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On the stability limit of non-linear resonances in multiple-degree-of-freedom vibrating systems

W. SZEMPLIŃSKA-STUPNICKA (WARSZAWA)

THE PAPER presents an analysis of stability of solutions derived in [1] and concerning non-linear, dissipative vibrating systems with many degrees of freedom; the solutions represent the so-called non-linear resonances, periodic and almost periodic (combined). The analysis is based on investigating the equations written in terms of variations which, in the case of the Ritz method, assume the form of a system of ordinary differential equations, their coefficients being periodic or almost periodic functions of time; in the case of the method of averaging the problem is reduced to a system of ordinary equations with constant coefficients.

Dla nieliniowego, dysypacyjnego układu drgającego o wielu stopniach swobody przeprowadzono analizę stateczności rozwiązań ustalonych, otrzymanych w pracy [1], a przedstawiających tzw. rezonanse nieliniowe. Wyprowadzono jednolite kryterium granicy pierwszego obszaru niestateczności wspólne dla wszystkich typów rezonansów nieliniowych-periodycznych i prawie-periodycznych (kombinowanych). Analiza opiera się na badaniu równań różniczkowych zwyczajnych o współczynnikach będących periodycznymi lub prawie-periodycznymi funkcjami czasu, a przy metodzie uśrednienia układu równań zwyczajnych o współczynnikach stałych.

Для нелинейной колебательной системы с затуханием с многими степенями свободы проведен анализ устойчивости установившихся решений, полученных в работе [1], а представляющих т. наз. нелинейные резонансы. Выведен однородный критерий границы первой области неустойчивости совместный для всех типов нелинейных резонансов — периодических и почти-периодических (комбинированных). Анализ опирается на исследованию уравнений в вариациях, которые при методе Ритца принимают форму системы обыкновенных дифференциальных уравнений с коэффициентами, являющимися периодическими или почти-периодическими функциями времени, а при методе усреднения — системы обыкновенных дифференциальных уравнений с постоянными коэффициентами.

1. Introduction

INVESTIGATED in a former paper by the author [1] were non-linear resonances of discrete dissipative systems with many degrees of freedom. A uniform approach to all types of resonances, periodic and almost periodic, was presented and the relations between the first order solutions resulting from the Ritz and averaging methods were given.

In the present paper, an analysis will be made of the stability of the solutions derived in [1], and a criterion will be given for the limits of the first instability region, uniform for all types of non-linear resonances. The analysis will be based on investigation of the corresponding variational equations which, in the case of the Ritz method, assume the form of a system of ordinary differential equations, their coefficients being periodic or almost periodic functions of time; in the case of the method of averaging, they constitute a system of ordinary equations with constant coefficients.

2. General equations

The mechanical model of the system consists of a chain of n concentrated masses $m_1, \ldots, m_n$ linked by means of massless springs and energy dissipating elements. The masses are acted on by harmonic forces directed along the axis of the chain. The equations of motion of the system have the form:

$$\varepsilon_i(t) \equiv m\ddot{x}_i + \sum_{k=0}^{n} K_{ik}(x_i - x_k) + \mu f_i(x_1, \ldots, x_n, \dot{x}_1, \ldots, x_n) - P_i \cos \Omega t = 0, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

where x_i denotes the displacement of m_i from the position of equilibrium, K_{ik} are the stiffnes coefficients of the linear part of elastic forces, μ is a small parameter, and $0 \ll \mu 1$.

The functions $f_i(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$ represent the non-linear part of the elastic forces and the damping forces; moreover, the relation $f_i(0, \ldots, 0, 0, \ldots, 0) = 0$ holds true. It is assumed that the functions may be represented in the form of finite power series.

In applying the method of averaging, it is convenient to use the equations of motion written in normal coordinates introduced by means of the transformation

$$x_i = \sum_{j=1}^{n} b_{0ij} \xi_{0j}, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Here, b_{0ij}, $i = 1, 2, \ldots, n$, is the j-th eigen function of the linearized system (at $\mu = 0$).

The Eqs. of motion (2.1) assume in that system the form

$$\varepsilon_j(t) \equiv M_{0j}\ddot{\xi}_{0j} + M_{0j}\omega_{0j}^2 \xi_{0j} + \mu F_j(\xi_{01}, \ldots, \xi_{0n}, \dot{\xi}_{01}, \ldots, \dot{\xi}_{0n}) - Q_{0j} \cos \Omega t = 0, \quad j = 1, 2, \ldots, n, \tag{2.3}$$

where

$$M_{0j} = \sum_{i=1}^{n} m_i b_{0ij}^2, \quad Q_{0j} = \sum_{i=1}^{n} P_i b_{0ij}, \quad F_j = \sum_{i=1}^{n} f_i b_{0ij}.$$

The solution describing the phenomenon of non-linear resonances in the first approximation is assumed in the form [1]

$$x_i = \sum_{s=1}^{p} r_s(b_{is} \cos \theta_s + e_{is} \sin \theta_s) + C_i \cos \Omega t, \quad i = 1, 2, \ldots, n, \tag{2.4}$$
$$\theta_s = \omega_s t + \Phi_s, \quad b_{1s} = 1, \quad e_{1s} = 0, \quad s = 1, 2, \ldots, p, \quad 1 \leqslant p \leqslant n,$$

or, in a simplified notation,

$$x_i = \sum_{s=1}^{p} (r_{1s}^{(1)} \cos \theta_s + r_{is}^{(2)} \sin \theta_s) + C_i \cos \Omega t, \quad i = 1, 2, \ldots, n. \tag{2.4}$$
$$r_{is}^{(2)} = 0, \quad s = 1, 2, \ldots, p.$$

The condition of existence of non-trivial solutions r_s yields the relation between Ω and ω_s,

$$\Omega = \frac{1}{N} \sum_{s=1}^{p} n_s \omega_s, \tag{2.5}$$

where $N, n_s = \mp 1, \mp 2, \ldots$

Periodic resonance occurs when $p = 1$ and $\Omega = \omega_s n_s/N$. The combined resonances ($p \geqslant 2$) represent almost-periodic vibrations, since the frequencies ω_s, $s = 1, 2, \ldots, p$, are generally assumed to be incommensurable numbers.

According to the Ritz method, the unknown coefficients of the solution (2.4), r_s, ω_s, b_{is}, e_{is} and $\Phi = \frac{1}{N}\sum_{s=1}^{p} n_s \Phi_s$, are determined from the equations

$$X_{is} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \varepsilon_i(t)\cos\theta_s\,dt = 0,$$

(2.6)
$$Y_{is} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \varepsilon_i(t)\sin\theta_s dt = 0,$$

$$Z \equiv \frac{1}{N}\sum_{s=1}^{p} n_s\omega_s - \Omega = 0, \quad s = 1, 2, \ldots, p, \quad i = 1, 2, \ldots, n.$$

When the averaging method is used, new variables $\bar{a}_j(t)$, $\bar{\Phi}_j(t)$ have to be introduced by means of the transformation

(2.7)
$$\begin{aligned} \xi_{0j} &= \bar{a}_j(t)\cos\bar{\theta}_j + d_j\cos\Omega t,\\ \dot{\xi}_{0j} &= -\bar{a}_j(t)\omega_{0j}\sin\bar{\theta}_j - d_j\Omega\sin\Omega t,\\ \theta_j &= \omega_{0j}t + \bar{\Phi}_j(t), \quad j = 1, 2, \ldots, n. \end{aligned}$$

The equations of motion (2.3) yield[1] the system of algebraic equations with the unknowns $a_1, \ldots, a_p$ and Φ,

$$(2.8) \quad \frac{da_s}{dt} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\mu F_s\sin\theta_s}{\omega_{0s}M_{0s}}dt = \mu D_s(a_1, \ldots, a_p, \Phi) = 0, \quad s = 1, 2, \ldots, p,$$

$$(2.8') \quad \frac{d\Phi}{dt} = \frac{\mu}{N}\sum_{s=1}^{p} n_s A_s(a_1, \ldots, a_p, \Phi) - \Omega + \frac{1}{N}\sum_{s=1}^{p} n_s\omega_{0s} = 0,$$

where

(2.9)
$$A_s = \frac{d\Phi_s}{dt} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{F_s\cos\theta_s}{a_s M_{0s}\omega_0}dt,$$

$$\frac{d\theta_s}{dt} = \omega_s = \omega_{0s} + \mu A_s, \quad \Phi = \frac{1}{N}\sum_{s=1}^{p} n_s\Phi_s.$$

3. Ritz method

The stability of the solutions obtained by means of the Ritz method — i.e., by means of the Eqs. (2.6) — is investigated by introducing a "perturbation" δx_i into the solution

[1] Variables $\bar{a}_j(t)$, $\bar{\Phi}_j(t)$ and a_j, Φ_j are related to each other by means of the equations [1]

$$\frac{d\bar{a}_j}{dt} = \frac{da_j}{dt} + \mu k_j(a_1, \ldots, a_p, \Phi, t), \quad \frac{d\bar{\Phi}_j}{dt} = \frac{d\Phi_j}{dt} + \mu L_j(a_1, \ldots, a_p, \Phi, t), \quad j = 1, 2, \ldots, n.$$

$x_i(t)$ and by inserting $x_i+\delta x_i$ into the equations of motion (2.1). Disregarding the terms which are non-linear in δx_i, we obtain the linear equations written in variations,

$$\delta\varepsilon_i(t) \equiv m_i \frac{d^2\delta x_i}{dt^2} + \sum_{k=0}^{n} K_{ik}(\delta x_i - \delta x_k) + \mu\delta f_i = 0,$$

(3.1)

$$\delta f_i = \sum_{k=1}^{n} \left(\frac{\partial f_i}{\partial x_k} \delta x_k + \frac{\partial f_i}{\partial \dot{x}_k} \delta \dot{x}_k \right), \quad i = 1, 2, \ldots, n.$$

Expanding the functions δf into generalized Fourier series, we obtain

(3.2) $$\delta f_i = \sum_{k=1}^{n} \delta x_k \Big[\sum_{s=1}^{P} p_s^{(k)} \cos 2\theta_s + g_s^{(k)} \sin 2\theta_s + \text{highei and mixed harmonics} \Big]$$
$$+ \sum_{k=1}^{n} \delta \dot{x}_k \Big[\sum_{s=1}^{P} \bar{p}_s^{(k)} \cos 2\theta_s + \bar{g}_s^{(k)} \sin 2\theta_s + \text{higher and mixed harmonics} \Big].$$

If the perturbations δx_i obtained by solving of the Eqs. (3.1) increase indefinitely in time, the solution $x_i(t)$ is defined as unstable. If all solutions $\delta x_i(t)$ decrease and tend to zero with $t \to \infty$, then the solution $x_i(t)$ is defined as asymptotically stable. A stationary solution δx_i corresponds to the boundary separating the stable and instable solutions.

Let us observe that the disturbance δx_i is imposed over the solution x_i at a constant frequency of the excitating force Ω, which implies a substantial difference between the periodic and combined resonances. In periodic resonances ($p = 1$), the constant frequency Ω corresponds to a constant frequency ω_s, since $\Omega = \omega_s n_s/N$. In combined resonances, however, the constant frequency Ω is not necessarily accompanied by constant frequencies $\omega_1, \omega_2, \ldots, \omega_p$, since they may be disturbed by $\delta\omega_s$, $s = 1, 2, \ldots, p$, and only the condition

(3.3) $$\frac{1}{N} \sum_{s=1}^{P} n_s \delta\omega_s = 0$$

must be fulfilled.

In the case of periodic resonances ($p = 1$), the system of equations (3.1) becomes a system of Hill's equations with dissipative terms. These equations may be solved by means of one of the approximate methods. The "small parameter method" is frequently applied making use of the assumption that the time-dependent terms are small by comparison with the constant terms [8–10]. Once it has been decided to solve the problem by means of the Ritz method, this simplification should be avoided and, using the FLOQUET theory and the results of [4], we shall seek a particular, stationary solution corresponding to the boundary of the stable and instable regions in the form of the Fourier series:

(3.4) $$\delta x_i = \delta r_{is}^{(1)} \cos k\theta_s + \delta r_{is}^{(2)} \sin k\theta_s, \quad i = 1, 2, \ldots, n,$$

where $k = 1, 3, 5, \ldots$ or $k = 0, 2, 4, \ldots$

The more terms are retained in the solution, the more instability regions will be obtained. Investigation of instability regions by this method is, in the case of a single Hill's equation, dealt with extensively in [7]. Assuming $k = 1$ in the Eq. (3.4), we obtain what is called the first instability region, and when higher harmonics are taken into account, higher order instability regions are found.

In the present paper, we shall use the notion of the "first instability region", both in the Ritz method and in the averaging method, thus the notion should be defined independently of the method of investigation of the stability problem and remain valid also for the almost-periodic solutions — i.e., for combined resonances.

The notion of the "first instability region" will be understood as a region at boundary of which the disturbed solution $x_i+\delta x_i$ is a function of the same form as the solution x_i whose stability is being investigated; the amplitudes, phase angles and frequencies of the individual harmonic components differ, however, from the corresponding values of the undisturbed solution by certain constants $\delta r_{is}^{(1)}$, $\delta r_{is}^{(2)}$, $\delta\omega_s$, $\delta\Phi_s$, and at least some of these constants are different from zero,

(3.5)
$$x_i+\delta x_i = \sum_{s=1}^{p} [(r_{is}^{(1)}+\delta r_{is}^{(1)})\cos(\theta_s+\delta\theta_s)+(r_{is}^{(2)}+\delta r_{is}^{(2)})\sin(\theta_s+\delta\theta)_s]+C_i\cos\Omega t,$$
$$\theta_s+\delta\theta_s = (\omega_s+\delta\omega_s)t+\Phi_s+\delta\Phi_s, \quad i = 1,2,\ldots,n, \quad s = 1,2,\ldots,p,$$

the relation (3.3) being satisfied.

In the case of a periodic ($p = 1$) resonance, $\delta\omega_s = 0$ and the variations δx_i oscillate with the frequency $\omega_s = \Omega N/n_s$.

At the limits of a higher order instability, the disturbed solution also contains harmonic components with frequencies different from those appearing in the solution for x_i: moreover, frequencies of these additional components determine the order of the instability region.

The role of the higher instability regions was investigated by the author in [3], on the example of a system with a single degree of freedom. It was established there that, provided the considerations are aimed at determining the stability of a solution having an assumed form, examination of the first instability region proves to be sufficient. The regions of higher order instability merely indicate the values, of parameters at which the form of the assumed solution is not an adequate one i.e. the assumed harmonie components are not dominating over "higher" harmonics.

In the subsequent considerations, we shall confine ourselves to the investigation of the first unstable region. The solutions of the variational system of the Eqs. (3.1) may, in the case of periodic resonances, be sought in the following form:

(3.6)
$$\delta x_i = \delta r_{is}^{(1)}\cos\theta_s+\delta r_{is}^{(2)}\sin\theta_s, \quad i = 1,2,\ldots,n,$$
$$\theta_s = \omega_s t+\Phi_s+\delta\Phi_s, \quad \delta r_{is}^{(2)} = 0.$$

Here $\delta r_{is}^{(1)}$, $\delta r_{is}^{(2)}$, $\delta\Phi_s$ are certain constants which do not identically vanish.

Let us substitute the expressions (3.6) in the Eqs. (3.1) and require them to satisfy the equations following from the Ritz method:

$$\begin{aligned} \delta X_{is} &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \delta\varepsilon_i(t)\cos\theta_s dt = 0, \quad i = 1, 2, \ldots, n, \\ \delta Y_{is} &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \delta\varepsilon_i(t)\sin\theta_s dt = 0, \end{aligned} \tag{3.7}$$

where $\delta\varepsilon_i(t)$ denote the residuals of the Eqs. (3.1) after substituting into them the solution (3.6).

These equations form a base for derivation of the criterion of the limit of the first unstable region limits in the case of periodic resonances. Constructing a formal extension of the above procedure to combined resonances, we might apply the principle of superposition and seek the solution in the form:

$$\delta x_i = \sum_{s=1}^{p}(\delta r_{is}^{(1)}\cos\theta_s + \delta r_{is}^{(2)}\sin\theta_s), \quad i = 1, 2, \ldots, n. \tag{3.8}$$

Such a procedure would not, however, account for the specific features of combined resonances, namely for the frequency variations $\delta\omega_s$, $s = 1, 2, \ldots, p$, according to the Eqs. (3.3). Let us therefore modify the approach to the problem and start from the definition of the limit of the first unstable region. Assuming the disturbed solution $x_i + \delta x_i$ in the form (3.5), the variation of the residuals of the equations of motion $\delta\varepsilon_i(t)$ are expanded into a Taylor series in the vicinity of the undisturbed solution defined by the values $r_{is}^{(1)(2)}$, ω_s, Φ, $s = 1, 2, \ldots, p$.

$$\delta\varepsilon_i(t) = \sum_{s=1}^{p}\left[\sum_{j=1}^{n}\left(\frac{\partial\varepsilon_i}{\partial r_{js}^{(1)}}\delta r_{js}^{(1)} + \frac{\partial\varepsilon_i}{\partial r_{js}^{(2)}}\delta r_{js}^{(2)}\right)\right] + \sum_{s=1}^{p}\frac{\partial\varepsilon_i}{\partial\omega_s}\delta\omega_s + \frac{\partial\varepsilon_i}{\partial\Phi}\delta\Phi, \tag{3.9}$$
$$i = 1, 2, \ldots, n, \ \delta r_{1s}^{(2)} = 0, \quad s = 1, 2, \ldots, p.$$

Consistent application of the Ritz method requires that the following equations be satisfied:

$$\begin{aligned} \delta X_{is} &\equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \delta\varepsilon_i(t)\cos\theta_s dt = 0, \\ \delta Y_{is} &\equiv \lim_{T\to\infty}\frac{1}{T}\int \delta\varepsilon_i(t)\sin\theta_s dt = 0, \\ \delta Z &= \frac{1}{N}\sum_{s=1}^{p} n_s\delta\omega_s = 0, \quad i = 1, 2, \ldots, n, \quad s = 1, 2, \ldots, p. \end{aligned} \tag{3.10}$$

Let us now observe that the relations

$$\lim_{T\to\infty}\frac{1}{T}\int \frac{\partial\varepsilon_i}{\partial r_{jk}}\cos\theta_s dt = \frac{\partial X_{is}}{\partial r_{jk}}, \quad \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\partial\varepsilon_i}{\partial r_{jk}}\sin\theta_s dt = \frac{\partial Y_{is}}{\partial r_{jk}}, \tag{3.11}$$

are true and hence the Eqs. (3.10) may be represented in the form:

$$\delta X_{is} \equiv \sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial X_{is}}{\partial r_{jk}^{(1)}} \delta r_{jk}^{(1)} + \frac{\partial X_{is}}{\delta r_{jk}^{(2)}} \partial r_{jk}^{(2)} \right) + \sum_{k=1}^{P} \frac{\partial X_{is}}{\partial \omega_k} \delta \omega_k + \frac{\delta X_{is}}{\partial \Phi} \delta \Phi = 0,$$

$$(3.12) \qquad \delta Y_{is} \equiv \sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial Y_{is}}{\partial r_{jk}^{(1)}} \delta r_{jk}^{(1)} + \frac{\partial Y_{is}}{\partial r_{jk}^{(2)}} \delta r_{jk}^{(2)} \right) + \sum_{k=1}^{P} \frac{\partial Y_{is}}{\partial \omega_k} \delta \omega_k + \frac{\partial Y_{is}}{\partial \Phi} \delta \Phi = 0,$$

$$\delta Z \equiv \sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial Z}{\partial r_{jk}^{(1)}} \delta r_{jk}^{(1)} + \frac{\partial Z}{\partial r_{jk}^{(2)}} \delta r_{jk}^{(2)} \right) + \sum_{k=1}^{P} \frac{\partial Z}{\partial \omega_k} \delta \omega_k + \frac{\partial Z}{\partial \Phi} \delta \Phi = 0,$$

where $i = 1, 2, \ldots, n$, $s = 1, 2, \ldots, p$.

The condition of existence of the non-vanishing solutions $\delta r_{jk}^{(1)\,(2)}$, $\delta \omega_s$, $\delta \Phi$, yields the requirement that the characteristic determinant should be equal to zero,

$$(3.13) \qquad \Delta = \frac{\partial (X_{11}, \ldots, X_{np}, Y_{11}, \ldots Y_{np}, Z)}{\partial (r_{11}^{(1)}, \ldots r_{np}^{(1)}, r_{11}^{(2)}, \ldots r_{np}^{(2)}, \omega_1, \ldots \omega_p, \Phi)} = 0,$$

It will be demonstrated that this condition is satisfied at those points of the resonance curves $r_s = r_s(\Omega)$, $s = 1, 2, \ldots, p$ at which the tangents are vertical. To this end, let us differentiate the Eqs. (2.6) with respect to the independent variable Ω.

$$\sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial X_{is}}{\partial r_{jk}^{(1)}} \frac{dr_{jk}^{(1)}}{d\Omega} + \frac{\partial X_{is}}{\partial r_{jk}^{(2)}} \frac{dr_{jk}^{(2)}}{d\Omega} \right) + \sum_{k=1}^{P} \frac{\partial X_{is}}{\partial \omega_k} \frac{d\omega_k}{d\Omega} + \frac{\partial X_{is}}{\partial \Phi} \frac{d\Phi}{d\Omega} + \frac{\partial X_{is}}{\partial \Omega} = 0,$$

$$(3.14) \qquad \sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial Y_{is}}{\partial r_{jk}^{(1)}} \frac{dr_{jk}^{(1)}}{d\Omega} + \frac{\partial Y_{is}}{\partial r_{jk}^{(2)}} \frac{dr_{jk}^{(2)}}{d\Omega} \right) + \sum_{k=1}^{P} \frac{\partial Y_{is}}{\partial \omega_k} \frac{d\omega_k}{d\Omega} + \frac{\partial Y_{is}}{\partial \Phi} \frac{d\Omega}{d\Phi} + \frac{\partial Y_{is}}{\partial \Omega} = 0,$$

$$\sum_{j=1}^{n} \sum_{k=1}^{P} \left(\frac{\partial Z}{\partial r_{jk}^{(1)}} \frac{dr_{jk}^{(1)}}{d\Omega} + \frac{\partial Z}{\partial r_{jk}^{(2)}} \frac{dr_{jk}^{(2)}}{d\Omega} \right) + \frac{\partial Z}{\partial \omega_k} \frac{d\omega_k}{d\Omega} + \frac{\partial Z}{\partial \Phi} \frac{d\Omega}{d\Phi} + \frac{\partial Z}{\partial \Omega} = 0,$$

$$i = 1, 2, \ldots n, \qquad s = 1, 2, \ldots p.$$

Solving this system for $dr_{jk}^{(1)\,(2)}/d\Omega$, $d\Phi/d\Omega$, $d\omega_s/d\Omega$, we obtain:

$$\frac{dr_{jk}^{(\beta)}}{d\Omega} = \frac{\Delta_{jk}^{(\beta)}}{\Delta}, \qquad \begin{array}{l} j = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, p, \beta = 1, 2, \end{array}$$

where Δ is the characteristic determinant of the system, and $\Delta_{jk}^{(\beta)}$ — the transformed determinant Δ in which the $j_k^{(\beta)}$-th column has been replaced by the column of terms $-\partial X_{11}/$ $/\partial \Omega, \ldots, -\partial X_{np}/\partial \Omega, \ldots, -\partial Y_{np}/\partial \Omega, -\partial Z/\partial \Omega$. Let us observe, moreover, that the characteristic determinant of the Eqs. (3.14) is identical with the characteristic determinant of the Eqs. (3.12) and may be expressed by the formula (3.13); thus, those points of resonance curves which have vertical tangents

$$\frac{dr_{jk}^{(1)(2)}}{d\Omega} = \infty, \qquad \begin{array}{l} j = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, p, \end{array}$$

correspond to zeros of the determinant (3.13) — i.e., they determine the limit of the first unstable region.

4. Method of averaging

The starting point for the investigation of stability of solutions obtained by the averaging method is the set of Eqs. (2.8). In order to simplify the notation, the right-hand side of the Eq. (2.8') is denoted by $\mu D\ _1$, and the general phase angle Φ by a_{p-1}. Equations (2.8), (2.8)' are now written in the abbreviated form as

(4.1) $$\frac{da_s}{dt} = \mu D_s(a_1, a_2, \ldots, a_{p+1}) = 0, \quad s = 1, 2, \ldots p+1.$$

Let $a_1, a_2, \ldots, a_{p+1}$ be the solutions of the system of Eqs. (4.1). Let us impose over these solutions certain perturbations at a constant frequency Ω, and consider the perturbed solution

(4.2) $$\tilde{a}_s(t) = a_s + \delta a_s, \quad s = 1, 2, \ldots, p+1.$$

It may be observed that, although the perturbations are directly imparted to the amplitudes $a_1, a_2, \ldots, a_{p+1}$ only, they are also indirectly transmitted to the frequencies $\omega_1, \ldots, \omega_s$, provided $p > 1$. It follows that, in accordance with the Eq. (2.9):

(4.3) $$\tilde{\omega}_s = \omega_{0s} + \mu A_s(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{p+1}) = \omega_{0s} + \mu A_s(a_1 +, \ldots, a_{p+1}) + \sum_{k=1}^{p+1} \frac{\partial A_s}{\partial a_k} \delta a_k.$$

Substituting the disturbed solutions into the Eqs. (4.1), and expanding D_s into a Taylor series in the neighbourhood of the undisturbed solution we obtain — the terms nonlinear with respect to δa being disregarded — the system of equations in variations:

(4.4) $$\frac{d\delta a_s}{dt} = \sum_{k=1}^{p+1} \frac{\partial D_s}{\partial a_k} \delta a_k, \quad s = 1, 2, \ldots, p+1.$$

The coefficients $\partial D_s/\partial a$ in the solution considered are constant, and hence the particular solution of the Eqs. (4.4) has the form:

(4.5) $$\delta a_s = \delta a_{s0} e^{\lambda t}, \quad s = 1, 2, \ldots, p+1.$$

Non-zero solutions for δa_{s_0} are obtained when the characteristic determinant of the system vanishes. Expansion of that determinant leads to a polynomial of order $p+1$ in λ,

(4.6) $$\Delta(\lambda) = b_0 \lambda^{p+1} + b_1 \lambda^p + \ldots b_p \lambda + b_{p+1} = 0.$$

According to the stability criterion, the solution of the set of Eqs. (4.1) is stable if the real parts of the roots of the Eq. (4.6) are negative, i.e., if the Routh-Hurwitz conditions [7] are satisfied. The limit separating the stable and unstable regions is reached when the real part of one of the roots λ changes its sign — that is, when it assumes the zero value. In applying this method to investigations of periodic resonances [5] — i.e., when $p = 1$ — the polynomial (4.6) was of second order,

(4.7) $$\lambda^2 + b_1 \lambda + b_2 = 0.$$

In the case of positive damping, the coefficient b_1 was always positive. Hence the condition $b_2 > 0$ was, according to the Routh-Hurwitz criterion, the only condition to be satisfied in order to make the solution stable. Zero value of that coefficient, $b_2 = 0$, corresponded to the vanishing real part of the root λ, and thus — to the limit of the first unstable region.

In the case of combined resonances ($p \geqslant 2$), the polynomial (4.6) is of a higher order than 2, and its roots may by either real numbers or complex conjugate numbers. However, the relation

$$b_{p+1} = \lambda_1 \cdot \lambda_2 \cdot \ldots \lambda_{p+1},$$

remains valid.

In this manner, independently of the order of the polynomial (4.6), the condition

$$b_{p+1} = 0 \tag{4.8}$$

corresponds to zero value of one of the real roots λ i.e., to the limit of the first unstable region.

The remaining Routh-Hurwitz conditions concern the behaviour of the real parts of the complex roots of the polynomial (4.6). It may be proved that vanishing of the real parts of the complex roots λ corresponds to the instability limits of higher orders. Let us investigate the behaviour of variations δx_i corresponding to a pair of imaginary roots

$$\lambda_{k1,2} = \mp i\bar{\lambda}_k, \quad \bar{\lambda}_k \text{ — a real, positive number}. \tag{4.9}$$

The solution of the equations written in variations is then:

$$\delta a_s(t) = \delta a_{s_0}^{(1)} \cos \bar{\lambda}_k t + \delta a_{s_0}^{(2)} \sin \bar{\lambda}_k t = \delta a_{s_0} \cos(\bar{\lambda}_k \cdot t + \delta_k), \quad s = 1, 2, \ldots, p. \tag{4.10}$$

Substituting this expression into the transformations (2.2) and (2.7), we arrive at the conclusion that the perturbation of x_i — that is δx_i — contains harmonic components with frequencies $\omega_s + \bar{\lambda}_k$, and $\omega_s - \bar{\lambda}_k$, $s = 1, 2, \ldots, p$. According to the definition, this corresponds to an instability limit of a higher order.

In order to derive the criterion of the first instability limit, it is necessary to differentiate the Eqs. (4.1) with respect to the independent variable Ω and to solve it with respect to $da_s/d\Omega$, as in the Ritz method. We then obtain:

$$\frac{da_s}{d\Omega} = \frac{1}{b_{p+1}} \frac{\partial(D_1, D_2, \ldots, D_{p+1})}{\partial(a_1, \ldots a_{s-1}, \Omega, a_{s+1}, \ldots, a_{p+1})}. \tag{4.11}$$

This implies that the points of the resonance curves $a_s = a_s(\Omega)$, $s = 1, 2, \ldots, p+1$, in which the tangents are vertical,

$$\frac{da_s}{d\Omega} = \infty, \quad s = 1, 2, \ldots, p+1, \tag{4.12}$$

correspond to the limit of the first instability region $b_{p+1} = 0$.

If the solution is represented in the form (2.4), the corresponding conditions assume the form

$$\frac{dr_s}{d\Omega} = \infty, \quad \frac{d\Phi}{d\Omega} = \infty, \quad s = 1, 2, \ldots, p. \tag{4.13}$$

5. Conclusions

In this paper it has been shown that for all types of non-linear resonances, periodic and combined, a uniform criterion exists for the limit of the first unstable region. Independently of whether the resonance curves $r_s = r_s(\Omega)$ and $a_s = a_s(\Omega)$ are found by means of the Ritz method or the averaging method, those points of resonance curves which are characterized by vertical tangents

$$\frac{dr_s}{d\Omega} = \infty, \quad \text{or} \quad \frac{da_s}{d\Omega} = \infty, \quad s = 1, 2, \ldots, p,$$

correspond to the limit of the first instability region.

It should be borne in mind, however, that the resonance curves obtained by the Ritz or the averaging method are close to each other only at very small amplitudes of motion, and at larger amplitudes they may differ substantially (cf. e.g. [2]).

Determination of a uniform criterion of the limit of the first instability region for periodic and combined resonances is possible only by taking into account the variations $\delta\omega_s$ of the frequencies ω_s, $s = 1, 2, \ldots, p$ appearing in the solution (2.4). This result could not be achieved by finding $\delta x_i = \delta x_i(t)$ from the formal solution of the variational Eq. (3.1). It may be concluded that in fact, the stability of individual harmonic components with frequencies ω_s should be considered as the orbital stability phenomena, since the "partial" solution

$$r_{is}^{(1)}\cos(\omega_s t + \Phi_s) + r_{is}^{(2)}\sin(\omega_s t + \Phi_s),$$

for which the disturbed solution at the instability limit takes the form

$$(r_{is}^{(1)} + \delta r_{is}^{(1)})\cos[(\omega_s + \delta\omega_s)t + \Phi_s + \delta\Phi_s] + (r_{is}^{(2)} + \delta r_{is}^{(2)})\sin[(\omega_s + \delta\omega_s)t + \Phi_s + \delta\Phi_s],$$

satisfies the orbital stability condition, and not the Liapunov condition of stability of motion. This conclusion is somewhat surprising since the notion of orbital stability is generally attributed to periodic motions of autonomous systems (in particular self-excited motions characterized by limit cycles), and not — as in the case considered — to stationary motions of a non-autonomous dissipative system. Note, however, that this remark concerns the stability of individual harmonic components, and not the general solution.

The method presented for investigation of instability limits of combined resonances consisting in the representation of a disturbed motion $x_i + \delta x_i$ in the form (3.5) enables us to determine only the first instability region. The analysis of higher order instability regions by means of the Ritz method requires further investigations.

Analysis of the problem of stability of combined resonances by means of the averaging method yields an interesting conclusion that, in such a case, the method makes it possible to detect the instability regions of orders higher than one.

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