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Dynamical bending of circular piece-wise nonhomogeneous plates

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IN THE PAPER is considered the dynamical behaviour of piecewise non-homogeneous circular plates simply supported along the contour and consisting of two parts with different mechanical properties: a circular interior and an annular exterior. The material of the plate is supposed to be perfectly rigid-plastic and satisfies the Tresca yield condition. The dynamical properties of such a plate prove to be better than those of homogeneous plates. A more detailed analysis concerns the problem of a plate loaded by a uniformly distributed pressure pulse.

Rozważono zagadnienie dynamiczne płyty odcinkami niejednorodnej, podpartej na obwodzie i składającej się z dwóch części o różnych własnościach mechanicznych: z kołowego wnętrza i z pierścienia kołowego. Materiał płyty jest doskonale sztywno-plastyczny i spełnia warunek Treski. Stwierdzono, że własności dynamiczne takiej płyty są lepsze aniżeli płyty jednorodnej. Przeprowadzono dokładniejszą analizę problemu w przypadku równomiernego udarowego obciążenia całej powierzchni płyty.

В работе исследуется динамический изгиб круглых шарнирно опертых по наружному контуру пластин, состоящих из двух частей с различными механическими свойствами: внутренней, круговой, и внешней, кольцевой. В некоторый момент пластинка внезапно нагружается не возрастающей во времени равномерно распределенной нагрузкой высокой интенсивности. Материал пластинки идеальный жестко-пластический, подчиняющийся условию пластичности Треска и ассоциированному с ним закону течения. Из-за ограниченности объема статьи приводится решение задачи лишь для случая, когда внутренняя часть пластинки в процессе движения остается жесткой. Указывается на существенное преимущество (в смысле уменьшения остаточных прогибов) кусочно-неоднородных пластин по сравнению с однородными пластинками того же радиуса и веса. Показано, что наибольшей динамической сопротивляемостью обладает ступенчатая пластинка с параметрами, соответствующими максимальной величине статической предельной нагрузки.

Notation

- a_0, b_0 radii of interior and exterior parts of a plate, respectively,
- W deflection,
- M_1, M_2 radial and circumferential bending moments, respectively,
- Q_r shearing force,
- $\gamma, 2\delta$ surface density and thickness of the ring-shaped part,
- σ_0 yield limit,
- P intensity of uniformly distributed load;
- t time.
- t_0 duration of rectangular pressure pulse,
- γ_0 surface density of the central part;
- r radial coordinate.

TO THE INVESTIGATION of dynamical bending of circular rigid-plastic plates have been devoted, numerous papers a survey of which may be found in [1–4]. The attention of the authors was attracted only by homogeneous plates. In this paper is investigated the dynamical behaviour of piece-wise nonhomogeneous plates simply supported along the exterior contour and consisting of two parts with different mechanical properties: the circularshaped interior one and ring-shaped exterior. Plates are supposed to be ideally rigid-plastic satisfying the Tresca yield condition. The further analysis not only gives a direct method of calculation of the behaviour of such plates under dynamical loads, but also indicates their essentially better dynamical resistance by comparison with homogeneous plates of the same radius and weight. It is shown that the maximum of dynamical resistance belongs to step-like plates with parameters corresponding to the maximum of static limit load. The dynamical behaviour of plates under the action of uniformly distributed rectangular pressure pulse is investigated in detail and the pecularities of the dynamic behaviour of plates under the action of arbitrary "blast-like" load [5] are discussed.

Static analysis of piece-wise nonhomogeneous plates is given in [6, 7].

1

Put the coordinate origin at the centre of a plate, direct the deflection axis along the direction of a load, the abscissa along the radius r.

The dimensionless equations of motion have the form [1]:

$$\begin{gathered}[xQ_x(x,\tau)]'+x[p(\tau)-\ddot{w}(x,\tau)]=0,\\ [xm_1(x,\tau)]'-m_2(x,\tau)-xQ_x(x,\tau)=0\\ m_i=M_i/\sigma_0\delta^2\ \dots\ (i=1,2),\quad w=W\gamma b_0^2/\sigma_0\delta^2t_0^2,\quad p=Pb_0^2/\sigma_0\delta^2.\\ Q_x=b_0Q_r/\sigma_0\delta^2,\quad \tau=t/t_0,\quad x=r/b_0.\quad \alpha=a_0/b_0\ \dots\quad 0\leqslant\alpha<1,\end{gathered}\tag{1.1}$$

where M_1 and M_2 are radial and circumferential bending moments, respectively, a_0 and b_0 are radii of the interior and exterior parts of a plate; γ, 2δ and σ_0 are, respectively, surface density, thickness and yield limit of the ring-shaped part of a plate, P — the intensity of uniformly distributed rectangular pressure pulse of duration t_0, t — time. Primes indicate differentiation with respect to x and dots with respect to τ.

In what follows the complete solution of the problem under consideration is ascertained only for the case in which the interior part of the plate remains rigid in the motion for want of space. Essential for our analysis is whether the inequality

$$k=\gamma_0/\gamma\geqslant 1\tag{1.2}$$

or inequalities

$$0\leqslant k<1\tag{1.3}$$

are satisfied. Here γ_0 indicates the surface density of the central part.

The necessary initial and boundary value conditions have the form:

$$\begin{gathered}w(x,0)=\dot{w}(x,0)=0,\quad w(1,\tau)=\dot{w}(1,\tau)=0,\quad w(\alpha,\tau)=w_\alpha(\tau),\\ \dot{w}(\alpha,\tau)=\dot{w}_\alpha(\tau),\quad m_1(1,\tau)=0,\quad \alpha Q_\alpha=-(\alpha^2/2)[p(\tau)-k\ddot{w}_\alpha(\tau)].\end{gathered}\tag{1.4}$$

The flow law implies that curvature velocities $\dot{\varkappa}_1$ and $\dot{\varkappa}_2$ up to the positive constants multipliers equal to

$$\dot{\varkappa}_i=\partial\mathscr{F}/\partial m_i\ \dots\ (i=1,2),\quad (\dot{\varkappa}_1=-\dot{w}'',\dot{\varkappa}_2=-\dot{w}'/x),\tag{1.5}$$

where $\mathscr{F} = \mathscr{F}(m_1, m_2)$ is a Tresca hexagon on the m_1, m_2-plane (see Fig. 1). Limit load corresponds to the plastic regime and is equal to (see [6, 7]):

$$p_0 = 6/(1-\alpha^3). \tag{1.6}$$

The plastic states corresponding to points A and B are realized for $x = \alpha$ and $x = 1$, respectively (Fig. 1).

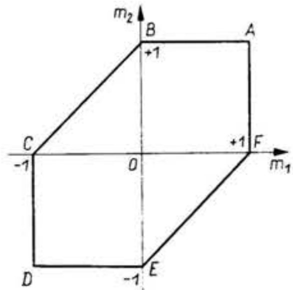


FIG. 1.

The complete solution of the indicated dynamical problem, depending on plate parameters and acting load, is the sum of particular solutions, corresponding to various sides and vertices of Tresca's hexagon (Fig. 1). Every stress profile is characterized by a definite system of inequalities for the functions $m_i(x, \tau)$ in various parts of the plate and for various times. In the interests of brevity, these inequalities have not been estimated for any part of the plate and any time. The estimations have been made only, where these inequalities are indeed not true, but assuming them to be fulfilled in other parts of the plate. The set of the values of parameters for which the corresponding system of inequalities is not satisfied implies transition of the plate to a different stress profile. For the same reason, of all possible plastic states only those have been selected which are realizable in fact. It should be remarked that continuity in x and τ of functions $w(x, \tau)$, $\dot{w}(x, \tau)$ and continuity in x of $m_1(x, \tau)$, $m_2(x, \tau)$, $Q_x(x, \tau)$ is used in the solution of the problem.

2

The plate is moving in regime $A-B$ (Fig. 1). Integrating the equation of motion (1.1) and taking into account (1.4)–(1.6), we obtain:

$$m_1 = 1-\frac{x-\alpha}{x(1-\alpha)}\Psi(x, p, k, \alpha), \quad m_2 = 1, \quad w = \frac{6\tau^2(p-p_0)}{p_0 A(k, \alpha)}\frac{1-x}{1-\alpha}, \tag{2.1}$$

$$\alpha \leqslant x \leqslant 1, \quad 0 \leqslant \tau \leqslant 1, \quad p_0 \leqslant p \leqslant p_*$$

$$m_1 = 1-\frac{x-\alpha}{x(1-\alpha)}\Psi(x, 0, k, \alpha), \quad m_2 = 1,$$

$$w = \frac{6(-\tau^2+2p\tau/p_0-p/p_0)}{A(k, \alpha)}\frac{1-x}{1-\alpha} \tag{2.2}$$

$$\alpha \leqslant x \leqslant 1, \quad 1 \leqslant \tau \leqslant \tau_f = p/p_0, \quad p_0 \leqslant p \leqslant p_*,$$

where

$$\Psi = 1+\frac{1-x}{1+\alpha+\alpha^2}\left[\frac{\varphi(x,k,\alpha)}{p_0A(k,\alpha)}(p-p_0)-(1+\alpha+x)\right],$$

$$\varphi = -(1+\alpha+\alpha^2)x^2-\alpha(1-\alpha)[6\alpha(k-1)+2\alpha+1]x-\alpha^2[6(k-1)(1-\alpha^2)-2\alpha^2-2\alpha+1],$$

$$A(k,\alpha) = (1-\alpha)[6\alpha^2(k-1)+3\alpha^2+2\alpha+1].$$

Solution (2.1) corresponds to the load acting in the time period $0 \leqslant \tau \leqslant 1$ and (2.2) — to the inertial moving unloading period $1 \leqslant \tau \leqslant \tau_f$. Necessary conditions for the regime A–B are

$$\text{(2.3)} \qquad \Psi(\alpha,p,k,\alpha) \geqslant 0, \quad (1-\alpha)\Psi'(1,p,k,\alpha)+\alpha \geqslant 0.$$

It may easily be shown that for k, satisfying the inequaily (1.2), the conditions (2.3) are in fact sufficient, and the second of the condition (2.3) follows from the first one, determining the characteristic load

$$p_* = 6k/(k-k_0)(1-\alpha)^2(1+2\alpha), \quad k_0 = (1+3\alpha)/2(1+2\alpha).$$

Time of motion τ_f is determined here and hereinafter from the condition $\dot{w}(x,\tau_f) = 0$. In sections 3–6 solutions are derived for the parameter k from (1.2).

3

In the load-action period, for loads $p_* \leqslant p \leqslant p_{01}$ the following stress profile is realized (Fig. 1):

$$\text{(3.1)} \qquad \alpha \leqslant x \leqslant \eta_0 \to A-F, \quad \eta_0 \leqslant x \leqslant 1 \to A-B.$$

By means of (1.5) and (1.4), we obtain the following velocity field:

$$\text{(3.2)} \qquad \dot{w}(x,\tau) = \begin{cases} \dot{w}_\alpha(\tau), & \alpha \leqslant x \leqslant \eta_0, \\ (1-x)(1-\eta_0)^{-1}\dot{w}_\alpha(\tau), & \eta_0 \leqslant x \leqslant 1. \end{cases}$$

From the Eqs. (1.1) by means of (3.1), (3.2) and (1.4), the following solution is derived:

$$m_1 = 1, \quad m_2 = 1-a\alpha^2(k-1)(\eta_0^2-x^2)/2\eta_0^2, \quad w = a\tau^2/2, \quad \alpha \leqslant x \leqslant \eta_0,$$

$$\text{(3.3)} \qquad m_1 = 1-a(x-\eta_0)^2[\eta_0^2(x^2-\eta_0^2)+2\alpha^2(k-1)(1-\eta_0)(x+2\eta_0)]/12x\eta_0^2(1-\eta_0),$$

$$m_2 = 1, \quad w = a\tau^2(1-x)/2(1-\eta_0), \quad \eta_0 \leqslant x \leqslant 1,$$

$$a = p[1+\alpha^2\eta_0^{-2}(k-1)]^{-1},$$

where the boundary of the regimes η_0 is connected with the parameters p, α and $k \geqslant 1$ by the relation

$$\text{(3.4)} \qquad p = 12(1-\eta_0)^{-2}[\eta_0^2+\alpha^2(k-1)][2\alpha^2(k-1)(1+2\eta_0)+\eta_0^2(1+\eta_0)]^{-1}.$$

Since $m_2(\alpha,\tau)$ decreases with increased p, the solution (3.3) is true for loads for which holds:

$$\Phi_1 = 1-(1/2)a\alpha^2(k-1)(1-\alpha^2\eta_0^{-2}) \geqslant 0$$

— that is, for loads

(3.5) $$p_* \leqslant p \leqslant p_{01},$$

where p_{01} is a root of the equation $\Phi_1 = 0$.

After unloading, one part of the plate $\alpha \leqslant x \leqslant \eta$ moves in regime A, another, $\eta \leqslant x \leqslant 1$ — in regime $A-B$ (Fig. 1). The boundary of regimes is a function of time $\eta = \eta(\tau)$. The solution has the form:

(3.6) $$\begin{gathered} m_1 = m_2 = 1, \quad w = a(2\tau-1)/2, \quad \alpha \leqslant x \leqslant \eta, \\ m_1 = 1-(x-\eta)^2[-x^2+2(1-\eta)x+\eta(4-3\eta)]/x(1-\eta)^3(1+3\eta), \\ m_2 = 1, \quad \dot{w} = a(1-x)(1-\eta)^{-1}, \quad \eta \leqslant x \leqslant 1, \\ 1 \leqslant \tau \leqslant \tau_\alpha, \quad p_* \leqslant p \leqslant p_{01}, \end{gathered}$$

where the function $\eta(\tau)$ is determined from equation

(3.7) $$a[\eta_0(1+\eta_0-\eta_0^2)-\eta(1+\eta-\eta^2)]-12(\tau-1) = 0.$$

Since $\dot{\eta} < 0$, at some moment $\eta(\tau_\alpha) = \alpha$. It follows that the solution (3.6) is true in the time interval

(3.8) $$1 \leqslant \tau \leqslant \tau_\alpha = 1+(a/12)[\eta_0(1+\eta_0-\eta_0^2)-\alpha(1+\alpha-\alpha^2)].$$

By means of (3.7) from (3.6) we obtain the equality:

(3.9) $$w(x, \tau) = \frac{1}{24}a^2(\eta_0-\eta)(1-x)(2+3\eta_0+3\eta)+\psi(x).$$

Obviously, for $\eta_0 \leqslant x \leqslant 1$

$$\psi(x) = w(x, 1) = \frac{1}{2}(1-x)(1-\eta_0)^{-1}.$$

In segment $\eta \leqslant x \leqslant \eta_0$, the function $\psi(x)$ is determined by the continuity $w(x, \tau)$ at $x = \eta$.

Then, from (3.6)–(3.9) is derived the equality:

$$\psi(x) = \frac{1}{24}a[12-a(\eta_0-x)^2(1+x+2\eta_0)], \quad \eta \leqslant x \leqslant \eta_0.$$

In the last interval of motion ($\tau_\alpha \leqslant \tau \leqslant \tau_f$), the plate is moving in the regime A–B (Fig. 1); the stress profile is determined from the solution (2.2). The distribution of residual deflections is given by:

(3.10) $$w(x, \tau_f) = \frac{1}{24}a^2(1-x)(1-\alpha)^{-1}[A(k, \alpha)+(1-\alpha)(\eta_0-\alpha)(3\eta_0+3\alpha+2)]+\psi(x),$$

$$\alpha \leqslant x \leqslant 1, \tau_f = \tau_\alpha+\frac{1}{24}aA(k, \alpha).$$

4

For loads from

(4.1) $$p_{01} \leqslant p \leqslant p_{02},$$

the stress profile is (Fig. 1):

(4.2) $$\alpha \leqslant x \leqslant u_0 \rightarrow E-F, \quad u_0 \leqslant x \leqslant \eta_0 \rightarrow F-A, \quad \eta_0 \leqslant x \leqslant 1 \rightarrow A-B.$$

From the flow law (1.5) for the side $E-F$ it follows that

$$\dot{w}(x, \tau) = \dot{A}_0(\tau)\ln x+\dot{B}_0(\tau),$$

where $\dot{A}_0(\tau)$ and $\dot{B}_0(\tau)$ are arbitrary functions of time. But since $0 \leqslant m_1(\alpha, \tau) < 1$, we have [1]

(4.3) $$\dot{w}'(\alpha, \tau) = 0.$$

Hence $\dot{A}_0(\tau) \equiv 0$. Therefore, the velocity field in this case is determined by the formulae (3.2).

Integration of (1.1) yields:

$$m_1 = 1 - \frac{1}{4} a u_0^2 \alpha^2 \eta_0^{-2}(k-1)(\xi - 1 - \ln \xi), \quad m_2 = m_1 - 1, \quad \alpha \leqslant x \leqslant u_0,$$

(4.4) $$m_1 = 1, \quad m_2 = (x^2 - u_0^2)(\eta_0^2 - u_0^2)^{-1}, \quad u_0 \leqslant x \leqslant \eta_0,$$
$$0 \leqslant \tau \leqslant 1, \quad p_{01} \leqslant p \leqslant p_{02},$$

where

(4.5) $$\xi = x^2 u_0^{-2}, \quad u_0^2 = \eta_0^2 [1 - 2/a\alpha^2(k-1)].$$

And η_0 is again determined from (3.4). The stress profile on the segment $\eta_0 \leqslant x \leqslant 1$ and the deflection function $w(x, \tau)$ on the whole plate are of the same form as in (3.3). Since $m_1(\alpha, \tau)$ decreases with increeses of p, therefore $p = p_{02}, . m_1(\alpha, \tau) = 0$. Hence stress profile (4.4) is realized only for loads, for which

$$\Phi_2 = 2\eta_0^2 u_0^{-2} - \alpha^2 u_0^{-2} - 1 - \ln(u_0^2 \alpha^{-2}) \geqslant 0$$

— i.e. for loads (4.1), where p_{02} is a root of the equation $\Phi_2 = 0$.

In the unloading interval $1 \leqslant \tau \leqslant \tau_f$, the dynamical behaviour of a plate is described by the formulae of the previous section for this interval of time. The expression for residual deflections coincides with (3.10).

5

For loads

(5.1) $$p_{02} \leqslant p \leqslant p_{03},$$

the following stress profile in the plate is realized;

(5.2) $$\alpha \leqslant x \leqslant v_0 \to D-E, \quad v_0 \leqslant x \leqslant u_0 \to E-F,$$
$$u_0 \leqslant x \leqslant \eta_0 \to F-A, \quad \eta_0 \leqslant x \leqslant 1 \to A-B;$$

the point D is not realized; the equality (4.3) is fulfilled as also is (3.2). Hence, as in Sec. 4, for $0 \leqslant \tau \leqslant 1$ we have

$$m_1 = -a\alpha^2(k-1)(v_0 - x)(3u_0^2 - x^2 - v_0 x - v_0^2)/6x\eta_0^2, \quad m_2 = -1, \quad \alpha \leqslant x \leqslant v_0,$$

$$m_1 = \frac{1}{4} a \eta_0^{-2} \alpha^2 (k-1)[v_0^2 - x^2 - u_0^2 \ln(v_0^2 x^{-2})], \quad m_2 = m_1 - 1, \quad v_0 \leqslant x \leqslant u_0.$$

Deflections of the whole plate and the stress profile at the segment $u_0 \leqslant x \leqslant 1$ are determined by formulae from the solutions (3.3) and (4.4) for corresponding segmental boundaries of the regimes η_0 and u_0, as earlier determined by the formulae (3.4) and (4.5); $v_0 = \sqrt{\lambda_0} u_0$, where λ_0 is the root of the equation

$$2\eta_0^2 u_0^{-2} - \lambda - 1 + \ln \lambda = 0$$

satisfying the inequality $0 < \lambda_0 < 1$. From the inequalities $-1 \leqslant m_1(\alpha, \tau) \leqslant 0$, the interval of loads (5.1) is derived for which

$$\Phi_3 = 6v_0\eta_0^2 - a\alpha^2(k-1)(v_0-\alpha)(3\eta_0^2 - v_0^2 - \alpha v_0 - \alpha^2) \geqslant 0,$$

where p_{03} is the root of the equation $\Phi_3 = 0$. After unloading up to the full stop of plate ($1 \leqslant \tau \leqslant \tau_f$), its dynamical behaviour is the same as in Secs. 3, 4 after unloading; for residual deflection, the formula (3.10) holds.

6

If $p \geqslant p_{03}$, then, though during loading the stress profile (5.2) holds, at $x = \alpha$ the point D is realized (Fig. 1), and (4.3) is not fulfilled; the velocity field in this case is different from (3.2) and is by the kind:

$$\dot{w}(x, \tau) = \begin{cases} \dot{w}_\alpha + \dot{z}(x-\alpha)(v-\alpha)^{-1}, & \alpha \leqslant x \leqslant v, \\ \dot{w}_\alpha + \dot{z}[1+\omega\ln(x/v)], & v \leqslant x \leqslant u, \\ \dot{w}_\alpha + \dot{z}[1+\omega\ln(u/v)], & u \leqslant x \leqslant \eta, \\ (1-x)(1-\eta)^{-1}\{\dot{w}_\alpha + \dot{z}[1+\omega\ln(u/v)]\}, & \eta \leqslant x \leqslant 1, \end{cases} \tag{6.1}$$

where

$$\omega = v(v-\alpha)^{-1}, \qquad \dot{z} = \dot{w}(v, \tau) - \dot{w}(\alpha, \tau).$$

Further analysis is carried out for an arbitrary "blast"-like load [5]. Therefore, from the very beginning of the load action, v, u, η are functions of time. Differentiating (6.1) by time, substituting the result in (1.1) and taking into account (1.4), the following system of five differential equations for w_α, z, v, u, η is derived:

$$\begin{aligned}
&2(v-\alpha)[3\alpha^2(k-1)+v^2+\alpha v+\alpha^2]\ddot{w}_\alpha + (v-\alpha)^2(v+\alpha)\ddot{z} - (v^2-\alpha^2)\dot{v}\dot{z} \\
&\qquad - 2[6v + p(v^3-\alpha^3)] = 0, \\
&3(v-\alpha)^2[u^2-v^2+2\alpha^2(k-1)\ln(u/v)]\ddot{w}_\alpha + (v-\alpha)\{[v(v^2+3u^2) \\
&\qquad + 2\alpha^3]\ln(u/v) - 3\alpha(u^2-v^2)\}\ddot{z} + \{[2(v^3-\alpha^3) - 3\alpha(u^2+v^2)]\ln(u/v) \\
&\qquad - 3(v-2\alpha)(u^2-v^2)\}\dot{v}\dot{z} - 3(v-\alpha)^2\{p(u^2-v^2) + 4[1+\ln(u/v)]\} = 0, \\
&6(v-\alpha)^2[u^2+\alpha^2(k-1)]\ddot{w}_\alpha + (v-\alpha)[v(3u^2+v^2) - 2\alpha(3u^2-\alpha^2) + 6vu^2\ln(u/v)]\ddot{z} \\
&\qquad + \{(2v-3\alpha)(v^2-3u^2) - 2\alpha[\alpha^2+3u^2\ln(u/v)]\}\dot{v}\dot{z} - 6(v-\alpha)^2(2+pu^2) = 0, \\
&(\eta^2-u^2)\{(v-\alpha)^2\ddot{w}_\alpha + (v-\alpha)[v-\alpha+v\ln(u/v)]\ddot{z} - [v-\alpha+\alpha\ln(u/v)]\dot{v}\dot{z} \\
&\qquad + (v/u)(v-\alpha)\dot{u}\dot{z}\} + (v-\alpha)^2[2-p(\eta^2-u^2)] = 0, \\
&(1-\eta)(1+3\eta)(\eta^2-u^2)\{(v-\alpha)\dot{w}_\alpha + [v-\alpha+v\ln(u/v)]\dot{z}\}\dot{\eta} \\
&\qquad - (v-\alpha)\{(\eta^2-u^2)[(1-\eta)^2(1+\eta)p - 12] + 2(1-\eta)^2(1+3\eta)\} = 0.
\end{aligned} \tag{6.2}$$

Observing the behaviour of the function $xQ_x(x, \tau)$ at the segment $\alpha \leqslant x \leqslant 1$, and the obvious identity

$$(xy)'' \equiv (x^2y')'/x$$

for every twice differentiable function $y = y(x)$, it is not difficult to show that the profile (5.2), corresponding to the system (6.2), is preserved under any loads exceeding p_{03}.

If a rectangular pulse is the loading plate, then during the loading period ($0 \leqslant \tau \leqslant 1$) we must set $\dot{v} = \dot{u} = \dot{\eta} = 0$, so that the system (6.2) becomes a system of algebraic equations with the unknowns $\ddot{w}_\alpha, \ddot{z}, v_0, u_0, \eta_0$. After unloading for the system (6.2), we should set $p \equiv 0$; numerical integration of the system under initial conditions at the time $\tau = 1$ is performed in the time interval $1 \leqslant \tau \leqslant \tau_0$, where π_0 is determined from the condition $\dot{z}(\tau_0) = 0$. At moment τ_0, the velocity field (6.1) goes into (3.2), and during the interval $\tau_0 \leqslant \tau \leqslant \tau_f$ the plate is moving with the same stress profile as in seetions 3, 4, 5 in the unloading interval ($1 \leqslant \tau \leqslant \tau_f$).

For an arbitrary "blast"-like load, numerical integration of the system (6.2) is performed in the time interval $0 \leqslant \tau \leqslant \tau_0$ with zero initial conditions for the functions $w_\alpha(\tau)$, $\dot{w}_\alpha(\tau)$, $z(\tau)$, $\dot{z}(\tau)$. Initial conditions for $v(\tau)$, $u(\tau)$, $\eta(\tau)$ are determined from the system of algebraic equations derived from (6.2) $\dot{v} = \dot{u} = \dot{\eta} = 0$, $p \geqslant p_{03}$. Though at $\tau = \tau_0$ the velocity field (6.1) goes to (3.2), the stress profile (5.2) does not vanish immediately, as for a rectangular pulse, but is preserved for some time, while the boundary v does not reach the rigid part of the plate. Subsequently the stress profile (4.2) is realized, while the boundary of u, in its turn, does not reach α. Then the transition of the plate to regime (3.1) is realized, and when $\eta = \alpha$ to regime $A-B$ (Fig. 1). If a peak of arbitrary "blast"-like load satisfies the inequalities (3.5), (4.1), (5.1), the analogous behaviour holds.

In conclusion of this section, we point out a number of problems, which are for α and k limit cases of the case considered. Above all, for any fived $\alpha \neq 0$ as $k \to 1$ the dynamics of the plate for $0 \leqslant \tau \leqslant \tau_\alpha$ is described by the formulae of the known solution by HOPKINS and PRAGER [1] for a hinge-supported homogeneous plate, and completely coincides with it if $\alpha \to 0$ and $k \to 1$. Furthermore, if $k \to \infty$, for $0 < \alpha < 1$, the dynamical state corresponds to dynamical bending of a ring-shaped plate, hinge-supported along the exterior contour and clamped along the interior one, if in addition to $\alpha \to 0$ the problem of dynamical bending of a circular homogeneous plate, hinge-supported along the exterior contour and fixed at the central point is derived. In both cases, the stress profile (5.2) is realized. Finally, for every pair $\alpha \neq 0$ and $k > 1$, the problem is equivalent to that of dynamical bending of a ring-shaped plate, hingesupported along the exterior countour and clamped along the interior one, if, furthermore, $\alpha \to 0$, then a homogeneous plate is supported in the centre by a pliable point.

7

Now, we investigated the dynamical behaviour of the plate for k satisfying the inequalities (1.3). The upper bound of peak load p_{**} in regime $A-B$ is not determinable here in an explicit way. But such a quantity $p^0 < p_{**}$ can be indicated that in the interval of loads

$$p_0 \leqslant p \leqslant p^0 = 2/\alpha^2(1-\alpha)(k^0-k), \quad k^0 = (1+5\alpha^2)/6\alpha^2 \tag{7.1}$$

the solution of Sec. 2 holds, since the necessary conditions (2.3) for loads (7.1) are at the same time sufficient for maintenance of the regime $A-B$ (Fig. 1) in the entire ring-shaped part of the plate. If the peak load rises, beginning from the load p_0, then, as was shown by numerical calculations, beginning from some $p^{1*} > p^0$, the monotonolity of $m_1(x, 0)$ is lost. For some $p > p^{1*}$, $m_1(x, 0)$ has the form shown at Fig. 2. Hence the conditions (2.3) for the regime A–B (Fig. 1) would be sufficient, if

$$m_1(x_1, 0) \geqslant 0, \quad m_1(x_2, 0) \leqslant 1, \tag{7.2}$$

where x_1 and x_2 (Fig. 2) are determined from the system of equations

$$m_1'(x_i, 0) = 0, \quad i = 1, 2, \quad \alpha_1 < x_1 < x_2 < 1. \tag{7.3}$$

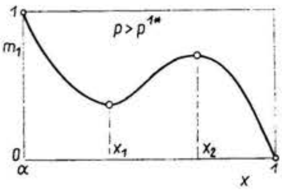


FIG. 2.

Depending on the parameters α, k, and growth of the peak load, it is possible that one or both of inequalities (7.2) are not satified. Denote by x_{1*}, and x_{2*} the values of x_1 and x_2 at which the equalities hold in one or both of the inequalities (7.2).

Then from (7.2), (7.3) and (2.1), we have

$$p_{**}/p_0 = \begin{cases} [\lambda(x_{2*})-1][\lambda(x_{2*})-\mu(x_{2*})]^{-1}, & \text{if} \quad m_1(x_1, 0) \geqslant 0, \\ 1+\dfrac{(1-x_{1*})[\alpha^3+x_{1*}(1+x_{1*})]}{(1+\alpha+\alpha^2)(x_{1*}-\alpha)[\lambda(x_{1*})-\mu(x_{1*})]} & \text{if} \quad m_1(x_2, 0) \leqslant 1; \end{cases}$$

$$\mu(x_{i*}) = (1-x_{i*})(1+\alpha+x_{i*})(1+\alpha+\alpha^2)^{-1},$$

$$\lambda(x_{i*}) = 1-[(x_{i*}-\alpha)f(x_{i*})+6k\alpha^2(1-\alpha)]A^{-1}(k, \alpha),$$

$$f(x_{i*}) = -x_{i*}^2+2(1-\alpha)x_{i*}+\alpha(4-3\alpha), \quad i = 1, 2,$$

where x_{1*} and x_{2*} are the roots $(\alpha < x_{1*} < x_{2*} < 1)$ of the polynomials.

$$x_{1*}^2(x_{1*}^2+2x_{1*}+3\alpha^3)+\alpha^3(1+2x_{1*})[1-3\alpha-6(1-\alpha)(1-k)] = 0,$$

$$x_{2*}^2(x_{2*}^2+2\alpha x_{2*}+3\alpha^2)-2\alpha^2(\alpha+2x_{2*})[\alpha+3(1-\alpha)(1-k)] = 0.$$

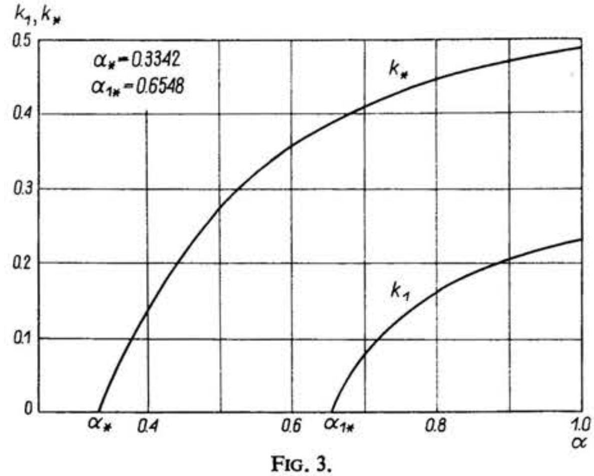


FIG. 3.

The set of values of α and k — at which with the increase of load the equalities hold in (7.2) — determines on the plane α, k a certain curve $k_*(\alpha)$ (Fig. 3). If a point with coordinates (α, k) lies above the curve $k_*(\alpha)$, or to the left of the straight line $\alpha = \alpha_* = 0.3342$, then along the rise of the peak load when $x = x_{2*}$, the moment m_1 achieves its limit positive value, though at $x = x_1$ it is still above zero (Fig. 4). Otherwise, if a point (α, k)

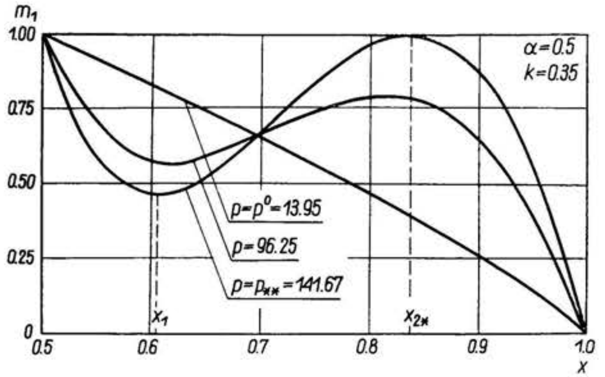


FIG. 4.

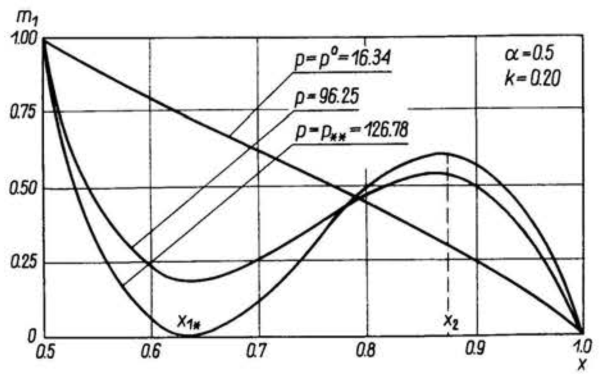


FIG. 5.

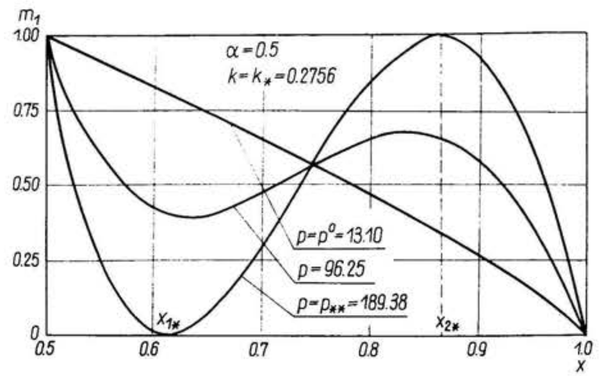


FIG. 6.

lies below the curve $k_*(\alpha)$ (Fig. 3), then at $x = x_{1*}$, m_1 becomes equal to zero, the time as at $x = x_2$ still being less than 1 (Fig. 5). On the curve $k_*(\alpha)$, as the load, rises the equalities in (7.2) are satisfied simultaneously (Fig. 6). In Figs. 7, 8 are plotted the dependences p^0, p_{**}, x_{i*}, $m_1(x_{i*}, 0)$ $(i = 1, 2)$ on the parameter of "washer's" mass k at $\alpha = 0.3$; 0.7. It is of interest to remark that on the curve $k_*(\alpha)$ (Fig. 3) p_{**} takes its maximum value.

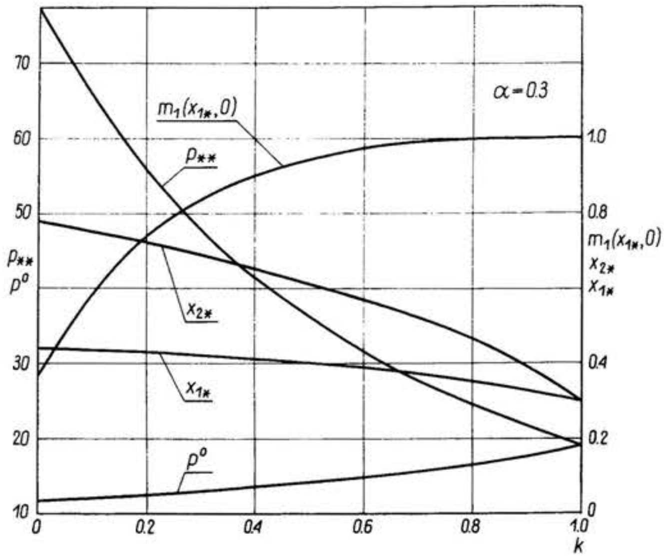


FIG. 7.

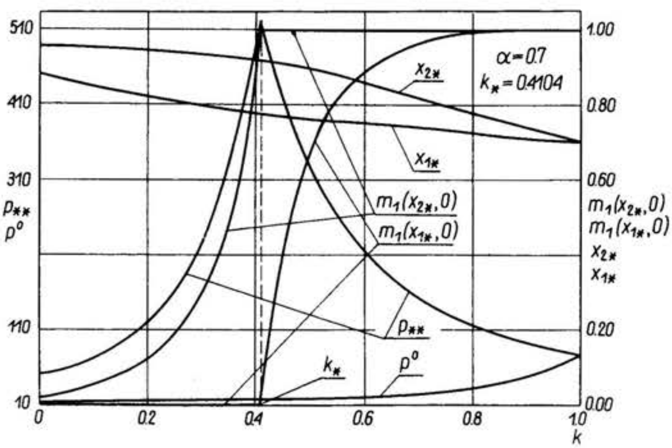


FIG. 8.

8

Let the parameters α and k satisfy the inequalities (Fig. 3):

$$0 \leqslant k < 1 \text{ at } 0 \leqslant \alpha < \alpha_*, \quad k_* < k < 1 \text{ at } \alpha_* \leqslant \alpha < 1. \tag{8.1}$$

In the interval of loads $p_{**} \leqslant p \leqslant p_{1*}$, the following velocity field exists:

$$\dot{w}(x, \tau) = \begin{cases} \dot{w}_\alpha - (x-\alpha)(\eta-\alpha)^{-1}(\dot{w}_\alpha - \dot{w}_\eta), & \alpha \leqslant x \leqslant \eta, \\ (1-x)(1-\eta)^{-1}\dot{w}_\eta, & \eta \leqslant x \leqslant 1, \end{cases} \tag{8.2}$$

$$\dot{w}_\eta = \dot{w}(\eta, \tau).$$

For m_1, we have side by side with (1.4) the following conditions:

$$m_1(\eta-0, \tau) = m_1(\eta+0, \tau) = 1, \quad m_1'(\eta-0, \tau) = m_1'(\eta+0, \tau) = 0. \tag{8.3}$$

It is obvious that $m_2 = 1$ in the whole segment $\alpha \leqslant x \leqslant 1$; the expressions for m_1 we determine from the motion equations (1.1) by means of (1.4), (1.6), (8.3)

(8.4)

$$m_1 = \begin{cases} 1 + (1-\alpha/x)\{\alpha Q_\alpha + (x-\alpha)[\ddot{A}(x^2+2\alpha x+3\alpha^2) + 2(\ddot{B}-p)(x+2\alpha)]/12\}, & \alpha \leqslant x \leqslant \eta, \\ 1 + (x-\eta)^2[\ddot{C}(x^2+2\eta x+3\eta^2) - 2(\ddot{C}+p)(x+2\eta)]/12x, & \eta \leqslant x \leqslant 1, \end{cases}$$

where $\ddot{A}$, $\ddot{B}$, $\ddot{C}$ are derivatives of the expressions:

$$\dot{A} = (\alpha-\eta)^{-1}(\dot{w}_\alpha - \dot{w}_\eta), \quad \dot{B} = (\eta-\alpha)^{-1}(\eta\dot{w}_\alpha - \alpha\dot{w}_\eta), \quad \dot{C} = (\eta-1)^{-1}\dot{w}_\eta. \tag{8.5}$$

The unknown functions $w_\alpha(\tau)$, $w_\eta(\tau)$, $\eta(\tau)$ are determined from the following system of differential equations:

$$a_{i1}\ddot{w}_\alpha + a_{i2}\ddot{w}_\eta + a_{i3}\eta = b_i, \quad i = 1, 2, 3$$

$$a_{11} = \eta^2 + 2\alpha\eta + 3\alpha^2 - 6\alpha^2(1-k), \quad a_{12} = \eta^2 - \alpha^2, \quad a_{13} = (\eta+\alpha)(\dot{w}_\alpha - \dot{w}_\eta),$$

$$b_1 = 2(\eta^2+\alpha\eta+\alpha^2)p, \quad a_{21} = \eta^2+\alpha\eta+\alpha^2-3\alpha^2(1-k), \quad a_{22} = (\eta-\alpha)(2\eta+\alpha), \tag{8.6}$$

$$a_{23} = (2\eta+\alpha)(\dot{w}_\alpha - \dot{w}_\eta), \quad b_2 = 3\eta^2 p, \quad a_{31} = 0, \quad a_{32} = (1-\eta)^2(1+3\eta),$$

$$a_{33} = (1-\eta)(1+3\eta)\dot{w}_\eta, \quad b_3 = 2[(1-\eta)^2(1+2\eta)p - 6].$$

When the rectangular pressure pulse acts on the plate, during its action $(0 \leqslant \tau \leqslant 1)$ we have to set $\eta = \eta_0 = \text{const}$. Then the system of differential equations (8.6) becomes the system of algebraic equations with the unknowns $\ddot{w}_\alpha$, $\ddot{w}_{\eta_0}$, η_0. For the independent parameter it is convenient to take η_0. Then $\ddot{w}_\alpha$, $\ddot{w}_{\eta_0}$, p are found from the system as functions of the parameter η_0.

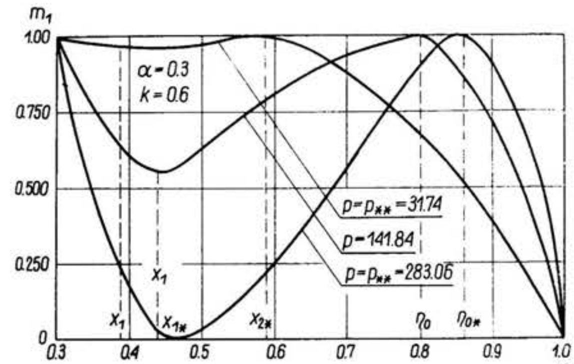


FIG. 9.

If the peak load rises from the value p_{**}, then $m_1(x_1, 0)$ decreases, as shown in Fig. 9. Obviously, the velocity fields (8.2) would be preserved while the first of the inequalities (7.2) holds. Denote by p_{1*}, η_{0*}, x_{1*} values of p, η_0 and x_1 at which the equality holds in the first inequality of (7.2). The qualities indicated are plotted in Fig. 10 as functions

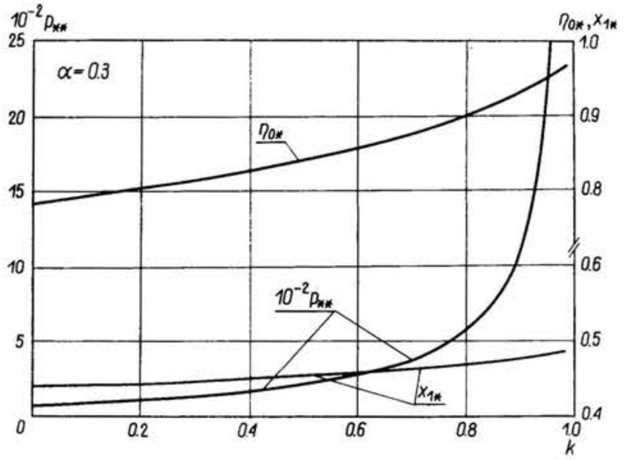


FIG. 10.

of k at $\alpha = 0.3$. For other values of α, the dependences are identical. Under zero initial conditions, from (8.2), (8.6) is derived the deflection in the loading phase ($0 \leqslant \tau \leqslant 1$):

$$(8.7)\qquad \dot{w}(x, \tau) = \begin{cases} [a_\alpha - (x-\alpha)(\eta_0-\alpha)^{-1}(a_\alpha - a_{\eta_0})]\,\tau^2/2, & \alpha \leqslant x \leqslant \eta_0, \\ a_{\eta_0}(1-x)(1-\eta_0)^{-1}\tau^2/2, & \eta_0 \leqslant x \leqslant 1, \end{cases}$$

$$a_\alpha \equiv \ddot{w}_\alpha, \quad a_{\eta_0} \equiv \ddot{w}_{\eta_0}.$$

After unloading at the segment $\alpha \leqslant x \leqslant \eta$, the regime A is realized, and at $\eta \leqslant x \leqslant 1$ the regime A–B (Fig. 1); $\eta = \eta(\tau)$. Clearly,

$$(8.8)\qquad m_1 = m_2 = 1, \quad \alpha \leqslant x \leqslant \eta; \quad m_2 = 1, \quad \eta \leqslant x \leqslant 1.$$

From the Eqs. (1.1), taking into account (8.7), (8.8), we obtain the deflection:

$$w(x, \tau) = [a_\alpha - (x-\alpha)(\eta_0-\alpha)^{-1}(a_\alpha - a_{\eta_0})](\tau - 1/2), \quad \alpha \leqslant x \leqslant \eta.$$

On the segment $\eta \leqslant x \leqslant 1$, m_1 coincides with the second line of (8.4), after setting in the expression for $\dot{C}$ from (8.5)

$$(8.9)\qquad \dot{w}_\eta \equiv \dot{w}(\eta, 1) = a_\alpha - (\eta-\alpha)(\eta_0-\alpha)^{-1}(a_\alpha - a_{\eta_0}), \quad \alpha \leqslant \eta \leqslant \eta_0.$$

By means of (8.9) we determine from the third equation of the system (8.6) the known relation (3.7) for $\eta(\tau)$. The characteristic time τ_α is determined by the formula (3.8). In the expressions (3.7), (3.8) it should be set:

$$a = [(1-\alpha)a_{\eta_0} - (1-\eta_0)a_\alpha](\eta_0-\alpha)^{-1}.$$

On the segment $\eta \leqslant x \leqslant 1$, the deflection equals to

$$w(x, \tau) = (a/24)(1-x)(\eta_0-\eta)[2a_0 + (\eta_0+\eta)(b_0+3a_0) + 2b_0(\eta_0^2 + \eta_0\eta + \eta^2)] + \psi(x),$$
$$a_0 = (\eta_0 a_\alpha - \alpha a_{\eta_0})(\eta_0-\alpha)^{-1}, \quad b_0 = -(a_\alpha - a_{\eta_0})(\eta_0-\alpha)^{-1}.$$

This is deduced analogously to (3.9); and $\psi(x)$ is equal to

$$\psi(x) = \begin{matrix} (1/2)(1-x)(1-\eta_0)^{-1}a_{\eta_0}, & \eta_0 \leqslant x \leqslant 1, \\ (1/24)[12(a_0+b_0x)-a^2(\eta_0-x)^2(1+2\eta_0+x)], & \eta \leqslant x \leqslant \eta_0. \end{matrix}$$

In the last stage of motion ($\tau_\alpha \leqslant \tau \leqslant \tau_f$), the stress profile is determined from the solution (2.2). The expression for residual deflection has the form:

$$w(x, \tau_f) = (1-x)(1-\alpha)^{-1}\{a_\alpha^2 A(k, \alpha)+a(1-\alpha)(\eta_0-\alpha)[2a_0+(\eta_0+\alpha)(b_0+3a_0) + 2b_0(\eta_0^2+\alpha\eta_0+\alpha^2)]\}/24+\psi(x), \quad \alpha \leqslant x \leqslant 1.$$

For arbitrary "blast"-like load, the numerical integration of the system (8.6) proceeds in the time interval $0 \leqslant \tau \leqslant \tau_0$; the initial conditions at $\tau = 0$ for the functions $w_\alpha(\tau)$, $w_\eta(\tau)$, $\dot{w}_\eta(\tau)$, $\dot{w}_u(\tau)$ are zero, and for $\eta(\tau)$ are $\eta(0) = \eta_0$, where η_0 is determined as the solution of the algebraic system of equations derived from (8.6) at $\dot{\eta} \equiv 0$, and $p = p(0)$, $p(0)$ is the peak load. The characteristic time τ_0 is determined from:

$$(1-\alpha)\dot{w}_\eta(\tau_0)-[1-\eta(\tau_0)]\dot{w}_\alpha(\tau_0) = 0. \tag{8.10}$$

To arrive at $w(x, \tau)$, it is obviously necessary to integrate (8.2) by τ in the interval $0 \leqslant \tau \leqslant \tau_0$

$$w(x, \tau) = \begin{cases} w_\alpha(\tau)-(x-\alpha)\int\limits_0^\tau \dfrac{\dot{w}_\alpha(\tau)-\dot{w}_\eta(\tau)}{\eta(\tau)-\alpha}d\tau, & \alpha \leqslant x \leqslant \eta, \\ (1-x)\int\limits_0^\tau [1-\eta(\tau)]^{-1}\dot{w}_\eta(\tau)\,d\tau+\psi(x), & \eta \leqslant x \leqslant 1. \end{cases} \tag{8.11}$$

For the segment $\eta_0 \leqslant x \leqslant 1$, $\psi(x) \equiv 0$, since $w(x, 0) = 0$. Further,

$$\psi(x) = w_\alpha^0(x)+(x-\alpha)\int\limits_x^{\eta_0} \frac{\dot{w}_\alpha^0(\eta)-\dot{w}_\eta^0(\eta)}{\dot{\eta}^0(\eta-\alpha)}d\eta+(1-x)\int\limits_x^{\eta_0}\frac{\dot{w}_\eta^0(\eta)}{\dot{\eta}^0(1-\eta)}d\eta, \quad \eta \leqslant x \leqslant \eta_0, \tag{8.12}$$

where $\dot{w}_\alpha^0(\eta)$, $w_\alpha^0(\eta)$, $\dot{w}_\eta^0(\eta)$, $\dot{\eta}^0(\eta)$ are known functions of η as a result of integration of the system (8.6). In the last stage of motion ($\tau_0 \leqslant \tau \leqslant \tau_f$), the regime A–B is realized in the plate (Fig. 1). For deflection, we have the equation

$$\ddot{w}(x, \tau) = 12(1-x)(p-p_0)/p_0(1-\alpha)A(k, \alpha), \quad \alpha \leqslant x \leqslant 1, \quad \tau_0 \leqslant \tau \leqslant \tau_f. \tag{8.13}$$

Thus, by means of (8.10)–(8.13), we have

$$w(x, \tau) = (1-x)(1-\alpha)^{-1}[(\tau-\tau_0)\dot{w}_\alpha(\tau_0)+12A^{-1}(k, \alpha)\times \int\limits_{\tau_0}^{\tau}\int\limits_{\tau_0}^{\tau}(pp_0^{-1}-1)d\tau d\tau]+w(x, \tau_0), \quad \alpha \leqslant x \leqslant 1, \tau_0 \leqslant \tau \leqslant \tau_f, \tag{8.14}$$

where τ_f is the root of the equation

$$\dot{w}_\alpha(\tau_0)+12A^{-1}(k, \alpha)\int\limits_{\tau_0}^{\tau_f}(pp_0^{-1}-1)d\tau = 0.$$

If the peak load exceeds p_{1*}, then for α and k from (8.1) the following stress profiles are realized in the plate (Fig. 1):

$$\begin{gathered}\alpha \leqslant x \leqslant \xi_1 \to A-B, \quad \xi_1 \leqslant x \leqslant \xi_2 \to B-(C)-B,\\ \xi_2 \leqslant x \leqslant \eta \to B-A, \quad \eta \leqslant x \leqslant 1 \to A-B,\end{gathered} \tag{8.15}$$

$$\begin{gathered}\alpha \leqslant x \leqslant \xi_1 \to A-B, \quad \xi_1 \leqslant x \leqslant \xi \to B-C,\\ \xi \leqslant x \leqslant \xi_2 \to C-B, \quad \xi_2 \leqslant x \leqslant \eta \to B-A,\\ \eta \leqslant x \leqslant 1 \to A-B.\end{gathered} \tag{8.16}$$

The stress profile (8.15) is realized in the interval of the loads $P_{1*} \leqslant P \leqslant P_{2*}$, where P_{2*} is determined from the condition $m_1(x_1, 0) = -1$ and the stress profile (8.16) — for loads with peak $P \geqslant P_{2*}$.

Solutions, corresponding to profiles (8.15) and (8.16) are not presented here because of their cumbersomeness

9

The parameters α and k satisfy the inequalities

$$\alpha_* < \alpha < 1, \quad 0 \leqslant k < k_* \tag{9.1}$$

— i.e., a point with coordinates α and k lies under the curve $k_*(\alpha)$ (Fig. 3). In the interval of the loads

$$p_{**} \leqslant p \leqslant p_1, \tag{9.2}$$

the following stress profile holds (Fig. 1):

$$\begin{gathered}\alpha \leqslant x \leqslant \xi_1 \to A-B, \quad \xi_1 \leqslant x \leqslant \xi_2 \to B-(C)-B,\\ \xi_2 \leqslant x \leqslant 1 \to B-(A)-B.\end{gathered} \tag{9.3}$$

From (1.5), (1.4), by means of the conditions [1]

$$\dot{w}'(\xi_i-0, \tau) = \dot{w}'(\xi_i+0, \tau), \quad i = 1, 2, \tag{9.4}$$

the following velocity field is derived:

$$\dot{w}(x, \tau) = \begin{cases} \dot{w}_\alpha \omega^{-1}(\alpha+\omega-x), & \alpha \leqslant x \leqslant \xi_1,\\ \dot{w}_\alpha \omega^{-1}\{\alpha+\omega-\xi_1[1+\ln(x/\xi_1)]\}, & \xi_1 \leqslant x \leqslant \xi_2,\\ (\xi_1 \dot{w}_\alpha/\xi_2\omega)(1-x), & \xi_2 \leqslant x \leqslant 1,\end{cases} \tag{9.5}$$
$$\omega = \xi_1\xi_2^{-1}[1+\xi_2\ln(\xi_2/\xi_1)]-\alpha.$$

By means of (1.4), (9.3)–(9.5), and also the conditions

$$m_1(\xi_i-0, \tau) = m_1(\xi_i+0, \tau) = 0, \quad i = 1, 2,$$

from (1.1) we derived the stress profile:

$$m_1 = \begin{cases} (1/4)(x^2-\xi_1^2)[\ddot{D}-p-(1-\ln x)\ddot{C}]+\{1+\xi_1 Q_{\xi_1}\\ \qquad +(\xi_1^2/2)[(1-\ln\xi_1)\ddot{C}+p-\ddot{D}]\}\ln(x/\xi_1), & \xi_1 \leqslant x \leqslant \xi_2,\\ x^{-1}(x-\xi_2)\{1+\xi_2 Q_{\xi_2}+(1/12)(x-\xi_2)[\ddot{E}(x^2+2x\xi_2+3\xi_2^2)\\ \qquad -2(\ddot{E}+p)(x+2\xi_2)]\}, & \xi_2 \leqslant x \leqslant 1;\end{cases} \tag{9.6}$$

$$m_2 = \begin{cases} 1, & \alpha \leqslant x - \leqslant \xi_1, \quad \xi_2 \leqslant x \leqslant 1, \\ 1+m_1, & \xi_1 \leqslant x \leqslant \xi_2, \end{cases}$$

$$\xi_1 Q_{\xi_1} = \alpha Q_\alpha + (1/6)(\xi_1-\alpha)[2\ddot{A}(\xi_1^2+\alpha\xi_1+\alpha^2)+3(\ddot{B}-p)(\xi_1+\alpha)],$$

$$\xi_2 Q_{\xi_2} = \xi_1 Q_{\xi_1} + (1/2)(\ddot{D}-p)(\xi_2^2-\xi_1^2)+(\ddot{C}/4)[\xi_2^2(2\ln\xi_2-1)-\xi_1^2(2\ln\xi_1-1)].$$

On the segment $\alpha \leqslant x \leqslant \xi_1$ the expression for m_1 coincides with the first line of (8.4). The quantities $\ddot{A}$, $\ddot{B}$, $\ddot{C}$, $\ddot{D}$, $\ddot{E}$ are derivatives of the expressions

$$\dot{A} = -\dot{w}_\alpha \omega^{-1}, \quad \dot{B} = \dot{w}_\alpha \omega^{-1}(\alpha+\omega), \quad \dot{C} = -\xi_1 \dot{w}_\alpha \omega^{-1},$$

$$\dot{D} = \dot{w}_\alpha \omega^{-1}[\alpha+\omega-\xi_1(1-\ln\xi_1)], \quad \dot{E} = -\xi_1 \dot{w}_\alpha/\xi_2\omega.$$

The functions of time $w_\alpha(\tau)$, $\xi_1(\tau)$, $\xi_2(\tau)$ satisfy the system of equations:

(9.7) $$\omega a_{i1}\ddot{w}_\alpha + \dot{w}_\alpha \xi_2^{-1}[(\xi_1-\alpha)a_{i2}\dot{\xi}_1 + \xi_1\xi_2^{-1}(1-\xi_2)a_{i3}\dot{\xi}_2] = \omega^2 b_i, \quad i = 1,2,3,$$

$$a_{11} = (\xi_1-\alpha)\{2\omega[\xi_1^2+\alpha\xi_1+\alpha^2-3\alpha^2(1-k)]-(\xi_1-\alpha)^2(\xi_1+\alpha)\}, \quad a_{12} = (\xi_1-\alpha)^2(\xi_1 +\alpha)[1-\xi_2+\xi_2\ln(\xi_2/\xi_1)], \quad a_{13} = -(\xi_1-\alpha)^3(\xi_1+\alpha), \quad b_1 = 2[(\xi_1^3-\alpha^3)p-6\xi_1],$$

$$a_{21} = 3\xi_1\xi_2^{-1}(\xi_2^2-\xi_1^2)-2[2\xi_1^3+\alpha^3+3\alpha^2\omega(1-k)]\ln(\xi_2/\xi_1), \quad a_{22} = 3(\xi_2^2-\xi_1^2)+2[(1 -\xi_2)(\xi_1-\alpha)(2\xi_1+\alpha)-3\xi_1^2-\xi_2(\xi_1^2+\alpha\xi_1+\alpha^2)\ln(\xi_2/\xi_1)]\ln(\xi_2/\xi_1), \quad a_{23} = 3\alpha(\xi_2^2-\xi_1^2) -(3\xi_1\xi_2^2+\xi_1^3+2\alpha^3)\ln(\xi_2/\xi_1), \quad b_2 = 3[(\xi_2^2-\xi_1^2)p-4\ln(\xi_2/\xi_1)], \quad a_{31} = \xi_1\xi_2^{-1}\times \times[3\xi_2(1+\xi_1)+(1-\xi_2)^2]-2[\alpha^3+2\xi_1^3+3\alpha^2\omega(1-k)], \quad a_{32} = 3(2-\xi_2)(\xi_2^2-\xi_1^2)+(1 -\xi_2)[2(2\xi_1^2-\alpha\xi_1-\alpha^2)+(1-\xi_2)(1+3\xi_2)]-2\xi_2(\xi_1^2+\alpha\xi_1+\alpha^2)\ln(\xi_2/\xi_1), \quad a_{33} = 3\xi_2^2\times \times(2\alpha-\xi_1)-\xi_1^3-2\alpha^3-(\xi_1-\alpha)(1-\xi_2)(1+3\xi_2)-\xi_1(1+2\xi_2+3\xi_2^2)\ln(\xi_2/\xi_1),$$

$$b_3 = 2[(1+\xi_2+\xi_2^2)p-6].$$

When beginning from the value $p = p_{**}$ the peak load rises, the quantity $m_1(x_1, 0)$ $(\xi_1 < x_1 < \xi_2)$ decreases monotonously, tending to -1, as has been shown by numerical calculations. The quantity $m_1(x_1, 0)$ $(\xi_2 < x < 1)$ can, depending on α and k, either increase monotonously up to 1 or stay constant or decrease, tending to zero. In this way, in the region (9.1) is determined a certain curve $k_1(\alpha)$ (Fig. 3) such that

(9.8) $$\frac{\partial}{\partial p(0)} m_1(x_2, 0) = 0 \quad \text{for} \quad (\alpha, k) \in k_1(\alpha).$$

For point (α, k) belonging to (9.1), but lying above the curve $k_1(\alpha)$ as is shown at Fig. 3, the characteristic load p_1 is determined from the condition

(9.9) $$m_1(x_2, 0) = 1.$$

If a point (α, k) lies under or on the curve $k_1(\alpha)$ [corresponding to fulfilment of (9.10)]

(9.10) $$0.6548 = \alpha_{1*} \leqslant \alpha < 1, \quad 0 \leqslant k \leqslant k_1(\alpha),$$

the p_1 is determined from the condition:

(9.11) $$\xi_2(0) = 1.$$

Denote by ξ_{1*}, ξ_{2*}, $m_1(x_{1*}, 0)$, x_{2*}, p_1 the characteristic values of the quantities ξ_1, $\xi_:$, $m_1(x_1\ 0)$, x_2, $p(0)$ at which the equality (9.9) is satisfied under the increase of load. These quantities are plotted in Fig. 11 depending on k at $\alpha = 0.5$. For α and k from (9.10) by the non positivity of the derivative (9.8) with rise of peak load, the equality (9.1) holds. Denote by $\xi_{,*}$, x_{1*}, $m_1(x_{1*}, 0)$, p_1 the values of the quantities obtained above. In Fig. (12), these quantities are plotted for k from (9.10) at $\alpha = 0.8$. For other values of α, the dependences are identical with those plotted in Figs. (11) and (12). It must be remarked also that in the whole interval of loads (9.2) in the region (9.1) $m_1(x_1, 0) > -1$.

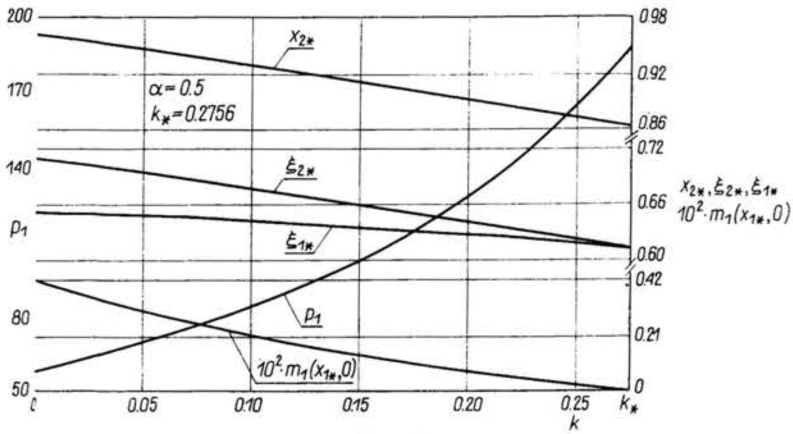


FIG. 11.

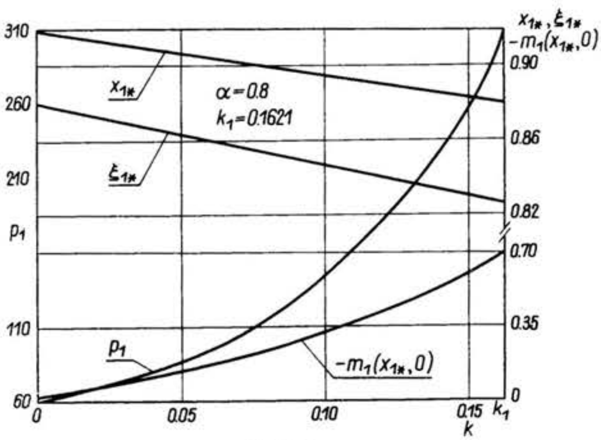


FIG. 12.

For a rectangular pressure pulse in the loading phase ($0 \leqslant \tau \leqslant 1$), in the system (9.7) it should put

$$\xi_1 = \xi_{01} = \text{const}, \quad \xi_2 = \xi_{02} = \text{const}. \tag{9.12}$$

For numerical solution of system obtained of transcendental equations the following reception nay be recommended. From any two of equations of the system, express $\ddot{w}_\alpha$

and p by ξ_{01} and ξ_{02}. Substituting these values into the third equation, we may solve it with respect to one of the unknowns ξ_{01} and ξ_{02}, considering the left unknown as an independent parameter. When motion proceeds by inertia, numerical integration of the system (9.7) is performed in the interval of time $1 \leqslant \tau \leqslant \tau_1$, where τ_1 is determined by the condition $\xi_1(\tau_1) = \xi_2(\tau_1)$. Further motion $(\tau_1 \leqslant \tau \leqslant \tau)$ proceeds in the regime A–B (Fig. 1).

For an arbitrary "blast"-like load [5], the system (9.7) is integrated in the time interval $0 \leqslant \tau \leqslant \tau_1$. The initial conditions at $\tau = 0$ for $w_\alpha(\tau)$ and $\dot{w}_\alpha(\tau)$ are zero conditions, and for $\xi_1(\tau)$ and $\xi_2(\tau)$ are determined from the system of transcendental equations derived from (9.7) under the conditions (9.10).

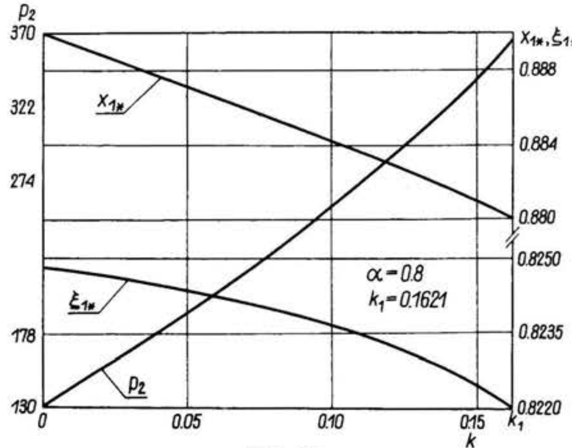


FIG. 13.

In conclusion of this section, it should be pointed out that under the rise of peak load from the value $p = p_1$, the stress profile (9.3) may pass on to the profile (8.15) and further into the profile (8.16), if the parameters α and k lie between the curves $k_*(\alpha)$ and $k_1(\alpha)$ in Fig. 3.

10

If α and k satisfy the inequalities (9.10), then in the interval of loads $p_1 \leqslant p \leqslant p_2$ the following stress profile holds (Fig. 1):

$$\alpha \leqslant x \leqslant \xi_1 \to A-B, \quad \xi_1 \leqslant x \leqslant 1 \to B-(C)-B. \tag{10.1}$$

The characteristic load p_2 is determined from the condition $m_1(x_1, 0) = -1$, $(\xi_1 < x_1 < 1)$ (see Fig. 13). The velocity field, and also $m_1(x, \tau)$, $m_2(x, \tau)$ are derived from (9.5), (9.6) by setting $\xi_2 \equiv 1$.

The unknown functions $w_\alpha(\tau)$, $\xi_1(\tau)$ are determined from the first two equations of the system (9.7) at $\xi_2 \equiv 1$. Numerical integration of the system derived is performed in the intervals of time $1 \leqslant \tau \leqslant \tau_{01}$, $0 \leqslant \tau \leqslant \tau_{01}$, respectively, for a rectangular pulse and an arbitrary "blast"-like load [5]. The characteristic time τ_{01}, in general different for different loads, is determined from the condition $m_1'(1, \tau_{01}) = 0$. In the time $\tau = \tau_{01}$ from the exterior contour of the plate its propagation begins inside a region *B*–(*A*)–*B* bounded from the interior side by the moving boundary $\xi_2(\tau)$, and from the exterior one — by the exterior contour of the plate. Thus in the entire ring-shaped part of the plate the stress profile (9.3) is realized, which, in turn, is preserved only up to the moment when the boundaries $\xi_1(\tau)$ and $\xi_2(\tau)$ merge on into the other. Further motion ($\tau_1 \leqslant \tau \leqslant \tau_f$) holds in the regime *A*–*B* (Fig. 1).

11

Let the $p \geqslant p_2$ parameters α and k satisfy (9.10). In the initial phase of motion, the following stress profile holds (fig. 1).

(11.1) $\alpha \leqslant x \leqslant \xi_1 \to A\text{–}B, \quad \xi_1 \leqslant x \leqslant \xi \to B\text{–}C, \quad \xi \leqslant x \leqslant 1 \to C\text{–}B.$

The velocity field may be expressed:

$$\dot{w}(x,\tau) = \begin{cases} \dot{w}_\xi + \dot{z}\omega^{-1}\{\xi_1[1+\ln(\xi/\xi_1)] - x\}, & \alpha \leqslant x \leqslant \xi_1, \\ \dot{w}_\xi + \xi_1 \dot{z}\omega^{-1}\ln(\xi/x), & \xi_1 \leqslant x \leqslant \xi, \\ \dot{w}_\xi (\ln x)/\ln\xi, & \xi \leqslant x \leqslant 1, \end{cases}$$

$$\dot{z} = \dot{w}_\alpha - \dot{w}_\xi, \quad \omega = \xi_1 - \alpha + \xi_1 \ln(\xi/\xi_1), \quad \dot{w}_\xi = \dot{w}(\xi, \tau).$$

The expression for m_1 coincides with the first line of (8.4) in the interval $\alpha \leqslant x \leqslant \xi_1$, and with the second line of (9.6) in the interval $\xi_1 \leqslant x \leqslant \xi$; in the interval $\xi \leqslant x \leqslant 1$ it has the form:

$$m_1 = -1 - (1/4)(x^2 - \xi^2)[p + (1 - \ln x)\ddot{E}] + (\xi^2/2)[p - (1 - \ln\xi)\ddot{E}]\ln(x/\xi).$$

$\ddot{A}, \ddot{B}, \ddot{C}, \ddot{D}, \ddot{E}$ are derivatives of the expressions

$$\dot{A} = -\dot{z}\omega^{-1}, \quad \dot{B} = \dot{w}_\xi + \xi_1 \dot{z}\omega^{-1}[1 + \ln(\xi/\xi_1)],$$

$$\dot{C} = -\xi_1 \dot{z}\omega^{-1}, \quad \dot{D} = \dot{w}_\xi + \xi_1 \dot{z}\omega^{-1}\ln\xi, \quad \dot{E} = \dot{w}_\xi(\ln\xi)^{-1}.$$

The functions $w_\alpha(\tau)$, $w_\xi(\tau)$, $\xi_1(\tau)$, $\xi(\tau)$ are determined from the system of equations:

(11.2) $\omega a_{i1}\ddot{w}_\alpha + \omega a_{i2}\ddot{w}_\xi + \dot{z}(\xi_1 - \alpha)a_{i3}\dot{\xi}_1 + \dot{z}\xi_1\xi^{-1}a_{i4}\dot{\xi} = \omega^2 b_i, \quad i = 1, \ldots, 4,$

$$a_{11} = (\xi_1 - \alpha)\{2\omega[\xi_1^2 + \alpha\xi_1 + \alpha^2 - 3\alpha^2(1-k)] - (\xi_1 - \alpha)^2(\xi_1 + \alpha)\},$$

$$a_{12} = (\xi_1 - \alpha)^2(\xi_1 + \alpha), \quad a_{13} = (\xi_1 - \alpha)^2(\xi_1 + \alpha)\ln(\xi/\xi_1), \quad a_{14} = (\xi_1 - \alpha)^3(\xi_1 + \alpha),$$

$$b_4 = 2[(\xi_1^3 - \alpha^3)p - 6\xi_1], \quad a_{21} = 3\xi_1(\xi^2 - \xi_1^2)2 - [3\alpha^2\omega(1-k) + 2\xi_1^3 + \alpha^3]\ln(\xi/\xi_1),$$

$$a_{22} = 3\alpha(\xi_1^2 - \xi^2) + (3\xi_1\xi^2 + \xi_1^3 + 2\alpha^3)\ln(\xi/\xi_1),$$

$$a_{23} = 3[\xi^2 - \xi_1^2 - 2\xi_1^2\ln(\xi/\xi_1)] - 2(\xi_1^2 + \alpha\xi_1 + \alpha^2)\ln^2(\xi/\xi_1), \quad a_{24} = a_{22},$$

$$b_2 = 3\{(\xi^2 - \xi_1^2)p - 4[1 + \ln(\xi/\xi_1)]\}, \quad a_{31} = 3\xi_1\xi^2 - \xi_1^3 - 2\alpha^3 - 6\alpha^2\omega(1-k),$$

$$a_{32} = 3\xi^2(\xi_1 - 2\alpha) + \xi_1^3 + 2\alpha^3 + 6\xi_1\xi^2\ln(\xi/\xi_1), \quad a_{33} = 3(\xi^2 - \xi_1^2) - 2(\xi_1^2 + \alpha\xi_1 + \alpha^2)\ln(\xi/\xi_1),$$

$$a_{34} = 3\xi^2(\xi_1 - 2\alpha) + \xi_1^3 + 2\alpha^3 + 6\xi_1\xi^2\ln(\xi/\xi_1),$$

$$b_3 = 6(\xi^2 p - 2), \quad a_{41} = 0, \quad a_{42} = \xi\omega^{-1}[1 - \xi^2 + 2\xi^2(1 - \ln\xi)\ln\xi]\ln\xi,$$

$$a_{43} = 0, \quad a_{44} = -\dot{w}_\xi\omega a_{42}/\dot{z}\xi_1\ln\xi, \quad b_4 = -\xi[4 + (1 - \xi^2 + 2\xi^2\ln\xi)p]\ln^2\xi.$$

As in previous cases for rectangular pulse, at the time of load action ($0 \leqslant \tau \leqslant 1$) we have a quasi-static problem, so that

$$\xi_1 = \xi_{01} = \text{const}, \quad \xi = \xi_0 = \text{const}.$$

The system (11.2) becomes a system of transcendental equations with respect to the unknowns $\ddot{w}_\alpha$, $\ddot{w}_\xi$, ξ_{01}, ξ_0. The latter is easily solved by reducing to one equation relative to one of the unknowns ξ_{01} or ξ_0.

Numerical integration of the system (11.2) for rectangular load is performed in the time interval

$$1 \leqslant \tau \leqslant \tau_*. \tag{11.3}$$

For an arbitrary "blast"-like load [5] unit in (11.3) should be replaced by zero. The necessary initial conditions at $\tau = 0$ are determined as in previously considered cases. Time τ_* is determined from the equation:

$$\omega\dot{w}_\xi + \xi_1 \dot{z} \ln \xi = 0.$$

Further motion in the interval $\tau^* \leqslant \tau \leqslant \tau_f$ proceeds by the same scheme as in the previous section in time intervals $1 \leqslant \tau \leqslant \tau_f$ and $0 \leqslant \tau \leqslant \tau_f$, respectively, for the first and the second types of load.

In conclusion, we would point out that if in the solution of this section we set

$$w_\xi \equiv \dot{w}_\xi \equiv \ddot{w}_\xi \equiv 0, \quad \xi \equiv 1,$$

then we obtain a solution corresponding to the dynamical bending of a piece-wise inhomogeneous plate with rigidly clamped exterior contour (with $0 \leqslant k \leqslant 1$). If, furthermore, $\alpha \to 0$, then we have the known solution [2].

It should be remarked also, that for α and k from (9.11) and $p > p_1$, the stress profiles (10.1) and (11.1) are similar in quality, as in the problem with local load investigated in [4].

12

Numerical calculations were performed by means of the electronic computer M–220, making use of previous results. In Fig. 14 is shown the influence of the mass parameter k on the maximum residual deflection in the plate. It will be seen that by enlarging the mass of the "washer", deflections may be reduced in an essential manner (making the dynamic resistance better); the time as static load capacity remains the same.

Figs. 15 and 16 enable us to compare the maximum residual deflections of piece-inhomogeneous and homogeneous plates of the same radii and volume, of the same material — i.e., when the jump of mechanical properties follows from the jump in the thickness the plate. The condition of volume equality $V = V^0$ is equivalent to the following

$$\delta^0/\delta \equiv \nu(\alpha, k) = 1 + \alpha^2(k-1), \tag{12.1}$$

where $2\delta^0$ is the thickness of a homogeneous plate. It is more convenient to express the dimensionless quantities by the parameters of a homogeneous plate, such that

$$p = \nu^2 p^0, \quad w = \nu w^0, \quad p^0 = pb_0^2/\sigma_0\delta^{02}, \quad w^0 = W\gamma^0 b_0^2/\sigma_0\,\delta^{02}t_0^2.$$

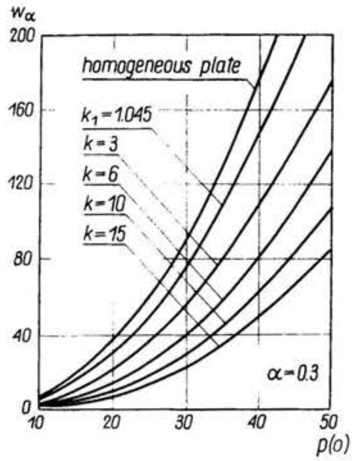


FIG. 14.

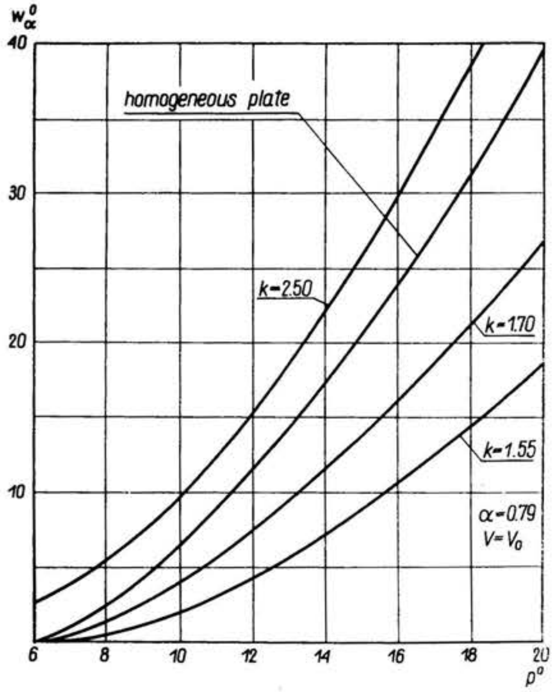
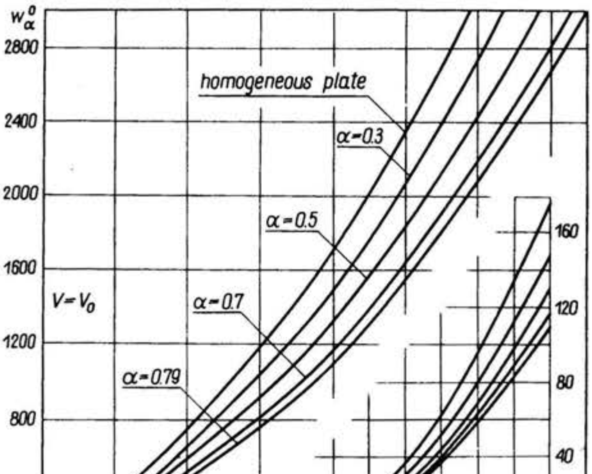


FIG. 15.



FG. 16.

Let M_{01} and M_{02} be limit moments of exterior and interior parts of a plate. Then at static load, the interior part of a plate remains rigid [7]:

$$\mu \leqslant \mu_1 = (1-\alpha^3)(1+\alpha^2-\alpha^3)^{-1}, \quad \mu = M_{01} M_{02}^{-1}. \tag{12.2}$$

Since $\mu k^2 = 1$, there follows the equivalence of (12.2) to the inequality: $k \geqslant k_{1*} = \mu_1^{-1/2}$.

Admitted were only such loads the central part of which remains rigid. Numerical calculations were performed for various α and k, satisfying (12.1). It should be remarked that deflections were maximum at $\alpha = 0.79$ and $k = k_{1*} = 1.49$ (Fig. 15) — i.e., for those parameters of the plate at which load carrying capacity is maximum [7].

From Fig. 15 it follows that redistribution of the plate material to the central part to improve its dynamic resistance is permissible only up to a certain limit, the exceeding of which leads to a negative result.

In Fig. 16 are shown the analogous dependences in the case in which the load capacity of both types of plates is the same — i.e., under the condition $\nu^2(1-\alpha^3) = 1$. It should be remarked that in this case, the maximum of dynamic resistance is reached at $\alpha = 0.79$.

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