Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 195–211, Warszawa 1973

The approximate methods in the theory of elastic lattice-type shells

M. KLEIBER WARSZAWA)

THE FOUNDATIONS of statics of elastic lattice-type shells were given in [1]. In the present paper a study is made of the possibility of simplifications of the basic system of equations formulated in [1]. Application of the theory is given.

W pracy [1] podana została metoda statycznej analizy sprężystych powłok prętowych. W niniejszej pracy wskazano na możliwości wprowadzenia pewnych uproszczeń do podstawowego układu równań opisującego problem. Opracowaną teorię zilustrowano prostym przykładem.

В работе [1] был предложен метод статического анализа упругих стержневых оболочек. Данная работа свидетельствует о возможности некоторого упрощения основной системы уравнений, описывающих рассматриваемую задачу. Разработанная теория иллюстрируется на простом примере.

THIS PAPER is continuation of an earlier paper [1] devoted to discussion of the static equations of the linear theory of elastic lattice-type shells and possible simplifications resulting from the assumption of small rise. With this assumption, making use of the fact that there are small parameters occurring in the higher-order difference operators, an asymptotic theory will now be presented together with the theory of edge effect.

1. The fundamental system of equations

It was shown in [1] that the fundamental system of equations for shells of small rise may be written in the following form ([1, 2])

$$\begin{gathered}
{}'\bar{\delta}_\Lambda{}'\bar{\delta}_\Psi[\bar{C}^{\Lambda\Psi\Lambda\Xi}\delta_\Phi(\epsilon_{\Omega\Xi}\gamma^\Omega-{}'\delta_\Xi u)]+\bar{\beta}_{\Lambda K}\left[l^\Xi_S\overset{*}{e}{}^{SK}\frac{1}{\overset{*}{a}}\epsilon^{\Lambda\Phi}{}'\bar{\delta}_\Phi{}'\bar{\delta}_\Xi\varphi\right] \\
+l^\Lambda_L\overset{*}{e}_K{}^L{}'\bar{\delta}_\Lambda m^K-f+G(m^\Lambda)=0,
\end{gathered}$$

$$(1.1)\qquad {}'\bar{\delta}_\Lambda[\bar{C}^{\Lambda\Xi\Phi K}\delta_\Phi(\epsilon_{\Omega\Xi}\gamma^\Omega-{}'\delta_\Xi u)]+\bar{\beta}^K_\Lambda m^\Lambda+e^K{}_M l^M_\Lambda A^\Lambda{}_\Phi\gamma^\Phi-\frac{a^K}{\overset{*}{a}}{}'\bar{\delta}_\Phi m^\Phi+m^K=0,$$

([1]) The meaning of the symbols introduced is the same as in [1].

([2]) We confine ourselves to a simple case, in which the lattice considered is composed of two families of bars only.

$$'\delta_\Lambda{}'\delta_\Psi[\tilde{a}^{\Lambda\Psi\Phi\Xi}{}'\bar{\delta}_\Phi(\epsilon_{\Omega\Xi}m^\Omega-{}'\bar{\delta}_\Xi\varphi)]+\beta_{\Lambda K}\left[l_S^\Xi \overset{*}{e}{}^{SK}\frac{1}{\overset{*}{a}}\epsilon^{\Lambda\Phi}{}'\delta_\Phi{}'\delta_\Xi u\right]+\overset{*}{G}(\gamma^\Lambda)=0,$$

$$'\delta_\Lambda[\tilde{a}^{\Lambda\Xi\Phi K}{}'\bar{\delta}_\Phi(\epsilon_{\Omega\Xi}m^\Omega-{}'\bar{\delta}_\Xi\varphi)]+\beta_\Lambda^K\gamma^\Lambda+e^K{}_M l_\Lambda^M c_\Phi{}^\Lambda m^\Phi-\frac{a^K}{\overset{*}{a}}{}'\delta_\phi\gamma^\phi=0,$$

where

$$G(m^\Lambda)=l_L^\Lambda \overset{*}{e}_K{}^{L}{}'\bar{\delta}_\Lambda\bar{\beta}_\Psi^K m^\Psi-\bar{\beta}_{\Lambda K}l_S^\Xi \overset{*}{e}{}^{SK}\frac{1}{\overset{*}{a}}\epsilon^{\Lambda\Phi}\epsilon_{\Omega\Xi}{}'\bar{\delta}_\Phi m^\Omega-l_L^\Lambda \overset{*}{e}_K{}^{L}\frac{a^K}{\overset{*}{a}}{}'\bar{\delta}_\Lambda{}'\bar{\delta}_\Phi m^\Phi,$$

$$\overset{*}{G}(\gamma^\Lambda)=l_L^\Lambda \overset{*}{e}_K{}^{L}{}'\delta_\Lambda\beta_\Psi^K\gamma^\Psi-\beta_{\Lambda K}l_S^\Xi \overset{*}{e}{}^{SK}\frac{1}{\overset{*}{a}}\epsilon^{\Lambda\Phi}\epsilon_{\Omega\Xi}{}'\delta_\Phi\gamma^\Omega-l_L^\Lambda \overset{*}{e}_K{}^{L}\frac{a^K}{\overset{*}{a}}{}'\delta_\Lambda{}'\delta_\Phi\gamma^\Phi,$$

$$\bar{C}^{\Lambda\Psi\Phi\Xi}=l_L^\Lambda \overset{*}{e}_K{}^{L}\tilde{C}^{\Psi\Xi\Phi K},$$

$$\tilde{a}^{\Lambda\Psi\Phi\Xi}=l_L^\Lambda \overset{*}{e}_K{}^{L}\tilde{a}^{\Psi\Xi\Phi K}.$$

The solution of a boundary-value problem of the theory of lattice shallow shells reduces to the determination of six functions $u(d)$, $\varphi(d)$, $m^\Lambda(d)$, $\gamma^\Lambda(d)$, $\Lambda=\mathrm{I},\mathrm{II}$, which satisfy, in the region D, $d\in D$, the set of difference equations (1.1) and the relevant edge conditions given in [1]. Symbols $\tilde{C}^{\Lambda\Psi\Phi\Xi}$, $\tilde{C}^{\Lambda\Psi\Lambda K}$, $\tilde{a}^{\Lambda\Psi\Phi\Xi}$, $\tilde{a}^{\Lambda\Psi\Phi K}$ in (1.1) denote the components of elastic rigidity tensors dependent on the geometric and material structure of the shell. These components are known for a given type of shell, see [3]. In the case of a lattice composed of n-families of bars, the components of elastic rigidity tensors have the form given by the formulae (2.2), (2.3), (4.7), (4.14) in [1].

Let us now study precisely the structure of these components. Let us confine ourselves to the case in which all bars joined in the node $d\in D$ form approximately a plane coinciding with the plane given by the vectors $g_K(d)$, $K=1,2$. cf. [1] (this condition is exactly fulfilled in a lattice composed of two families of bars and in plane problems). Moreover, disregarding the influence of shear forces on bending of the bars, the formulae $(4.7)_{1,2,3}$ and (4.14), [1] can be written in the following form:

$$(1.2)\qquad
\begin{aligned}
A^{\Lambda\Phi KL}&=\delta^{\Lambda\Phi}\left[\frac{E_\Lambda A_\Lambda}{l_\Lambda}\underline{t}^{K\Lambda}\underline{t}^{L\Lambda}+\frac{12E_\Lambda J'_\Lambda}{l_\Lambda^3}{}''t^{K\Lambda}{}''t^{L\Lambda}\right],\\
A^{\Lambda\Phi K}&=0,\\
A^{\Lambda\Phi}&=\delta^{\Lambda\Phi}\left[\frac{12E_\Lambda J''_\Lambda}{l_\Lambda^3}{}'t^{\Lambda}{}'t^{\Lambda}\right],\\
C^{\Lambda\Phi KL}&=\delta^{\Lambda\Phi}[C_\Lambda\underline{t}^{K\Lambda}\underline{t}^{L\Lambda}+E_\Lambda J''_\Lambda{}''t^{K\Lambda}{}''t^{L\Lambda}]\frac{1}{l_\Lambda},\\
C^{\Lambda\Phi K}&=0,\\
C^{\Lambda\Phi}&=\delta^{\Lambda\Phi}\left[\frac{E_\Lambda J'_\Lambda}{l_\Lambda}{}'t^{\Lambda}{}'t^{\Lambda}\right].
\end{aligned}$$

2. The asymptotic theory

Let us denote by $'i_\Lambda$, $''i_\Lambda$ the radia of inertia of the cross-section of the Λ — bar with respect to the axes given by the vectors $'t_k^{\Lambda}(d)$, $''t_k^{\Lambda}(d)$, respectively,

$$'i_\Lambda = \sqrt{\frac{J'_\Lambda}{A_\Lambda}}, \quad ''i_\Lambda = \sqrt{\frac{J''_\Lambda}{A_\Lambda}}.$$

Let us also denote

$$'\lambda_\Lambda = \frac{'i_\Lambda}{l_\Lambda}, \quad ''\lambda_\Lambda = \frac{''i_\Lambda}{l_\Lambda}.$$

Denoting by $'\lambda, ''\lambda, l$ the arithmetic means of the quantities $'\lambda'_\Lambda$, $''\lambda_\Lambda$, l_Λ, $\Lambda = \mathrm{I}, \mathrm{II}$, respectively, we have then:

$$\begin{aligned} \frac{E_\Lambda J'_\Lambda}{l_\Lambda} &= l^2 \frac{12 E_\Lambda J'_\Lambda}{l_\Lambda^3} \frac{1}{12}\left(\frac{l_\Lambda}{l}\right)^2, \\ \frac{12 E_\Lambda J'_\Lambda}{l_\Lambda^3} &= \frac{E_\Lambda A_\Lambda}{l_\Lambda} 12'\lambda^2 \left(\frac{'\lambda_\Lambda}{'\lambda}\right)^2, \\ \frac{12 E_\Lambda J''_\Lambda}{l_\Lambda^3} &= \frac{1}{l^2} \frac{E_\Lambda J''_\Lambda}{l_\Lambda} 12 \left(\frac{l}{l_\Lambda}\right)^2. \end{aligned} \tag{2.1}$$

From these formulae results the existence of two small parameters $'\lambda$ and l. The parameter $'\lambda$ is much smaller than 1 and the parameter l, for lattices sufficiently dense, is much smaller than the global dimensions of the shell. We assume, moreover, a long "wave" of loading as compared with l. By virtue of (1.2) and (2.1), it is convenient to represent the components of elastic rigidity tensors in the form:

$$\begin{aligned} A^{\Lambda\Phi KL} &= {'A}^{\Lambda\Phi KL} + {'\lambda}^2\, {''A}^{\Lambda\Phi KL}, \\ A^{\Lambda\Phi} &= l^{-2}\, {'A}^{\Lambda\Phi}, \\ C^{\Lambda\Phi} &= l^2\, {'C}^{\Lambda\Phi}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} {'A}^{\Lambda\Phi KL} &= \delta^{\Lambda\Phi}\left[\frac{E_\Lambda A_\Lambda}{l_\Lambda} \underline{t}^{K\Lambda} \underline{t}^{L\Lambda}\right], \\ {''A}^{\Lambda\Phi KL} &= \delta^{\Lambda\Phi}\left[\frac{E_\Lambda A_\Lambda}{l_\Lambda} 12 \left(\frac{'\lambda_\Lambda}{'\lambda}\right)^2 {''t}^{K\Lambda}\, {''t}^{L\Lambda}\right], \\ {'C}^{\Lambda\Phi} &= \delta^{\Lambda\Phi}\left[\frac{12 E_\Lambda J'_\Lambda}{l_\Lambda^3} \cdot \frac{1}{12}\left(\frac{l_\Lambda}{l}\right)^2 {'t}^{\Lambda}\, {'t}^{\Lambda}\right], \\ {'A}^{\Lambda\Phi} &= \delta^{\Lambda\Phi}\left[\frac{F_\Lambda J'_\Lambda}{l_\Lambda} \cdot 12 \left(\frac{l}{l_\Lambda}\right)^2 {'t}^{\Lambda}\, {'t}^{\Lambda}\right]. \end{aligned} \tag{2.3}$$

From the relations (1.2), (2.2) and (2.3), determining the structure of the tensors ${'A}^{\Lambda\Phi KL}$, ${''A}^{\Lambda\Phi KL}$, ${'C}^{\Lambda\Phi}$, ${'A}^{\Lambda\Phi}$, it follows that for each value of l and $'\lambda$ the components ${'C}^{\Lambda\Omega}$ are of the same order as the components $A^{\Lambda\Phi KL}$, the components ${'A}^{\Lambda\Phi KL}$ of the same order as the components $''A^{\Lambda\Phi KL}$, and the components $'A^{\Lambda\Phi}$ of the same order as the components $C^{\Lambda\Phi KL}$. By virtue of (2.2), the stress strain relations will now be represented in the form:

$$
\begin{aligned}
t^{K\Lambda} &= ('A^{\Lambda\Phi KL} + '\lambda^2 \, ''A^{\Lambda\Phi KL})\gamma_{L\Phi}, \\
t_\Lambda &= \frac{1}{l^2}\, 'A_{\Lambda\Phi}\dot{\gamma}_\Phi, \\
m^{K\Lambda} &= C^{\Lambda\Phi KL}\varkappa^{L\Phi}, \\
m^\Lambda &= l^2\, 'C^{\Lambda\Phi}\dot{\varkappa}_\Phi.
\end{aligned}
$$

Assuming $l \to 0$, $'\lambda \neq 0$, we arrive at the asymptotic theory of lattice-type shells, cf. [2, 5, 8]. In accordance with (2.2), the Eqs. (1.1) reduce to the form:

$$\gamma^\Lambda = 0, \quad m^\Lambda = 0,$$

$$
\begin{aligned}
(2.4) \qquad & '\bar{\delta}_\Lambda{}'\bar{\delta}_\Psi(\tilde{C}^{\Lambda\Psi\Phi\Xi}{}'\delta_\Xi{}'\delta_\Xi u) + \bar{\beta}_{\Lambda K}\left(l^\Xi_S \dot{e}^{KS}\frac{1}{\dot{a}}\epsilon^{\Lambda\Phi}{}'\bar{\delta}_\Phi{}'\bar{\delta}_\Xi\varphi\right) + l^\Lambda_L \dot{e}^L{}_K{}'\bar{\delta}_\Lambda m^K + f = 0, \\
& '\delta_\Lambda{}'\delta_\Psi(\tilde{a}^{\Lambda\Psi\Phi\Xi}{}'\bar{\delta}_\Phi{}'\bar{\delta}_\Xi\varphi) + \beta_{\Lambda K}\left(l^\Xi_S \dot{a}^{KS}\frac{1}{\dot{a}}e^{\Lambda\Phi}{}'\delta_\Phi{}'\delta_\Xi u\right) = 0.
\end{aligned}
$$

The Eqs. (2.4) are called the equations of the asymptotic theory. They constitute the system of partial difference equations of the 8th order in two unknown functions u and φ. Since the asymptotic theory of lattice-type shells leads to the equalities $m^\Lambda = 0$, $\gamma^\Lambda = 0$, its application would make sense only if m^Λ and γ^Λ are sufficiently small within the entire region D. Application of the asymptotic theory also requires the number of boundary conditions to be reduced from six to four. In these boundary conditions only the unknown functions u and φ should occur. If the applicability conditions of the asymptotic theory are not satisfied, we can take into account the theory of edge effect. The equation of the edge effect together with the equations of the asymptotic theory (2.4) enable us to obtain the approximate solution of the boundary-value problem for the set of equations (1.1).

3. The edge effect theory

The approximate theory of the edge effect in a lattice-shell will consist in supplementing solutions u, φ of the set (2.4) by four functions m^Λ, γ^Λ, which are to satisfy the following conditions:

1. They have finite values at the edge of shell and fulfil, jointly with functions u, φ, the relevant boundary conditions.

2. They have values approaching zero at each point of the region of the shell except the part adjacent to the edge.

For the sake of simplicity, we assume that in the part adjacent to the edge both families of bars under consideration are mutually orthogonal and the bars of the first family ($\Lambda = I$) are straight and normal to the boundary. Solution of the more general problem is also possible, but we should take into account a more general model of discrete media, cf. [2, 9]. All the further considerations will be referred to the region adjacent to the edge. In agreement with the conditions imposed on the functions m^Λ and γ^Λ, they should vanish rapidly with increasing distance from the edge. In this connection, we assume:

$$
(3.1) \qquad
\begin{aligned}
'\delta_I{}'\delta_I\gamma^\Lambda &\sim {}'\delta_I\gamma^\Lambda \sim \gamma^\Lambda, \\
'\delta_I{}'\delta_I m^\Lambda &\sim {}'\delta_I m^\Lambda \sim m^\Lambda,
\end{aligned}
$$

where the symbol "$\sim$" means that the values on both sides of it are of the same order of magnitude (that is, neither of them can be rejected in the general consideration as small compared with the other). Let us further assume that all loads acting across the edge, and the loads of the surface of the shell vary slowly along the edge — that is,

$$\alpha^{2}\,'\delta_{\mathrm{II}}\,'\delta_{\mathrm{II}}\Omega \sim \alpha\,'\delta_{\mathrm{II}}\Omega \sim \Omega,$$

where Ω is an arbitrary load and α is a dimensionless parameter with value much greater than 1. Then, in the close neighbourhood of the edge, it may be assumed that

$$\begin{aligned} &\alpha^{2}\,'\delta_{\mathrm{II}}\,'\delta_{\mathrm{II}}\gamma^{\Lambda} \sim \alpha\,'\delta_{\mathrm{II}}\gamma^{\Lambda} \sim \gamma^{\Lambda},\\ &\alpha^{2}\,'\delta_{\mathrm{II}}\,'\delta_{\mathrm{II}}m^{\Lambda} \sim \alpha\,'\delta_{\mathrm{II}}m^{\Lambda} \sim m^{\Lambda}. \end{aligned} \tag{3.2}$$

Let us compare the quantities γ^{I} and γ^{II}. For this purpose, we assume in $(6.7)_1$, [1] $\bar{b}^{\Lambda K} = 0$ and $A^{\Lambda}_{\cdot\Phi} = \mathrm{const}$. In view of $A^{\mathrm{I}}{}_{\mathrm{I}} = A^{\mathrm{II}}{}_{\mathrm{II}} = 0$, we obtain:

$$A^{\mathrm{I}}{}_{\mathrm{II}}\,'\delta_{\mathrm{I}}\gamma^{\mathrm{II}} + A^{\mathrm{II}}{}_{\mathrm{I}}\,'\delta_{\mathrm{II}}\gamma^{\mathrm{I}} + f = 0. \tag{3.3}$$

From this we obtain $\delta_{\mathrm{I}}\gamma^{\mathrm{II}} \sim \delta_{\mathrm{II}}\gamma^{\mathrm{I}}$ (assuming e.g. in (3.3) $f = 0$) and from (3.2) we obtain

$$\gamma^{\mathrm{I}} \sim \alpha\gamma^{\mathrm{II}} \quad \text{or} \quad \gamma^{\mathrm{I}} \gg \gamma^{\mathrm{II}}. \tag{3.4}$$

Likewise, from $(6.7)_2$ in [1] we obtain

$$m^{\mathrm{I}} \sim \alpha m^{\mathrm{II}} \quad \text{or} \quad m^{\mathrm{I}} \gg m^{\mathrm{II}}. \tag{3.5}$$

From the analysis of the boundary conditions, cf. [2], we find that:

$$\begin{aligned} &'\delta_{\mathrm{I}}\,'\delta_{\mathrm{I}}\gamma^{\mathrm{I}} \sim \alpha\,'\delta_{\Lambda}\,'\delta_{\Phi}\,'\delta_{\Xi}u \quad \text{or} \quad '\delta_{\mathrm{I}}\,'\delta_{\mathrm{I}}\gamma^{\mathrm{I}} \gg\, '\delta_{\Lambda}\,'\delta_{\Phi}\,'\delta_{\Xi}u,\\ &'\delta_{\mathrm{I}}\,'\delta_{\mathrm{I}}m^{\mathrm{I}} \sim \alpha\,'\delta_{\Lambda}\,'\delta_{\Phi}\,'\delta_{\Xi}\varphi \quad \text{or} \quad '\delta_{\mathrm{I}}\,'\delta_{\mathrm{I}}m^{\mathrm{I}} \gg\, '\delta_{\Lambda}\,'\delta_{\Phi}\,'\delta_{\Xi}\varphi. \end{aligned} \tag{3.6}$$

All the above considerations will remain valid if we replace the symbol "$'\delta_{\Lambda}$" by the symbol "$'\bar{\delta}_{\Lambda}$". It is also assumed that the functions u, φ which are integrals of the set of equations (2.4) do not increase rapidly in the domain of the shell because the solution for (2.4) has no asymptotic features. It will be also assumed that the conditions (3.6), which are satisfied at the edge, remain valid in the region adjacent to the edge. By writing out the Eqs. (1.1), and bearing in mind all the assumptions cited above, we can formulate the following set of four equations([3]):

$$\begin{aligned} &\tilde{C}^{\mathrm{I\,II\,I}1}\,'\bar{\delta}_{\mathrm{I}}\,'\delta_{\mathrm{I}}\gamma^{\mathrm{I}} + \bar{\beta}^{1}_{\mathrm{I}}m^{\mathrm{I}} + e^{1}{}_{2}l^{2}_{\mathrm{II}}A^{\mathrm{II}}{}_{\mathrm{I}}\gamma^{\mathrm{I}} - \frac{a^{1}}{\dot{a}}\,'\bar{\delta}_{\mathrm{I}}m^{\mathrm{I}} + m^{1} = 0,\\ &\tilde{C}^{\mathrm{I\,I\,I}2}\,'\bar{\delta}_{\mathrm{I}}\,'\delta_{\mathrm{I}}\gamma^{\mathrm{II}} - \tilde{C}^{\mathrm{I\,II\,II}2}\,'\bar{\delta}_{\mathrm{I}}\,'\delta_{\mathrm{II}}\gamma^{\mathrm{I}} - \tilde{C}^{\mathrm{II\,II\,I}2}\,'\delta_{\mathrm{I}}\,'\bar{\delta}_{\mathrm{II}}\gamma^{\mathrm{I}} - \bar{\beta}^{2}_{\mathrm{II}}m^{\mathrm{II}} +\\ &\qquad - e^{2}{}_{1}l^{1}_{\mathrm{I}}A^{\mathrm{I}}{}_{\mathrm{II}}\gamma^{\mathrm{II}} + \frac{a^{2}}{\dot{a}}\,'\bar{\delta}_{\mathrm{I}}m^{\mathrm{I}} + \tilde{C}^{\Lambda\Xi\Phi 2}\,'\bar{\delta}_{\Lambda}\,'\delta_{\Phi}\,'\delta_{\Xi}u + m^{2} = 0, \end{aligned} \tag{3.7}$$

([3]) In (3.7) it was taken into consideration that for the given structure of the shell only the following components of the tensors $\tilde{C}^{\Lambda\Phi\Xi K}$, $\tilde{C}^{\Lambda\Phi\Xi\Omega}$, $\tilde{a}^{\Lambda\Phi\Xi K}$, $\tilde{a}^{\Lambda\Phi\Xi\Omega}$ are not equal to zero:

for $\tilde{C}^{\Lambda\Phi\Xi K}$ — $\tilde{C}^{\mathrm{I\,II\,I}1}$, $\tilde{C}^{\mathrm{I\,II}1} = \tilde{C}^{\mathrm{I\,I\,II\,I}2}$, $\tilde{C}^{\mathrm{II\,I\,II}2}$, $\tilde{C}^{\mathrm{I\,I\,I}2}$, $\tilde{C}^{\mathrm{I\,II\,II}2} = \tilde{C}^{\mathrm{II\,I\,I}1}$, $\tilde{C}^{\mathrm{II\,II\,II}1}$,

for $\tilde{C}^{\Lambda\Phi\Xi\Omega}$ — $\tilde{C}^{\mathrm{II\,I\,II}}$, $\tilde{C}^{\mathrm{II\,I\,II}} = \tilde{C}^{\mathrm{I\,II\,II}}$, $\tilde{C}^{\mathrm{II\,I}}$, $\tilde{C}^{\mathrm{II\,II\,I}}$, $\tilde{C}^{\mathrm{I\,II\,II}} = \tilde{C}^{\mathrm{II\,II\,I}}$, $\tilde{C}^{\mathrm{II\,II\,II}}$,

for $\tilde{a}^{\Lambda\Phi\Xi K}$, $\tilde{a}^{\Lambda\Phi\Xi\Omega}$ — similarly.

$$\text{(3.7) [cont.]}\quad \tilde{a}^{\mathrm{I}\,\mathrm{II}\,\mathrm{I}\,1}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{I}}+\beta_{\mathrm{I}}^{1}\gamma^{\mathrm{I}}+e^{1}_{\cdot\,2}l_{\mathrm{II}}^{2}c_{\mathrm{I}}{}^{\mathrm{II}}m^{\mathrm{I}}-\frac{a^{1}}{\dot{a}}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{I}}=0,$$

$$\tilde{a}^{\mathrm{I}\,\mathrm{II}\,2}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{II}}-\tilde{a}^{\mathrm{I}\,\mathrm{II}\,\mathrm{II}\,2}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{II}}m^{\mathrm{I}}-\tilde{a}^{\mathrm{II}\,\mathrm{II}\,\mathrm{I}\,2}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{II}}m^{\mathrm{I}}-\beta_{\mathrm{II}}^{2}\gamma^{\mathrm{II}}+$$
$$-e^{2}{}_{1}l_{\mathrm{I}}^{1}c_{\mathrm{II}}{}^{\mathrm{I}}m^{\mathrm{II}}+\frac{a^{2}}{\dot{a}}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{I}}+\tilde{a}^{\Lambda\Xi\Phi 2}{}'\delta_{\Lambda}{}'\bar{\delta}_{\Phi}{}'\bar{\delta}_{\Xi}\varphi=0.$$

The Eqs. $(3.7)_{1,3}$ are the fundamental equations of the edge effect. General solutions of them will easily be obtained if we take into account that all the coefficients in (3.7) should be considered constant, in agreement with the assumptions of the theory of the edge effect, cf. [2, 8]. In this way, by virtue of the Eqs. $(3.7)_{1,3}$, we obtain as solutions the functions γ^{I} and m^{I} which play an essential role in the theory of edge effect. The sets of Eqs. (2.4) and $(3.7)_{1,3}$ are of 8th and 4th order, respectively. In this way, we can fulfil all the six relevant boundary conditions. In agreement with this, the remaining unknowns — i.e. γ^{II} and m^{II} — should be calculated from the algebraic set of equations (if the functions γ^{II} and m^{II} were the unknowns in the difference equations, we should obtain then additional constants without any additional boundary conditions). To obtain the equations for these functions, we apply the method given below. From the Eqs. $(6.7)_1$ and $(6.7)_2$ in [1], we derive the relations:

$$A^{\Lambda}{}_{\Phi}{}'\bar{\delta}_{\Lambda}\gamma^{\Phi}+\bar{\beta}_{\Lambda K}\left[l_{S}^{\Xi}\dot{e}^{SK}\frac{1}{\dot{a}}\boldsymbol{\epsilon}^{\Lambda\Phi}(\boldsymbol{\epsilon}_{\Omega\Xi}{}'\bar{\delta}_{\Phi}m^{\Omega}-{}'\bar{\delta}_{\Phi}{}'\bar{\delta}_{\Xi}\varphi)\right]=0,$$
$$\text{(3.8)}$$
$$c_{\Phi}{}^{\Lambda}{}'\delta_{\Lambda}m^{\Phi}+\beta_{\Lambda K}\left[l_{S}^{\Xi}\dot{e}^{SK}\frac{1}{\dot{a}}\boldsymbol{\epsilon}^{\Lambda\Phi}(\boldsymbol{\epsilon}_{\Omega\Xi}{}'\delta_{\Phi}\gamma^{\Omega}-{}'\delta_{\Phi}{}'\delta_{\Xi}u)\right]=0.$$

Representing these equations in an explicit form, taking the difference with respect to $\Lambda=I$, and bearing in mind the simplifications made above, we arrive at:

$$A^{\mathrm{I}}{}_{\mathrm{II}}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{II}}+A^{\mathrm{II}}_{\cdot\,\mathrm{I}}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{II}}\gamma^{\mathrm{I}}+\bar{\beta}_{\mathrm{II}\,2}\left[l_{2}^{\mathrm{I}}\dot{e}^{12}\frac{1}{\dot{a}}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}(m^{\mathrm{II}}+{}'\bar{\delta}_{\mathrm{I}}\varphi)\right]$$
$$+\bar{\beta}_{\mathrm{II}}\left[l_{2}^{\mathrm{II}}\dot{e}^{21}\frac{1}{\dot{a}}{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{II}}(-m^{\mathrm{I}}+{}'\bar{\delta}_{\mathrm{II}}\varphi)\right]=0,$$
$$\text{(3.9)}$$
$$c_{\mathrm{II}}{}^{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}m^{\mathrm{II}}+c_{\mathrm{I}}{}^{\mathrm{II}},\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{II}}+\beta_{\mathrm{II}\,2}\left[l_{1}^{\mathrm{I}}\dot{e}^{12}\frac{1}{\dot{a}}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}(\gamma^{\mathrm{II}}+{}'\delta_{\mathrm{I}}u)\right]$$
$$+\beta_{\mathrm{II}}\left[l_{2}^{\mathrm{II}}\dot{e}^{21}\frac{1}{\dot{a}}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{II}}(-\gamma^{\mathrm{I}}+{}'\delta_{\mathrm{II}}u)\right]=0.$$

Let us observe that the functions u, φ, γ^{I} and m^{I} in $(3.7)_{2,4}$ and (3.9) ar known [they can be calculated from (2.4) and $(3.7)_{1,3}$]. Considering $(3.7)_{2,4}$ and (3.9) as an algebraic set of equations with m^{II}, γ^{II}, ${}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{II}}$ and ${}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}\gamma^{\mathrm{II}}$ as unknowns, and assuming that all the remaining quantities involved have been found, we can obtain expressions for m^{II} and γ^{II} in terms of m^{I}, γ^{I}, μ and φ. The approximate method presented for computing lattice shells consists in replacing the set of Eqs. (1.1) by the Eqs. (2.4), the Eqs. $(3.7)_{1,3}$ and the above expressions for m^{II} and γ^{II}. The order of the set of equations remains unaltered — i.e. 12th, which allows for rigorous satisfaction of all the six boundary conditions.

The solution of the boundary value problem thus found is a sufficient approximation only if the assumptions of the theory of the edge effect are satisfied — that is, if the influence of the functions $\gamma^{\Lambda}, m^{\Lambda}$ on the state of stress and strain is negligible, but in the region adjacent to the edge. At the same time, because the edge effect theory is connected with the existence of the small parameter, the more dense the lattice bars, the more precise will be the solution of the problem.

In many cases, as was shown in [3, 2], the difference equations of the problem under consideration can be written in the form of partial differential equations. From all the equations given above we obtain the differential equations of the "continous" model of a lattice-type shell, cf. [2]. Those equations were obtained also in a completely different way in [4–6].

4. Cylindrical lattice-type shells

In 3-dimensional Euclidean space with rectangular Cartesian coordinates (z^k), $k = 1, 2, 3$, let the radius-vector be given in the form

$$r = [hx_1, r_0 \sin \alpha x_2, r_0 \cos \alpha x_2],$$

where $\alpha = 2\pi/k$, k — an arbitrary integer, h — an arbitrary real number.

If coordinates x_K, $K = 1, 2$ run over the sequence $1, 2, 3, \ldots$ then the radius-vector indicates the place in the physical space occupied by the successive nodes of the lattice shell under consideration, see Fig. 1. A more general case in which $h = h(x_K)$, $\alpha = \alpha(x_K)$

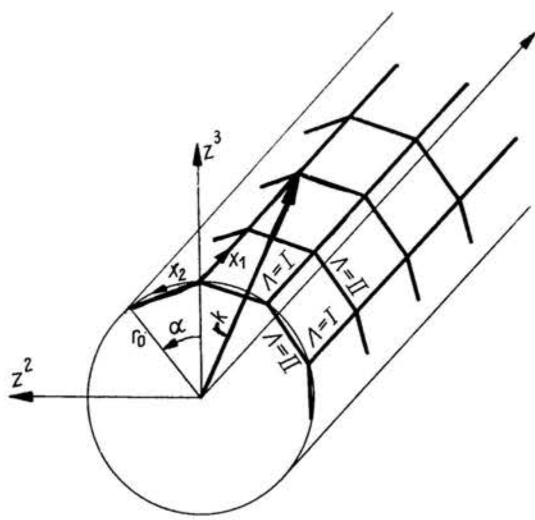

FIG. 1.

is also possible. We then obtain an irregular lattice of bars created on a cylindrical surface. The lengths of the bars in the respective families are in our case:

$$l_{\mathrm{I}} = h. \qquad l_{\mathrm{II}} = 2r_0 \sin\frac{\pi}{k}.$$

The increments of the radius-vector along the directions $\Lambda = \mathrm{I}$ and $\Lambda = \mathrm{II}$ are given by the expressions:

$$
(4.2)\qquad \begin{aligned}
&\Delta_{\mathrm{I}} r(x) = r(x+t_{\mathrm{I}})-r(x) = h\cdot[1,0,0],\\
&\Delta_{\mathrm{II}} r(x) = r(x+t_{\mathrm{II}})-r(x) = 2r_0\sin\frac{\alpha}{2}\left[0,\cos\alpha\left(x_2+\frac{1}{2}\right), -\sin\alpha\left(x_2+\frac{1}{2}\right)\right].
\end{aligned}
$$

Let the vector base be assumed in the form, cf. [2]:

$$g_K(x) = \delta_K^{\Lambda}\Delta_{\Lambda} r(x),$$

$n(x)$ — along the radius r_0.

We then obtain

$$
(4.3)\qquad \begin{aligned}
&g_1(x) = h\cdot[1,0,0],\\
&g_2(x) = 2r_0\sin\frac{\alpha}{2}\left[0,\cos\alpha\left(x_2+\frac{1}{2}\right), -\sin\alpha\left(x_2+\frac{1}{2}\right)\right],\\
&n(x) = d\cdot[0,\sin\alpha x_2,\cos\alpha x_2].
\end{aligned}
$$

By virtue of (4.3) and the formulae (3.3)—(3.8) in [1], we can calculate all the necessary geometrical quantities:

$$a_{MN} = \begin{vmatrix} h^2 & 0 \\ 0 & 4r_0^2\sin^2\dfrac{\alpha}{2}\end{vmatrix}, \qquad a^{MN} = \begin{vmatrix} \dfrac{1}{h^2} & 0 \\ 0 & \dfrac{1}{r_0^2\sin^2\alpha}\end{vmatrix},$$

$$a_M = \left[0,-2dr_0\sin^2\frac{\alpha}{2}\right], \qquad a^M = \left[0,\frac{1}{2r_0 d\cos^2\dfrac{\alpha}{2}}\right],$$

$$a = d^2, \qquad \mathring{a} = \frac{1}{\alpha^2\cos^2\dfrac{\alpha}{2}},$$

$$g = (r_0 hd\sin\alpha)^2, \qquad \mathring{g} = \frac{1}{(r_0 hd\sin\alpha)^2},$$

$$g^1 = \frac{1}{h}[1,0,0],$$

$$g^2 = \frac{1}{r_0\sin\alpha}[0,\cos\alpha x_2,-\sin\alpha x_2],$$

$$\mathring{n} = \frac{1}{d\cos\dfrac{\alpha}{2}}\left[0,\sin\alpha\left(x_2+\frac{1}{2}\right),\cos\alpha\left(x_2+\frac{1}{2}\right)\right],$$

$$\Delta_{\Lambda} g_L = -\delta_L^2\delta_{\Lambda}^{\mathrm{II}}4r_0\sin^2\frac{\alpha}{2}[0,\sin\alpha(x_2+1),\cos\alpha(x_2+1)],$$

$$\Delta_{\Lambda} n = \delta_{\Lambda}^{\mathrm{II}}2d\sin\frac{\alpha}{2}\left[0,\cos\alpha\left(x_2+\frac{1}{2}\right),-\sin\alpha\left(x_2+\frac{1}{2}\right)\right],$$

$$\bar{\Delta}_\Lambda g_L = -\delta_L^2 \delta_\Lambda^{II} 4 r_0 \sin^2\frac{\alpha}{2}[0, \sin\alpha x_2, \cos\alpha x_2],$$

$$\bar{\Delta}_\Lambda n = \delta_\Lambda^{II} 2d \sin\frac{\alpha}{2}\left[0, \cos\alpha\left(x_2-\frac{1}{2}\right), -\sin\alpha\left(x_2-\frac{1}{2}\right)\right],$$

$$\Delta_\Lambda g^L = -\delta_2^L \delta_\Lambda^{II} \frac{1}{r_0\cos\frac{\alpha}{2}}\left[0, \sin\alpha\left(x_2+\frac{1}{2}\right), \cos\alpha\left(x_2+\frac{1}{2}\right)\right],$$

$$\Delta_\Lambda \overset{\bullet}{n} = \delta_\Lambda^{II} \frac{2\operatorname{tg}\frac{\alpha}{2}}{d}[0, \cos\alpha(x_2+1), -\sin\alpha(x_2+1)], \tag{4.4}$$

$$\bar{\Delta}_\Lambda g^L = -\delta_2^L \delta_\Lambda^{II} \frac{1}{r_0\cos\frac{\alpha}{2}}\left[0, \sin\alpha\left(x_2-\frac{1}{2}\right), \cos\alpha\left(x_2-\frac{1}{2}\right)\right],$$

$$\bar{\Delta}_\Lambda \overset{\bullet}{n} = \delta_\Lambda^{II} \frac{2\operatorname{tg}\frac{\alpha}{2}}{d}[0, \cos\alpha x_2, -\sin\alpha x_2],$$

$$G_{\Lambda L}^K = -\overset{\bullet}{G}{}_{L\Lambda}^K = -\delta_2^K \delta_L^2 \delta_\Lambda^{II} 4 \sin^2\frac{\alpha}{2},$$

$$\overset{\bullet}{G}{}_{\Lambda L}^K = -G_{L\Lambda}^K = 0,$$

$$b_{\Lambda K} = -\bar{h}_{\Lambda K} = -\delta_K^2 \delta_\Lambda^{II} 4\frac{r_0}{d}\sin^2\frac{\alpha}{2},$$

$$b_\Lambda^K = -\bar{h}_\Lambda^K = \delta_2^K \delta_\Lambda^{II}\frac{d}{r_0},$$

$$b_\Lambda = -\bar{h}_\Lambda = 0,$$

$$h_{\Lambda K} = -\bar{b}_{\Lambda K} = \delta_K^2 \delta_\Lambda^{II} 4\frac{r_0}{d}\sin^2\frac{\alpha}{2},$$

$$h_\Lambda^K = -\bar{b}_\Lambda^K = -\delta_2^K \delta_\Lambda^{II}\frac{d}{r_0},$$

$$h_\Lambda = -\bar{b}_\Lambda = -\delta_\Lambda^{II} 4\sin^2\frac{\alpha}{2},$$

$$e_{12} = r_0 h d\sin\alpha, \qquad e^1{}_2 = \frac{r_0 d\sin\alpha}{h},$$

$$e_1{}^2 = \frac{hd}{r_0\sin\alpha}, \qquad e^{12} = \frac{d}{r_0 h\sin\alpha},$$

$$\overset{\bullet}{e}_{12} = \frac{2r_0 h\operatorname{tg}\frac{\alpha}{2}}{d}, \qquad \overset{\bullet}{e}{}^1{}_2 = \frac{2r_0\operatorname{tg}\frac{\alpha}{2}}{hd},$$

$$\overset{\bullet}{e}_1{}^2 = \frac{h}{r_0 d\sin\alpha}, \qquad \overset{\bullet}{e}{}^{12} = \frac{2r_0 h\operatorname{tg}\frac{\alpha}{2}}{d}.$$

(Using the procedure making the "discrete" and "continous" models of a lattice shell coincide, cf. [3, 2], from (4.4) we can obtain the known formulae of differential geometry). By virtue of (4.4) and assuming axially symmetric state of strain, the basic system of Eqs. (1.1) may be rewritten in the following form:

$$
\begin{aligned}
&K_1{}'\bar{\delta}_{\mathrm{I}}^2{}'\delta_{\mathrm{I}}^2u+K_1{}'\bar{\delta}_{\mathrm{I}}^2{}'\delta_{\mathrm{I}}\gamma^{\mathrm{II}}+K_2{}'\bar{\delta}_{\mathrm{I}}^2\varphi+K_3{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{II}}+K_4{}'\bar{\delta}_{\mathrm{I}}^3m^{\mathrm{I}}+f+\mathring{e}^{1}_{\cdot 2}{}'\bar{\delta}_{\mathrm{I}}m^2 = 0,\\
&K_6{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{I}}+K_7\gamma^{\mathrm{I}}+m^1 = 0,\\
(4.5)\qquad &K_9{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{II}}+K_9\bar{\delta}_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}\delta_{\mathrm{I}}^2u+K_{10}m^{\mathrm{II}}+K_{11}\gamma^{\mathrm{II}}+K_{12}{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{I}}+m^2 = 0,\\
&K_{13}{}'\delta_{\mathrm{I}}^{2}{}'\bar{\delta}_{\mathrm{I}}^2+K_{13}{}'\delta_{\mathrm{I}}^{2}{}'\bar{\delta}_{\mathrm{I}}m^{\mathrm{II}}+K_2{}'\delta_{\mathrm{I}}^2u+K_3{}'\delta_{\mathrm{I}}\gamma^{\mathrm{II}}+K_4{}'\delta_{\mathrm{I}}^2\gamma^{\mathrm{I}} = 0,\\
&K_{14}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}m^{\mathrm{I}}+K_{15}m^{\mathrm{I}} = 0,\\
&K_{16}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}m^{\mathrm{II}}+K_{16}{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}^2\varphi+K_{10}\gamma^{\mathrm{II}}+K_{17}m^{\mathrm{II}}+JK_{12}{}'\delta_{\mathrm{I}}\gamma^{\mathrm{I}} = 0,
\end{aligned}
$$

where $\delta_A^n\Phi(x) = \underbrace{\delta_A \ldots \delta_A}_{\text{n-times}}\Phi(x)$, $\Phi(x)$ — an arbitrary function,

$$
(4.6)\qquad
\begin{aligned}
&K_1 = \tilde{C}^{\mathrm{IIII}}, && K_2 = \frac{\sin\alpha}{h}, && K_3 = \frac{\sin\alpha\,\mathrm{tg}^2\frac{\alpha}{2}}{h}, && K_4 = \frac{2\,\mathrm{tg}\frac{\alpha}{2}}{h},\\
&K_6 = \tilde{C}^{\mathrm{IIIII}}, && K_7 = \frac{hdA^{\mathrm{II}}{}_{\cdot\mathrm{I}}}{r_0\sin\alpha}, && K_9 = \tilde{C}^{\mathrm{III2}}, && K_{10} = -\frac{d}{r_0},\\
&K_{11} = \frac{hdA^{\mathrm{I}}{}_{\mathrm{II}}}{r_0\sin\alpha}, && K_{12} = \frac{d}{2r_0}, && K_{13} = \tilde{a}^{\mathrm{III,I}}, && K_{14} = \tilde{a}^{\mathrm{IIIII}},\\
&K_{15} = \frac{hdc_{\mathrm{I}}{}^{\mathrm{II}}}{r_0\sin\alpha}, && K_{16} = \tilde{a}^{\mathrm{III2}}, && K_{17} = \frac{hdc_{\mathrm{II}}{}^{\mathrm{I}}}{r_0\sin\alpha}.
\end{aligned}
$$

The Eqs. (4.5) constitute the system of six partial difference equations of the 12th order in six unknown functions $\mu, \varphi, m^A, \gamma^A$. In the equality $(4.5)_2$ (in $(4.5)_5$ similarly), we have one unknown function only. The solutions of those equations may be given in the form, cf. [2]:

$$
(4.7)\qquad
\begin{aligned}
\gamma^{\mathrm{I}}(x) &= C_1\left(\frac{2-\beta_1+\sqrt{\beta_1^2-4\beta_1}}{2}\right)^x+C_2\left(\frac{2-\beta_1-\sqrt{\beta_1^2-4\beta_1}}{2}\right)^x+\gamma_S^{\mathrm{I}}(x),\\
m^{\mathrm{I}}(x) &= C_3\left(\frac{2-\beta_2+\sqrt{\beta_2^2-4\beta_2}}{2}\right)+C_4\left(\frac{2-\beta_2-\sqrt{\beta_2^2-4\beta_2}}{2}\right)^x
\end{aligned}
$$

where γ_S^{I} is a particular integral of the nonhomogeneous Eqs. $(4.5)_2$, and we have introduced the notations:

$$
\beta_1 = \frac{K_7}{K_6} = \frac{hdA^{\mathrm{II}}{}_{\mathrm{I}}}{r_0\sin\alpha C^{\mathrm{IIIII}}},\qquad \beta_2 = \frac{K_{15}}{K_{14}} = \frac{hdc_{\mathrm{I}}{}^{\mathrm{II}}}{r_0\sin\alpha\cdot\tilde{a}^{\mathrm{IIIII}}},\qquad x = x_1.
$$

We assume $K_3 = 0$ because this quantity, for small angles α, is negligibly small in relation to the other coefficients. Computing $\bar{\delta}_{\mathrm{I}}^2\delta_{\mathrm{I}}\gamma^{\mathrm{II}}$ and $\bar{\delta}_{\mathrm{I}}\delta_{\mathrm{I}}^2m^{\mathrm{II}}$ from $(4.5)_1$ and $(4.5)_4$, respectively, and using $(4.5)_{3,6}$, we arrive at the following expressions:

$$
(4.8)\qquad
\begin{aligned}
A_1{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}^2\varphi+A_2{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}^2u+A_3{}'\bar{\delta}_{\mathrm{I}}\varphi+A_4{}'\delta_{\mathrm{I}}u &= \mathscr{L}_1,\\
B_1{}'\delta_{\mathrm{I}}{}'\bar{\delta}_{\mathrm{I}}^2\varphi+B_2{}'\bar{\delta}_{\mathrm{I}}{}'\delta_{\mathrm{I}}^2u+B_3{}'\bar{\delta}_{\mathrm{I}}\varphi+B_4{}'\delta_{\mathrm{I}}u &= \mathscr{L}_2,
\end{aligned}
$$

where

(4.9)
$$A_1 = \frac{K_9 K_2}{K_1} + K_{10} = \frac{\tilde{C}^{\mathrm{I}\mathrm{I}\mathrm{I}2}\sin\alpha}{h\tilde{C}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}} - \frac{d}{r_0}, \qquad A_2 = K_{11} = \frac{hdA^{\mathrm{I}}{}_{\mathrm{II}}}{r_0 \sin\alpha},$$
$$A_3 = \frac{K_{11} K_2}{K_1} = \frac{dA^{\mathrm{I}}{}_{\mathrm{II}}}{r_0 \tilde{C}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}}, \qquad A_4 = \frac{K_{10} K_2}{K_{13}} = -\frac{d\sin\alpha}{r_0 h \tilde{a}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}},$$
$$B_1 = K_{17} = \frac{hdc_{\mathrm{II}}{}^{\mathrm{I}}}{r_0 \sin\alpha}, \qquad B_2 = \frac{K_{16} K_2}{K_{13}} + K_{10} = \frac{\tilde{a}^{\mathrm{I}\mathrm{I}\mathrm{I}2}\sin\alpha}{h\tilde{a}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}} - \frac{d}{r_0},$$
$$B_3 = \frac{K_{10} K_2}{K_1} = -\frac{d\sin\alpha}{r_0 h \tilde{C}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}}, \qquad B_4 = \frac{K_{17} K_2}{K_{13}} = \frac{dc_{\mathrm{II}}{}^{\mathrm{I}}}{r_0 \tilde{a}^{\mathrm{I}\mathrm{I}\mathrm{I}\mathrm{I}}},$$
$$\mathscr{L}_1 = \left(K_{12} - \frac{K_9 K_4}{K_1}\right)'\bar{\delta}_{\mathrm{I}}^2{}'\delta_{\mathrm{I}} m^{\mathrm{I}} - \frac{K_{11} K_4}{K_1}{}'\bar{\delta}_{\mathrm{I}} m^{\mathrm{I}} - \frac{K_{10} K_4}{K_{13}}{}'\delta_{\mathrm{I}} \gamma^{\mathrm{I}} + K_{10} C_6 + K_{11} C_5,$$
$$\mathscr{L}_2 = \left(K_{12} - \frac{K_{10} K_4}{K_{13}}\right)'\delta_{\mathrm{I}}^2{}'\bar{\delta}_{\mathrm{I}} \gamma^{\mathrm{I}} - \frac{K_{17} K_4}{K_{13}}{}'\delta_{\mathrm{I}} \gamma^{\mathrm{I}} - \frac{K_{10} K_4}{K_1}{}'\bar{\delta}_{\mathrm{I}} m + K_{10} \cdot C_5 + K_{17} \cdot C_7,$$

($\mathscr{L}_1$, $\mathscr{L}_2$ are treated as known functions). Denoting $E\varphi(x) = \varphi(x+t_{\mathrm{I}})$, $E^n\varphi(x) = \varphi(x+n \cdot t_{\mathrm{I}})$, the Eqs. (4.8) can be written in the following operator form:

(410).
$$\varphi_1(E)\varphi(x) + \psi_1(E)u(x+1) = \mathscr{L}_1(x+2),$$
$$\varphi_2(E)\varphi(x) + \psi_2(E)u(x+1) = \mathscr{L}_2(x+2),$$

where

(4.11)
$$\varphi_1(E) = A_1 E^3 + (A_3 - 3A_1)E^2 + (3A_1 - A_3)E - A_1,$$
$$\psi_1(E) = A_2 E^3 + (A_0 - 3A_2)E^2 + (3A_2 - A_4)E - A_2,$$
$$\varphi_2(E) = B_1 E^3 + (B_3 - 3B_1)E^2 + (3B_1 - B_3)E - B_1,$$
$$\psi_2(E) = B_2 E^3 + (B_0 - 3B_2)E^2 + (3B_2 - B_4)E - B_2.$$

This set of equations may also be given in the form, cf. [7]:

(4.12)
$$[\varphi_2(E)\psi_1(E) - \varphi_1(E)\psi_2(E)]u(x+1) = \varphi_2(E)\mathscr{L}_1(x+2) - \varphi_1(E)\mathscr{L}_2(x+2),$$
$$[-\psi_2(E)\varphi_1(E) + \psi_1(E)\varphi_2(E)]\varphi(x) = -\psi_2(E)\mathscr{L}_1(x+2) + \psi_1(E)\mathscr{L}_2(x+2).$$

Each equation of the set (4.12) is a nonhomogeneous difference equation of 6th order in one unknown function. These equations may easily be resolved, on reducing the homogeneous equations to algebraic equations of 6th grade and because the right-hand sides of (4.12) are of the $A+Ba^x$ type, cf. [2, 7]. From (4.12), we derive the solution with six constants. Together with the six constants obtained earlier from (4.7), we can rigorously satisfy all the six boundary conditions. As may be seen, the closed form solutions for more complicated lattices are difficult to obtain. In those cases, we should apply numerical computational procedures. Having the equations of asymptotic method and the equations of edge effect, let us try to compare them with those obtained above. The Eqs. (2.4), (3.7), and (3.9) have now the form:

(4.13)
$$K_1{}'\bar{\delta}_{\mathrm{I}}^2 \delta_{\mathrm{I}}^2 u + K_2 \bar{\delta}_{\mathrm{I}}^2 \varphi + f + \overset{*}{e}{}^{1}_{\cdot 2}{}'\delta_{\mathrm{I}} m^2 = 0,$$
$$K_{13} \bar{\delta}_{\mathrm{I}}^2 \delta_{\mathrm{I}}^2 \varphi + K_2 \delta_{\mathrm{I}}^2 u = 0,$$

$$K_6{}'\bar{\delta}_I\delta_I\gamma^I+K_7\gamma^I+m^1 = 0, \tag{4.14}$$
$$K_{14}{}'\bar{\delta}_I\delta_I m^I+K_{15}m^I = 0,$$

$$K_9{}'\bar{\delta}_I{}'\delta_I\gamma^{II}+K_9{}'\bar{\delta}_I{}'\delta_I^2u+K_{10}m^{II}+K_{11}\gamma^{II}+K_{12}{}'\bar{\delta}_I m^I+m^2 = 0, \tag{4.15}$$
$$K_{16}{}'\bar{\delta}_I{}'\delta_I m^{II}+K_{16}\delta_I{}'\bar{\delta}_I^2\varphi+K_{10}\gamma^{II}+K_{17}m^{II}+K_{12}{}'\delta_I\gamma^I = 0,$$

$$K_{18}{}'\bar{\delta}_I\delta_I\gamma^{II}+K_2{}'\bar{\delta}_I{}'\delta_I m^{II}+K_2{}'\delta_I{}'\bar{\delta}_I^2\varphi = 0, \tag{4.16}$$
$$K_{19}{}'\bar{\delta}_I\delta_I m^{II}+K_2{}'\bar{\delta}_I\delta_I\gamma^{II}+K_2{}'\delta_I^2\bar{\delta}_I u = 0,$$

where $K_{18} = A^I{}_{II}$, $K_{19} = c_{II}{}^I$.

The Eqs. (4.14) and (4.15) are identical as before, and the Eqs. (4.13) now have simpler form and are independent of one another. The basic unknowns of the theory of edge effect γ^I and m^I are given by the formulae (4.7). The relations (4.13) may be rewritten in the form:

$$'\delta_I^4u(x)+\gamma u(x+2) = -\frac{1}{K_1}[f(x+2)+\mathring{e}^1{}_2{}'\delta_I m^2(x+2)]+C_1x+C_2, \tag{4.17}$$
$$\delta_I^4\varphi(x) = -\frac{K_2}{K_{13}}\delta_I^2u(x+2),$$

where $\gamma = -\dfrac{K_2K_2}{K_1K_{13}}$, C_1, C_2 — arbitrary constant coefficients. The solution of $(4.17)_1$ is given by the expression:

$$u(x) = C_3r_1^x+C_4r_2^x+C_5r_3^x+C_6r_4^x+u_s(x),$$

where

$$r_{1,3} = 1\mp\frac{\sqrt{\gamma}}{2}+\sqrt{\frac{\gamma}{4}\mp\sqrt{\gamma}},\quad r_{2,4} = 1\mp\frac{\sqrt{\gamma}}{2}-\sqrt{\frac{\gamma}{4}\mp\sqrt{\gamma}},$$

$u_s(x)$ — a particular integral of the nonhomogeneous equation $(4.17)_1$, C_3, C_4, C_5, C_6 — an arbitrary set of constant coefficients. In the cases of external load of the type $a^x\varphi(x)$, a — an arbitrary real number, $\varphi(x)$ — an arbitrary polynomial, it is easy to obtain solution of the Eqs. $(4.17)_2$ within accuracy of six constant coefficients $C, i = 1, 2, \ldots, 6$ and two new constants C_7, C_8. Because the functions γ^{II}, m^{II} are solutions of the algebraic set of equations, the final solution will have 12 constants. This will enable the relevant boundary conditions to be rigorously satisfied. Comparing the systems (4.5) and (4.13)–(4.16), we can make sure that the solutions obtained on the basis of the two theories will differ only slighty.

Let us now consider a lattice shell similar to the previous one making use of a "continuous" model of lattice structure, cf. [4–6]. On a given surface π of the cylinder, we introduce an orthogonal system of coordinates in which the boundary of the domain of the parameters x^K coincides with the parametric line $x^1 = 0$, whereas the parametric lines $x^2 =$ const are straight lines and normal to the boundary. Such coordinates are referred to as normal coordinates, cf. [8]. These coordinates fulfil the condition that the curvature lines of the shell should coincide with the parametric lines, cf. [6]. The equations of the surface under consideration may now be presented in the form:

$$z^1 = x^1, \quad z^2 = r_0 \sin\frac{x^2}{r_0}, \quad z^3 = r_0 \cos\frac{x^2}{r_0}, \tag{4.19}$$

where x^2 is the length of an arc on π in the section $x^1 = \text{const}$. Such coordinates x^K are equivalent, after the unfolding of a surface π, to the rectangular Cartesian coordinates. It follows that the covariant, contravariant and physical components of tensors having the same indices, are equal one to another. We confine ourselves from now on to the axially symmetric external loads, which enables the problem to be considered as one-dimensional. Moreover, let us assume that the principal axes of orthotropy are tangent to the parametric lines of the system of normal coordinates (it is obvious that the concept of orthotropy concerns the structure of tensors of elastic rigidity, not the material of which the lattice shell is made). The basic system of equations can now be obtained by means of the equations given in [4–6] or directly from (4.5). This set of equations has the following form, cf. [4–6]:

$$\begin{gathered}
\underset{\sim}{C}^{1112}(u_{,1}+\tilde{\gamma}_2)_{,111}+\frac{1}{r_0}\varphi_{,11} = h^2{}_{,1}+b,\\
\underset{\sim}{a}^{1112}(\varphi_{,1}+m_2)_{,111}+\frac{1}{r_0}u_{,11} = 0,\\
\underset{\sim}{C}^{1112}(u_{,1}+\tilde{\gamma}_2)_{,11}-\frac{1}{r_0}m_2+\tilde{A}^{12}\tilde{\gamma}_2 = h^2,\\
\underset{\sim}{a}^{1112}(\varphi_{,1}+m_2)_{,11}-\frac{1}{r_0}\tilde{\gamma}_2+\tilde{c}^{12}m_2 = 0,\\
\underset{\sim}{C}^{1211}\tilde{\gamma}_{1,11}+\tilde{A}^{21}\tilde{\gamma}_1 = -h^1,\\
\underset{\sim}{a}^{1211}m_{1,11}+\tilde{c}^{21}m_1 = 0.
\end{gathered} \tag{4.20}$$

The Eqs. $(4.20)_{5,6}$ are identical with the equations of the edge effect in the theory of discs and plates, respectively, cf. [8]. The general integrals m^1 and $\tilde{\gamma}^1$ of these equations can be written, within the framework of the edge effect, in the form:

$$\begin{aligned}
m_1(x^1) &= C_1\exp\left[-\sqrt{\frac{c^{11}}{a^{2121}}}\cdot x^1\right]+C_2\exp\left[\sqrt{\frac{c^{11}}{a^{2121}}}\cdot x^1\right],\\
\tilde{\gamma}_1(x^1) &= C_3\exp\left[-\sqrt{\frac{A^{22}}{C^{1111}}}\cdot x^1\right]+C_4\exp\left[\sqrt{\frac{A^{22}}{C^{1111}}}\cdot x^1\right].
\end{aligned} \tag{4.21}$$

If $x^1 \geqslant 0$ in the region of the shell, we should assume $C_2 = C_4 = 0$, cf. [8]. Let us consider the Eqs. $(4.20)_{1-4}$. Computing $\tilde{\gamma}_{2,111}$ and $m_{2,111}$ from $(4.20)_{1,2}$ and using $(4.20)_{3,4}$, we arrive at the following expressions:

$$\begin{aligned}
u^{\mathrm{IV}}+N_1\varphi^{\mathrm{IV}}+N_2\varphi'' &= \mathscr{L}_1,\\
\varphi^{\mathrm{IV}}+N_3u^{\mathrm{IV}}+N_4u'' &= \mathscr{L}_2,
\end{aligned} \tag{4.22}$$

where

$$\Phi^{(n)} = \Phi_{\underbrace{,1\ldots 11}_{n\text{-times}}}, \quad \Phi \text{ — an arbitrary function},$$

$$N_1 = \frac{1}{\tilde{A}^{12} r_0}, \quad N_2 = \frac{\frac{1}{r_0{}^2} - \tilde{A}^{12}\tilde{c}^{12}}{\tilde{C}^{1112}\tilde{A}^{12}\tilde{c}^{12}} \cdot \frac{1}{r_0}, \quad N_3 = \frac{1}{\tilde{c}^{12} r_0},$$

$$N_4 = -\frac{\frac{1}{r_0{}^2} - \tilde{A}^{12}\tilde{c}^{12}}{\underset{\sim}{a}^{1112}\tilde{A}^{12}\tilde{c}^{12}} \frac{1}{r_0},$$

$$\mathscr{X}_1 = \frac{-(h^2_{,1}+b)\left(\frac{1}{r_0{}^2} - \tilde{A}^{12}\tilde{c}^{12}\right)}{\tilde{C}^{1112}\tilde{A}^{12}\tilde{c}^{12}} + \frac{b''}{\tilde{A}^{12}}. \quad \mathscr{X}_2 = \frac{b''}{\tilde{A}^{12}\tilde{c}^{12} r_0}.$$

From (4.22) we obtain:

$$\begin{aligned} u^{\mathrm{IV}}(1-N_1N_3)+u''(-N_1N_2-N_2N_3)+u(-N_2N)\mathfrak{b} \\ = \mathscr{X}_1+N_1\mathscr{X}_2+N_2\iint \mathscr{X}_2\,dx_1\,dx_2+D_1x_1+D_2, \end{aligned} \tag{4.23}$$

$$\varphi'' = \iint \mathscr{X}_2 dx_1\,dx_1 - N_3u'' - N_4u.$$

Solving $(4.23)_1$, we obtain:

$$u(x^1) = e^{\gamma x^1}(A\cos\delta x_1 + B\sin\delta x_1) + e^{-\gamma x^1}(C\cos\delta x_1 + D\sin\delta x_1) + u_s(x^1) + Ex^1 + F, \tag{4.24}$$

where

$$(C^{1212}c^{22} - a^{2222}A^{11})^2 - 4r_0{}^2(A^{11}c^{12})^2 a^{2222}C^{1212} < 0$$

$$\begin{aligned} \gamma &= \sqrt{\frac{1}{2}\sqrt{\frac{\frac{1}{A^{11}c^{22}} + r_0^2}{r_0{}^4 C^{1212}a^{2222}}} + \frac{1}{4r_0}\left(\frac{1}{a^{2222}A^{11}} + \frac{1}{C^{1212}c^{12}}\right)}, \\ \delta &= \sqrt{\frac{1}{2}\sqrt{\frac{\frac{1}{A^{11}c^{22}} + r_0{}^2}{r_0{}^4 C^{1212}a^{2222}}} - \frac{1}{4r_0}\left(\frac{1}{a^{2222}A^{11}} + \frac{1}{C^{1212}c^{22}}\right)}. \end{aligned} \tag{4.25}$$

Substituting (4.24) into $(4.23)_2$, we obtain

$$\begin{aligned} \varphi(x^1) = e^{\gamma x^1}[(Af_1 - Bf_2)\cos\delta x_1 + (Bf_1 + Af_2)\sin\delta x^1] \\ + e^{-\gamma x^1}[(Cf_1 - Df_2)\cos\delta x^1 + (Df_1 + Cf_2)\sin\delta x^1] + \varphi_s(x^1) + Gx^1 + H, \end{aligned} \tag{4.26}$$

where

$$f_1 = \frac{1}{(\gamma^2+\delta^2)^2}\left[\frac{\gamma^2-\delta^2}{ra^{2222}} - \frac{\gamma^2-\delta^2}{r^3a^{2222}A^{11}c^{22}} - \frac{(\gamma^2+\delta^2)^2}{rc^{22}}\right],$$

$$f_2 = \frac{1}{(\gamma^2+\delta^2)^2}\left[\frac{2\gamma\delta}{ra^{2222}} - \frac{2\gamma\delta}{r^3a^{2222}A^{11}c^{22}}\right],$$

and $A, B, \dots, H$ in (4.24) and (4.26) are constant coefficients. The functions $\tilde{\gamma}_2$ and m_2 will be computed from $(4.20)_{3,4}$:

(4.27)
$$\tilde{\gamma}'_2 = -\frac{\frac{u''}{r}+\tilde{c}^{12}b-\tilde{c}^{12}\frac{1}{r}\varphi''}{\frac{1}{r^2}-\tilde{A}^{12}\tilde{c}^{12}},$$
$$m'_2 = \frac{\frac{1}{r}b-\frac{1}{r}\varphi''-\frac{1}{r}\tilde{A}^{12}u''}{\frac{1}{r^2}-\tilde{A}^{12}c^{12}},$$

where u, φ are given by (4.24) and (4.26). The Eqs. (4.21), (4.24), (4.26) and (4.27) are the solutions of the problem considered. The coefficients $C_1, \dots, C_4, A, B, \dots, H$ may be computed from the relevant boundary conditions, cf. [4–6].

We obtain several simplifications using the asymptotic method and the theory of edge effect. The basic set of equations now has the form:

(4.28)
$$\underset{\sim}{C}^{1112}u_{,1111}+\frac{1}{r_0}\varphi_{,11} = h^2_{,1}+b,$$
$$a^{1112}\varphi_{,1111}+\frac{1}{r_0}u_{,11} = 0,$$
$$\underset{\sim}{C}^{1211}\tilde{\gamma}_{2,11}+\tilde{A}^{21}\tilde{\gamma}_1 = -h^1,$$
$$\underset{\sim}{a}^{1211}m_{1,11}+\tilde{c}^{21}m_1 = 0,$$

and the algebraic set of equations for computing the unknown functions $\tilde{\gamma}_2$ and m_2 has the form:

(4.29)
$$\underset{\sim}{C}^{1112}(u_{,1}+\tilde{\gamma}_2)_{,11}-\frac{1}{r_0}m_2+\tilde{A}^{12}\tilde{\gamma}_2 = h^2,$$
$$\underset{\sim}{a}^{1112}(\varphi_{,1}+m_2)_{,11}-\frac{1}{r_0}\tilde{\gamma}_2+\tilde{c}^{12}m_2 = 0,$$
$$\frac{1}{r_0}\tilde{\gamma}_{2,11}-\tilde{c}^{12}m_{2,11} = -\frac{1}{r_0}u_{,111},$$
$$\frac{1}{r_0}m_{2,11}-\tilde{A}^{12}\tilde{\gamma}_{2,11} = -\frac{1}{r_0}\varphi_{,111}+b_{,1}.$$

The Eqs. $(4.28)_{3,4}$ are the same as the Eqs. $(4.20)_{5,6}$. From $(4.28)_{1,2}$ we obtain

(4.30)
$$u^{\mathrm{IV}}+\alpha^4 u = \frac{h^2_{,1}+b}{C^{1212}}+Ex^1+F,$$
$$\varphi = \frac{1}{r_0 a^{2222}}\iint u(x^1)dx_1\,dx_1,$$

where

$$\alpha^4 = \frac{1}{r_0{}^2 a^{2222}C^{1212}}.$$

The solution of the Eq. $(4.28)_1$ may be expressed in the form:

$$u(x^1) = e^{\beta x^1}(A\cos\beta x^1 + B\sin\beta x^1) + e^{-\beta x^1}(C\cos\beta x^1 + D\sin\beta^1 x^1) + u_s(x^1) + Ex^1 + F, \tag{4.31}$$

where

$$\beta = \frac{1}{\sqrt{2}}\sqrt[4]{\frac{1}{r_0{}^2 a^{2222} C^{1212}}}\ (^4).$$

Substituting (4.31) into $(4.30)_2$, we obtain:

$$\varphi(x^1) = -\sqrt{\frac{C^{1212}}{a^{2222}}}\,[e^{\beta x^1}(B\cos\beta x^1 - A\sin\beta x^1) + e^{-\beta x^1}(D\cos\beta x^1 - C\sin\beta x^1)] + \varphi_s(x^1) + Gx^1 + H. \tag{4.32}$$

The functions $\tilde{\gamma}_2$ and m_2 may be presented in the form, cf. [2]:

$$\tilde{\gamma}_2 = \frac{h^2 \tilde{c}^{12}\left(\dfrac{1}{r_0} - \tilde{c}^{12}\tilde{A}^{12}\right) + f_3 u''' + f_4\varphi''',}{\left(\dfrac{1}{r_0^2} - \tilde{c}^{12}\tilde{A}^{12}\right)^2},$$

$$m_2 = \frac{f_5\cdot\varphi''' + f_6 u'''}{\left(\dfrac{1}{r_0^2} - \tilde{c}^{12}\tilde{A}^{12}\right)}, \tag{4.33}$$

where

$$f_3 = -\underset{\sim}{C}^{1112}(\tilde{c}^{12})^2\tilde{A}^{12} - \frac{\underset{\sim}{a}^{1112}A^{12}}{r_0},$$

$$f_4 = \underset{\sim}{C}^{1112}(\tilde{c}^{12})^2\frac{1}{r_0} - \frac{\underset{\sim}{a}^{1112}A^{12}\tilde{c}^{12}}{r_0},$$

$$f_5 = -\underset{\sim}{a}^{1112}(\tilde{A}^{12})^2\tilde{c}^{12} - \frac{\underset{\sim}{C}^{1112}\tilde{c}^{12}}{r_0},$$

$$f_6 = \underset{\sim}{a}^{1112}(\tilde{A}^{12})^2\frac{1}{r_0} - \frac{C^{1112}\tilde{c}^{12}A^{12}}{r_0}.$$

As before, the final solution is given within accuracy of 12 constant coefficients. These may be computed from the relevant boundary conditions.

References

1. M. KLEIBER, *Statics of elastic lattice-type shells*, Arch. Mech., **25**, 2, 1973.
2. M. KLEIBER, *Statyka sprężystych powłok siatkowych* [Statics of elastic lattice-type shells]. Doctoral thesis, Technical University, Warsaw 1971.
3. CZ. WOŹNIAK, *Discrete elastic Cosserat media*, Arch. Mech., **25**, 1, 1973.
4. M. KLEIBER, CZ. WOŹNIAK, *On equations of the linear theory of elastic lattice-type shells*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 3, 1971.

(4) This solution, in the case of the isotropic lattice, is identical with that of isotropic cylindrical shell given in [10].

5. M. KLEIBER, CZ. WOŹNIAK, *The equations of shallow lattice shells*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 4, 1971.
6. M. KLEIBER, CZ. WOŹNIAK, *The edge effect in the theory of lattice shells*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 4, 1971.
7. CH. JORDAN, *Calculus of finite differences*, Chelsea Publ. Comp., New York 1950.
8. CZ. WOŹNIAK, *Siatkowe dźwigary powierzchniowe* [Lattice-type plates and shells], PWN, Warszawa 1970.
9. S. KONIECZNY, F. PIETRAS, CZ. WOŹNIAK, *O liniowych zagadnieniach dyskretnej teorii sprężystości* [On linear problems in discrete theory of elasticity], Rozpr. Inż., **20**, 2, 1972.
10. S. TIMOSHENKO, S. WOINOWSKI-KRIEGER, *Teoria płyt i powłok* [Theory of plates and shells], Arkady, 1962.

INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
POLISH ACADEMY OF SCIENCES.

Received November 30, 1971.