Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 213–225, Warszawa 1973

Theory of vibratory bending of unsymmetrical sandwich plates

Y.V.K.S. RAO and B.C. NAKRA (NEW DELHI)

THE BASIC equations of vibratory bending of unsymmetrical sandwich plates are developed by means of variational methods. Taken into consideration are the effects of flexural and membrane energies in the faces, transverse shear in the core, and rotatory, translatory and transverse inertias in both core and faces. The frequency equation obtained for simply supported edge conditions gives 5 families of modes which are designated according to displacement ratios. Values of frequency parameters are plotted for varying non-dimensional parameters of the sandwich plate.

Opierając się na metodach wariacyjnych wyprowadzono podstawowe równania zginania wibracyjnego niesymetrycznych płyt sandwiczowych. Uwzględniono efekty energii zginania i efekty membranowe w warstwach, poprzeczne ścinanie w rdzeniu oraz obrotowe translacje i poprzeczne efekty inercji zarówno w warstwach jak i w rdzeniu. Równanie częstości otrzymane dla przypadku zamocowanego brzegu daje pięć rodzin postaci drgań, które są przypisane odpowiednim współczynnikom przemieszczenia. Wartości parametrów częstości wykreślono w funkcji różnych zmiennych, bezwymiarowych parametrów płyty sandwiczowej.

На основе вариационных методов выведены основные уравнения теории вибрационного изгиба несимметричных пластинок типа сэндвич. При этом учитываются такие эффекты, как влияние изгибной и мембранной энергий в несущих слоях поперечный сдвиг в сердцевине, а также вращательная, трансляционная и поперечная инерции, как в сердцевине, так и в несущих слоях. Уравнение частот, выведенное для свободно опертой пластины, дает пять семейств мод, которые связываются с соответствующими коэффициентами перемещений. Даны графики значений частотных параметров в функции безразмерных геометрических параметров пластины.

1. Introduction

FLEXURAL vibrations of sandwich plates have been investigated by several authors [1–6]. The work reported by YU [4, 5] and KOVARIK and SLAPAK [2] is applicable to symmetrical sandwich structures. The analysis as reported in [1, 2, 3] is valid at low frequencies, since only transverse inertia effects have been included and rotary and translatory effects have been ignored. Though these effects have been included by CHANG and FANG [6], their analysis treats faces as membranes only. In the present work, the equations for flexural vibrations of unsymmetrical sandwich plates have been derived without any such restriction. It may be noted from a recent survey [7] of work on sandwich structures that work on effects of translatory and rotary inertia has received less attention. These effects are found to be of importance only at very high frequencies for homogeneous beams and plates [8]. RAO and NAKRA [9] have shown that these are of considerable importance at relatively lower frequencies in case of sandwich beams.

In the present investigation, the influence of all these inertia effects are included for the case of a sandwich plate with unsymmetrical faces and the frequencies corresponding to various families of modes are determined for varying values of non-dimensional parameters.

2. Equations of motion

The plate configuration is shown in Fig. 1. The assumptions made in the foregoing analysis are:

1. Plane transverse to the middle plane before bending, remains plane and perpendicular to the middle plane after bending;

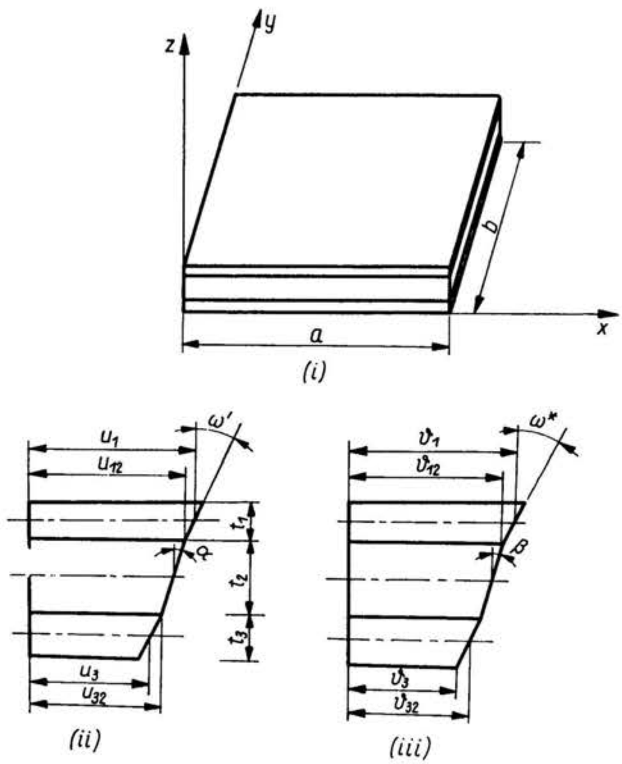

FIG. 1. Geometry of sandwich plate

i) plate configuration plate, ii) variation of displacement u, iii) variation of displacement v.

2. Transverse displacement at a section does not vary along thickness;

3. The longitudinal displacements u and v at a transverse section are assumed to vary as shown in Fig. 1;

4. All displacements are small;

5. There is perfect continuity at the interfaces and no slip occurs there while the plate is bending;

6. Extension effect in the core is ignored and stresses σ_x, σ_y and τ_{xy} are considered negligible in the core.

Noting that subscript 1 denotes the upper face, 2 the core and 3 — the lower face, we obtain:

$$
(2.1)\qquad \begin{aligned} u_{12} &= u_1 - w'\frac{t_1}{2}, \quad & u_{32} &= u_3 + w'\frac{t_3}{2},\\ v_{12} &= v_1 - w^*\frac{t_1}{2}, \quad & v_{32} &= v_3 + w^*\frac{t_3}{2}, \end{aligned}
$$

where w is the transverse displacement, $w' = \partial w/\partial x$ and $w^* = \partial w/\partial y$.

The distortion angles in the core are determined by the equation

$$
(2.2)\qquad \begin{aligned} &\gamma_{xz} = \alpha - \frac{\partial w}{dx}, \quad \gamma_{yz} = \beta - \frac{\partial w}{dy}, \quad \text{or}\\ &\gamma_{xz} = \frac{u_1 - u_3}{t_2} - \frac{w'c}{t_2}, \quad \gamma_{yz} = \frac{v_1 - v_3}{t_2} - w^*\frac{c}{t_2}, \end{aligned}
$$

where

$$
c = t_2 + \frac{t_1 + t_3}{2}.
$$

For the stresses, we have

$$
(2.3)\qquad \tau_{xz} = G_2\gamma_{xz}, \quad \tau_{yz} = G_2\gamma_{yz},
$$

where G_2 is the shear modulus of the material of the core. Let z be the distance measured along the $0z$-axis from the middle surface of each face. Then the tangential displacements in face 1 and face 3 are

$$
(2.4)\qquad \begin{aligned} &u = u_1 - zw' \quad \text{and} \quad v = v_1 - zw^* \quad \text{in face 1} \quad \text{and}\\ &u = u_3 - zw' \quad \text{and} \quad v = v_3 - zw^* \quad \text{in face 3}. \end{aligned}
$$

The strain components in the faces are given by

$$
(2.5)\qquad \begin{aligned} &\varepsilon_{xx} = u_1' - zw'', \quad \varepsilon_{yy} = v_1^* - zw^{**}, \quad \gamma_{xy} = u_1^* + v_1' - 2zw'^*, \quad \text{in face 1 and}\\ &\varepsilon_{xx} = u_3' - zw'', \quad \varepsilon_{yy} = v_3^* - zw^{**} \quad \gamma_{xy} = u_3^* + v_3' - 2zw'^*, \quad \text{in face 3,} \end{aligned}
$$

where $'$ denotes differentiation with respect to x and $*$ with respect to y.

The corresponding stress components in face 1 are

$$
(2.6)\qquad \begin{aligned} \sigma_{xx} &= \frac{E_1}{1-\nu_1^2}[u_1' + \nu_1 v_1^* - z(w'' + \nu_1 w^{**})],\\ \sigma_{yy} &= \frac{E_1}{1-\nu_1^2}[v_1^* + \nu_1 u_1' - z(w^{**} + \nu_1 w'')],\\ \tau_{xy} &= G_1(u_1^* + \nu_1' - 2zw'^*), \quad \text{and those in face 3 are}\\ \sigma_{xx} &= \frac{E_3}{1-\nu_3^2}[u_3' + \nu_3 v_3^* - z(w'' + \nu_3 w^{**})],\\ \sigma_{yy} &= \frac{E_3}{1-\nu_3^2}[v_3^* + \nu_3 u_3' - z(w^{**} + v_3 w'')],\\ \tau_{xy} &= G_3\,(u_3^* + v_3' - 2zw'^*), \end{aligned}
$$

where G_1 and G_3 are the shear moduli, ν_1 and ν_3 the Poisson's ratios, and E_1 and E_3 are the elastic moduli of the faces 1 and 3. The strain energy of the sandwich plate is U, where

$$2U = \iiint (\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+\tau_{xy}\gamma_{xy}+\tau_{xz}\gamma_{xz}+\tau_{yz}\gamma_{yz})\,dx\,dy\,dz. \tag{2.7}$$

Substituting (2.2), (2.3), (2.5) and (2.6) in (2.7) and performing the integration over the thickness of each face and core, we obtain

$$\begin{aligned} U = \iint \Bigg[& \frac{E_1 t_1}{2(1-\nu_1^2)}\left\{u_1'^2+\nu_1 u_1' v_1^* + v_1^{*2}+\nu_1 v_1^* u_1' + \frac{(1-\nu_1)}{2}(u_1^{*2}+v_1'^2+2u_1^* v_1')\right\} \\ & + \frac{E_3 t_3}{2(1-\nu_3^2)}\left\{u_3'^2+\nu_3 u_3' v_3^* + v_3^{*2}+\nu_3 v_3^* u_3' + \frac{(1-\nu_3)}{2}(u_3^{*2}+v_3'^2+2u_3^* v_3')\right\} \\ & + \frac{E_1 t_1^3}{24(1-\nu_1^2)}\{w''^2+2v_1 w'' w^{**}+w^{**2}+2(1-\nu_1)w'^{*2}\} \\ & + \frac{E_3 t_3^3}{24(1-\nu_3^2)}\{w''^2+2\nu_3 w'' w^{**}+w^{**2}+2(1-\nu_3)w'^{*2}\} + \frac{G_2 t_2}{2}\left\{\left(\frac{u_1-u_3}{t_2}\right)^2+\left(\frac{v_1-v_3}{t_2}\right)^2\right. \\ & \left. + (w'^2+w^{*2})\left(\frac{C}{t_2}\right)^2 - 2\frac{C}{t_2}\left(w'\frac{u_1-u_3}{t_2}+w^*\frac{v_1-v_3}{t_2}\right)\right\}\Bigg]dx\,dy. \end{aligned} \tag{2.8}$$

Assuming that the plate is loaded with a normal load of intensity q, the potential energy of it is given by

$$V = -\iint qw\,dx\,dy. \tag{2.9}$$

Now, the kinetic energy of the plate is T, where

$$\begin{aligned} 2T = (\varrho_1 t_1+\varrho_2 t_2+\varrho_3 t_3)\iint \dot{w}^2 dx\,dy + \iiint \varrho_i(\dot{u}_i+\dot{\bar{u}}_i z_i)^2 dx\,dy\,dz_i \\ + \iiint \varrho_i(\dot{v}_i+\dot{\bar{v}}_i z_i)^2 dx\,dy\,dz_i, \end{aligned} \tag{2.10}$$

where $\bar{u}_i$ and $\bar{v}_i$ are the rotations about middle, $i = 1, 2, 3$ denoting the layers 1, 2 and 3 and the dot represents differentiation w.r.t. t, the time variable.

The rotations in the layers 1 and 3 are

$$\bar{u}_1 = \bar{u}_3 = w' \quad \text{and} \quad \bar{v}_1 = \bar{v}_3 = w^*. \tag{2.11}$$

The displacement components in the core are

$$u_2 = \frac{u_1+u_3}{2}+w'\varepsilon_1, \quad v_2 = \frac{v_1+v_3}{2}+w^*\varepsilon_1, \tag{2.12}$$

where $\varepsilon_1 = (t_3-t_1)/4$ and the rotations in the core

$$\bar{u}_2 = \alpha = \frac{u_1-u_3}{t_2}-\frac{w'}{t_2}\varepsilon_2, \quad \bar{v}_2 = \beta = \frac{v_1-v_3}{t_2}-\frac{w^*}{t_2}\varepsilon_2, \tag{2.13}$$

where $\varepsilon_2 = (t_1+t_3)/2$.

Substituting (2.11), (2.12) and (2.13) in (2.10), we have

$$(2.14)\quad T=\frac{\varrho}{2}\iint \dot{w}^2\,dx\,dy+\frac{1}{2}\iint\Big[\varrho_1 t_1\dot{u}_1^2+\varrho_3 t_3\dot{u}_3^2+\dot{w}'^2\frac{\varrho_1 t_1^3+\varrho_3 t_3^3}{12}+\varrho_1 t_1\dot{v}_1^2$$
$$+\varrho_3 t_3\dot{v}_3^2+\dot{w}^{*2}\frac{\varrho_1 t_1^3+\varrho_3 t_3^3}{12}+\varrho_2 t_2\left\{\left(\frac{\dot{u}_1+\dot{u}_3}{2}+\dot{w}'\varepsilon_1\right)^2+\left(\frac{\dot{v}_1+\dot{v}_3}{2}+\dot{w}^*\varepsilon_1\right)^2\right\}$$
$$+\frac{\varrho_2 t_2}{2}\{(\dot{u}_1-\dot{u}_3-\dot{w}'\varepsilon_2)^2+(\dot{v}_1-\dot{v}_3-\dot{w}^*\varepsilon_2)^2\}\,dx\,dy,$$

where $\varrho=\varrho_1 t_1+\varrho_2 t_2+\varrho_3 t_3$.

According to Hamilton's principle, the stationery value of $\overline{\Phi}$ is equivalent to the equilibrium problem, where

$$\delta\overline{\Phi}=\int_{t_1}^{t_2}(\delta t-\delta U-\delta V)\,dt=0,$$

where t_1 and t_2 are any two instants of time.

Performing the variation term by term, the following equations of motion are obtained for arbitrary virtual displacements:

$$\gamma_1\left\{u_1''+\frac{(1+\nu_1)}{2}v_1'^*+\frac{(1-\nu_1)}{2}u_1^{**}\right\}+\gamma_2\left\{\frac{c}{t_2^2}w'-\frac{(u_1-u_3)}{t_2^2}\right\}-\varrho_1 t_1\ddot{u}_1-\frac{\varrho_2 t_2}{3}\left\{\ddot{u}_1+\frac{\ddot{u}_3}{2}+\frac{(t_3-2t_1)}{4}\ddot{w}'\right\}=0,$$

$$\gamma_1\left\{v_1^{**}+\frac{(1+\nu_1)}{2}u_1'^*+\frac{(1-\nu_1)}{2}v_1''\right\}+\gamma_2\left\{\frac{c}{t_2^2}w^*-\frac{(v_1-v_3)}{t_2^2}\right\}-\varrho_1 t_1\ddot{v}_1-\frac{\varrho_2 t_2}{3}\left\{\ddot{v}_1+\frac{\ddot{v}_3}{2}+\frac{(t_3-2t_1)}{4}\ddot{w}^*\right\}=0,$$

$$\gamma_3\left\{u_3''+\frac{(1+\nu_3)}{2}v_3'^*+\frac{(1-\nu_3)}{2}u_3^{**}\right\}-\gamma_2\left\{\frac{c}{t_2^2}w'-\frac{(u_1-u_3)}{t_2^2}\right\}-\varrho_3 t_3\ddot{u}_3-\frac{\varrho_2 t_2}{3}\left\{\ddot{u}_3+\frac{\ddot{u}_1}{2}+\frac{(2t_3-t_1)}{4}\ddot{w}'\right\}=0,$$

$$(2.16)\quad \gamma_3\left\{v_3^{**}+\frac{(1+\nu_3)}{2}u_3'^*+\frac{(1-\nu_3)}{2}v_3''\right\}-\gamma_2\left\{\frac{c}{t_2^2}w^*-\frac{(v_1-v_3)}{t_2^2}\right\}-\varrho_3 t_3\ddot{v}_3-\frac{\varrho_2 t_2}{3}\left\{\ddot{v}_3+\frac{\ddot{v}_1}{2}+\frac{2t_3-t_1}{4}\ddot{w}^*\right\}=0,$$

$$(D_1+D_3)\nabla^4 w-\gamma_2\frac{c}{t_2}\left\{\frac{c}{t_2}(w''+w^{**})+\frac{(u_1'-u_3')}{t_2}+\frac{(v_1^*-v_3^*)}{t_2}\right\}-\frac{\varrho_1 t_1^3+\varrho_3 t_3^3}{12}+(\ddot{w}''+\ddot{w}^{**})$$
$$+\varrho\ddot{w}-\varrho_2 t_2\left\{\frac{t_3-2t_1}{12}(\ddot{u}_1'+\ddot{v}_1^*)+\frac{2t_3-t_1}{12}(\ddot{u}_3'+\ddot{v}_3^*)+\left(\varepsilon_1^2+\frac{\varepsilon_2^2}{12}\right)(\ddot{w}''+\ddot{w}^{**})\right\}-q=0,$$

where

$$\gamma_1 = \frac{E_1 t_1}{1-\nu_1^2}, \quad \gamma_3 = \frac{E_3 t_3}{1-\nu_3^2}, \quad \gamma_2 = G_2 t_2, \quad D_1 = \frac{E_1 t_1^3}{12(1-\nu_1^2)} \quad \text{and} \quad D_3 = \frac{E_3 t_3^3}{12(1-\nu_3^2)}.$$

The boundary conditions from the line integrals can be obtained as i) along $x = 0$ and $x = a$

$$\begin{aligned}
&\text{either} && \text{or} \\
&u_1 = 0, && \gamma_1(u_1' + \nu_1 v_1^*) = 0, \\
&v_1 = 0, && \frac{\gamma_1(1-\nu_1)}{2}(v_1' + u_1^*) = 0, \\
&u_3 = 0, && \gamma_3(u_3' + \nu_3 v_3^*) = 0, \\
&\nu_3 = 0, && \frac{\gamma_3(1-\nu_3)}{2}(v_3' + u_3^*) = 0, \\
&w = 0, && D_1\{w''' + (2-\nu_1)w'^{**}\} + D_3\{w''' + (2-\nu_3)w'^{**}\} \\
& && \quad - \gamma_2 \frac{c}{t_2}\left\{w\frac{c}{t_2} - \frac{(u_1-u_3)}{t_2}\right\} + \frac{\varrho_1 t_1^3 + \varrho_3 t_3^3}{12}\ddot{w}' \\
& && \quad + \varrho_2 t_2 \left\{\frac{(t_3-2t_1)}{12}\ddot{u}_1 + \frac{(2t_3-t_1)}{12}\ddot{u}_3 + \ddot{w}'\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{12}\right)\right\} = 0, \\
&w' = 0, && D_1(w'' + \nu_1 w^{**}) + D_3(w'' + \nu_3 w^{**}) = 0,
\end{aligned} \tag{2.17}$$

and ii) along $y = 0$ and $y = b$

$$\begin{aligned}
&\text{either} && \text{or} \\
&u_1 = 0, && \frac{\gamma_1(1-\nu_1)}{2}(u_1^* + \nu_1') = 0, \\
&v_1 = 0, && \gamma_1(v_1^* + \nu_1 u_1') = 0, \\
&u_3 = 0, && \frac{\gamma_3(1-\nu_3)}{2}(u_3^* + v_3') = 0, \\
&v_3 = 0, && \gamma_3(v_3^* + \nu_3 u_3') = 0, \\
&w = 0, && D_1\{w^{***} + (2-\nu_1)w''^{*}\} + D_3\{w^{***} + (2-\nu_3)w''^{*}\} \\
& && \quad - \gamma_2 \frac{c}{t_2}\left\{w^*\frac{c}{t_2} - \frac{(v_1-v_3)}{t_2}\right\} + \frac{\varrho_1 t_1^3 + \varrho_3 t_3^2}{12}\ddot{w}^* \\
& && \quad + \varrho_2 t_2 \left\{\frac{t_3-2t_1}{12}\ddot{v}_1 + \frac{2t_3-t_1}{12}\ddot{v}_3 + \ddot{w}^*\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{12}\right)\right\} = 0, \\
&w^* = 0, && D_1(w^{**} + \nu_1 w'') + D_3(w^{**} + \nu_3 w'') = 0
\end{aligned} \tag{2.18}$$

and for free edge, reaction at the corner

$$\{D_1(1-\nu_1) + D_3(1-\nu_3)\}w'^* = 0.$$

3. Solution for simply supported case

The boundary conditions when all the edges of the plate are simply supported from Eqs. (2.17) and (2.18) are

$$
\begin{array}{lll}
 & \text{along } x = 0 \text{ and } x = a & \text{along } y = 0 \text{ and } y = b \\
 & \text{i} \quad u_1' + \nu_1 v_1^* = 0, & \text{vii} \quad u_1^* + v_1' = 0, \\
 & \text{ii} \quad v_1' + u_1^* = 0, & \text{viii} \quad v_1^* + \nu_1 u_1' = 0, \\
 & \text{iii} \quad u_3' + \nu_3 v_3^* = 0, & \text{ix} \quad u_3^* + v_3' = 0, \\
(3.1) & \text{iv} \quad v_3' + u_3^* = 0, & \text{x} \quad v_3^* + \nu_3 u_3' = 0, \\
 & \text{v} \quad w = 0, & \text{xi} \quad w = 0, \\
 & \text{vi} \quad D_1(w'' + \nu_1 w^{**}) + D_3(w'' + \nu_3 w^{**}) = 0, & \text{xii} \quad D_1(w^{**} + \nu_1 w'') \\
 & & \qquad + D_3(w^{**} + \nu_3 w'') = 0.
\end{array}
$$

The conditions listed above excepting ii) and vii), which can be neglected as per the discussion in [10], are satisfied if the solution is assumed in the series form as:

$$
(3.2) \qquad
\begin{aligned}
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t, \\
u_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{1mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t, \\
u_3 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{3mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t, \\
v_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{1mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \omega t, \\
v_3 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{3mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \omega t, \text{ and} \\
q &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega t, \text{ taking}
\end{aligned}
$$

harmonic excitation of frequency ω. Substitution of (3.2) in Eqs. (2.16) yields a system of algebraic equations which upon non-dimensionalisation give rise to:

$$
(3.3) \qquad
\begin{aligned}
&(AA + \lambda AB) W_{mn} - (AC - \lambda AD) U_{1mn} + (AE + \lambda AF) U_{3mn} - V_{1mn} AG = 0, \\
&(XA + XB\lambda) W_{mn} - U_{1mn} AG - (AH - AD\lambda) V_{1mn} + (AE + \lambda AF) V_{3mn} = 0, \\
&(\lambda AK - AA) W_{mn} + (AE + \lambda AF) U_{1mn} - (AL - \lambda AM) U_{3mn} - ANV_{3mn} = 0, \\
&(\lambda XC - XA) W_{mn} - ANU_{3mn} + (AE + \lambda AF) V_{1mn} - (AP - AM\lambda) V_{3mn} = 0. \\
&(AQ - \lambda AR) W_{mn} + (AA - \lambda AB) U_{1mn} - (AA + \lambda AK) U_{3mn} + (XA - XB\lambda) V_{1mn} \\
&\qquad + (XA + XC\lambda) V_{3mn} = \overline{Q}_{mn},
\end{aligned}
$$

where

$$AA = \delta_{2.3}\left(1+\frac{1+\theta_{1.3}}{2\theta_{2.3}}\right), \quad AB = \frac{\gamma_{2.3}}{\gamma_{1.3}}\frac{\theta_{2.3}}{12}(1-2\theta_{1.3}),$$

$$AC = \frac{\alpha_{1.3}\theta_{1.3}}{1-\psi_{1.3}^2\nu_3^2}\left[m\beta+\frac{(1-\psi_{1.3}\nu_3)}{2}\gamma^2\beta\frac{n^2}{m}\right]+AE, \quad AD = \frac{\theta_{1.3}}{\beta m}+YA, \quad AE = \frac{\delta_{2.3}}{\theta_{2.3}\beta},$$

$$AF = \frac{YA}{2}, \quad YA = \frac{\gamma_{2.3}}{\gamma_{1.3}}\frac{\theta_{2.3}}{3\beta m}, \quad AG = \frac{\alpha_{1.3}\theta_{1.3}n\beta\gamma}{2(1-\psi_{1.3}\nu_3)}, \quad YB = \frac{n}{m}\gamma, \quad XA = AAYB,$$

$$XB = ABYB, \quad AH = \frac{\alpha_{1.3}\theta_{1.3}}{1-\psi_{1.3}^2\nu_3^2}\left[\frac{n^2}{m}\gamma^2\beta+\frac{(1-\psi_{1.3}\nu_3)}{2}\beta m\right]+AE, \quad \bar{Q}_{mn} = \frac{Q_{mn}}{E_3\left(\dfrac{m\pi}{a}\right)},$$

$$AK = \frac{\gamma_{2.3}}{\gamma_{1.3}}\frac{\theta_{2.3}}{12}(2-\theta_{1.3}), \quad AL = \frac{1}{1-\nu_3^2}\left[m\beta+\frac{1-\nu_3}{2}\gamma^2\beta\frac{n^2}{m}\right]+AE,$$

$$AM = \frac{1_1}{\gamma_{1.3}\beta m}+YA, \quad AN = \frac{n\beta\gamma}{2(1-\nu_3)}, \quad XC = AKYB,$$

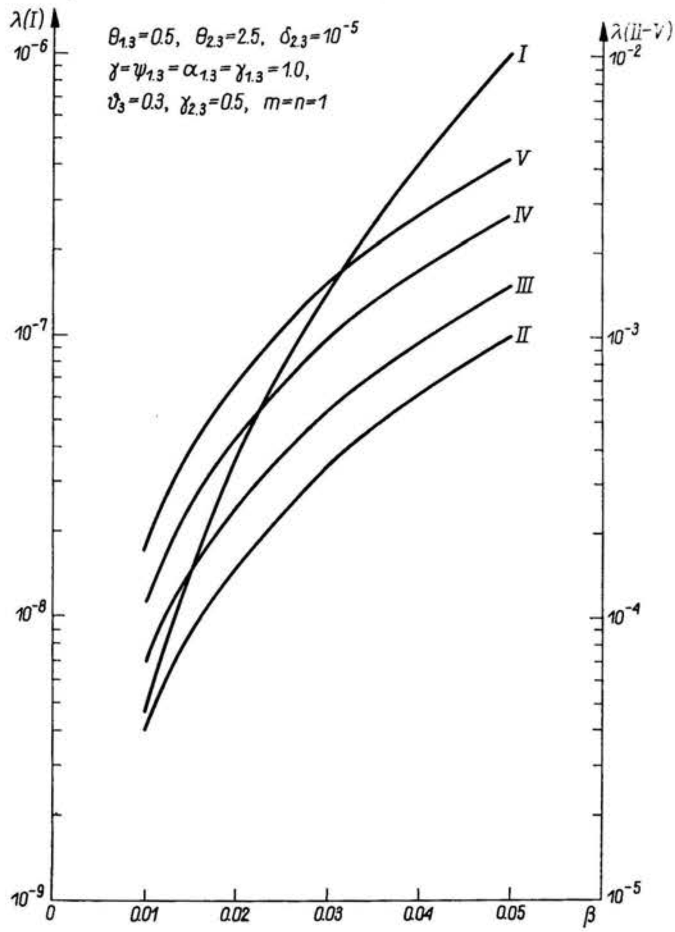

FIG. 2. Variation of λ with β.

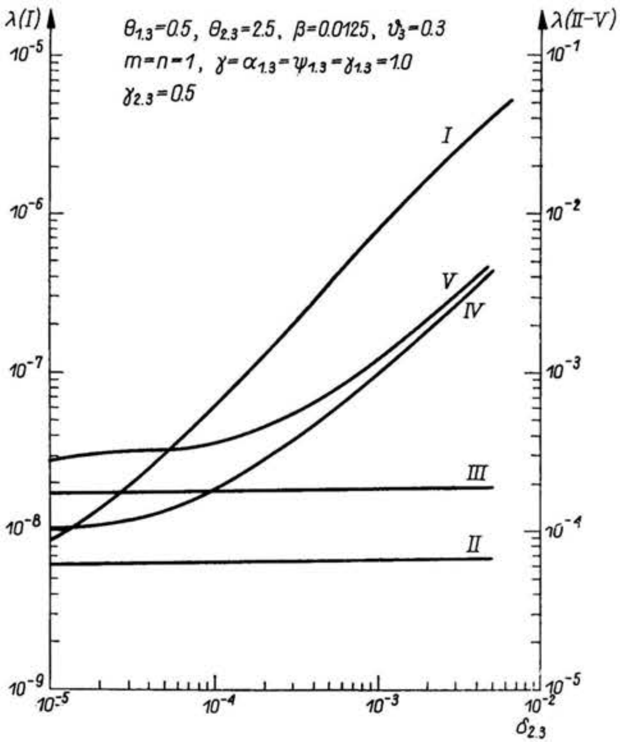

FIG. 3. Variation of λ with $\delta_{2,3}$.

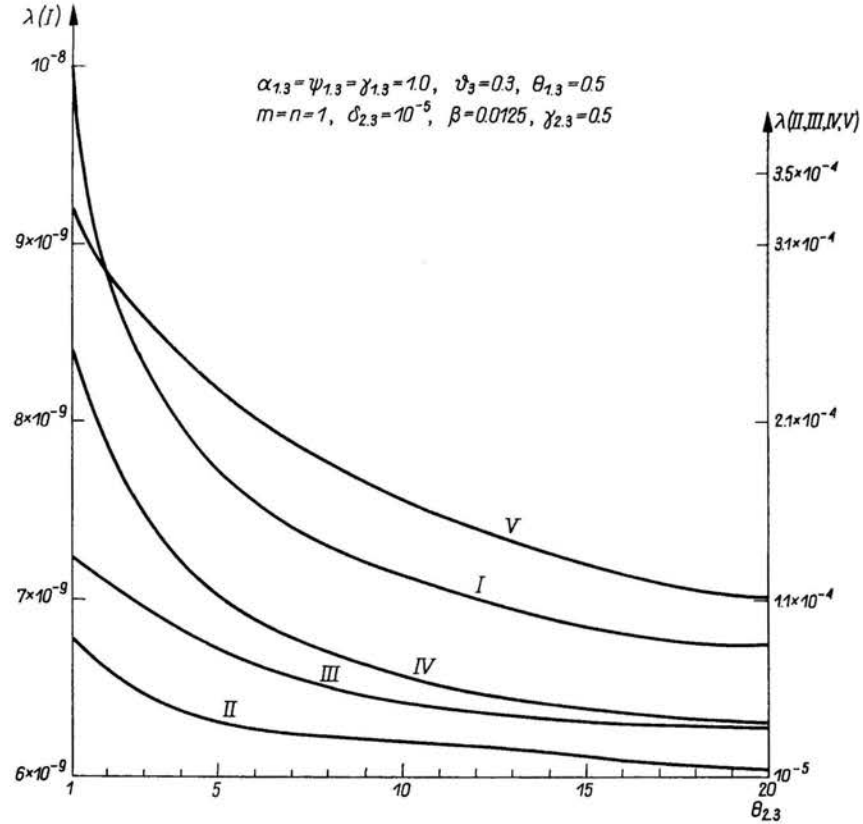

FIG. 4. Variation of λ with $\theta_{2,3}$.

$$AP = \frac{1}{(1-\nu_3^2)}\left[\frac{n^2}{m}\beta\gamma^2+\frac{(1-\nu_3)}{2}m\beta\right]+AE,$$

$$AQ = m^3\beta^3\left[\left\{\frac{\alpha_{1.3}\theta_{1.3}^3}{12(1-\psi_{1.3}^2\nu_3^2)}+\frac{1}{12(1-\nu_3^2)}\right\}\left\{1+2\left(\frac{n}{m}\right)^2+\left(\frac{n}{m}\right)^4\gamma^4\right\}\right.$$
$$\left.+\frac{1}{\beta^2}\left(\frac{1}{m^2}+\frac{n^2}{m^4}\gamma^2\right)\delta_{2.3}\theta_{2.3}\left(1+\frac{1+\theta_{1.3}}{2\theta_{2.3}}\right)^2\right],$$

$$AR = \frac{m^3\beta}{12}\left(\frac{1}{m^2}+\frac{n^2}{m^4}\gamma^2\right)\left\{\left(\theta_{1.3}^3+\frac{1}{\gamma_{1.3}}\right)+\frac{\theta_{2.3}\gamma_{2.3}}{\gamma_{1.3}}(1+\theta_{1.3}^2-\theta_{1.3})\right\}$$
$$+\frac{1}{\beta m}\left(\theta_{1.3}+\theta_{2.3}\frac{\gamma_{2.3}}{\gamma_{1.3}}+\frac{1}{\gamma_{1.3}}\right),$$

$$\alpha_{1.3}=\frac{E_1}{E_3},\quad \alpha_{2.3}=\frac{E_2}{E_3},\quad \delta_{2.3}=\frac{G_2}{E_3},\quad \theta_{1.3}=\frac{t_1}{t_3},\quad \theta_{2.3}=\frac{t_2}{t_3},$$

$$\beta=\frac{\pi t_3}{a},\quad \psi_{1.3}=\frac{\nu_1}{\nu_3},\quad \gamma=\frac{a}{b}.$$

For the case of the free vibrations, i.e. $Q_{mn}=0$, the system of Eqs. (3.3) reduces to a fifth order polynomial in λ. The roots of this polynomial have been computed on I.C.L. 1909 Computer for varying values of $\delta_{2.3}, \beta, \theta_{1.3}$ and $\theta_{2.3}$ and are drawn in Figs. 2–5.

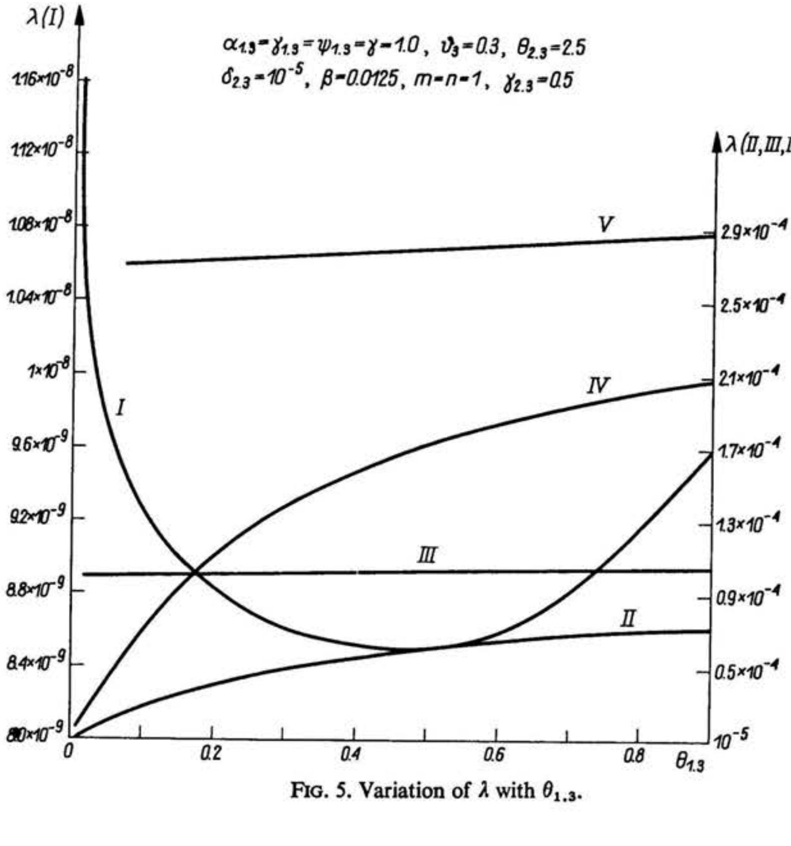

FIG. 5. Variation of λ with $\theta_{1.3}$.

4. Discussion

For a given combination of modal numbers m, n, 5 values of λ are obtained and thus 5 families of modes would exist. For each family of mode, the values of displacement ratios U_{1mn}/W_{mn}, U_{3mn}/W_{mn}, V_{1mn}/W_{mn} and V_{3mn}/W_{mn} can be computed from Eqs. (3.3) and the corresponding family of mode may be designated from the nature of these ratios.

In Table 1, values of frequencies corresponding to the five families of modes for various values of m and n are given, while in Table 2, the displacement ratios for the five families for $m = n = 1$ are given. It was observed that the mode corresponding to the lowest

Table 1. Frequencies of sandwich plates

$\alpha_{1,3} = \gamma = \gamma_{1,3} = \psi_{1,3} = 1.0, \quad \gamma_{2,3} = 0.5, \quad \delta_{2,3} = 0.00001, \quad \beta = 0.0125, \quad \theta_{1,3} = 0.7,$
$\nu_3 = 0.3, \quad \theta_{2,3} = 10$

Modal number		Non-dimensional frequency parameter λ for various families of modes					
m	n	$\mathrm{I}\times 10^8$		$\mathrm{II}\times 10^4$	$\mathrm{III}\times 10^4$	$\mathrm{IV}\times 10^4$	$\mathrm{V}\times 10^3$
		Transverse inertia terms	all inertia terms				
1	1	0.730	0.730	0.295	0.623	0.842	0.176
1	3	7.224	7.220	1.473	3.069	4.207	0.875
2	5	45.64	45.623	4.270	8.878	12.197	2.534
5	5	125.75	125.67	7.362	15.297	21.032	4.369
4	7	207.16	206.99	9.37	19.884	27.342	5.679
3	8	258.83	258.59	10.748	22.329	30.707	6.377
2	9	347.099	346.73	12.515	25.998	35.754	7.425
5	8	379.38	378.96	13.103	27.221	37.437	7.775
7	7	457.27	456.72	14.428	29.973	41.223	8.562
1	10	484.84	484.24	14.870	30.889	42.484	8.822

Table 2. Displacement ratios for various families of modes

$\alpha_{1,3} = \gamma_{1,3} = \gamma = \psi_{1,3} = 1.0, \quad \gamma_{2,3} = 0.5, \quad \theta_{1,3} = 0.7$
$\theta_{2,3} = 10, \quad \nu_3 = 0.3, \quad \delta_{2,3} = 0.00001, \quad \beta = 0.0125, \quad m = n = 1$

Number of mode family	$\frac{U_1}{W}$	$\frac{U_{3mn}}{W_{mn}}$	$\frac{V_{1mn}}{W_{mn}}$	$\frac{V_{3mn}}{W_{mn}}$
I	-0.159×10^{-2}	-0.407×10^{-3}	0.159×10^{-2}	-0.38×10^{-3}
II	-0.353×10^{7}	-0.212×10^{7}	0.353×10^{7}	0.212×10^{7}
III	0.438×10^{5}	-0.517×10^{5}	-0.438×10^{5}	0.517×10^{5}
IV	0.107×10^{4}	0.635×10^{3}	0.107×10^{4}	0.635×10^{3}
V	0.991×10^{3}	-0.117×10^{4}	0.991×10^{3}	-0.117×10^{4}

frequency is of predominantly flexural type, the corresponding values of all displacement ratios are small. This mode is also obtained when only transverse inertia effects are included. In Table 1, the frequencies for the first family of modes are given for both the cases, viz. when all inertia effects are included and when only transverse inertia effects are considered, the frequencies obtained in the two cases are not significantly different.

Because of inclusion of rotary and translatory inertia, in addition to transverse inertia, four additional families of modes are obtained. For families of modes designated as belonging to II and IV families, the displacement ratios U_{1mn}/W_{mn} and U_{3mn}/W_{mn} are of same sign and so are the ratios V_{1mn}/W_{mn} and V_{3mn}/W_{mn}. All these ratios are large and so these families of modes may be classified as of predominantly extensional types. For III and V families of modes, the ratios U_{1mn}/W_{mn} and U_{3mn}/W_{mn} are of opposite signs and so are V_{1mn}/W_{mn} and V_{3mn}/W_{mn}. These ratios are large and these families of modes may be classified as of predominantly thickness shear types. It was found that for the parameters listed in Table 1, frequencies for the first family, viz. flexural modes, for $m = 10$, $n = 12$ are $\lambda = 0.274\times10^{-4}$ and for $m = 10, n = 13$ are $\lambda = 0.332\times10^{-4}$, while the frequency for the second family, viz. extensional mode corresponding to $m = n = 1$ is 0.295×10^{-4}. Thus, beyond flexural mode corresponding to modal numbers $m = 10$, $n = 12$, the higher families of modes exist. To get an idea of the relative frequencies, the frequencies for various families of modes for $m = n = 1$ and for parameters given in Table 1 were computed, taking $t_3 = 1''/8$, $E_3 = 10^7$ lb/in^2 and $\varrho_3 = 0.000259$ lb in^{-4} sec^2. These frequencies are seen to be: 21.4, 1360, 1990, 2310 and 3320c.p.s. for the five families of modes, respectively. It may be noted that the frequencies for higher families of modes are not very high and are thus of practical interest.

In Fig. 2, the frequencies corresponding to various modes are plotted against β. The frequencies for all the five families of modes are seen to increase with increase of β. As seen from Fig. 3, a stiffer core would increase all the frequencies, though the increase would be very small for II and III families of modes. Increasing the thickness of the core, as seen from Fig. 4, lowers the frequencies for all the families of modes. This is due to the fact that increase in generalized stiffness due to increase of $\theta_{2.3}$ would be less marked compared to increase in generalized mass of the sandwich plate. As seen from Fig. 5, if one of the faces is considerably thicker compared to the other, i.e. for lower values of $\theta_{1.3}$, frequencies for higher families of modes are reduced, though for the flexural mode, the frequency decreases up to a particular value of $\theta_{1.3}$ and increases thereafter.

5. Conclusions

Equations of bending vibrations of unsymmetrical sandwich plates have been derived using variational methods. Frequency equation for simply supported edge conditions gives five families of modes, which have been designated according to the nature of displacement ratios. Inclusion of rotary and translatory inertia effects, in addition to transverse inertia, gives rise to higher families of modes which occur at frequencies of practical interest. These modes occur at reduced frequencies when $\theta_{2.3}$ is large, $\delta_{2.3}$ and β are small and the sandwich plate has a high degree of unsymmetry.

References

1. V. V. BOLOTIN, *Vibration of layered elastic plates*, Proc. Vibr. Probl., **4**, 4, 1963.
2. V. KOVARIK and P. SLAPAK, *Natural transverse vibrations of sandwich plates*, Acta Technica CSAV, **12**, 2, 187–219, 1967.
3. R. A. DITARANTO, and J. R. MCGRAW, Vibratory bending of damped laminated plates, J. Eng. Ind., 1081–1090, 1969.
4. Y. Y. YU, *A new theory of elastic sandwich plates. One-dimensional case*, J. Appl. Mech., **26**, 415–421, 1959.
5. Y. Y. YU, *Flexural vibrations of elastic sandwich plates*, J. Aero. Sci., **27**, 272–282, 1960.
6. C. C. CHANG and B. T. FANG, *Transient and periodic response of a loaded sandwich panel*, J. Aero. Sci., 382–396, 1961.
7. L. M. HABIB, *A survey of modern development in the analysis of sandwich structures*, Appl. Mech. Rev., **13**, 2, 1965.
8. R. D. MINDLIN, *Influence of rotary inertia and shear on flexural motions of isotropic elastic plates*, J. Appl. Mech., **18**, 31–38, 1951.
9. Y. V. K. S. RAO and B. C. NAKRA, *Influence of rotary and translatory inertia on the vibrations of unsymmetrical sandwich beams*. Paper presented at 15th Conference of Indian Society of Theoretical and Applied Mechanics held at Sindri, India, 1970.
10. N. J. HOFF, Bending and buckling of rectangular sandwich plates, N.A.C.A. Tech. Note, 2225, 1950.

DEPARTMENT OF MECHANICAL ENGINEERING,
INDIAN INSTITUTE OF TECHNOLOGY,
HAUZ KHAS, NEW DELHI-29, INDIA.

Received December 1, 1971.