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Coupled thermoelastic vibrations of plates

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IN THE PAPER considered is the case of transversal vibrations of plates produced by non-stationary temperature field. The fields of temperature and deformations are assumed to be coupled, the transversal and longitudinal vibrations are mutually independent, the temperature varies harmonically in time and linearly across the thickness of the plate. Examples concerning rectangular and circular plates resting on elastic foundations are given.

W pracy rozpatrzono problem poprzecznych drgań płyty, wywołanych niestacjonarnym polem temperatury. Założono, że pole temperatury i pole deformacji są wzajemnie sprężone, drgania poprzeczne i podłużne są niezależne, pole temperatury harmonicznie zmienia się w czasie oraz że temperatura zmienia się liniowo wzdłuż grubości.

В работе рассмотрена проблема поперечных колебаний плиты, вызванных нестационарным полем температуры. Предполагается, что поле температуры и поле деформации взаимно сопряжены, поперечные и продольные колебания независимы, поле температуры изменяется гармонически во времени, а температура изменяется линейным образом вдоль толщины плиты.

Introduction

IN THIS paper, consideration is given to the problem of transversal vibrations of plates under the influence of nonstationary temperature field using coupled thermoelasticity.

The main themes are similar to those treated by B. A. BOLEY and A. D. BARBER [8], who dealt with a rectangular plate simply supported on a contour, and subject to a uniform step heat input over one face.

Also A. D. KOVALENKO [7] considered a similar problem for a circular plate.

Finally, J. IGNACZAK and W. NOWACKI [1] solved the problem of harmonic vibrations of infinite plates produced by heating.

In the present paper the following assumptions are made:

— The temperature field and strain field are coupled.

— The longitudinal vibrations of the plate are independent of the flexural vibrations.

— The temperature field changes harmonically with time.

— Temperature changes linearly with the thickness of the plate.

Using the finite integral transform technique, we give solutions to the governing equations for a rectangular plate and for a circular plate subject to several types of boundary conditions.

1. General equations

Governing Eqs. of the problem are found from [1, 4] in the form

$$\nabla_1^4 w + \frac{\varrho h}{N}\,\partial_t^2 w + kw + (1+\nu)\alpha_t \nabla_1^2 \tau = 0, \tag{1.1}$$

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t - \varepsilon\right)\tau + \xi^* \partial_t \nabla_1^2 w + \frac{6}{\lambda h^2}(\bar{q} - \bar{p}) = 0, \tag{1.2}$$

where

$\nabla_1^2 = \partial_1^2 + \partial_2^2$ denotes the Laplacian operator,
∂_t the symbol of time derivative,
w deflection of the plate in the x_3-direction,
ν Poisson's ratio,
α_t coefficient of thermal expansion,
ϱ the plate density per unit area of the middle surface,
h thickness of the plate,
$\xi^* = \gamma T_0/\lambda_0$ — coefficient determining the coupling of the temperature and strain field,
$\gamma = 3K\alpha_t$, K — bulk modulus,
T_0 temperature of the plate in natural state (the strains and stresses being equal to zero),
λ_0 coefficient of heat conduction,
$\varkappa = \lambda_0/C_\varepsilon$, C_ε — specific heat at constant strain,
$N = Eh^3/12(1-\nu^2)$ — flexural rigidity of a plate,
E modulus of elasticity,
$\varepsilon = \frac{12}{h^2}$, $k = \frac{C}{N}$, C – the foundation coefficient.

On the basis of the assumption that the temperature changes linearly along the thickness of the plate, we have: $\tau = \frac{1}{h}(T_1 - T_2)$, where T_1 and T_2 are the upper and the lower face temperatures, respectively. The temperature function $T(x_1, x_2, x_3, t)$ is assumed in the form

$$T(x_1, x_2, x_3, t) = \tau_0(x_1, x_2, t) + x_3\tau(x_1, x_2, t), \tag{1.3}$$

$$\tau(x_1, x_2, t) = \frac{12}{h^3}\int_{-h/2}^{h/2} T(x_1, x_2, x_3, t)x_3 dx_3. \tag{1.4}$$

Moreover, $T(x_1, x_2, x_3, t)$ should satisfy the heat equation [2]:

$$\nabla^2 T - \frac{1}{\varkappa}T - \xi^* \partial_t \varepsilon_{kk} = 0 \tag{1.5}$$

with the boundary conditions

$$\lambda \left.\frac{\partial T}{\partial x_3}\right|_{x_3 = \frac{h}{2}} = \bar{q}(x_1, x_2, t); \quad \lambda \left.\frac{\partial T}{\partial x_3}\right|_{x_3 = -\frac{h}{2}} = -p(x_1, x_2, t). \tag{1.6}$$

The strains ε_{ij} occurring in (1.5) are connected with the displacements in the following manner [1, 3]:

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij} = \frac{1}{2}(u'_{i,j} + u'_{j,i}) - x_3 w_{,ij}, \tag{1.7}$$

where u_i' denotes the displacements due to uniform tension of the middle surface, and $u_i'' = -x_3 w_{,i}$ stands for the displacements due to the deflection of the plate denoted by $u_3'' = w$. In the case in which the temperature changes harmonically with time $T(x_1, x_2, t) = U(x_1, x_2)e^{i\omega t}$, and all the other magnitudes are of the same form:

$$w(x_1, x_2, t) = W(x_1, x_2)e^{i\omega t}, \qquad \tau(x_1, x_2, t) = \theta(x_1, x_2)e^{i\omega t},$$

the Eqs. (1.1) and (1.2) take the form

$$(\nabla_1^4 - \beta^2 + k)W + (1+\nu)\alpha_t \nabla_1^2 \theta = 0, \tag{1.8}$$

$$i\omega\xi^* \nabla_1^2 W + (\nabla_1^2 - i\eta - \varepsilon)\theta = \frac{6}{\lambda h^2}(p-q), \tag{1.9}$$

where

$$\bar{p}(x_1, x_2, t) = p(x_1, x_2)e^{i\omega t}; \qquad \bar{q}(x_1, x_2, t) = q(x_1, x_2)e^{i\omega t}, \tag{1.10}$$

$$\beta^2 = \frac{\omega^2 \varrho h}{N}, \qquad \eta = \frac{\omega}{\varkappa}.$$

2. A rectangular plate simply supported on the edges and additionally on the line $x_1 = \xi_1$

Let the plate be under the influence of the vibrations forced by the temperature field, changing harmonically with time, and additionally loaded by a normal force $R(x_2, t)$

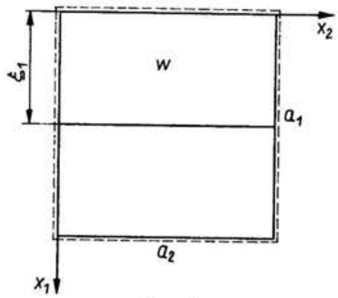

FIG. 1.

along the line $x_1 = \xi_1$, changing also harmonically with time, with the same frequency as the temperature field (Fig. 1),

$$R(x_2, t) = r(x_2)e^{i\omega t}. \tag{2.1}$$

Let the thermal boundary conditions be:

$$\lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3 = \frac{h}{2}} = q(x_1, x_2)e^{i\omega t}; \qquad \lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3 = -\frac{h}{2}} = -p(x_1, x_2)e^{i\omega t}, \tag{2.2}$$

$$T = 0 \text{ on the edges } x_1 = 0, a_1; \; x_2 = 0, a_2. \tag{2.3}$$

In this case, the Eqs. (1.8) and (1.9) should be modified to take the form

$$(\nabla_1^4 - \beta^2 + k)W + (1+\nu)\alpha_t \nabla_1^2 \theta = \frac{r(x_2)}{N}\delta(x_1 - \xi_1), \tag{2.4}$$

(2.5) $$i\omega\xi^* \nabla_1^2 W + (\nabla_1^2 - i\eta - \varepsilon)\theta = \frac{6}{\lambda h^2}(p-q).$$

Let us define the double finite sine Fourier transform of a function $f(x_1, x_2)$ by (cf. [6])

(2.6) $$f_s^*(\alpha_n, \beta_m) = \int_0^{a_1}\int_0^{a_2} f(x_1, x_2)\sin\alpha_n x_1 \sin\beta_m x_2\, dx_1\, dx_2.$$

The inverse transform of $f_s^*(\alpha_n, \beta_m)$ is then given by

(2.7) $$f(x_1, x_2) = \frac{4}{a_1 a_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_s^*(\alpha_n; \beta_m)\sin\alpha_n x_1 \sin\beta_m x_2,$$
$$\alpha_n = \frac{n\pi}{a_1}, \quad \beta_m = \frac{m\pi}{a_2}.$$

Taking into account the following expressions

(2.8) $$\int_0^{a_1}\int_0^{a_2} \nabla_1^2(f)\sin\alpha_n x_1 \sin\beta_m x_2\, dx_1\, dx_2 = -(\alpha_n^2+\beta_m^2) f_s^*(\alpha_n, \beta_m),$$

(2.9) $$(q_{nm}^*, p_{nm}^*) = \int_0^{a_1}\int_0^{a_2} (q, p)\sin\alpha_n x_1 \sin\beta_m x_2\, dx_1\, dx_2,$$

(2.10) $$r(x_2) = \frac{2}{a_2}\sum_{m=1}^{\infty} R_m^* \sin\beta_m x_2,$$

and performing the transforms on (2.4), (2.5), we obtain

(2.11) $$(\Delta_{mn}^2 - \beta^2 + k) W_{mn}^* - (1+\nu)\alpha_t \Delta_{mn}\theta_{mn}^* = \frac{R_m^*}{N}\sin\alpha_n \xi_1,$$

(2.12) $$i\omega\xi^* \Delta_{mn} W_{mn}^* + (\Delta_{mn} + i\eta + \varepsilon)\theta_{mn}^* = \frac{6}{\lambda h^2}(q_{nm}^* - p_{nm}^*),$$

where

$$\Delta_{mn} = \alpha_n^2 + \beta_m^2, \quad W_{mn}^* = W_s(\alpha_n, \beta_m), \quad \theta_{nm}^* = \theta_s(\alpha_n, \beta_m).$$

If we find the solution W_{nm}^*, θ_{nm}^* of (2.11) and (2.12), then, by applying the inverse transform, we obtain the solution for the deflection of the plate in the form:

(2.13) $$w(x_1, x_2, t) = \frac{4e^{i\omega t}}{a_1 a_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}^* \sin\alpha_n x_1 \sin\beta_m x_2,$$

where

$$W_{mn}^* = \frac{D_{m,n}^{(1)}}{D_{m,n}},$$

(2.14) $$D_{m,n}^{(1)} = \frac{R_m^*}{N}(\Delta_{mn} + i\eta + \varepsilon)\sin\alpha_n \xi_1 + \frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_{nm}^* - p_{nm}^*)\Delta_{mn},$$

(2.15) $$D_{m,n} = (\Delta_{mn}^2 - \beta^2 + k)(\Delta_{mn} + i\eta + \varepsilon) + i\omega\xi^*(1+\nu)\alpha_t \Delta_{mn}^2.$$

According to the formulation of our problem, R_m is to be determined from the condition:

$$W|_{x_1=\xi_1} = 0, \text{ i.e.,}$$

(2.16) $$\sum_{n=1}^{\infty}\left[\frac{R_m}{N}(\Delta_{mn}+i\eta+\varepsilon)\sin\alpha_n\xi_1+\frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_{nm}^*-p_{nm}^*)\Delta_{mn}\right]\frac{\sin\alpha_n\xi_1}{D_{m,n}}=0.$$

Substituting R_m from (2.16) into (2.13), we obtain the final form of the deflection function.

From (2.13) it is easily seen that the boundary condtitions for a simply supported plate are satisfied:

$$w=0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x_1^2}=\frac{\partial^2 w}{\partial x_2^2}=0 \quad \text{for} \quad x_1=0, a_1, \quad x_2=0, a_2.$$

3. A rectangular plate simply supported on the edges $x_1 = a_1$, $x_2 = 0, a_2$ and clamped along the edge $x_1 = 0$

Let the plate be under the influence of the vibrations forced by the temperature field, changing harmonically with time and loaded by the moments $M(x_2, t)$ along the line $x_1 = \xi_1$ (Fig. 2).

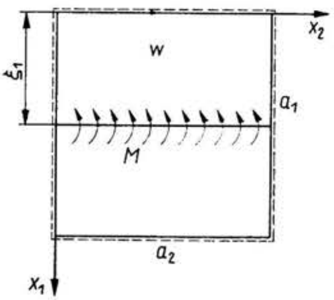

FIG. 2.

Let the moments $M(x_2, t)$ change harmonically with time, with the same frequency as the temperature field:

(3.1) $$M(x_2, t) = m(x_2)e^{i\omega t}.$$

Let the thermal boundary conditions be (2.2) and (2.3). In this case, the equations to be solved take the form:

(3.2) $$(\nabla_1^4-\beta^2+k)W+(1+\nu)\alpha_t\nabla_1^2\theta = -m(x_2)\frac{\partial}{\partial x_1}[\delta(x_1-\xi_1)],$$

(3.3) $$i\omega\xi^*\nabla_1^2 W+(\nabla_1^2-i\eta-\varepsilon)\theta = \frac{6}{\lambda h^2}(p-q).$$

Let us perform the Fourier transform (2.6) on the Eqs. (3.2), (3.3). Taking into account the expressions (2.8), (2.9) and assuming the expansion:

$$m(x_2) = \frac{2}{a_2} \sum_{m=1}^{\infty}{}' M_m^* \sin \beta_m x_2, \tag{3.4}$$

we obtain:

$$(\varDelta_{mn}^2 - \beta^2 + k) W_{mn}^* - (1+\nu) \alpha_t \varDelta_{mn} \theta_{mn}^* = M_m^* \alpha_n \cos \alpha_n \xi_1, \tag{3.5}$$

$$i\omega \xi^* \varDelta_{mn} W_{mn}^* + (\varDelta_{mn} + i\eta + \varepsilon) \theta_{mn}^* = \frac{6}{\lambda h^2} (q_{mn}^* - p_{mn}^*). \tag{3.6}$$

If we find the solution W_{mn}^*, θ_{mn}^* of this algebraic system, then we obtain, by applying the inverse sine transform, the solution for the deflection of the plate in the form:

$$w(x_1, x_2, t) = \frac{4e^{i\omega t}}{a_1 a_2} \sum_{n=1}^{\infty}{}' \sum_{m=1}^{\infty}{}' W_{mn}^* \sin \alpha_n x_1 \sin \beta_m x_2, \tag{3.7}$$

where

$$W_{mn}^* = \frac{D_{m,n}^{(1)}}{D_{m,n}},$$

$$D_{m,n}^{(1)} = M_m^* (\varDelta_{mn} + i\eta + \varepsilon) \alpha_n \cos \alpha_n \xi_1 + \frac{6}{\lambda h^2} (1+\nu) \alpha_t (q_{mn}^* - p_{mn}^*) \varDelta_{mn} \tag{3.8}$$

and $D_{m,n}$ is given by (2.15).

Let the moment $m(x_2)$ act along the line $x_1 = \xi_1 = 0$. From (3.4) and from the additional condition

$$\left. \frac{\partial W}{\partial x_1} \right|_{x_1=0} = 0, \text{ i.e.,}$$

$$\sum_{n=1}^{\infty}{}' \left[M_m^* (\varDelta_{mn} + i\eta + \varepsilon) \alpha_n + \frac{6}{\lambda h^2} (1+\nu) \alpha_t (q_{mn}^* - p_{mn}^*) \varDelta_{mn} \right] \frac{\alpha_n}{D_{m,n}} = 0, \tag{3.9}$$

we find M_m^*. Substituting M_m^* from (3.9) into (3.7), we arrive at the final form of w for a plate simply supported on the edges $x_1 = a_1$, $x_2 = 0, a_2$, and clamped along the edge $x_1 = 0$.

4. The rectangular plate clamped on the whole edge (Fig. 3)

The solution of the above problem can be obtained using the solution (3.7) in the following manner. From (3.8) we have:

$$D_{m,n}^{(1)} = M_m^* (\varDelta_{mn} + i\eta + \varepsilon) \alpha_n + \frac{6}{\lambda h^2} (1+\nu) \alpha_t (q_{mn}^* - p_{mn}^*) \varDelta_{mn} \quad \text{for} \quad \xi_1 = 0 \quad \text{and} \tag{4.1}$$

$$D_{m,n}^{(1)} = M_m^* (\varDelta_{mn} + i\eta + \varepsilon) (-1)^n \alpha_n + \frac{6}{\lambda h^2} (1+\nu) \alpha_t (q_{mn}^* - p_{mn}^*) \varDelta_{mn} \quad \text{for} \quad \xi_1 = a_1. \tag{4.2}$$

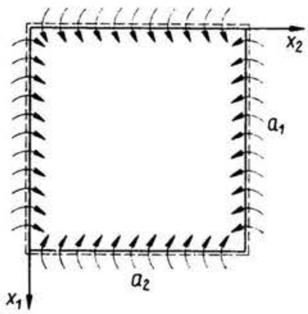

FIG. 3.

If we introduce the notations

(4.3)
$$M_1 = M|_{x_1=0} = \frac{2}{a_2}\sum_{m=1}^{\infty} A_m^* \sin\beta_m x_2,$$
$$M_2 = M|_{x_1=a_1} = \frac{2}{a_2}\sum_{m=1}^{\infty} C_m^* \sin\beta_m x_2,$$
$$M_3 = M|_{x_2=0} = \frac{2}{a_1}\sum_{n=1}^{\infty} B_n^* \sin\alpha_n x_1,$$
$$M_4 = M|_{x_2=a_2} = \frac{2}{a_1}\sum_{n=1}^{\infty} D_n^* \sin\alpha_n x_1,$$

where M_i $(i = 1, 2, 3, 4)$ stands for the moments acting on the edges of the plate, then the solution for the plate clamped on the whole edge takes the form:

(4.4)
$$w(x_1, x_2, t) = \frac{4e^{i\omega t}}{a_1 a_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \hat{W}_{mn}^* \sin\alpha_n x_1 \sin\beta_m x_2,$$

where

(4.5)
$$\hat{W}_{mn}^* = \frac{\hat{D}_{m,n}}{D_{m,n}},$$
$$\hat{D}_{m,n} = \{[A_m - (-1)^n C_m^*]\alpha_n + [B_n^* - (-1)^m D_n^*]\beta_m\}\varrho_{mn} + \vartheta_{mn},$$

and we adopt the following notations:

$$\Delta_{mn} + i\eta + \varepsilon = \varrho_{mn} \quad \text{and} \quad \vartheta_{mn} = \frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_{mn}^* - p_{mn}^*)\Delta_{mn}.$$

The constants A_m^*, B_n^*, C_m^*, D_n^* may be obtained from the condtions:

(4.6)
$$\left.\frac{\partial w}{\partial x_1}\right|_{\substack{x_1=0\\ x_1=a_1}} = 0 \quad \text{and} \quad \left.\frac{\partial w}{\partial x_2}\right|_{\substack{x_2=0\\ x_2=a_2}} = 0.$$

If the functions $q(x_1, x_2)$ and $p(x_1, x_2)$ are symmetric with respect to the axis $x_1 = a_1/2$ and $x_2 = a_2/2$, we have:

$$A_m^* = C_m^* \quad \text{and} \quad B_n^* = D_n^* \quad \text{for} \quad m, n = 1, 3, 5, \ldots.$$

If $q(x_1, x_2)$ is symmetric and $p(x_1, x_2)$ is antisymmetric with respect to the axis $x_1 = a_1/2$, $x_2 = a_2/2$, we have:

$$A_m^* = C_m^*;\; B_n^* = -D_n^* \quad \text{for} \quad n = 1, 3, 5, \ldots \quad \text{and} \quad m = 2, 4, 6, \ldots.$$

If $q(x_1, x_2)$ is antisymmetric and $p(x_1, x_2)$ is symmetric with respect to the axis $x_1 = a_1/2$ and $x_2 = a_2/2$, we have:

$$A_m^* = -C_m^*;\; B_n^* = D_n^* \quad \text{for} \quad n = 2, 4, 6, \ldots \quad \text{and} \quad m = 1, 3, 5 \ldots.$$

We consider only the first case $A_m^* = C_m^*$ and $B_n^* = D_n^*$. The conditions (4.6) will be satisfied, if:

$$\text{(4.7)} \qquad \left.\frac{\partial w}{\partial x_1}\right|_{x_1=0} = 0 \quad \text{and} \quad \left.\frac{\partial w}{\partial x_2}\right|_{x_2=0} = 0.$$

In this case, the expression (4.5) takes the form:

$$\text{(4.8)} \qquad D_{m,n}^{(1)} = 2(A_m^*\alpha_n + B_n^*\beta_m)\varrho_{mn} + \vartheta_{mn}, \qquad m, n = 1, 3, 5, \ldots.$$

The boundary conditions (4.7) are reduced to:

$$\text{(4.9)} \qquad \sum_{n=1,3,5}^{\infty} [2(A_m^*\alpha_n + B_n^*\beta_m)\varrho_{mn} + \vartheta_{mn}] \frac{\alpha_n}{D_{m,n}} = 0,$$

$$\text{(4.10)} \qquad \sum_{m=1,2,3}^{\infty} [2(A_m^*\alpha_n + B_n^*\beta_m)\varrho_{mn} + \vartheta_{mn}] \frac{\beta_m}{D_{m,n}} = 0.$$

From (4.9) and (4.10), it results that

$$\text{(4.11)} \qquad 2A_m^* \sum_{n=1,3,5}^{\infty} \frac{\alpha_n^2 \varrho_{mn}}{D_{m,n}} + 2\beta_m \sum_{n=1,3,5}^{\infty} \frac{B_n^* \alpha_n \varrho_{mn}}{D_{m,n}} = -\sum_{n=1,3,5} \frac{\alpha_n \vartheta_{mn}}{D_{m,n}},$$

$$\text{(4.12)} \qquad 2\alpha_n \sum_{m=1,3,5}^{\infty} \frac{A_m^* \beta_m \varrho_{mn}}{D_{m,n}} + 2B_n^* \sum_{m=1,3,5}^{\infty} \frac{\beta_m \varrho_{mn}}{D_{m,n}} = -\sum_{m=1,3,5}^{\infty} \frac{\beta_m \vartheta_{mn}}{D_{m,n}}.$$

This system of equations can be reduced to the form [5]:

$$\text{(4.13)} \qquad A_m^* + \sum_{i=1,3,5}^{\infty} A_i^* G_{mi} = \sum_{i=1,3,5}^{\infty} K_{mi}.$$

Substituting $B_n^* = D_n^* = 0$ into the solution (4.4), we can obtain the solution for a plate clamped on the edges $x_1 = 0, a_1$ and simply supported on the edges $x_2 = 0, a_2$.

5. A rectangular plate simply supported on the entire edge and additionally at a single interior point

Let the plate be under the influence of the vibrations forced by the temperature field, changing harmonically with time and loaded by the force $R(t)$ at a single interior point (ξ_1, ξ_2) (Fig. 4). Let the force $R(t)$ change harmonically with time with the same frequency as the temperature field:

$$R(t) = R^* e^{i\omega t}. \tag{5.1}$$

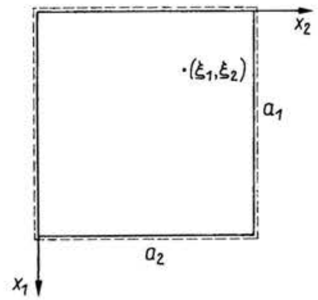

FIG. 4.

Let the thermal boundary conditions be (2.2), (2.3). In this case, the Eqs. (1.8), (1.9) should be modified to the form:

$$(\nabla_1^4 - \beta^2 + k) W + (1+\nu)\alpha_t \nabla_1^2 \theta = \frac{R^*}{N} \delta(x_1 - \xi_1)\delta(x_2 - \xi_2), \tag{5.2}$$

$$i\omega\xi^* \nabla_1^2 W + (\nabla_1^2 - i\eta - \varepsilon)\theta = \frac{6}{\lambda h^2}(p-q). \tag{5.3}$$

Let us perform the double finite sine Fourier transform on the Eqs. (5.2), (5.3). Taking into account the expressions (2.8) and (2.9), we find the solution in the form:

$$w(x_1, x_2, t) = \frac{4e^{i\omega t}}{a_1 a_2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}^* \sin\alpha_n x_1 \sin\beta_m x_2, \tag{5.4}$$

where $W_{mn} = D_{m,n}^{(1)}/D_{m,n}$ is given by (2.15),

$$D_{m,n}^{(1)} = \frac{R^*}{N}(\Delta_{mn} + i\eta + \varepsilon)\sin\alpha_n \xi_1 \sin\beta_m \xi_2 + \frac{6}{\lambda h^2}(1+\nu)\alpha_t (q_{mn}^* - p_{mn}^*)\Delta_{mn}, \tag{5.5}$$

and R^* may be found from the additional condition:

$$W\Big|_{\substack{x_1=\xi_1\\ x_2=\xi_2}} = 0, \quad \text{i.e.,}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{R^*}{N}(\Delta_{mn} + i\eta + \varepsilon)\sin\alpha_n\xi_1 \sin\beta_m\xi_2 + \right.$$
$$\left. + \frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_{mn}^* - p_{mn}^*)\Delta_{mn}\right]\frac{\sin\alpha_n\xi_1 \sin\beta_m\xi_2}{D_{m,n}} = 0. \tag{5.6}$$

Substituting R^* from (5.6) into (5.4), we obtain the expression for the deflection of the plate in the case under consideration.

6. A circular plate simply supported on the edge

Let the plate be under the influence of vibrations forced by the temperature field, changing harmonically with time. Let the thermal boundary conditions be:

$$\lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3=h/2} = q(r)e^{i\omega t}, \quad \lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3=-h/2} = -p(r)e^{i\omega t} \tag{6.1}$$

and

$$T(r,t)\Big|_{r=a} = 0. \tag{6.2}$$

In the case of axial symmetry, the Eqs. (1.8) and (1.9) take the form:

$$(\nabla_r^4 - \beta^2 + k)W + (1+\nu)\alpha_r \nabla_r^2 \theta = 0, \tag{6.3}$$

$$i\omega\xi^* \nabla_r^2 W + (\nabla_r^2 - i\eta - \varepsilon)\theta = \frac{6}{\lambda h^2}(p-q), \tag{6.4}$$

where

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

Let us define the finite Hankel transform of a function $f(r)$ by (cf. [6])

$$f^*(\alpha_n) = \int_0^a r f(r) J_0(\alpha_n r)\, dr. \tag{6.5}$$

The inverse transform of $f^*(\alpha_n)$ is then given by

$$f(r) = \frac{2}{a^2}\sum_{n=1}^{\infty} f^*(\alpha_n)\frac{J_0(\alpha_n r)}{[J_0'(\alpha_n a)]^2}, \tag{6.6}$$

where a is the radius of the plate and $a\alpha_n$ are the zeros of the Bessel function of the first kind and of zero order: $J_0(a\alpha_n) = 0$, and $J_0'(r)_a = dJ_0(r)/dr$. We also have:

$$\int_0^a r\nabla_r^2 f(r) J_0(\alpha_n r)\, dr = -\alpha_n^2 f(\alpha_n). \tag{6.7}$$

On the basis of (6.5), we can write

$$(q_n^*, p_n^*) = \int_0^a (q,p) r J_0(\alpha_n r)\, dr. \tag{6.8}$$

If we apply the finite transform (6.5) to the Eqs. (6.3), (6.4) and take into account (6.7) and (6.8), we obtain:

$$(\alpha_n^4 - \beta^2 + k) W_n^* - (1+\nu)\alpha_t \alpha_n^2 \theta_n^* = 0, \tag{6.9}$$

$$i\omega\xi^* \alpha_n^2 W_n^* + (\alpha_n^2 + i\eta + \varepsilon)\theta_n^* = \frac{6}{\lambda h^2}(q_n^* - p_n^*). \tag{6.10}$$

Solving (6.9) and (6.10) with respect to W_n^* and θ_n^*, as in the previous case, we arrive at the solution for the deflection of the plate:

(6.11) $$w(r,t) = \frac{12(1+\nu)\alpha_t e^{i\omega t}}{\lambda h^2 a^2} \sum \frac{\alpha_n^2(q_n^* - p_n^*)}{D_n} \frac{J_0(\alpha_n r)}{[J_0'(\alpha_n a)]^2},$$

where

(6.12) $$D_n = (\alpha_n^4 - \beta^2 + k)(\alpha_n^2 + i\eta + \varepsilon) + i\omega\xi^*(1+\nu)\alpha_t\alpha_n^4.$$

The solution in the form (6.11) can be treated as an approximate solution, since the boundary conditions are satisfied only approximately.

On the boundary we have

(6.13) $$w|_{r=a} = 0, \quad \nabla_r^2 w|_{r=a} = 0,$$

instead of

(6.14) $$w\Big|_{r=a} = 0, \quad \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\Big|_{r=a} = 0,$$

hence we have assumed

$$\frac{1-\nu}{r}\frac{\partial w}{\partial r}\Big|_{r=a} \approx 0.$$

7. A circular plate simply supported on the edge and additionally at the center

Let the plate be under the influence of vibrations forced by the temperature field changeing harmonically with time and additionally loaded by a force $P(t)$ applied along a circle of radius $r = \xi < a$. Let $p(t)$ change harmonically with time, with the same frequency as the temperature field:

(7.1) $$P(t) = 2\pi P^* e^{i\omega t},$$

where P^* is a constant to be determined later.

Let the thermal boundary condition be (6.1), (6.2). In this case, the Eqs. (1.8), (1.9) should be replaced by:

(7.2) $$(\nabla_r^4 - \beta^2 + k)W + (1+\nu)\alpha_t\nabla_r^2\theta = \frac{2\pi P^*}{N}\frac{\delta(r-\xi)}{2\pi r},$$

(7.3) $$i\omega\xi^*\nabla_r^2 W + (\nabla_r^2 - i\eta - \varepsilon)\theta = \frac{6}{\lambda h^2}(p-q).$$

Performing the finite Hankel transform (6.5), (6.6) on (7.2), (7.3), and taking into account (6.7), (6.8) we find, as in the previous case, that the solution for the deflection of the plate takes the form:

(7.4) $$w(r,t) = \frac{2e^{i\omega t}}{a^2}\sum_{n=1}^{\infty} W_n^* \frac{J_0(\alpha_n r)}{[J_0'(\alpha_n a)]^2},$$

where $W_n^* = D_n^{(1)}/D_n$ and D_n is given by (6.12). Moreover,

$$D_n^{(1)} = \frac{P^*}{N}(\alpha_n^2+i\eta+\varepsilon)J_0(\alpha_n\xi)+\frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_n^*-p_n^*)\alpha_n^2 \tag{7.5}$$

and P may be found from the additional condition $w(\xi, t) = 0$. If the plate is supported at its center, the additional condition is:

$$w(\xi, t)|_{\xi\to 0} = 0. \tag{7.6}$$

When $\xi \to 0$, $J_0(\alpha_n\xi) \to 1$ and (7.6) reduces to:

$$\frac{\frac{P^*}{N}(\alpha_n^2+i\eta+\varepsilon)+\frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_n^*-p_n^*)\alpha_n^2}{D_n[J_0'(\alpha_n a)]^2} = 0. \tag{7.7}$$

Substituting P^* from (7.7) into (7.4), we arrive at the final form of $w(r, t)$ for a plate simply supported on the edge and additionally supported in the center. In this case, we have also satisfied the approximate boundary conditions having the form (6.13).

8. A circular plate clamped on the edge

Let the plate be vibrating under temperature field changing harmonically with time and also subject to the moments $M(t)$ uniformly distributed along a circle of radius $r = a$. Let the moments $M(t)$ change harmonically with time, with the same frequency as the temperature field

$$M(t) = 2\pi M^* e^{i\omega t}, \tag{8.1}$$

where M^* is a constant to be determined later. Let the thermal boundary conditions be (6.1), (6.2). In this case, the Eqs. (1.8), (1.9) should be replaced by:

$$(\nabla_r^4-\beta^2+k)W+(1+\nu)\alpha_t\nabla_r^2\theta = -2\pi M^*\frac{\partial}{\partial r}\left[\frac{\delta(r-a)}{2\pi r}\right], \tag{8.2}$$

$$i\omega\xi^*\nabla_r^2W+(\nabla_r^2-i\eta-\varepsilon)\theta = \frac{6}{\lambda h^2}(p-q). \tag{8.3}$$

Performing the finite Hankel transform (6.5), (6.6) on (8.2), (8.3), and taking into account (6.7), (6.8), we find the solution in the form:

$$w(r,t) = \frac{2e^{i\omega t}}{a^2}\sum_{n=1}^{\infty} W_n^*\frac{J_0(\alpha_n r)}{[J_0'(\alpha_n a)]^2}, \tag{8.4}$$

where $W_n^* = D_n^{(1)}/D_n$, D_n is given by (6.12),

$$D_n = -M^*(\alpha_n^2+i\eta+\varepsilon)\alpha_n J_0'(\alpha_n a)+\frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_n^*-p_n^*)\alpha_n^2, \tag{8.5}$$

and M may be found from the additional condition $\partial w/\partial r|_{r=a} = 0$. This condition takes the form:

$$\frac{\left[-M^*(\alpha_n^2+i\eta+\varepsilon)\alpha_n J_0'(\alpha_n a)+\frac{6}{\lambda h^2}(1+\nu)\alpha_t(q_n^*-p_n^*)\alpha_n^2\right]\alpha_n}{D_n J_1(\alpha_n a)}=0. \tag{8.6}$$

Substituting M^* from (8.6) into (8.4), we arrive at the final form of $w(r, t)$ for a plate clamped on the edge. In this case, we have also satisfied the approximate boundary conditions having the form (6.13).

References

1. J. IGNACZAK and W. NOWACKI, *Transversal vibrations of a plate produced by heating*, Arch. Mech. Stos., **13**, 5, 1961.
2. W. NOWACKI, *Dynamiczne zagadnienia termosprężystości* [Dynamic problems of thermoelasticity, in Polish], 1966.
3. S. TIMOSHENKO, *Theory of plates and shells*, New York 1940.
4. K. MARGUERRE, *Thermo-elastische Plattegleichungen*, ZAMM, **6**, 1935.
5. W. NOWACKI, *Free vibrations and buckling of a rectangular plate with all edges built in*, Arch. Mech. Stos., **6**, 4, 1954.
6. N. I. SNEDDON, *Fourier transforms*, New York 1951.
7. А. Д. КОВАЛЕНКО [A. D. Kovalenko], *Основы термоупругости* [Foundations of thermoelasticity, in Russian], Киев [Kiev] 1970.
8. B. A. BOLEY and A. D. BARBER. *Dynamic response of beams and plates to rapid heating*, J. Appl. Mech., **24**, 413, 1957.

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