE A3. There exists a function $\nabla \mathsf{p}^\# : \mathscr{A} \to V_{10}$ such that for each $\Lambda \in \mathscr{A}$:

$$\lim_{\sigma \to 0^+} \frac{1}{\sigma} \{\mathsf{p}^\#(\Lambda + \sigma\Omega) - \mathsf{p}^\#(\Lambda)\} = \nabla \mathsf{p}^\#(\nabla) \cdot \Omega \quad \text{for} \quad \Omega \in V_{10}. \tag{5.11}$$

Note that we here assume only that the gradient $\nabla \mathsf{p}^\#(\Lambda)$ exists for Λ in $\mathscr{A}$. This is similar to our earlier assumptions concerning $\mathsf{D}_\Gamma \mathfrak{p}$, $\mathsf{D}_g \mathfrak{p}$, $\mathsf{d}_\Gamma \mathfrak{p}$ and $\mathsf{d}_g \mathfrak{p}$.

The last postulate gives [24]:

THEOREM 7. *If* $\Lambda \in \mathscr{A}$, *then*

$$\mathsf{D}_\Gamma \mathfrak{p}(\Lambda^\dagger, 0^\dagger) = \nabla \mathsf{p}^\#(\Lambda). \tag{5.12}$$

Proof. Let $\Lambda \in \mathscr{A}$. From the proof of Remark, we know that $(\Lambda[\delta, \Omega], 0^\dagger)$ is in $\hat{\mathscr{D}}$ for all δ in some interval $(0, b)$ and for arbitrary $\Omega \in V_{10}$. Moreover, $(\Lambda + \delta\Omega) \in \mathscr{A}$. By Theorem 6, we have

$$\mathfrak{p}(\Lambda[\delta, \Omega], 0^\dagger) \geqslant \mathsf{p}^\#(\Lambda + \delta\Omega). \tag{5.13}$$

Let us define two expressions

$$\mathscr{F}_\delta[\Omega] \equiv \frac{1}{\delta}\{\mathsf{p}^\#(\Lambda + \delta\Omega) - \mathsf{p}^\#(\Lambda)\}, \tag{5.14}$$

$$\mathscr{H}_\delta[\Omega] \equiv \frac{1}{\delta}\{\mathfrak{p}(\Lambda[\delta, \Omega], 0^\dagger) - \mathfrak{p}(\Lambda^\dagger, 0^\dagger)\}.$$

In view of Theorem 6 and (5.3_1), we have:

$$\mathscr{F}_\delta[\Omega] \leqslant \mathscr{H}_\delta[\Omega] \quad \text{for} \quad \delta \in (0, b). \tag{5.15}$$

Note that by Postulate P4

$$\lim_{\delta \to 0} \mathscr{H}_\delta[\Omega] = \mathsf{D}_\Gamma \mathfrak{p}(\Psi\Lambda^\dagger, 0^\dagger) \cdot \Omega, \tag{5.16}$$

([23]) The set $\mathscr{A}$ was defined by (5.4). A similar remark was proved in [7].

([24]) Cf. COLEMAN [1], COLEMAN and MIZEL [5], COLEMAN and OWEN [7].

because the functions $(\Phi_\delta^{(\Omega)}, 0^\dagger)$, defined by (5.9), form a regular family of functions vanishing rapidly with δ. On the other hand, Postulate A3 yields

$$\lim_{\delta\to 0}\mathscr{F}_\delta[\Omega] = \nabla \mathsf{p}^{\#}(\Lambda)\cdot\Omega. \tag{5.17}$$

It is clear from (5.15)–(5.17) that

$$\{\mathsf{D}_\Gamma \mathfrak{p}(\Lambda^\dagger, 0^\dagger) - \nabla\mathsf{p}^{\#}(\Lambda)\}\cdot\Omega \geqslant 0, \tag{5.18}$$

and since this relation must hold for every $\Omega \in V_{10}$, (5.12) follows directly.

Since $\mathsf{D}_\Gamma \mathfrak{p}(\Lambda^\dagger, 0^\dagger) = \mathfrak{S}(\Lambda^\dagger, 0^\dagger) = \mathsf{S}^{\#}(\Lambda)$, therefore a consequence of Theorem 7 is

Theorem 8. *The equilibrium functions* $\mathsf{p}^{\#}$ and $\mathsf{S}^{\#}$ *obey the classifical formula* ([25])

$$\mathsf{S}^{\#}(\Lambda) = \nabla\mathsf{p}^{\#}(\Lambda). \quad \textit{for each } \Lambda\in\mathscr{A}. \tag{5.19}$$

We shall prove an interesting theorem([26]).

Theorem 9. *In a state of equilibrium*, $\mathsf{D}_g\mathfrak{p} = 0$ *and therefore* $\mathsf{q}^{\#} = 0$.

Proof. Let $(\Lambda^\dagger, 0^\dagger)$ describe a state of equilibrium. We define a regular family of functions vanishing rapidly with δ as $(0^\dagger, h_\delta)$, where $h_\delta(s) \equiv \int_s^\delta k d\lambda \chi_{[0,\delta]}(s)$ for arbitrary $k\in V_3$, $\delta\in(0,\beta)$, $\beta > 0$.

It follows from Postulate P4 that

$$\mathfrak{p}(\Lambda^\dagger, 0^\dagger + h_\delta) - \mathfrak{p}(\Lambda^\dagger, 0^\dagger) = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, 0^\dagger)\cdot k\delta + o(\delta) \quad \text{for} \quad \delta\in(0,a).\ a\leqslant\beta. \tag{5.20}$$

But from Theorem 6, we have

$$\mathfrak{p}(\Lambda^\dagger, 0^\dagger + h_\delta) \geqslant \mathfrak{p}(\Lambda^\dagger, 0^\dagger). \quad \text{for each } \delta\in(0,\beta). \tag{5.21}$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\{\mathfrak{p}(\Lambda^\dagger, 0^\dagger + h_\delta) - \mathfrak{p}(\Lambda^\dagger, 0^\dagger)\} = \mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, 0^\dagger)\cdot k, \tag{5.22}$$

then, by (5.21)

$$\mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, 0^\dagger)\cdot k \geqslant 0, \tag{5.23}$$

and since this relation must hold for every $k\in V_3$, $\mathsf{D}_g\mathfrak{p}(\Lambda^\dagger, 0^\dagger) = 0$. Hence, by Theorem 2 and $(5.3)_3$, we obtain $\mathsf{q}^{\#} = 0$.

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([25]) Cf. Coleman [1], Coleman and Mizel [5], Coleman and Gurtin [3]. Gurtin and Pipkin [10], Coleman and Owen [7], Nunziato [13].

([26]) This result has been obtained here in a manner different from [3, 10, 13].

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