Statics of elastic lattice-type shells

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IN THE PRESENT paper the problem of statics of elastic lattice-type shells is dealt with. This problem is considered on the basis of the theory of discrete elastic Cosserat media. The basic system of equations of the problem is studied. Compatibility conditions, static-geometric analogy and stress function are introduced and the equations of shallow lattice shells are shown.

W pracy rozpatrzono szczegółowo zagadnienie statyki sprężystych powłok prętowych. Do analizy problemu wykorzystano równania teorii dyskretnego ośrodka Cosseratów. Sformułowano podstawowy układ równań, wykazano zachodzenie analogii statyczno-geometrycznej oraz możliwość wprowadzenia funkcji naprężeń. Przyjmując pewne założenia upraszczające z ogólnego układu równań otrzymano układ opisujący mało wyniosłe powłoki prętowe,

В работе рассмотрена подробно статическая задача теории упругости стержневых оболочек. Используются основные уравнения теории дискретной упругой среды Коссера. Сформулирована определяющая система уравнений, показана статико-геометрическая аналогия, доказана возможность введения функции напряжений. Для некоторых упрощающих предложений из основной системы уравнений получена система, описывающая пологие стержневые оболочки.

LOAD-carrying structures constituting a regular lattice lying on a surface are usually called lattice-type surface structures or, in short, lattice-type shells. In the present paper, such structures will be defined as those composed of thin elastic bars, having the following properties:

1) the axes of bars constitute a triangulation of a given surface;

2) the stress state in a particular bar segment may be described by the shear forces, longitudinal force, bending moments and twisting moment;

3) the bars are rigidly connected at nodes;

4) the bar segments between any two nodes are prismatic, homogeneous, isotropic and linearly elastic.

In the literature on structural mechanics, the problem of statics of lattice-type structures has been investigated in numerous papers, in particular, by Polish authors in [1–6]. However, the methods so far developed in problems of this type do not make it possible to obtain the basic system of equations in sufficiently simple and clear form. The aim of the present considerations is to formulate and analyse the statical equations of linear theory of lattice-type shells. We begin with the equations of discrete elasticity given in [7–12]. The proposed method appears advantageous in that it enables the known general theorems of discrete elasticity to be employed. Moreover, the resemblance of the equations

considered to the equations of the continous shells theory suggest in many cases the application of considerations and methods of solution analogous to those known in the classical theory.

1. The general equations of discrete elasticity

Let D be a countable or finite set of elements d with a difference structure of order m given by the set of functions $f_A: D_A \to D_{-A}, A = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{m}$, cf. [7–12]. Denoting by $u^k = u^k(d), v^k = v^k(d)$ the components of a displacement vector of a mass centre of the body $d \in D$ and the components of an infinitesimal rotation vector of this body, respectively[1], the elastic potential of the system considered can be expressed in the form [2]

$$(1.1) \qquad \varepsilon = \frac{1}{2} A_{kl}^{A\Phi} \gamma_A^k \gamma_\Phi^l + B_{kl}^{A\Phi} \gamma_A^k \varkappa_\Phi^l + \frac{1}{2} F_{kl}^{A\Phi} \varkappa_A^k \varkappa_\Phi^l,$$

where $A_\Phi^A, B_{kl}^{A\Phi}, F_{kl}^{A\Phi}$ represent the elastic properties of the system,

$$(1.2) \qquad \gamma_A^k = \Delta_A u^k + \varepsilon_{\cdot pr}^k v^r \Delta_A \psi^p, \quad \varkappa_A^{ki} = \Delta_A v^k, \quad d \in D_A,$$

and $z^k = \psi^k(d)$ are coordinates of the place in the physical space occupied by the centre of mass of the body d. In static problems, the basic system of equations has then the form [7]:

1) constitutive equations

$$(1.3) \qquad \begin{aligned} t_k{}^A &= A_{kl}^{A\Phi} \gamma_\Phi^l + B_{kl}^{A\Phi} \varkappa_\Phi^l, \\ m_k{}^A &= F_{kl}^{A\Phi} \varkappa_\Phi^l + B_{lk}^{\Phi A} \gamma_\Phi^l; \end{aligned}$$

2) equilibrium equations

$$(1.4) \qquad \begin{aligned} \bar\Delta_A t_k{}^A + f_k &= 0, \\ \bar\Delta_A m_k{}^A + \varepsilon_{kp\cdot}^r t_r{}^A \Delta_A \psi^p + n_k &= 0, \end{aligned}$$

where $t_k{}^A, m_k{}^A$ and $\gamma_A^k, \varkappa_A^k$ will be called the components of stress and strain, respectively. Equations of equilibrium will be transformed into the symmetric form if the following strain components are introduced:

$$(1.5) \qquad \begin{aligned} \eta_A^k(d) &= \Delta_A u^k(d) + \frac{1}{2} \varepsilon_{\cdot pr}^k \Delta_A \psi^p(d) [v^r(d) + v^r(f_A d)], \\ \varkappa_A^k(d) &= \Delta_A v^k(d), \quad d \in D_A. \end{aligned}$$

The elastic potential is now expressed by:

$$(1.6) \qquad \pi = \frac{1}{2} A_{kl}^{A\Phi} \eta_A^k \eta_\Phi^l + H_{kl}^{A\Phi} \eta_A^k \varkappa_\Phi^l + C_{kl}^{A\Phi} \varkappa_A^k \varkappa_\Phi^l,$$

$$H_{kl}^{A\Phi} = B_{kl}^{A\Phi} - \frac{1}{4} A_{kr}^{A\Phi} \varepsilon_{\cdot pl}^r l^{\Phi p} - \frac{1}{4} A_{kr}^{A\Phi} \varepsilon_{\cdot pl}^r l^{Ap},$$

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[1] All these components are related to the inertial Cartesian coordinate system (z^k) in physical space.
[2] The argument d of the functions considered will be omitted from now on.

$$(1.7) \qquad C_{kl}^{A\Phi} = F_{kl}^{A\Phi} + \frac{1}{4} A_{st}^{A\Phi} \varepsilon_{\cdot pk}^{t} \varepsilon_{\cdot rl}^{s} l^{\Phi p} l^{Ar},$$

$$l^{Ak} = \Delta_A \psi^k.$$

The constitutive equations will be given by:

$$(1.8) \qquad \begin{aligned} t_k^A &= A_{kl}^{A\Phi} \eta_{/\Phi}^l + H_{kl}^{A\Phi} \varkappa_{\Phi}^l, \\ g_k^A &= C_{kl}^{A\Phi} \varkappa_{\Phi}^l + H_{lk}^{\Phi A} \eta_{/\Phi}^l. \end{aligned}$$

By virtue of

$$(1.9) \qquad g_k^A = m_k^A - \frac{1}{2} t_k^A \varepsilon_{\cdot pr}^l l_A^p, \qquad \bar{\varphi}(d) \equiv \varphi(f_{-A} d)$$

($\varphi: D \to R$ is the arbitrary function, the summation convention with respect to A does not hold), we can transform the equations of equilibrium (1.4) to the form:

$$(1.10) \qquad \begin{aligned} \frac{1}{2} (\bar{\Delta}_A t_k^A + \Delta_A \bar{t}_k^A) + f_k &= 0, \\ \frac{1}{2} (\bar{\Delta}_A g_k^A + \Delta_A \bar{g}_k^A) + \frac{1}{2} \varepsilon_{kl}^m (l_A^l t_m^A + \bar{l}_A^l \bar{t}_m^A) + n_k &= 0. \end{aligned}$$

The Eqs. (1.5), (1.8) and (1.10) are the alternative form of the basic equations of the discrete elastic media. All the equations given above can be transformed to a more general form after introducing, for each $d \in D$, the separate Cartesian coordinate system in physical space, cf. [7].

2. Spatial systems of bars

Let set D be a set of rigid nodes of the lattice composed of thin, linearly elastic bars. The difference structure on the set D will be determined if we assume $d' = f_A d$ when the nodes d and d' are connected by a bar. This bar will be called the A — bar. Let the dimensions of nodes be disregarded and external loads be assumed to act on the nodes only. Let us calculate the tensors of elastic rigidity $A_{kl}^{A\Phi}$, $B_{kl}^{A\Phi}$, $F_{kl}^{A\Phi}$, $H_{kl}^{A\Phi}$, $C_{kl}^{A\Phi}$. To this end, let us denote:

$t_k^A(d)$	the components of the unit vector normal to the cross-section of the bar connecting the nodes d and $f_A d$,
$'t_k^A(d), ''t_k^A(d)$	the components of the unit vectors directed along the principal axes of this cross-section,
$l_A(a)$	length of the A-bar,
$A_A'(d)$	the cross-sectional area of the A-bar,
$C_A(d)$	the torsional rigidity of the A-bar,
$J_A'(d), J_A''(d)$	the moments of inertia with respect to the axes given by the vectors $'t_k^A(d)$, $''t_k^A(d)$, respectively,
$E_A(d)$	Young's modulus of the A-bar,
$M_A(d)$	the twisting moment in the middle of the A-bar,

$'M_A(d), ''M_A(d)$ the bending moments in the middle of the Λ-bar with respect to the axes given by the vectors $'t_k{}^A, ''t_k{}^A$, respectively,

$P_A(d)$ the longitudinal force in the Λ-bar,

$'P_A(d), ''P_A(d)$ the shear forces in the Λ-bar with respect to axes given by the vectors $'t_k{}^A(d)$, $''t_A(d)$, respectively,

$\gamma'_A(d), \gamma''_A(d)$ factors of the influence of the shear forces $'P_A(d), ''P_A(d)$ on bending of the Λ-bar.

According to the known formulae of the theory of structures, we obtain:

$$M_A = \frac{C_A}{l_A} \Delta_A v^k t_k{}^A,$$

$$'M_A = \frac{E_A J'_A}{l_A} \Delta_A v^{k'} t_k{}^A, \qquad ''M_A = \frac{E_A J''_A}{l_A} \Delta_A v^{k''} t_k{}^A,$$

(2.1)

$$P_A = \frac{E_A A_A}{l_A} \Delta_A u^k t_k{}^A,$$

$$''P_A = \frac{12 E_A J''_A}{(1+12\gamma'_A)l_A^2} \left[\frac{\Delta_A u^{k'} t_k{}^A}{l_A} + \frac{2v^k + \Delta_A v^k}{2} \, ''t_k{}^A \right],$$

$$''P_A = \frac{12 E_A J'_A}{(1+12\gamma''_A)l_A^2} \left[\frac{\Delta'_A u^{k''} t_k{}^A}{l_A} - \frac{2v^k + \Delta^l_A v^k}{2} \, 't_k{}^A \right].$$

Calculating the elastic potential ε in the element $d \in D$ (which is a potential of m Λ-bars joining the node d with the nodes $f_A d, A = I, II, \ldots, m$) and bearing in mind (1.1), we finally obtain cf. [13]:

$$A_{kl}^{A\Phi} = \delta^{A\Phi} \left\{ E_A A_A \underline{t}_k{}^A \underline{t}_l{}^A + \frac{12 E_A J''_A}{(1+12\gamma'_A)l_A^2} \, 't_k{}^{A} 't_l{}^A + \frac{12 E_A J'_A}{(1+12\gamma''_A)l_A^2} \, ''t_k{}^{A} ''t_l{}^A \right\} \frac{1}{l_A},$$

(2.2) $$B_{kl}^{A\Phi} = \delta^{A\Phi} \left\{ \frac{6 E_A J'_A}{(1+12\gamma''_A)l_A^2} \, 't_l{}^{A} '' t_k{}^A - \frac{6 E_A J''_A}{(1+12\gamma'_A)l_A^2} \, 't_k{}^{A} '' t_l{}^A \right\},$$

$$F_{kl}^{A\Phi} = \delta^{A\Phi} \left\{ C^A \underline{t}_k{}^A \underline{t}_l{}^A + E_A J'_A \left[1 + \frac{3}{1+12\gamma''_A} \right] 't_{kA} 't_{lA} \right.$$

$$\left. + E_A J''_A \left[1 + \frac{3}{1+12\gamma'_A} \right] ''t_k{}^{A} ''t_l{}^A \right\} \frac{1}{l_A}.$$

Likewise, from (1.7) we obtain:

$$H_{kl}^{A\Phi} = 0,$$

(2.3)

$$C_{kl}^{A\Phi} = \delta^{A\Phi} \left\{ C_A \underline{t}_k{}^A \underline{t}_l{}^A + E_A J'_A \, 't_k{}^{A} 't_l{}^A + E_A J''_A \, ''t_k{}^{A} ''t_l{}^A \right\} \frac{1}{l_A}.$$

Having those tensors of elastic rigidity, we are able to solve the basic set of equations given by (1.2), (1.3), (1.4), or (1.5), (1.8), (1.10).

Quantities $M_A, 'M_A, ''M_A, P_A, 'P_A, ''P_A$ can be determined directly from (2.1) or from the following relations:

$$t_k{}^A \underline{t}_l{}^A \delta^{kl} = P_A,$$

$$t_k{}^{A'} \underline{t}_l{}^A \delta^{kl} = P_A,$$

$$t_k{}^{A''} \underline{t}_l{}^A \delta^{kl} = ''P_A,$$

$$m_k{}^A \underline{t}_l{}^A \delta^{kl} = M_A^0, \qquad g_k{}^A \underline{t}_l{}^A \delta^{kl} = M_A,$$

$$m^{A'} \underline{t}_l{}^A \delta^{kl} = 'M_A^0, \qquad g_k{}^{A'} \underline{t}_l{}^A \delta^{kl} = 'M_A,$$

$$m_k{}^{A''} \underline{t}_l{}^A \delta^{kl} = ''M_A^0, \qquad g_k{}^{A''} \underline{t}_l{}^A \delta^{kl} = ''M_A,$$

where the symbol "o" is referred to the cross-section at the node $f_A d$ of the given A-bar.

3. The difference geometry of lattice shells

The difference description of lattice shells will now be formulated on the basis of concepts introduced in [8]. Let the considered structure be composed of three families of bars, see Fig. 1. By means of the difference structure given on the set of nodes, the direc-
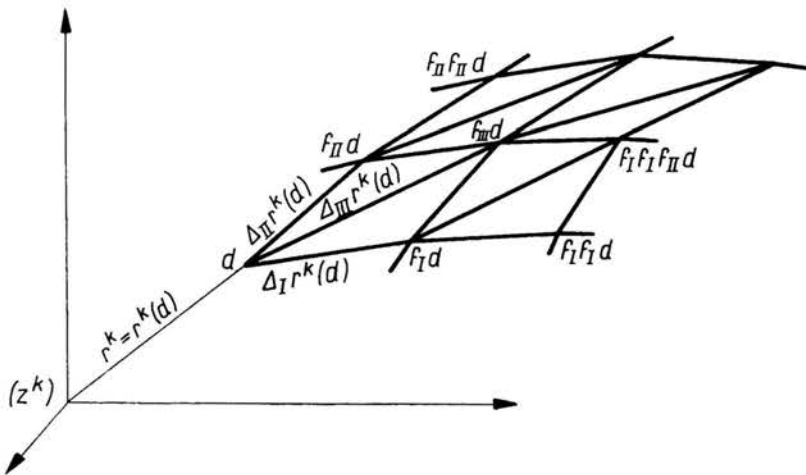

FIG. 1.

tions I, II, III are selected according to those families of bars. Let us denote by z^k, $k = 1, 2, 3$ the rectangular, Cartesian coordinates in the physical space. The radius-vector of the lattice of nodes will be denoted by the symbol $r^k = r^k(d)$.

We have then $\Delta_A r^k = l_A^k$. Let us assume the vector base $e_\alpha(d)$, $\alpha = 1, 2, 3$ in each vector space $V_d^3(d)$ assigned to the node $d \in D$. In the covector space $V_d^{*3}(d)$, the base will be denoted by $e^\alpha(a)$, $\alpha = 1, 2, 3$. Let us denote:

(3.1)
$$\Delta_A e_\alpha(d) = G_{A\alpha}^\beta(d) e_\beta(d), \qquad \bar{\Delta}_A e_\alpha(d) = G_{\alpha A}^\beta(d) e_\beta(d),$$

$$\Delta_A e^\alpha(d) = \overset{*}{G}{}_{A\beta}^\alpha(d) e^\beta(d), \qquad \bar{\Delta}_A e^\alpha(d) = \overset{*}{G}{}_{\beta A}^\alpha(d) e(d),$$

(3.1)
$$\alpha, \beta = 1, 2, 3, \quad \Lambda = I, II, III,$$
$$e_1 = g_1, \quad e_2 = g_2, \quad e_3 = n,$$
$$e^1 = g^1, \quad e^2 = g^2, \quad e^3 = \overset{*}{n}.$$

For each $d \in D$, the vector base g_K, n and $g^K, \overset{*}{n}$ ($K = 1, 2$) will be defined as follows:

(3.2)
$$g_K(d) = H_K^\Lambda(d)\Lambda_\Lambda r(d), \quad \Lambda = I, II, III, \quad K = 1, 2$$
$$H_K^{III}(d) = 0, \quad r[H_K^\Lambda] = 2,$$

$n(d)$ — an arbitrary vector linearly independent of $g_K(d)$,

$$g^K(d) = \frac{\epsilon^{KL}g_L(d) \times n(d)}{\sqrt{g(d)}}, \quad \overset{*}{n}(d) = \frac{g_1(d) \times g_2(d)}{\sqrt{g(d)}},$$

where the symbol ϵ_{KL} is that of Ricci and g is a determinant of a matrix of the metric tensor. These formulae strictly connect the general coordinates introduced with the spatial configuration of the bars. Let us denote the components of metric tensor by

(3.3)
$$a_{MN} = g_M g_N, \quad a^{MN} = g^M g^N,$$
$$a_M = g_M n, \quad a^M = g^M \overset{*}{n},$$
$$a = nn, \quad \overset{*}{a} = \overset{*}{n}\overset{*}{n},$$

and define Ricci's bivector by

(3.4)
$$e_{KL} = \varepsilon^{klm}g_{Kk}g_{Ll}n_m,$$
$$\overset{*}{e}{}^{KL} = \varepsilon^{klm}g^K{}_k g^L{}_l \overset{*}{n}_m.$$

The formulae (3.2) can now be written in the form:

(3.5)
$$g^{Km} = \frac{1}{2}\overset{*}{e}{}^{KL}\varepsilon_{kl\cdot}{}^m g_L{}^k n^l,$$
$$\overset{*}{n}{}^m = \frac{1}{2}\overset{*}{e}{}^{KL}\varepsilon_{kl\cdot}{}^m g_K{}^k g_L{}^l.$$

Let us denote:

$$g = \det \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a \end{vmatrix}, \quad \overset{*}{g} = \det \begin{vmatrix} a^{11} & a^{12} & a^1 \\ a^{21} & a^{22} & a^2 \\ a_1 & a^2 & \overset{*}{a} \end{vmatrix},$$

$$G_{\Lambda K}^3 \equiv b_{\Lambda K}, \quad G_{K\Lambda}^3 \equiv \bar{b}_{\Lambda K},$$
$$\overset{*}{G}{}_{\Lambda K}^3 \equiv h_{\Lambda K}, \quad \overset{*}{G}{}_{K\Lambda}^3 \equiv \bar{h}_{\Lambda K},$$

(3.6)
$$G_{3\Lambda}^K \equiv b_\Lambda^K, \quad G_{3\Lambda}^K \equiv \bar{b}_\Lambda^K,$$
$$\overset{*}{G}{}_{\Lambda 3}^K \equiv h_\Lambda^K, \quad \overset{*}{G}{}_{3\Lambda}^K \equiv \bar{h}_\Lambda^K,$$
$$G_{\Lambda 3}^3 \equiv b_\Lambda, \quad G_{3\Lambda}^3 \equiv \bar{b}_\Lambda,$$
$$\overset{*}{G}{}_{3\Lambda}^3 \equiv h_\Lambda, \quad \overset{*}{G}{}_{3\Lambda}^3 \equiv \bar{h}_\Lambda.$$

Making use of (3.6), we obtain from the (3.1) the following expressions:

$$\Delta_A g_K{}^k = G_{AK}^L g_L{}^k + b_{AK} n^k,$$

$$\Delta_A n^k = b_A^L g_L{}^k + b_A n^k,$$

$$\Delta_A g^{Kk} = \overset{*}{G}{}_{AL}^K g^{Lk} + h_A^K \overset{*}{n}{}^k,$$

$$(3.7) \qquad \Delta_A \overset{*}{n}{}^k = h_{AL} g^{LK} + h_A \overset{*}{n}{}^k,$$

$$\bar{\Delta}_A g_K{}^k = G_{KA}^L g_L{}^k + \bar{b}_{AK} n^k,$$

$$\bar{\Delta}_A n^k = \bar{b}_A^L g_L{}^k + \bar{b}_A n^k,$$

$$\bar{\Delta}_A g^{Kk} = \overset{*}{G}{}_{LA}^K g^{Lk} + \bar{h}_A^K \overset{*}{n}{}^k,$$

$$\bar{\Delta}_A \overset{*}{n}{}^k = \bar{h}_{AL} g^{Lk} + \bar{h}_A \overset{*}{n}{}^k,$$

for each $d \in D_A$ or $d \in D_{-A}$, respectively. From (3.7), we obtain:

$$b_{AK} = \Delta_A g_K{}^k \overset{*}{n}{}^l \delta_{kl}, \qquad \bar{b}_{AK} = \bar{\Delta}_A g_K{}^k \overset{*}{n}{}^l \delta_{kl},$$

$$b_A^K = \Delta_A n^k g^{Kl} \delta_{kl}, \qquad \bar{b}_A^K = \bar{\Delta}_A n^k g^{Kl} \delta_{kl},$$

$$b_A = \Delta_A n^k \overset{*}{n}{}^l \delta_{kl}, \qquad \bar{b}_A = \bar{\Delta}_A n^k \overset{*}{n}{}^l \delta_{kl},$$

$$(3.8) \qquad h_{AK} = \Delta_A \overset{*}{n}{}^k g_K{}^l \delta_{kl}, \qquad \bar{h}_{AK} = \bar{\Delta}_A \overset{*}{n}{}^k g_K{}^l \delta_{kl},$$

$$h_A^K = \Delta_A g^{KL} n^l \delta_{kl}, \qquad \bar{h}_A^K = \bar{\Delta}_A g^{Kk} n^l \delta_{kl},$$

$$h_A = \Delta_A \overset{*}{n}{}^k n^l \delta_{kl}, \qquad \bar{h}_A = \bar{\Delta}_A \overset{*}{n}{}^k n^l \delta_{kl},$$

$$G_{AK}^L = \Delta_A g_K{}^k g^{Lk} \delta_{kl}, \qquad G_{KA}^L = \bar{\Delta}_A g_K{}^k g^{Ll} \delta_{kl},$$

$$\overset{*}{G}{}_{LA}^K = \Delta_A g^{Ks} g_L{}^l \delta_{kl}, \qquad \overset{*}{G}{}_{LA}^K = \bar{\Delta}_A g^{Kl} g_L{}^l \delta_{kl}.$$

For the sake of further simplicity, let us denote the following operators:

$$\beta_A^K \varphi(d) = b_A^K(d) \varphi(f_A d), \qquad \bar{\beta}_A^K \varphi(d) = \bar{b}_A^K(d) \varphi(f_{-A} d),$$

$$\beta_{AK} \varphi(d) = b_{AK}(d) \varphi(f_A' d), \qquad \bar{\beta}_{AK} \varphi(d) = \bar{b}_{AK}(d) \varphi(f_{-A}' d),$$

$$(3.9) \qquad \beta_A \varphi(d) = b_A(d) \varphi(f_A d), \qquad \bar{\beta}_A \varphi(d) = \bar{b}_A(d) \varphi(f_{-A} d),$$

$$\eta_A^K \varphi(d) = h_A^K(d) \varphi(f_A d), \qquad \bar{\eta}_A^K \varphi(d) = \bar{h}_A^K(d) \varphi(f_{-A} d),$$

$$\eta_{AK} \varphi(d) = h_{AK}(d) \varphi(f_A d), \qquad \bar{\eta}_{AK} \varphi(d) = \bar{h}_{AK}(d) \varphi(f_{-A} d),$$

$$\eta_A \varphi(d) = h_A(d) \varphi(f_A d), \qquad \bar{\eta}_A \varphi(d) = \bar{h}_A(d) \varphi(f_{-A} d),$$

where $\varphi(d)$, $d \in D_A$ is an arbitrary real-valued function.

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4. Equations of lattice shells in general coordinates

All the equations given in Secs. 1 and 2 will now be presented in the general coordinates introduced in Sec. 3. Let the components of quantities considered in the general coordinates be defined as follows:

$$t_k{}^A = t^{KA}g_{Kk} + t^A n_k, \qquad f_k = f^K g_{Kk} + f n_k,$$

$$m_k{}^A = m^{KA}g_{Kk} + m^A n_k, \qquad m_k = m^K g_{Kk} + m n_k,$$

$$l_A^m = l_A^K g_K{}^m + l_A n^m,$$

(4.1)
$$\gamma_A^k = \gamma_{KA}g^{Kk} + \overset{*}{\gamma}_A \overset{*}{n}{}^k = \gamma^K{}_A g_K{}^k + \gamma_A n^k,$$

$$\varkappa_A^k = \varkappa_{KA}g^{Kk} + \overset{*}{\varkappa}_A \overset{*}{n}{}^k = \varkappa^K{}_A g_K{}^k + \varkappa_A n^k,$$

$$u^k = u_K g^{Kk} + \overset{*}{u}\,\overset{*}{n}{}^k = u^K g_K{}^k + u n^k,$$

$$v^k = v_K g^{Kk} + \overset{*}{v}\,\overset{*}{n}{}^k = v^k g_K{}^k + v n^k.$$

Making use of the relations (3.9) and denoting, according to (3.3) and (3.4):

$$\varepsilon_{km.}{}^n g_M{}^m g_{Nn} g^{Lk} = a^L e_{MN},$$

$$\varepsilon_{km.}{}^n g_{Nn} g^{Lk} n^m = e_N{}^L,$$

$$\varepsilon_{km.}{}^n g_M{}^m n_n g^{Lk} = e^L{}_M,$$

$$\varepsilon_{km.}{}^n n^m n_n g^{Lk} = 0,$$

(4.2)
$$\varepsilon_{km.}{}^n g_M{}^m g_{Nn} \overset{*}{n}{}^k = \overset{*}{a} e_{MN},$$

$$\varepsilon_{km.}{}^n g_{Nn} n^m \overset{*}{n}{}^k = a^S e_{NS},$$

$$\varepsilon_{km.}{}^n g_M{}^m n_n \overset{*}{n}{}^k = a^S e_{SM},$$

$$\varepsilon_{km.}{}^n n^m n_n \overset{*}{n}{}^k = 0,$$

after simple calculations which can be found in [13], we obtain equations of equilibrium (1.4) in the form:

$$'\bar{\delta}_A t^{KA} + \bar{\beta}_A^K t^A + f^K = 0,$$

$$'\bar{\delta}_A t^A + \bar{\beta}_{AK} t^{KA} + \bar{\beta}_A t^A + f = 0,$$

(4.3)
$$'\bar{\delta}_A m^{KA} + \bar{\beta}_A^K m^A + e^K{}_M l_A^M t^A + (e_N{}^K l_A + a^K e_{MN} l_A^M) t^{NA} + m^K = 0,$$

$$'\bar{\delta}_A m^A + \bar{\beta}_{AK} m^{KA} + \bar{\beta}_A m^A + e_{MN} \overset{*}{a} l_A^M t^{NA} + a^S e_{SM} l_A^M t^A + a^S e_{NS} l_A t^{NA} + m = 0,$$

where the symbol $'\bar{\delta}_A$ denotes the absolute difference operator in the bundles of two-dimensional subspaces of spaces $V_d^3(d)$, and

$$'\bar{\delta}_A w^K(d) = \bar{\Delta}_A w^K(d) + G_{LA}^K(d) w^L(f_{-A} d), \qquad '\bar{\delta}_A w(d) = \bar{\Delta}_A w(a),$$

$w^\alpha = (w^K, w)$ — an arbitrary vector field defined on D, see [8]. The geometric equation will now be presented in the form:

$$\gamma_{K\Lambda} = {}'\delta_\Lambda u_K + \eta_{\Lambda K}\overset{*}{u} + [l_\Lambda^S a^M e_{KS} + l_\Lambda e^M{}_K]v_M + [l_\Lambda^S \overset{*}{a}e_{KS} + l_\Lambda a^S e_{SK}]v,$$

(4.4)
$$\overset{*}{\gamma}_\Lambda = {}'\delta_\Lambda \overset{*}{u} + \eta_\Lambda^{K} u_K + \eta_\Lambda \overset{*}{u} + l_\Lambda^K e^M{}_K v_M + l_\Lambda^K a^N e_{NK}\overset{*}{v},$$

$$\varkappa_{K\Lambda} = {}'\delta_\Lambda v_K + \eta_{\Lambda K}\overset{*}{v},$$

$$\overset{*}{\varkappa}_\Lambda = {}'\delta_\Lambda \overset{*}{v} + \eta_\Lambda^K v_K + \eta_\Lambda \overset{*}{v},$$

where

$$'\delta_\Lambda u_K(d) = \Lambda_\Lambda u_K(d) + \overset{\bullet}{G}{}_{\Lambda K}^L(d)u_\Lambda(f_\Lambda a),$$

$$'\delta_\Lambda u(d) = \Lambda_\Lambda u(d).$$

Denoting

$$A_{kl}^{\Lambda\Phi}g^{Kk}g^{Ll} = A^{\Lambda\Phi KL}, \qquad B_{kl}^{\Lambda\Phi}g^{Kk}g^{Ll} = B^{\Lambda\Phi KL},$$

$$A_{kl}^{\Lambda\Phi}g^{Kk}\overset{*}{n}^l = A^{\Lambda\Phi K}, \qquad B_{kl}^{\Lambda\Phi}g^{Kk}\overset{*}{n}^l = B^{\Lambda\Phi K},$$

$$A_{kl}^{\Lambda\Phi}\overset{*}{n}^k\overset{*}{n}^l = A^{\Lambda\Phi}, \qquad B_{kl}^{\Lambda\Phi}\overset{*}{n}^k\overset{*}{n}^l = B^{\Lambda\Phi},$$

(4.5)
$$F_{kl}^{\Lambda\Phi}g^{Kk}g^{Ll} = F^{\Lambda\Phi KL},$$

$$F_{kl}^{\Lambda\Phi}g^{Kk}\overset{*}{n}^l = F^{\Lambda\Phi K},$$

$$F_{kl}^{\Lambda\Phi}\overset{*}{n}^k\overset{*}{n}^l = F^{\Lambda\Phi},$$

from (1.3), we obtain:

$$t^{K\Lambda} = A^{\Lambda\Phi KL}\gamma_{L\Phi} + A^{\Lambda\Phi K}\overset{*}{\gamma}_\Phi + B^{\Lambda\Phi KL}\varkappa_{L\Phi} + B^{\Lambda\Phi K}\overset{*}{\varkappa}_\Phi,$$

(4.6)
$$t^\Lambda = A^{\Lambda\Phi L}\gamma_{L\Phi} + A^{\Lambda\Phi}\overset{*}{\gamma}_\Phi + B^{\Lambda\Phi L}\varkappa_{L\Phi} + B^{\Lambda\Phi}\overset{*}{\varkappa}_\Phi,$$

$$m^{K\Lambda} = F^{\Lambda\Phi KL}\varkappa_{L\Phi} + F^{\Lambda\Phi K}\overset{*}{\varkappa}_\Phi + B^{\Phi\Lambda KL}\gamma_{L\Phi} + B^{\Phi\Lambda K}\overset{*}{\gamma}_\Phi,$$

$$m^\Lambda = F^{\Lambda\Phi L}\varkappa_{L\Phi} + F^{\Lambda\Phi}\overset{*}{\varkappa}_\Phi + B^{\Phi\Lambda}\gamma_{L\Phi} + B^{\Phi\Lambda}\overset{*}{\gamma}_\Phi.$$

Making use of (2.2) and (4.6) and denoting

$$\underline{t}_k{}^\Lambda = \underline{t}^{K\Lambda}g_{Kk} + \underline{t}^\Lambda n_k,$$

$$'t_k{}^\Lambda = 't^{K\Lambda}g_{Kk} + 't^\Lambda u_k,$$

$$''t_k{}^\Lambda = ''t^{K\Lambda}g_{Kk} + ''t^\Lambda n_k,$$

we arrive at the following expressions:

$$A^{\Lambda\Phi KL} = \delta^{\Lambda\Phi}\left[E_\Lambda A_\Lambda \underline{t}^{K\Lambda}\underline{t}^{L\Lambda} + \frac{12E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Lambda^2}\,t^{K\Lambda'}t^{L\Lambda} + \frac{12E_\Lambda J_\Lambda'}{(1+12\gamma_\Lambda'')l_\Lambda^2}\,{}''t^{K\Lambda}{}''t^{L\Lambda}\right]\frac{1}{l_\Lambda}.$$

$$A^{\Lambda\Phi K} = \delta^{\Lambda\Phi}\left[E_\Lambda A_\Lambda \underline{t}^{K\Lambda}\underline{t}^\Lambda + \frac{12E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Lambda^2}\,{}'t^{K\Lambda'}t^\Lambda + \frac{12E_\Lambda J_\Lambda'}{(1+12\gamma_\Lambda'')l_\Lambda^2}\,{}''t^{K\Lambda}{}''t^\Lambda\right]\frac{1}{l_\Lambda},$$

$$A^{\Delta\Phi} = \delta^{\Delta\Phi}\left[E_\Lambda A_\Lambda t^\Lambda \underline{t}^\Lambda + \frac{12E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Lambda^2}{}'t^{\Lambda'}t^\Lambda + \frac{12E_\Lambda J_\Lambda'}{(1+12\gamma_\Lambda'')l_\Lambda^2}{}''t^{\Lambda''}t^\Lambda\right]\frac{1}{l_\Lambda},$$

$$B^{\Delta\Phi KL} = \delta^{\Delta\Phi}\left[\frac{6E_\Lambda J_\Lambda'}{(1+12\gamma_\Lambda')l_\Lambda^2}{}''t^{K\Lambda'}t^{L\Lambda} - \frac{6E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Lambda^2}{}'t^{K\Lambda''}t^{L\Lambda}\right],$$

$$B^{\Delta\Phi K} = \delta^{\Delta\Phi}\left[\frac{6E_\Lambda J_\Lambda'}{(1+12\gamma_\Lambda')l_\Lambda^2}{}''t^{K\Lambda'}t^\Lambda - \frac{6E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Lambda^2}{}'t^{K\Lambda''}t^\Lambda\right],$$

(4.7) $$B^{\Delta\Phi} = \delta^{\Delta\Phi}\left[\frac{6E_\Lambda J_\Lambda}{(1+12\gamma_\Lambda'')l_\Lambda^2}{}''t^{\Lambda'}t^\Lambda - \frac{6E_\Lambda J_\Lambda''}{(1+12\gamma_\Lambda')l_\Theta^2}{}'t^{\Lambda''}t^\Lambda\right],$$

$$F^{\Delta\Phi KL} = \delta^{\Delta\Phi}\left\{C_\Lambda \underline{t}^{K\Lambda} t^{L\Lambda} + E_\Lambda J_\Lambda'\left[1 + \frac{3}{1+12\gamma_\Lambda''}\right]'t^{K\Lambda'}t^{L\Lambda}\right.$$
$$\left. + E_\Lambda J_\Lambda''\left[1 + \frac{3}{1+12\gamma_\Lambda'}\right]''t^{K\Lambda''}t^{L\Lambda}\right\}\frac{1}{l_\Lambda},$$

$$F^{\Delta\Phi K} = \delta^{\Delta\Phi}\left\{C_\Lambda \underline{t}^{K\Lambda} \underline{t}^\Lambda + E_\Lambda J_\Lambda'\left[1 + \frac{3}{1+12\gamma_\Lambda''}\right]'t^{K\Lambda'}t^\Lambda\right.$$
$$\left. + E_\Lambda J_\Lambda''\left[1 + \frac{3}{1+12\gamma_\Lambda'}\right]''t^{K\Lambda''}t^\Lambda\right\}\frac{1}{l_\Lambda},$$

$$F^{\Delta\Phi} = \delta^{\Delta\Phi}\left\{C_\Lambda \underline{t}^\Lambda \underline{t}^\Lambda + E_\Lambda J_\Lambda'\left[1 + \frac{3}{1+12\gamma_\Lambda''}\right]'t^{\Lambda'}t^\Lambda + E_\Lambda J_\Lambda''\left[1 + \frac{3}{1+12\gamma_\Lambda'}\right]''t^{\Lambda''}t^\Lambda\right\}\frac{1}{l_\Lambda}.$$

The Eqs. (4.3), (4.4) and (4.6) form the basic set of equations for lattice-type shells. The solution of this set of equations should fulfil the relevant boundary conditions. Those conditions will be written, see [13] in the form:

$$u_K = \hat{a}_K, \qquad \dot{u} = \dot{a}, \qquad v_K = b_K, \qquad \dot{v} = \dot{b}, \qquad d \in \partial D_q,$$

(4.8) $$B^L = -f^L, \qquad B = -f, \qquad B_\cdot{}^L = -m^L, \qquad B_\cdot = -m, \qquad d \in \partial D_f,$$

$$\partial D_q \cup \partial D_f = \partial D,$$

where

$$B^L = \sum_{\Lambda \in L_d} t^{L\Lambda} - \sum_{\Lambda \in \bar{L}_d}[\bar{t}^{K\Lambda}\bar{g}_{Kk}g^{Lk} + \bar{t}^\Lambda \bar{n}_k g^{Lk}],$$

$$B = \sum_{\Lambda \in L_d} t^\Lambda - \sum_{\Lambda \in \bar{L}_d}[\bar{t}^{K\Lambda}\bar{g}_{Kk}\overset{*}{n}{}^k + \bar{t}^\Lambda \bar{n}_k \overset{*}{n}{}^k],$$

(4.9)

$$B_\cdot{}^L = \sum_{\Lambda \in L_d} m^{L\Lambda} - \sum_{\Lambda \in \bar{L}_d}[\bar{m}^{K\Lambda}\bar{g}_{Kk}g^{Lk} + \bar{m}^\Lambda \bar{n}_k g^{Lk}] + (e_{MK}l_\Lambda^M a^L + e_K{}^L l_\Lambda)t^{K\Lambda} + e^L{}_K l_\Lambda^K t^\Lambda,$$

$$B_\cdot = \sum_{\Lambda \in L_d} m^\Lambda - \sum_{\Lambda \in \bar{L}_d}[\bar{m}^{K\Lambda}\bar{g}_{Kk}\overset{*}{n}{}^k + \bar{m}^\Lambda \bar{n}_k \overset{*}{n}{}^k] + e_{MK}l_\Lambda^M t^{K\Lambda},$$

where $\Lambda \in L_d$ if $d \in D_\Lambda$ and $\Lambda \in \bar{L}_d$ if $d \in D_{-\Lambda}$.

Now, the geometrical relations (4.4) will be presented in a somewhat different way. Making use of the quantities $\gamma^K{}_A$, $\varkappa^K{}_A$, γ_A, $\varkappa_A$ instead of the quantities γ_{KA}, $\varkappa_{KA}$, $\overset{*}{\gamma}_A$, $\overset{*}{\varkappa}_A$, we arrive at the following formulae:

(4.10)
$$\gamma^K{}_A = {}'\delta_A u^K + \beta^K_A u + (l^A_S a^K e_{SM} + l_A e_M{}^K)v^M + l^S_A e^K_S v,$$
$$\gamma_A = \delta'_A u + \beta_{AL} u^K + \beta_A u + (l^S_A \overset{\textbf{.}}{a} e_{SM} + l_A a^S e_{MS})v^M + l^S_A a^N e_{NS} v,$$
$$\varkappa^K{}_A = {}'\delta_A v^K + \beta^K_A v,$$
$$\varkappa_A = {}'\delta_A v + \beta_{AK} v^K + \beta_A v.$$

Between $\gamma^K{}_A$, γ_A, $\varkappa^K{}_A$, $\varkappa_A$ and γ_{KA}, $\overset{*}{\gamma}_A$, $\varkappa_{KA}$, $\overset{*}{\varkappa}_A$ the following connections take place:

(4.11)
$$\varkappa^K{}_A = \varkappa_{LA} a^{KL} + \overset{*}{\varkappa}_A a^K,$$
$$\varkappa_A = \varkappa_{LA} a^L + \overset{*}{\varkappa}_A a,$$
$$\gamma^K{}_A = \gamma_{LA} a^{KL} + \overset{*}{\gamma}_A a^K,$$
$$\gamma_A = \gamma_{AL} a^L + \overset{*}{\gamma}_A a.$$

Let us now consider the plane problem, and assume that the plane on which we form a structure is the plane of elastic symmetry. From (4.3), (4.4), (4.6) it is easy to obtain two independent sets of equations referring to problems of lattice-type discs (also called plane problems) and plates, respectively. Assuming in (3.8) $n^k = \overset{*}{n}{}^k$, we obtain:

$$b_{AK} = b^K_A = b_A = \bar{b}_{AK} = \ldots = \bar{h}_A = 0,$$

and from (4.7):

$$B^{A\Phi KL} = B^{A\Phi} = A^{A\Phi K} = F^{A\Phi K} = 0.$$

The set of equations for the plate problem has the form:

(4.12)
$$t^A = A^{A\Phi} \overset{*}{\gamma}_\Phi + B^{A\Phi L} \varkappa^L_\Phi,$$
$$m^{KA} = F^{A\Phi KL} \varkappa_{L\Phi} + B^{\Phi AK} \overset{*}{\gamma}_\Phi,$$
$$'\bar{\delta}_A t^A + f = 0, \qquad '\bar{\delta}_A m^{KA} + e^K_M l^M_A t^A + m^K = 0,$$
$$\overset{*}{\gamma}_A = {}'\delta_A \overset{*}{u} + l^K_A e^M_K v_M, \qquad \varkappa_{KA} = {}'\delta_A v_K,$$

and for the plane problem, the form:

(4.13)
$$t^{KA} = A^{A\Phi KL} \gamma_{L\Phi} + B^{A\Phi K} \overset{*}{\varkappa}_\Phi,$$
$$m^A = F^{A\Phi} \overset{*}{\varkappa}_\Phi + B^{\Phi AL} \gamma_{L\Phi},$$
$$'\bar{\delta}^A t^{KA} + f^K = 0, \qquad '\bar{\delta}_A m^A + e_{MN} l^M_A t^{NA} + m = 0,$$
$$\gamma_{KA} = {}'\delta_A u_K + l^S_A e_{KS} \overset{*}{v}, \qquad \overset{*}{\varkappa}_A = {}'\delta_A \overset{*}{v}.$$

All our considerations in Sec. 4 apply to the set of equations (1.2), (1.3), (1.4). Obviously, the analogous formulae can be obtained from the alternative set of equations (1.5), (1.8),

(1.10). In the constitutive equations, the following tensors of elastic rigidity will then appear:

$$C^{A\Phi KL} = \delta^{A\Phi}[C_A \underline{t}^{KA} \underline{t}^{LA} + E_A J_A' \, 't^{KA}\, 't^{LA} + E_A J_A'' \, ''t^{KA}\, ''t^{LA}]\frac{1}{l_A},$$

(4.14)
$$C^{A\Phi K} = \delta^{A\Phi}[C_A \underline{t}^{LA} \underline{t}^A + E_A J_A' \, 't^{KA}\, 't^A + E_A J_A'' \, ''t^{KA}\, ''t^A]\frac{1}{l_A},$$

$$C^{A\Phi} = \delta^{A\Phi}[C_A \underline{t}^A \underline{t}^A + E_A J_A' \, 't^A\, 't^A + E_A J_A'' \, ''t^A\, ''t^A]\frac{1}{l_A}.$$

5. Compatibility conditions, static-geometric analogy and stress functions

Let us assume regularity of the difference structure considered. The following identities, see [8], then hold:

$$'\delta_{[A}'\delta_{\Phi]}u(d) = 0,$$

$$'\delta_{[A}'\delta_{\Phi]}v(d) = 0,$$

(5.1)
$$'\delta_{[A}'\delta_{\Phi]}u^K(d) = \beta^K_{[A}b_{\Phi]L}(d)u^L(f_A f_\Phi d),$$

$$'\delta_{[A}'\delta_{\Phi]}v^K(d) = \beta^K_{[A}b_{\Phi]L}(d)v^L(f_A f_\Phi d).$$

Let us introduce the symbols:

$$\epsilon^{A\Phi} = \begin{cases} +1 & \text{when } \Phi - A = 1 \quad \text{or} \quad \Phi - A = -2, \\ -1 & \text{when } A - \Phi = 1 \quad \text{or} \quad A - \Phi = -2, \\ 0 & \text{in other cases,} \end{cases}$$

$$\gamma^{KA} = \epsilon^{A\Phi}\gamma^K_\Phi, \qquad \varkappa^{KA} = \epsilon^{A\Phi}\varkappa^K_\Phi,$$

$$\gamma^A = \epsilon^{A\Phi}\gamma_\Phi, \qquad \varkappa^A = \epsilon^{A\Phi}\varkappa_\Phi,$$

and let us assume $\Delta_A l^k_\Phi \ll l^k_\Phi$ for each pair A, Φ and each $d \in D_{A,\Phi}$. By virtue of (5.1), introducing components of the strain state connected with components of the displacement state through relations (4.10), we obtain the following system of equations:

$$'\delta_A \varkappa^{KA} + \beta^K_A \varkappa^A = 0,$$

$$'\delta_A \varkappa^A + \beta_{AK}\varkappa^{KA} + \beta_A \varkappa^A = 0,$$

(5.
$$'\delta_A \gamma^{KA} + \beta^K_A \gamma^A + e^K_M l^M_A \varkappa^A + [a^K e_{MN} l^M_A + e_N{}^K l_A]\varkappa^{NA} = 0,$$

$$'\delta_A \gamma^A + \beta^{AK}\gamma^{KA} + \beta_A \gamma^A + [\overset{\circ}{a}e_{MN}l^M_A + a^S e_{NS} l_A]\varkappa^{NA} + a^S e_{SM} l^M_A \varkappa^A = 0.$$

The Eqs. (5.2) are called the conditions of compatibility of the linear theory of elastic lattice shells. For the plane problem, the system (5.2) splits up into two independent systems. These are the compatibility equations referring to the discs and plates, respectively. Now, let us recall the form of the equations of equilibrium (4.3) without external forces. From a confrontation of these equations with the compatibility equation just derived, there

follows an analogy of their structure. We can change (5.2) into (4.3), and vice-versa, using the formal scheme:

(5.3)
$$'\bar{\delta}_A \leftrightarrow '\delta_A, \quad t^{KA} \leftrightarrow \varkappa^{KA}, \quad t^A \leftrightarrow \varkappa^A, \quad m^{KA} \leftrightarrow \gamma^{KA},$$
$$m^A \leftrightarrow \gamma^A, \quad \bar{\beta}_A^K \leftrightarrow \beta_A^K, \quad \bar{\beta}_{AK} \leftrightarrow \beta_{AK}, \quad \bar{\beta}_A \leftrightarrow \beta_A.$$

According to this scheme, we can write the formulae:

$$m^{KA} = \epsilon^{A\Phi}['\bar{\delta}_\Phi \varphi^K + \bar{\beta}_\Phi^K \varphi + (l_\Phi^S a^K e_{SM} + e_M{}^K l_\Phi)v^M + e^K{}_S l_\Phi^S \psi],$$

(5.4)
$$m^A = \epsilon^{A\Phi}['\bar{\delta}_\Phi \varphi + \bar{\beta}_{\Phi K}\varphi^K + \bar{\beta}_\Phi \varphi + a^N e_{NS} l_\Phi^S \psi + (\dot{a}e_{SM} l_A^S + a^R e_{MR} l_A)\psi^M],$$

$$t^{KA} = \epsilon^{A\Phi}['\bar{\delta}_\Phi \psi^K + \bar{\beta}_\Phi^K \psi],$$

$$t^A = \epsilon^{A\Phi}['\bar{\delta}_\Phi \psi + \bar{\beta}_{\Phi K}\psi^K + \bar{\beta}_\Phi \psi],$$

where the functions φ, φ_i^K, ψ, ψ^K are said to be the stress functions in the surface problem of discrete elasticity. Making use of (5.4), we are able to satisfy the homogeneous equilibrium equations (4.3). In this way, we can extend the scheme (5.3) by the formulae:

(5.5)
$$\varphi^K \leftrightarrow u^K, \quad \varphi \leftrightarrow u, \quad \psi^K \leftrightarrow v^K, \quad \psi \leftrightarrow v.$$

To each equation of the theory in which the components of stress state or stress functions occur there corresponds an equation in which occur the components of the state of strain or components of the state of displacement, according to (5.3) and (5.5). From the relations given above, two fundamental systems of equations can be derived. In one of them, as the basic unknowns we shall take the functions u^K, u, v^K, v and in the other — the functions φ^K, φ, ψ^K, ψ. The analogy in the form of those equations can easily be observed, see [7].

6. Equations of shallow lattice shells

In this section, various approximations of lattice shell equations derived in previous sections will be considered for the case of shallow shells. Some advantages resulting from the presentation of the above given equations in the general coordinates (Sec. 3) will now be seen. Let us confine ourselves to the cases in which the direction of the vector n differs only slightly from the direction of vector $g_1 \times g_2$. The mathematical consequence of this assumption consists in adapting the approximations $a^K, a_K \ll a^{KL}, a_{KL}$. Moreover, to the system of equations formulated above the other following aproximations will be applied, resulting also from the assumption of the shallow shells theory, and being known as a generalization of simplifications familiar from the classical (i.e. continous) theory of shallow shells, cf. [14–16]:

1) the functions u^K, u_K will be treated as considerably smaller than the functions u, $\dot{u}$;

2) the functions v, $\dot{v}$ are also much smaller than the functions v_K, v^K;

3) the components l_A are much smaller than the components l_A^K;

4) the quantities b_A, $\bar{b}_A$, h_A, $\bar{h}_A$ are much smaller than the quantities b_{KA}, $\bar{b}_{KA}$, b_A^K, $\bar{b}_K^A$, h_{AK}, $\bar{h}_{AK}$, h_A^K, $\bar{h}_A^K$.

In view of those simplifications, we reduce the geometric equations (4.11) to the form, see [13]:

$$\gamma_A^K = {'}\delta_A u^K + \beta_A^K u + l_A^S a^K e_{SM} v^M + l_A^S e^K s v,$$

(6.1)
$$\gamma_A = {'}\delta_A u + l_A^S \overset{*}{a} e_{SM} v^M,$$

$$\varkappa^K_A = {'}\delta_A v^K,$$

$$\varkappa_A = {'}\delta_A v + \beta_{AK} v^K.$$

From the Eqs. (5.4), after application of approximations similar to 1) and 2) concerning stress functions, we obtain:

$$t^{KA} = \epsilon^{A\Phi}{'}\bar\delta_\Phi \psi^K,$$

(6.2)
$$t^A = \epsilon^{A\Phi}[{'}\bar\delta_\Phi \psi + \bar\beta_{\Phi K}\psi^K],$$

$$m^{KA} = \epsilon^{A\Phi}[{'}\bar\delta_\Phi \varphi^K + \bar\beta_\Phi^K \varphi + l_\Phi^S a^K e_{SM}\psi^M + l_\Phi^S e^K s\psi],$$

$$m^A = \epsilon^{A\Phi}[{'}\bar\delta_\Phi \varphi + l_\Phi^S e_{SM}\overset{*}{a}\psi^M].$$

The equations of equilibrium (4.3) take the form:

$${'}\bar\delta_A t^{KA} + f^K = 0,$$

(6.3)
$${'}\bar\delta_A t^A + \bar\beta_{AK} t^{KA} + f = 0,$$

$${'}\bar\delta_A m^{KA} + \bar\beta_A^K m^A + e^K_M l_A^M t^A + [e_N{}^K l_A + a^K e_{MN} l_A^M] t^{NA} + m = 0,$$

$${'}\delta_A m^A + \overset{*}{a} e_{MN} l_A^M t^{NA} + m = 0.$$

Let us confine ourselves to a simple case in which the lattice considered is composed of two families of bars only. Denoting by $[l_S^\Phi]$ the 2×2 matrices, and assuming $l_A^S l_S^\Phi = \delta_A^\Phi$ from (6.1)$_2$ and (6.2)$_4$, we arrive at the equations

$$\psi^M = [\epsilon_{A\Phi} m^A - {'}\bar\delta_\Phi \varphi] l_S^\Phi \overset{*}{e}_{SM} \frac{1}{\overset{*}{a}},$$

(6.4)
$$v^M = [\epsilon_{A\Phi}\gamma^A - {'}\delta_\Phi u] l_S^\Phi \overset{*}{e}{}^{SM} \frac{1}{\overset{*}{a}}.$$

Taking into account (6.4)$_1$, from (6.2)$_1$ we obtain:

$$t^{KA} = \epsilon^{A\Phi}[\epsilon_{\Omega\Xi}{'}\bar\delta_\Phi m^\Omega - {'}\bar\delta_\Phi{'}\bar\delta_\Xi \varphi] l_S^\Xi \overset{*}{e}{}^{SK} \frac{1}{\overset{*}{a}},$$

(6.5)
$$\varkappa^{KA} = \epsilon^{A\Phi}[\epsilon_{\Omega\Xi}{'}\delta_\Phi \gamma^\Omega - {'}\delta_\Phi{'}\delta_\Xi u] l_S^\Xi \overset{*}{e}{}^{SK} \frac{1}{\overset{*}{a}}.$$

The fundamental system of equations assumes, from (6.1) and (6.2), the form:

$${'}\bar\delta_A t^A + \bar\beta_{AK} t^{KA} + f = 0,$$

(6.6)
$${'}\bar\delta_A m^{KA} + \bar\beta_A^K m^A + e^K_{.M} l_A^M t^A + a^K e_{MN} l_A^M t^{NA} + m^K = 0,$$

$${'}\delta_A \varkappa^A + \beta_{AK}\varkappa^{KA} = 0,$$

$${'}\delta_A \gamma^{KA} + \beta_A^K \gamma^A + e^K_{.M} l_A^M \varkappa^A + a^K e_{MN} l_A^M \varkappa^{NA} = 0.$$

The other equations of equilibrium and conditions of compatibility are fulfilled because of (6.5). Let us take the functions u, φ, γ^A and m^A as the fundamental unknowns, which appears very expedient for the asymptotic approach applied further on. Then, let us substitute the relation $t^A = A^A_{.\Phi}\gamma^\Phi$, $A^A_{.\Phi} = A^{AE}\epsilon_{\Phi E}$ into $(6.6)_1$, thus eliminating t^A. Taking into account the relation $m^{KA} = C^A_{.\Phi.L}\varkappa^{L\Phi} = C^A_{\Phi}{}^K_L\epsilon^{\Phi E}\delta_E v^L$, $C^A_{\Phi}{}^K_L = C^{AEK}{}_L\epsilon_{\Phi E}$ and (6.1), we eliminate m^{KA}, by means of the relations (6.5), we eliminate t^{KA} and, finally, similar relations are applied to $\varkappa^{KA}$, $\varkappa^A$ and γ^{KA} Accordingly, we arrive at the following system of equations:

$$'\bar{\delta}_A[A^A_{.\Phi}\gamma^\Phi] + \bar{\beta}_{AK}\left[l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}e^{A\Phi'}\bar{\delta}_\Phi(\epsilon_{\Omega E}m^\Omega - '\bar{\delta}_E\varphi)\right] + f = 0,$$

$$'\delta_A[c_\Phi{}^A m^\Phi] + \beta_{AK}\left[l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}\epsilon^{A\Phi'}\delta^\Phi(\epsilon_{\Phi E}\gamma^\Omega - '\delta_E u)\right] = 0,$$

(6.7)

$$'\delta_A[\bar{C}^{AE\Phi K'}\delta_\Phi(\epsilon_{\Omega E}\gamma^\Omega - '\delta_E u)] + \bar{\beta}^K_A m^A + e^K{}_M l^M_A A^A_{.\Phi}\gamma^\Phi - \frac{a^K}{\overset{\bullet}{a}}'\bar{\delta}_\Phi m^\Phi + m^K = 0,$$

$$'\delta_A[\tilde{a}^{AE\Phi K'}\bar{\delta}_\Phi(\epsilon_{\Omega E}m^\Omega - '\bar{\delta}_E\varphi)] + \beta^K_A\gamma^A + e^K_{.M} l^M_A c_\Phi{}^A m^\Phi - \frac{a^K}{\overset{\bullet}{a}}'\delta_\Phi\gamma^\Phi = 0,$$

where

$$\bar{C}^{A\Omega E K} = C^A_{\Phi}{}^K_L\epsilon^{\Phi E}l_S^\Theta\overset{\bullet}{e}{}^{SL}\frac{1}{\overset{\bullet}{a}}, \qquad \tilde{a}^{A\Omega E K} = a_\Phi{}^A_L{}^K\epsilon^{\Phi E}l_S^\Omega\overset{\bullet}{e}{}^{SL}\frac{1}{\overset{\bullet}{a}}.$$

This system consists of 6 partial finite difference equations of the 12th order in six unknown functions u, φ, γ^A, m^A, $A = I, II$. Introducing the notations

$$G(m^A) = l^A_L\overset{\bullet}{e}_K{}^{L'}\delta_A\bar{\beta}^K_\Psi m^\Psi - \bar{\beta}_{AK}l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}\epsilon^{A\Phi}\epsilon_{\Omega E}{}'\delta_\Phi m^\Omega - l^A_L\overset{\bullet}{e}_K{}^L\frac{a^K}{\overset{\bullet}{a}}{}'\delta_A{}'\bar{\delta}_\Phi m^\Phi,$$

$$\overset{\bullet}{G}(\gamma^A) = l^A_L\overset{\bullet}{e}_K{}^{L'}\delta_A\beta^K_\Psi\gamma^\Psi - \beta_{AL}l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}\epsilon^{A\Phi}\epsilon_{\Omega E}{}'\delta_\Phi\gamma^\Omega - l^A_L\overset{\bullet}{e}_K{}^L\frac{a^K}{\overset{\bullet}{a}}{}'\delta_A{}'\delta_\Phi\gamma^\Phi,$$

$$\bar{C}^{A\Psi\Phi E} = l^A_L\overset{\bullet}{e}_K{}^L\tilde{C}^{\Psi E\Phi K}, \qquad \tilde{a}^{A\Psi\Phi E} = l^A_L\overset{\bullet}{e}_K{}^L\tilde{a}^{\Psi E\Phi K},$$

we represent (6.7) in the following form:

$$'\bar{\delta}_A{}'\bar{\delta}_\Psi[\tilde{C}^{A\Psi\Phi E'}\delta_\Phi(\epsilon_{\Omega E}\gamma^\Omega - '\delta_E u)] + \beta_{AK}\left[l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}\epsilon^{A\Phi'}\delta_\Phi{}'\delta_E\varphi\right]$$

$$+ l^A_K\overset{\bullet}{e}_K{}^{L'}\bar{\delta}_A m^K - f + G(m^A) = 0,$$

(6.8) $\quad '\bar{\delta}_A[\tilde{C}^{AE\Phi K'}\delta_\Phi(\epsilon_{\Omega E}\gamma^\Omega - '\delta_E u)] + \bar{\beta}^K_A m^A + e_K{}^M l^M_A A^A_{.\Phi}\gamma^\Phi - \dfrac{a^K}{\overset{\bullet}{a}}\bar{\delta}_\Phi m^\Phi + m^K = 0,$

$$'\delta_A{}'\delta_\psi[\tilde{a}^{A\Psi\Phi E'}\bar{\delta}_\Phi(\epsilon_{\Omega E}m^\Omega - '\bar{\delta}_E\varphi)] + \beta_{AK}\left[l_S^E\overset{\bullet}{e}{}^{SK}\frac{1}{\overset{\bullet}{a}}\epsilon^{A\Phi'}\delta_\Phi{}'\delta_E u\right] + \overset{\bullet}{G}(\gamma^A) = 0,$$

$$'\delta_A[\tilde{a}^{AE\Phi K'}\bar{\delta}_\Phi(\epsilon_{\Omega E}m^\Omega - '\bar{\delta}_E\varphi)] + \beta^K_A\gamma^A + e_K{}^M l^M_A c_\Phi{}^A m^\Phi - \frac{a^K}{\overset{\bullet}{a}}\delta_\Phi\gamma^\Phi = 0.$$

The solution of the boundary-value problem consists in finding such six functions $u, \varphi, \gamma^A, m^A$ which within the domain D satisfy the system (6.8), and at the boundary of this region appropriate boundary conditions. The number of these conditions, in the general case, is equal to six and their form is dependent on the manner of loading and the support conditions of the edges of the lattice-type shell. The very complicated form of the basic set of equations leads to a search for possibilities of simplifications of this system. In the next paper [17], we shall consider these possibilities, resulting from the occurrence of small parameters in difference operators of higher order in the fundamental set of equations.

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