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Unsteady multidimensional isentropic flows described by linear Riemann invariants

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THE OBJECT of the present paper consisted in investigation of effective possibilities of construction of flows by means of generalized Riemann invariants. We restricted our attention to the most simple situation, when these invariants are linear. The basic notions are given in Introduction. The problem of determination of linear Riemann invariants is reduced to investigation of integrability of the Pfaff's system. The necessary mathematical algorithm and theorems are given in Sec. 2. All kinds of possible solutions of maximal functional freedom and their classification are given in Sec. 3. Some interesting properties of these solutions are also discussed. Attention is payed to configurations of simple waves which interact by linear superposition. In the last Section, an example is given of the solutions without functional freedom. This Section also illustrates how mechanism of prolongation works for the Pfaff's system considered.

Celem pracy było zbadanie efektywnych możliwości konstruowania przepływów metodą uogólnionych inwariantów Riemanna. Ograniczono się do najprostszej sytuacji, gdy inwarianty te są liniowe. Podstawowe pojęcia podano we wstępie. Problem wyznaczania liniowych inwariantów Riemanna sprowadza się do badania całkowalności układów Pfaffa. Potrzebny aparat matematyczny przytoczono w rozdziale 2. W rozdziale 3 podano możliwe rozwiązania o maksymalnym stopniu swobody funkcyjnej, sklasyfikowano je i przedyskutowano ciekawe własności. Na uwagę zasługują konfiguracje fal prostych superponujących się liniowo. W ostatnim rozdziale podano przykład rozwiązań bez swobody funkcyjnej. Rozdział ten pokazuje także działanie mechanizmu przedłużania dla rozważanych układów Pfaffa.

В работе исследуются возможности эффективного построения течений газа методом обобщенных инвариантов Риманна. Мы ограничимся до наиболее простого случая, когда эти инварианты линейны. Основные понятия и определения даются во введении. Задача определения линейных инвариантов Риманна сводится к исследованию интегрируемости системы Пфаффа. Необходимый математический аппарат представлен в главе 2. Все возможные решения с максимальным функциональным произволом и их классификация приведены в главе 3. Рассматриваются также некоторые замечательные свойства этих решений, между прочем были обнаружены конфигурации простых волн взаимодействующих по принципу линейной суперпозиции. В последней главе приводится пример решения без функционального произвола. Эта глава также иллюструет как "работает механизм" продолжения для рассматриваемых Пфаффовых систем.

1. Generalized Riemann invariants

1.1. Nonelliptic systems of partial differential equations

IN THIS PAPER, we shall consider quasi-linear systems of differential equations of the first order of the form

$$(1.1)\qquad A^i_{\nu\alpha}(x, u)\frac{\partial u^\alpha}{\partial x^i} = 0, \qquad \begin{aligned} i &= 1, \ldots, n,\\ \alpha &= 1, \ldots, m,\\ \nu &= 1, \ldots, l, \end{aligned}$$

where $x = (x^1, \ldots, x^n) \in E$, $u = (u^1, \ldots, u^m) \in H$. In the formula (1.1) and subsequent formulas we use, unless otherwise stated, the summation convention over repeated upper and lower indices. The Euclidean spaces E and H will be called the physical and the hodograph space, respectively. The solution $u = u(x)$ of (1.1), defined in a region $D \subset E$ and with the values in a region $\Omega \subset H$, may be interpreted as a transformation of D into Ω; in particular, the region called the hodograph of the solution Ω may degenerate into a hypersurface or a line in H. We shall call Ω the hodograph of the solution. Transformation du tangent to a solution of (1.1), given by the $n \times m$ matrix $du = \left\| \frac{\partial u^\alpha}{\partial x^i} \right\|$, maps the space E' tangent to E at the point x into H' tangent to H at the point $u = u(x)$. Conversely, any C^1- regular transformation of $D \subset E$ in $\Omega \subset H$ will be a solution of (1.1) if its tangent map $d\varphi$ satisfies (1.1) for $(x, u(x)) \in D \times \Omega$.

Such a point of view makes it possible to give an algebraic characterisation of the geometrical properties of the solutions in terms of du. For instance, if (1.1) has the solutions of the simple wave type

$$u^\alpha = u^\alpha(f(x^1, \ldots, x^n)), \quad \alpha = 1, \ldots, n, \tag{1.2}$$

then

$$\frac{\partial u^\alpha}{\partial x^i} = \dot{u}^\alpha \frac{\partial f}{\partial x^i}, \quad \dot{u}^\alpha = \frac{du^\alpha}{df}, \tag{1.3}$$

and we see that in this case du is factorised (split) and that the rank $\left\| \frac{\partial u^\alpha}{\partial x^i} \right\| = 1$. This kind of splitting of du for a solution $u = u(x)$ may occur either in a region of E or at separate points. Therefore it is convenient to introduce the following notion:

DEFINITION 1. A triplet (x_0, u_0, P), with the matrix $P: E'(x_0) \to H'(u_0)$ is said to be an integral element (solution "at the point") of (1.1), if

$$A^i_{\nu\alpha}(x_0, u_0) P^\alpha_i = 0, \quad \nu = 1, \ldots, l. \tag{1.4}$$

For a given solution $u = u(x)$, with $x \in D$, all triplets $x, u(x), du|_x$ define in D a distribution of integral elements of (1.1) over the space E. In more modern language, such distribution is a section of a bundle of jets.

Determination of all integral elements for a system of the form (1.1) is an elementary problem of linear algebra. The algebraic solutions will depend also on the coordinates x and u of the product space $E \times H$. Thus we shall be concerned not with jets but rather with distributions of integral elements over the space $E \times H$; namely

$$p = p(x, u),$$

for which the conditions of integrability are given by the Frobenius theorem

$$p^\alpha_{[i} p^\beta_{j]}, u + p^\beta_{[i,j]} = 0.$$

This, however, is not the general situation, since the algebraic problem posed may introduce new arbitrary parameters, say η and the distributions of integral elements will be of the form

$$p = p(x, u, \eta), \quad \text{where} \quad \eta = (\eta^1, \ldots, \eta^k).$$

The problem of integrability of such distributions of integral elements is more complicated and needs a recourse to Cartan's algorithm outlined in Sec. 2.

Referring now to the simple waves, we may say that an integral element $(x, u, P(x, u))$ is simple if the matrix P may be split into a tensor product of two non-vanishing vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in E$ and $\boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^m) \in H$. On substitution of $p_i^\alpha = \lambda_i \gamma^\alpha$ into the Eqs. (1.1), we obtain:

$$A^i_{\nu\alpha}(x, u)\lambda_i \gamma^\alpha = 0, \quad i = 1, \ldots, n. \tag{1.5}$$

Thus, in order to have non-vanishing $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$, we must require

$$\operatorname{rank}||A^i_{\nu\alpha}(x, u)\lambda_i|| < m = \dim H \quad \text{or} \quad \operatorname{rank}||A^i_{\nu\alpha}(x, u)\gamma^\alpha|| < n = \dim E. \tag{1.6}$$

In particular, for $n = m$, the first of these conditions takes the known form

$$\det||A^i_{\nu\alpha}\lambda_i|| = 0, \quad \alpha, \nu = 1, \ldots, m, \tag{1.7}$$

which expresses the fact that the vector $\boldsymbol{\lambda}$ is normal to a characteristic surface of (1.1) in the space E. This fact implies that for hyperbolic systems (1.1) simple integral elements always exist. The systems (1.1) which have simple elements are called *non-elliptic systems* and the corresponding non-vanishing real valued vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ are called *characteristic vectors* in the spaces E and H, respectively.

The idea of relating the algebraic properties of (x, u, P) with the study of the geometrical properties of certain solutions of the systems (1.1) is due to M. BURNAT (cf. [1 and 2]; also further references). In particular, by means of such algebraic-geometrical considerations, also the concept of Riemann invariants has been generalized in [1] to multidimensional systems with $n > 2$. A classification of the systems (1.1) in terms of algebraic properties of (x, u, P) is proposed in [3]. In particular, such classification enables a better understanding of the physical processes described by differential systems.

1.2. More general integral elements

The next natural step after introducing simple integral elements for (1.1), is to consider a linear combination of such elements, taking into account that (1.1) is a homogeneous system. We shall say that two vectors $\boldsymbol{\lambda} \in E$ and $\boldsymbol{\gamma} \in H$ taken at the point $(x, u) \in E \times H$ are knotted, and write $\boldsymbol{\lambda} \rightleftharpoons \boldsymbol{\gamma}$ if $\boldsymbol{\lambda} \neq 0$, $\boldsymbol{\gamma} \neq 0$ and (1.5) holds true. Evidently, from $\boldsymbol{\lambda} \rightleftharpoons \boldsymbol{\gamma}$ it follows that both vectors are characteristic ones. For a given $\boldsymbol{\gamma}$ vector, there may correspond a linear subspace $\Lambda(\boldsymbol{\gamma})$ of knotted vectors $\boldsymbol{\lambda}$. In fact, if for the vector $\boldsymbol{\gamma} \rightleftharpoons \boldsymbol{\lambda}^1$, $\boldsymbol{\gamma} \rightleftharpoons \boldsymbol{\lambda}^2$, then an arbitrary combination of $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ is also knotted with $\boldsymbol{\gamma}$. From a more general point of view, $\boldsymbol{\lambda}$ vectors are elements of the dual space E^*; thus they may be regarded as covectors and, therefore, they may be identified, as in Sec. 1.3, with differential forms.

If we have a finite sequence of pairwise knotted vectors $\boldsymbol{\lambda}^a \rightleftharpoons \boldsymbol{\gamma}_a$, $a = 1, \ldots, k$, then an element with the matrix $P = ||p_i^\alpha||$ in which

$$p_i^\alpha = \eta^1 \gamma_1^\alpha \lambda_i^1 + \eta^2 \gamma_2^\alpha \lambda_i^2 + \ldots + \eta^k \gamma_k^\alpha \lambda_i^k \tag{1.8}$$

and $\eta^1, \ldots, \eta^k$ are arbitrary functions of (x, u) is obviously an integral element.

From here on, we shall consider such a system (1.1), *and only such a system, in which the coefficients* $A^i_{\nu\alpha}$ *are independent of* x. For the sake of simplicity, we also assume that $l = m = n$. Thus we have

$$A^i_{\nu\alpha}(u)\frac{\partial u^\alpha}{\partial x^i} = 0, \quad \nu, i, \alpha = 1, \dots, n. \tag{1.9}$$

The conditions (1.6) defining characteristic vectors in E and H, respectively, take the form

$$\det\|A^i_{\nu\alpha}(u)\lambda_i\| = 0, \quad \det\|A^i_{\nu\alpha}(u)\gamma^\alpha\| = 0. \tag{1.10}$$

Now, we can introduce the following class of the solutions $u = u(x)$ of (1.1) which have the property that their derivatives may be decomposed in the form

$$\frac{\partial u^\alpha}{\partial x^i} = \eta^1\gamma^\alpha_1\lambda^1_i + \dots + \eta^k\gamma^\alpha_k\lambda^k_i, \tag{1.11}$$

where we shall assume that $\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^k$ are linearly independent and also $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_k$ knotted to them are linearly independent. Integral elements of this kind are by BURNAT called, for their interesting geometrical properties, free integral elements. In particular, it is shown in [2] that the corresponding free solutions may be obtained by integration of an overdeterminate (in general) linear system of partial differential equations, which results from the change of the roles of dependent and independent variables.

The solutions corresponding to (1.11) are called in [6] k-waves, since they may be interpreted as an interaction of k simple waves. These simple waves correspond to simple elements which are involved in (1.11). A generalization in which $\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^k$ may be dependent is considered in [7]. In this paper, we shall deal with the situations both of which lead to the notion of Riemann invariants. We use the definition given in [6, 7].

1.3. Generalized Riemann invariants

DEFINITION 2. We say that a solution of the k-waves type is constructed by means of Riemann invariants if it may be represented in the form

$$u^\alpha = u^\alpha(R^1, \dots, R^k), \quad \alpha = 1, \dots, n \tag{1.12}$$

in which $R^a = R^a(x^1, \dots, x^n)$. Moreover, the vectors $\boldsymbol{\gamma}_a$ given by the formulae

$$\boldsymbol{\gamma}_a = \left(\frac{\partial u^1}{\partial R^a}, \dots, \frac{\partial u^n}{\partial R^a}\right) \tag{1.13}$$

are characteristic vectors in H space, and $\operatorname{grad} R^a$ belongs to the subspace knotted with $\boldsymbol{\gamma}_a$. Thus

$$\left(\frac{\partial R^a}{\partial x^1}, \dots, \frac{\partial R^a}{\partial x^n}\right) \in \Lambda^a = \Lambda(\boldsymbol{\gamma}_a) \tag{1.14}$$

at the point $u(R^1(x), \dots, R^k(x))$. Thus the functions $u^\alpha = u^\alpha(R^1, \dots, R^k)$ describing the hodograph should be the solutions of the equations:

$$\det\left\|A^i_{\nu\alpha}(u(R))\frac{\partial u^\alpha}{\partial R^a}\right\| = 0, \quad \begin{array}{l} i = 1, \dots, n, \\ a = 1, \dots, k, \end{array} \tag{1.15}$$

and the parameters $R^1, \ldots, R^k$ on the hodograph surface — hereinafter called generalized Riemann invariants (for reasons explained in [1, 2]) — must be the solutions of Pfaff's systems:

$$\theta^a \equiv dR^a - \eta_1^a \lambda_1^a - \ldots - \eta_{p(a)}^a \lambda_{p(a)}^a = 0, \quad a = 1, \ldots, k, \tag{1.16}$$

where $p(a)$ is the dimension of the space Λ^a and where the differential forms λ_q^a are given by

$$\underset{q}{\lambda^a} = \underset{q}{\lambda_i^a} dx^i, \quad a = 1, \ldots, k, \tag{1.17}$$

$q = 1, \ldots, p(a)$ span the space Λ^a.

In Sec. 2.4, we shall impose on these Pfaff forms what are called conditions of involution. Confining ourselves to certain subspaces of Λ^a, we shall under this condition obtain different types of Riemann invariants. We shall preserve the notation as in (1.16) understanding by $p(a)$ the dimension of subspace of Λ^a considered. It is proved in [7] that the corresponding solution will depend on one function of $p(1)$ arguments, one function of $p(2)$ arguments, ... and one function of $p(k)$ arguments, provided that the conditions of involution are satisfied (k denotes number of $\boldsymbol{\gamma}$ vectors taken into account). The corresponding coordinates system $R^1, \ldots, R^k$ in the hodograph manifold will be called the system of Riemann invariants with $\big(p(1), \ldots, p(k)\big)$ degree of freedom.

Let us observe that our definition of Riemann invariants in which we emphasize integrability conditions (condition of involution) represents a generalization of definitions adopted in [1 and 2] and it appears to be more natural.

1.4. Linear Riemann invariants

The choice of such distributions of characteristic vectors $\boldsymbol{\gamma}_a \rightleftharpoons \boldsymbol{\lambda}^a$ over H and free parameters η^r, for which there exist the functions $u^a = u^a(R^1, \ldots, R^k)$ and $R^a = R^a(x^1, \ldots, \ldots, x^n)$ satisfying (1.13) and (1.16), respectively, contains two integrability problems. Without dwelling on any general discussion which is given in [6 and 7], let us observe that for many differential systems encountered in mathematical physics, the condition $(1.10)_2$ defining characteristic vectors $\boldsymbol{\gamma}_a$ in the hodograph space H does not involve an explicit dependence on the hodograph variables $u^1, \ldots, u^n$. In such cases, it seems reasonable to ask whether there exist Riemann invariants linear with respect to these variables $u^1, \ldots, u^n$, since we may attempt to satisfy $(1.10)_2$ by a configuration of $\boldsymbol{\gamma}$ vectors constant in the space H. The hodograph surface will be reduced then to a k-dimensional hyperplane in H.

This approach eliminates the problem of integration of the Eqs. (1.15) and therefore the existence of linear Riemann invariants and the corresponding solutions $u = u(x)$ will be studied by investigation of the integrability of the Eqs. (1.16).

Another reason which makes linear Riemann invariants of interest is based on physical considerations. In fact, the Eqs. (1.12) and (1.16) may be interpreted as a rule of interaction of simple waves, which in their turn have an immediate mechanical meaning.

Although discussion of interactions may be pursued in the general case of non-linear Riemann invariants, it is obvious that certain facts are more easily observed in the case of solutions constructed by means of linear Riemann invariants. In particular, certain curious features of interaction, as observed in the subsequent Sections, prove that our restriction to linear Riemann invariants was reasonable.

2. Existence of Riemann invariants as an involutivity problem for Pfaff systems

2.1. Cartan's algorithm of investigation of involutivity of Pfaff system

Let us pay attention to two essential points of our problem of integration of the systems (1.16).

The system (1.16) involves free parameters $\eta^1, \ldots, \eta^k$, the differentials of which do not enter into the system under consideration.

We are interested only in such n-dimensional solutions as may be described in the form $R^a = R^a(x^1, \ldots, x^n)$, where $x^1, \ldots, x^n$ are independent variables.

The theory of such integrations problems with free parameters and prescribed independent variables was created by E. CARTAN in 1904 [4, 5].

Algebraically, the problem may be elucidated by closing the system

(2.1) $$\theta^a(x, R, \eta) \equiv dR^a - G_i^a(x, R, \eta)dx^i = 0, \quad a = 1, \ldots, \sigma$$

by the requirement that exterior derivatives of the forms θ^a should vanish on the solution of (2.1) ([1])

(2.2) $$d\theta^a = 0 \quad (\mathrm{mod}\ \theta^a = 0),$$

where $d\theta^a$ may be written in the form

(2.3) $$d\theta^a \equiv \alpha_{ij}^a dx^i \wedge dx^j + \beta_{\mu j}^a \pi^\mu \wedge dx^j \ (^2)\ (\mathrm{mod}\theta^a = 0),$$

where π^μ — ($\mu = 1, \ldots, q$) are new forms (usually differentials of free parameters).

Let us define the Cartan number

(2.4) $$Q = \mathfrak{s}_1 + 2\mathfrak{s}_2 + \ldots + n\mathfrak{s}_n,$$

where what are called the reduced characters $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ are given by the formula

(2.5) $$\mathfrak{s}_1 + \mathfrak{s}_2 + \ldots + \mathfrak{s}_p = \mathrm{rank}\begin{bmatrix} \beta_{\mu j}^\alpha \xi_1^j \\ \beta_{\mu j}^\alpha \xi_2^j \\ \cdots \\ \beta_{\mu j}^\alpha \xi_p^j \end{bmatrix}, \quad 1 \leqslant p \leqslant n-1,$$

where the rank is calculated for general values of the variables $\xi_1^1, \ldots, \xi_1^n, \ldots, \xi_p^1, \ldots, \xi_p^n$, which means that there exists such a neighbourhood U of the point $\xi_1^1, \ldots, \xi_p^n$ in the space $\mathbf{R}^{n\cdot p}$ that the rank is constant over U. The last character is defined by

$$\mathfrak{s}_n = q - \mathfrak{s}_1 - \mathfrak{s}_2 - \ldots - \mathfrak{s}_{n-1},$$

where q is the number of free parameters η.

([1]) A manifold given by $R = R(x)$, $\eta = \eta(x)$, $(x = x^1, \ldots, x^n)$ in the space $R^{\sigma+q+n}$ is a solution of (2.1), if the forms θ^a restricted to this manifold vanish identically. In other words, the substitution of $R = R(x)$, $\eta = \eta(x)$ and $dR^a = \dfrac{\partial R^a}{\partial x^j}dx^j$ in (2.1) gives an identity with arbitrary $dx^1, \ldots, dx^n$.

([2]) In practice, it may be convenient to take instead of $dx^1, \ldots, dx^n$ other bases of differentials, say $\omega^1, \ldots, \omega^n$ $(\omega^i = \omega_j^i dx^j)$.

A family of integral manifolds of (2.1) defined in a certain neighbourhood of a point (x, R, η) will be called the general solution of this system if:

1) the set τ of tangent spaces to these manifolds at the point (x, R, η) depends on Q parameters (τ is a Q-dimensional manifold);

2) there exists such a neighbourhood of the point (x, R, η) in which the reduced characters $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ are preserved.

Since $dx^1, \ldots, dx^n$ are required to be the differentials of the independent variables $x^1, \ldots, x^n$, therefore the forms π^μ (usually $d\eta$) should on a solution be expressed linearly in dx^i. Take a point (x, R, η), then

(2.6) $$\pi^\mu = l^\mu_j dx^j, \quad \mu = 1, \ldots, q \text{ at } (x, R, \eta).$$

The parameters l^μ_j should be chosen by substitution of (2.6) in (2.2) which, taking into account (2.3), leads to a linear system of equations for l^μ_j:

(2.7) $$\alpha^a_{ij} + \beta^a_{\mu i} l^\mu_j = 0.$$

Two cases are possible:

1) the system is contradictory;

2) the system has solutions which obviously form a linear space of a certain dimension — say, N $\big(N = N(x, R, \eta)\big)$.

In the case 1), our problem has no solution which passes through (x, R, η). CARTAN has proved that always

(2.8) $$N \leqslant Q.$$

If the equality sign holds and the point (x, R, η) is a regular point, which means that this equality and the reduced characters are preserved in a certain neighbourhood of (x, R, η), then the system (2.1) is said to be in involution and a general solution of it exists. It may be constructed by successive integration of Cauchy-Kovalevska systems. The reduced characters are in this case called simply characters and they determine the degree of freedom of such a general solution. Namely, the general solution depends on $\sigma = \mathfrak{s}_0$ arbitrary constants, $\mathfrak{s}_1$ arbitrary functions of one argument, $\mathfrak{s}_2$ arbitrary functions of two arguments, ..., and $\mathfrak{s}_n$ arbitrary functions of n arguments ([3]).

If $0 < N < Q$ or (x, R, η) is not a regular point, the system (2.1) must be prolonged by addition of the equations (2.6) and eventual constraints among x, R and η. Thus the prolonged system has the form

(2.4) $$\begin{aligned} \theta^a &= \theta^a(x, R, \eta) = 0, \quad & a &= 1, \ldots, k, \\ \theta^\mu &= \theta^\mu(x, R, \eta, \alpha) = 0, \quad & \mu &= 1, \ldots, q, \end{aligned}$$

with new free parameters α. Applying to this system the same procedure, we shall (if there is no contradiction) calculate a new set of reduced characters and new values of numbers Q and N, say $Q^{(1)}$ and $N^{(1)}$, and apply the criterion of involutivity $N^{(1)} = Q^{(1)}$ at the regular point. If $N^{(1)} \neq Q^{(1)}$, or if the point (x, R, η) is not regular, the procedure should be repeated. After a finite number of steps — say L — we shall arrive either at a contradiction, case 1), or at an involutive system.

([3]) This theorem has been proved by CARTAN and KÄHLER using the analycity assumption.

2.2. Condition of involution

In this Section, we shall consider the necessary and the sufficient condition under which our Pfaff system is in involution. We shall begin with a simple case in which for a given $\boldsymbol{\gamma}(u)$ we take only one knotted $\boldsymbol{\lambda}(u)$. This system defining Riemann invariants is then

$$\theta^a \equiv dR^a - \eta^a \lambda^a = 0. \tag{2.10}$$

It has the advantage that free parameters η^a appear in each equation separately and in a linear manner. Exterior derivatives of θ^a have the form:

$$d\theta^a = \lambda^a \wedge d\eta^a - \eta^a d\lambda^a. \tag{2.11}$$

Since

$$d\lambda^a = \frac{\partial \lambda_i^a}{\partial R^b} dR^b \wedge dx^i = dR^b \wedge \left(\frac{\partial \lambda_i^a}{\partial R^b} dx^i\right),$$

therefore denoting

$$\lambda^a{}_{,R^b} = \frac{\partial \lambda_i^a}{\partial R^b} dx^i \tag{2.12}$$

and taking into account (2.10), we shall write the closed system in the form:

$$\begin{aligned} &\theta^a \equiv dR^a - \eta^a \lambda^a = 0, \\ &d\theta^a \equiv -(d\eta^a \wedge \lambda^a + \eta^a \eta^b \lambda^b \wedge \lambda^a{}_{,R^b}) = 0. \end{aligned} \tag{2.13}$$

Taking the exterior product of the last equation by λ^a, we obtain a set of constraints on free parameters:

$$\eta^a \sum_b \Delta_b^a \eta^b = 0, \tag{2.14}$$

where three-forms Δ_b^a are given as

$$\Delta_b^a = \lambda^a \wedge \lambda^{(b)} \wedge \lambda^a{}_{,R^{(b)}} \; (^4) \qquad \text{no summation over } (b). \tag{2.15}$$

The constraint (2.14) will be eliminated, if and only if, we assume that

$$\Delta_b^a = 0. \tag{2.16}$$

At the same time [6], by the suppresion of the constraint (2.14), makes the system (2.11) at once involutive (with prescribed independent variables $x^1, \ldots, x^n$); no prolongations are needed; and the general solution depends on k arbitrary functions of one variable.

For this reason we shall call (2.16) the condition of involution. A more general situation with a subspace of $\boldsymbol{\lambda}$ knotted with a given $\boldsymbol{\gamma}$ will be discussed in the next Section (see also [7, 8] for more exhaustive analysis).

(4) This condition is another expression of the fact that for $a \neq c$ we have $\lambda^c{}_{,R^a} \in [\lambda^c, \lambda^a]$.

3. Flows with hodographs satisfying the condition of involution

3.1. The system considered. Characteristic vectors

The flows considered in this section are the solutions of the system:

$$
(3.1)\quad
\begin{aligned}
&c_t+uc_x+vc_y+wc_z+\frac{\varkappa-1}{2}c(u_x+u_y+w_z) = 0,\\
&u_t+uu_x+vu_y+wu_z+\frac{2}{\varkappa-1}cc_x = 0,\\
&v_t+uv_x+vv_y+wv_z+\frac{2}{\varkappa-1}cc_y = 0,\\
&w_t+uv_x+vw_y+ww_z+\frac{2}{\varkappa-1}cc_z = 0,
\end{aligned}
$$

where $\varkappa$ — denotes the adiabatic exponent.

Characteristic vectors $\boldsymbol{\gamma} = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ in the hodograph space $H(c, u, v, w)$ are given by the condition

$$
(3.2)\quad \gamma^0\left[\left(\frac{2}{\varkappa-1}\right)^2(\gamma^0)^2-(\gamma^1)^2-(\gamma^2)^2-(\gamma^3)^2\right] = 0
$$

and they lie either on the cone C:

$$
(3.3)\quad \left(\frac{2}{\varkappa-1}\right)^2(\gamma^0)^2-(\gamma^1)^2-(\gamma^2)^2-(\gamma^3)^2 = 0
$$

or on the three-dimensional hyperplane P:

$$
(3.4)\quad \gamma^0 = 0.
$$

For solutions described by linear Riemann invariants, the k-dimensional plane hodograph is given by the equations

$$
(3.5)\quad c = \gamma_a^0 R^a, \quad u = \gamma_a^1 R^a, \quad v = \gamma_a^2 R^a, \quad w = \gamma_a^3 R^a, \quad a = 1, \ldots, k,
$$

where the constant coefficients $\gamma_a^0, \ldots, \gamma_a^3$ give the components of $\boldsymbol{\gamma}$-characteristic vectors $\boldsymbol{\gamma}_a = (\gamma_a^0, \gamma_a^1, \gamma_a^2, \gamma_a^3)$ satisfying (3.3) or (3.4).

For abbreviation, we shall denote by an asterisk the "spatial" part of the vectors from E or H. For example:

$$
\overset{*}{\boldsymbol{\gamma}}_a = (\gamma_a^1, \gamma_a^2, \gamma_a^3);
$$

$\overset{*}{\mathbf{h}} = (u, v, w)$ — velocity vector, where $\mathbf{h} = (a, u, v, w)$ — radius vector in H.

We shall also use the following notation:

$$
\langle\boldsymbol{\gamma}|\mathbf{h}\rangle = \frac{2}{\varkappa-1}c\gamma^0+u\gamma^1+v\gamma^2+w\gamma^3 = \frac{2}{\varkappa-1}\gamma^0 h^0+\overset{*}{\mathbf{h}}\cdot\overset{*}{\boldsymbol{\gamma}},
$$

which represents a certain scalar product in four-dimensional space. By $[\overset{*}{\mathbf{a}}, \overset{*}{\mathbf{b}}]$, we shall denote the vector product and by $[\overset{*}{\mathbf{a}}, \overset{*}{\mathbf{b}}, \overset{*}{\mathbf{c}}]$ — the mixed product in three-dimensional space ($[\overset{*}{\mathbf{a}}, \overset{*}{\mathbf{b}}, \overset{*}{\mathbf{c}}] = \det|\overset{*}{\mathbf{a}}, \overset{*}{\mathbf{b}}, \overset{*}{\mathbf{c}}|$).

With this convention, we may write the characteristic vector $\boldsymbol{\gamma}$ and the knotted vectors $\boldsymbol{\lambda}$ in the form [8].

$$C \ni \boldsymbol{\gamma} = \left(\frac{\varkappa-1}{2}, \dot{\mathbf{e}}\right) \rightleftharpoons \boldsymbol{\lambda} = (\langle \mathbf{h}|\boldsymbol{\gamma}\rangle, -\dot{\boldsymbol{\gamma}}), \tag{3.6}$$

$\dot{\mathbf{e}}$ — arbitrary versor ($\dot{\mathbf{e}}^2 = 1$)

$$P \ni \boldsymbol{\gamma} = (0, \dot{\mathbf{e}}) \rightleftharpoons -\boldsymbol{\lambda} = ([\dot{\mathbf{m}}, \dot{\boldsymbol{\gamma}}, \dot{\mathbf{h}}], [\dot{\boldsymbol{\gamma}}, \dot{\mathbf{h}}])$$

$\dot{\mathbf{m}}$ — arbitrary vector.

In the second formula, for each vector $\boldsymbol{\gamma} \in P$ we have a two-dimensional space of the characteristic vectors $\boldsymbol{\lambda}$ knotted with $\boldsymbol{\gamma}$. Therefore, we may take the vectors

$$\boldsymbol{\lambda} = ([\dot{\mathbf{m}}, \dot{\boldsymbol{\gamma}}, \dot{\mathbf{h}}], [\dot{\boldsymbol{\gamma}}, \dot{\mathbf{m}}])$$

and

$$\tilde{\boldsymbol{\lambda}} = ([\dot{\tilde{\mathbf{m}}}, \dot{\boldsymbol{\gamma}}, \dot{\mathbf{h}}], [\dot{\boldsymbol{\gamma}}, \dot{\tilde{\mathbf{m}}}])$$

as the two independent vectors knotted with $\boldsymbol{\gamma} \in P$. In these formulae $\mathbf{m}$ and $\dot{\mathbf{m}}$ are so chosen that $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$ are independent.

The algebraical form of simple elements related to these vectors provides certain physical information on the possible flows; e.g., the solution which involves only elements of the first type is a potential one, while the remaining ones introduce the vorticity [8].

3.2. Pfaff's system for Riemann invariants. Condition of involution

We consider here the more general situation signalized in the Section 2.2. In the case of elements defined by the cone C, we have only one direction $\boldsymbol{\lambda}$ for the given characteristic vector $\boldsymbol{\gamma}$, and therefore the Pfaff forms connected with these elements should be given by

$$dR^a = \eta^a \lambda^a.$$

However, in the case of elements defined by the plane P, the situation is richer and we shall have a form of the type

$$dR^b = \eta^b \lambda^b,$$

where we decide to take only one $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{\gamma})$ knotted with given $\boldsymbol{\gamma} \in P$, as also the form

$$dR^c = \eta^c \lambda^c + \tilde{\eta}^c \tilde{\lambda}^c,$$

in the case in which we use the entire space Λ knotted with the vector $\boldsymbol{\gamma}$.

Accordingly, the Pfaff system connected with the Riemann invariants will have the form

$$\begin{aligned} dR^c &= \eta^c \lambda^c, & c &= 1, \ldots, l, \\ dR^p &= \eta^p \lambda^p + \tilde{\eta}^p \tilde{\lambda}^p, & p &= l+1, \ldots, k. \end{aligned} \tag{3.7}$$

To close the system (3.7), we have to adjoint the equations:

$$\begin{aligned} d\eta^c \wedge \lambda^c + \eta^c d\lambda^c &= 0, \\ d\eta^p \wedge \lambda^p + d\tilde{\eta}^p \wedge \tilde{\lambda}^p + \eta^p d\lambda^p + \tilde{\eta}^p d\tilde{\lambda}^p &= 0. \end{aligned} \tag{3.8}$$

Taking the exterior product of $(3.8)_1$ by λ^c and of $(3.8)_2$ by $\lambda^p \wedge \lambda$, we have the following constraints on free parameters

$$\lambda^c \wedge d\lambda^c = 0, \tag{3.9}$$
$$\tilde{\lambda}^p \wedge \lambda^p \wedge (\eta^p d\lambda^p + \tilde{\eta}^p d\tilde{\lambda}^p) = 0.$$

The exterior derivative of the form λ^a is given by the formula

$$d\lambda^a = dR^b \wedge \lambda^a{}_{,R^b} + dR^a \wedge \lambda^a{}_{,R^{\breve{a}}} \underset{\text{mod (3.7)}}{=} \eta^b \lambda^b \wedge \lambda^a{}_{,R^b} + (\eta^q \lambda^q + \tilde{\eta}^q \tilde{\lambda}^q) \wedge \lambda^a{}_{,R^q}, \tag{3.10}$$

where $a = 1, \ldots, k$, and $b = 1, \ldots, l$, $q = l+1, \ldots, k$.

On substitution of (3.10) into (3.9), we have

$$\overset{\bullet}{\Delta}{}^d_c \eta^c + \overset{\bullet}{\Delta}{}^d_p \eta^p + \overset{\bullet}{\Delta}{}^d_{\tilde{p}} \tilde{\eta}^p = 0, \tag{3.11}$$
$$\eta^p(\Delta^p_c \eta^c + \Delta^p_q \eta^q + \Delta^p_{\tilde{q}} \tilde{\eta}^q) + \tilde{\eta}^p(\Delta^{\tilde{p}}_c \eta^c + \Delta^{\tilde{p}}_q \eta^q + \Delta^{\tilde{p}}_{\tilde{q}} \tilde{\eta}^q) = 0,$$

where $c, d = 1, \ldots, l$, $p, q = l+1, \ldots, k$, and the tilde sign over an index indicates that the corresponding form is related to $\tilde{\boldsymbol{\lambda}}$ vectors. In (3.11), $\overset{\bullet}{\Delta}$ are three-forms and Δ are four-forms. Their components are given below. From (3.11), we see that these constraints do not appear if we assume that all the forms $\overset{\bullet}{\Delta}$ and Δ vanish. As is shown in [7], also in this case, analogously as in the case discussed in Sec. 3.2, with elimination of these constraints the system (3.7) becomes in involution with prescribed independent variables t, x, y, z, and its general solution depends on l arbitrary functions of one argument, and $k-l$ arbitrary functions of two arguments.

Therefore, we shall attempt to satisfy the conditions of involution

$$\overset{\bullet}{\Delta} = 0, \quad \Delta = 0$$

by appropriate choice of the $\boldsymbol{\gamma}$ vectors. A case in which these conditions are not satisfied is also considered. The appropriate Pfaff forms (3.7) must be prolonged in this case, leading to restriction of the functional freedom in the solution (all the functional freedoms are excluded in the solution considered except simple waves).

In what follows, the vectors $\boldsymbol{\gamma}$ appearing in our considerations will be assumed to be constant vectors; the forms (3.13) *and* (3.14) *are calculated under this assumption.*

Before performing the calculation of the components of the forms $\overset{\bullet}{\Delta}$ and Δ in our specific cases, let us write the hodograph of the solution as:

$$\mathbf{h} = \boldsymbol{\gamma}_c R^c + \boldsymbol{\gamma}_\alpha R^\alpha + \boldsymbol{\gamma}_A R^A + \mathbf{h}_0,$$

where $\boldsymbol{\gamma}_c \in C$, $c = 1, \ldots, l$, $\boldsymbol{\gamma}_\alpha \in P$, $\alpha = l+1, \ldots, \nu$, and also $\boldsymbol{\gamma}_A \in P$, $A = \nu+1, \ldots, k$ and $\mathbf{h}_0$ is a constant vector. The indices are distinguished in order to make clear that the corresponding equations in Pfaff's system are different — namely,

$$dR^c = \eta^c \lambda^c, \tag{3.12}$$
$$dR^\alpha = \eta^\alpha \lambda^\alpha,$$
$$dR^A = \eta^A \lambda^A + \tilde{\eta}^A \tilde{\lambda}^A.$$

Since $\lambda^a_{,R^c} = \dfrac{\partial \lambda^a}{\partial R^c} dt$ ($\lambda^a_1, \lambda^a_2, \lambda^a_3$ — being constants), therefore in the forms $\overset{\bullet}{\Delta}$ the component corresponding to $dx \wedge dy \wedge dz$ is equal to zero. The remaining components are equal to components of a three-dimensional vector.

We shall write the following expressions for $\overset{\bullet}{\Delta}$ forms (the equality sign means equality of independent components of both sides) ([5]). *In these expressions the summation convention is eliminated. The index p becomes* α *or* A *— that is,* $p = l+1, \ldots, k$;

$$
(3.13)\qquad
\begin{aligned}
\overset{\bullet}{\Delta}{}^{c}_{d} &= \lambda^{c} \wedge \lambda^{d} \wedge \lambda^{c}_{,R^{d}} = \langle \boldsymbol{\gamma}_{c} | \boldsymbol{\gamma}_{d} \rangle \, [\dot{\boldsymbol{\gamma}}_{d}, \dot{\boldsymbol{\gamma}}_{c}],\\
\overset{\bullet}{\Delta}{}^{c}_{p} &= \lambda^{c} \wedge \lambda^{p} \wedge \lambda^{c}_{,R^{p}} = \langle \boldsymbol{\gamma}_{p} | \boldsymbol{\gamma}_{c} \rangle \, [[\dot{\boldsymbol{\gamma}}_{p}, \dot{\mathbf{m}}^{p}], \dot{\boldsymbol{\gamma}}_{c}],\\
\overset{\bullet}{\Delta}{}^{c}_{\tilde{A}} &= \lambda^{c} \wedge \tilde{\lambda}^{A} \wedge \lambda^{c}_{,R^{A}} = \langle \boldsymbol{\gamma}_{A} | \boldsymbol{\gamma}_{c} \rangle \, [[\dot{\boldsymbol{\gamma}}_{A}, \dot{\tilde{\mathbf{m}}}{}^{A}], \boldsymbol{\gamma}_{c}],\\
\overset{\bullet}{\Delta}{}^{\alpha}_{c} &= \lambda^{\alpha} \wedge \lambda^{c} \wedge \lambda^{\alpha}_{,R^{c}} = [\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}, \dot{\boldsymbol{\gamma}}_{c}] \, [[\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}], \dot{\boldsymbol{\gamma}}_{c}],\\
\overset{\bullet}{\Delta}{}^{\alpha}_{p} &= \lambda^{\alpha} \wedge \lambda^{p} \wedge \lambda^{\alpha}_{,R^{p}} = [\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}, \dot{\boldsymbol{\gamma}}_{p}] \, [[\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}], [\dot{\mathbf{m}}^{p}, \dot{\boldsymbol{\gamma}}_{p}]],\\
\overset{\bullet}{\Delta}{}^{\alpha}_{\tilde{A}} &= \lambda^{\alpha} \wedge \tilde{\lambda}^{A} \wedge \lambda^{\alpha}_{,R^{A}} = [\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}, \dot{\boldsymbol{\gamma}}_{A}] \, [[\dot{\mathbf{m}}^{\alpha}, \dot{\boldsymbol{\gamma}}_{\alpha}], [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\boldsymbol{\gamma}}_{A}]].
\end{aligned}
$$

The one component of four-forms Δ is given on the right-hand of the Eqs:

$$
(3.14)\qquad
\begin{aligned}
\Delta^{A}_{c} &= \tilde{\lambda}^{A} \wedge \lambda^{A} \wedge \lambda^{c} \wedge \lambda^{A}_{,R^{c}} = \underline{[\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}]} \, [\dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}_{c}] \, \langle \boldsymbol{\gamma}_{c} | \boldsymbol{\gamma}_{A} \rangle,\\
\Delta^{A}_{p} &= \tilde{\lambda}^{A} \wedge \lambda^{A} \wedge \lambda^{p} \wedge \lambda^{A}_{,R^{p}} = \underline{[\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}]} \, [\dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}_{p}] \, [\dot{\mathbf{m}}^{p}, \dot{\boldsymbol{\gamma}}_{p}, \dot{\boldsymbol{\gamma}}_{A}],\\
\Delta^{\bar{A}}_{\tilde{B}} &= \tilde{\lambda}^{A} \wedge \lambda^{A} \wedge \tilde{\lambda}^{B} \wedge \lambda^{A}_{,R^{B}} = \underline{[\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}]} \, [\dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}_{B}] \, [\dot{\tilde{\mathbf{m}}}{}^{B}, \dot{\boldsymbol{\gamma}}_{B}, \dot{\boldsymbol{\gamma}}_{A}],\\
\Delta^{\tilde{A}}_{c} &= \tilde{\lambda}^{A} \wedge \lambda^{A} \wedge \lambda^{c} \wedge \tilde{\lambda}^{A}_{,R^{c}} = \underline{[\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}]} \, [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}_{c}] \, \langle \boldsymbol{\gamma}_{c} | \boldsymbol{\gamma}_{A} \rangle,\\
\Delta^{\tilde{A}}_{p} &= \tilde{\lambda}^{A} \wedge \lambda^{A} \wedge \lambda^{p} \wedge \tilde{\lambda}^{A}_{,R^{p}} = [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}] \, [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}_{p}] \, [\dot{\tilde{\mathbf{m}}}{}^{p}, \dot{\boldsymbol{\gamma}}_{p}, \dot{\boldsymbol{\gamma}}_{A}],\\
\Delta^{\tilde{A}}_{\tilde{B}} &= \tilde{\lambda}^{A} \wedge \tilde{\lambda}^{A} \wedge \tilde{\lambda}^{B} \wedge \tilde{\lambda}^{A}_{,R^{B}} = [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}] \, [\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\boldsymbol{\gamma}}_{A}, \dot{\boldsymbol{\gamma}}^{B}] \, [\dot{\tilde{\mathbf{m}}}{}^{B}, \dot{\boldsymbol{\gamma}}_{B}, \dot{\boldsymbol{\gamma}}_{A}].
\end{aligned}
$$

The terms $[\dot{\tilde{\mathbf{m}}}{}^{A}, \dot{\mathbf{m}}^{A}, \dot{\boldsymbol{\gamma}}_{A}]$ underlined do have to vanish, since we have assumed that $\boldsymbol{\lambda}^{A}$ and $\tilde{\boldsymbol{\lambda}}^{A}$ are pairs of independent vectors. Therefore, these terms may be disregarded in the investigation of the conditions of involution.

Note that the diagonal terms $\overset{\bullet}{\Delta}{}^{i}_{i}$ and Δ^{A}_{A} are in these formulas equal to zero. Also, by the (anti)symmetry properties, the vanishing of the terms $\overset{\bullet}{\Delta}{}^{c}_{d}$, $\Delta^{\tilde{A}}_{\tilde{B}}$, $\Delta^{A}_{\tilde{B}}$ implies the vanishing of $\overset{\bullet}{\Delta}{}^{d}_{c}$, $\Delta^{\tilde{B}}_{\tilde{A}}$, $\Delta^{\tilde{A}}_{p}$. This reduces the number of independent conditions of involution.

([5]) Three-forms $\overset{\bullet}{\Delta}$ are calculated by means of the appropriate isomorphism between the exterior product and the determinants, e.g.

$$
\overset{\bullet}{\Delta}{}^{a}_{b} = \begin{vmatrix} \mathbf{e}_{t} & \mathbf{e}_{x} & \mathbf{e}_{y} & \mathbf{e}_{z} \\ \lambda^{a}_{0} & \lambda^{a}_{1} & \lambda^{a}_{2} & \lambda^{a}_{3} \\ \lambda^{b}_{0} & \lambda^{b}_{1} & \lambda^{b}_{2} & \lambda^{b}_{3} \\ \lambda^{a}_{0,R^{b}} & \lambda^{a}_{1,R^{b}} & \lambda^{a}_{2,R^{b}} & \lambda^{a}_{3,R^{b}} \end{vmatrix},
$$

where $\mathbf{e}_{t}, \mathbf{e}_{x}, \mathbf{e}_{y}, \mathbf{e}_{z}$ are the unit vectors of the axes t, x, y, z.

Analogously, with this convenient notation four-forms Δ are given by $\Delta^{A}_{p} = \det(\tilde{\lambda}^{A}, \lambda^{A}, \lambda^{p}, \lambda^{A}_{,R^{p}})$

3.3. The solutions and their classification

The solutions will be classified according to two principles. First, according to the number of Riemann invariants used (that is, the number of interacting simple waves) and then, according to the type of vectors $\boldsymbol{\gamma}$ used in the construction. To distinguish all possible cases, we introduce the symbolic notation. For example — for construction of double waves we may take two independent $\boldsymbol{\gamma}$ from the case C; this case will be denoted $C_1 - C_1$. If one of the vectors $\boldsymbol{\gamma}$ is of C-type and the other one of P-type, we shall denote the corresponding solution either by $C_1 - P_2$ if the whole two-dimensional space $\Lambda(\gamma)$ is taken for a given $\boldsymbol{\gamma} \in P$, or by $C_1 - P_1$ if we take only one direction $\boldsymbol{\lambda} \in \Lambda$. Thus the subscripts in this symbolic notation indicate the number of free parameters introduced in Pfaff's system (3.7) and the letters C or P indicate the type of $\boldsymbol{\gamma}$-vectors used in construction of the solutions.

Subsequent formulations of the conditions of involution will be based on the requirement that the corresponding $\dot{\Delta}$ and Δ forms should vanish. Thus, for instance, if we take into consideration the case $C_1 - P_1 - P_2$, we have as the hodograph

$$\mathbf{h} = \boldsymbol{\gamma}_0 R^0 + \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + \mathbf{h}_0,$$

where $\boldsymbol{\gamma}_0 \in C$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in P$; then all $\dot{\Delta}$ and Δ forms with indices $0, 1, 2$ should vanish, in order to satisfy conditions of involution for the Pfaff system

$$dR^0 = \eta^0 \lambda^0, \quad dR^1 = \eta^1 \lambda^1, \quad dR^2 = \eta^2 \lambda^2 + \tilde{\eta}^2 \tilde{\lambda}^2.$$

The corresponding forms may easily be enumerated by taking all possible combinations of indices. Note that three-forms arise by taking the lower case letter superscripts $(\dot{\Delta}_i^a, \dot{\Delta}_i^{\tilde{a}})$ and four-forms by taking capital letters superscripts (Δ_i^A). In our case, we have $\dot{\Delta}_1^0, \dot{\Delta}_2^0$, $\dot{\Delta}_{\tilde{2}}^0, \dot{\Delta}_0^1, \dot{\Delta}_2^1, \dot{\Delta}_{\tilde{2}}^1$ as three-forms and $\Delta_0^2, \Delta_0^{\tilde{2}}, \Delta_1^2, \Delta_1^{\tilde{2}}$ as four-forms.

3.4.1. Solutions defined by C-characteristic vectors. The following cases should be distinguished depending on the dimension of the hodograph.

a) *Simple waves* C_1. The solution of a simple wave type involves one Riemann invariant only and the Pfaff form connected with this invariant is

$$dR = \eta \lambda\big(h(R)\big), \tag{3.15}$$

where $\boldsymbol{\lambda}$ is a characteristic vector knotted with the vector $\boldsymbol{\gamma}(R)$ tangent to a characteristic curve Γ given by $h = h(R)$.

The Pfaff form (3.15) has always the solution which may be given by the implicit relation

$$R = \varphi\big(\lambda_i(R) x^i\big), \tag{3.15'}$$

where φ is an arbitrary differentiable function of one variable.

When the characteristic curve Γ is a straight line

$$c = \frac{\varkappa - 1}{2} R + c_0, \quad u = e^1 R + u_0, \quad u = e^2 R + v_0, \quad w = e^3 R + w_0$$

(where $(e^1)^2+(e^2)^2+(e^3)^3 = 1$), the solution is given by those relations and the implicit formula (3.15) with the following vector

$$\boldsymbol{\lambda}(R) = (\langle\boldsymbol{\gamma}|\boldsymbol{\gamma}\rangle R+\langle\mathbf{h}_0|\boldsymbol{\gamma}\rangle,\ -\mathring{\boldsymbol{\gamma}}) = \left(\frac{\varkappa+1}{2}R+c_0+\mathring{\mathbf{h}}_0\cdot\mathring{\mathbf{e}},\ -\mathring{\mathbf{e}}\right).$$

Of course, we can take $\mathring{\mathbf{h}}_0 = 0$, making use of the appropriate Galillelian transformation. Therefore, the only constant $h_0^0 = c_0$ has an intrinsic physical meaning. For this reason in what follows, we shall put $u_0 = v_0 = w_0 = 0$.

The solution represents a plane wave, with constant values of flow parameters on a hyperplane orthogonal to the direction of wave propagation given by $\mathring{\boldsymbol{\gamma}}$. The condition of involution $\mathring{\Delta} = 0$ is obviously satisfied automatically.

b) *Double waves* C_1-C_1. In agreement with our notation, we take

$$\mathbf{h} = \boldsymbol{\gamma}_1 R^1+\boldsymbol{\gamma}_2 R^2+\mathbf{h}_0, \quad \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in C$$

as the double wave hodograph. The choice of $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ will be controlled by the conditions of involution, which take the form

$$\mathring{\Delta}_2^1 = 0, \quad \mathring{\Delta}_1^2 = 0$$

and reduce to the following

$$\mathring{\Delta}_2^1 = \langle\boldsymbol{\gamma}_1|\boldsymbol{\gamma}_2\rangle[\mathring{\boldsymbol{\gamma}}_2, \mathring{\boldsymbol{\gamma}}_1] = 0, \tag{3.16}$$

since we have $\mathring{\Delta}_2^1 = -\mathring{\Delta}_1^2$.

The condition (3.16) implies that either $[\mathring{\boldsymbol{\gamma}}, \mathring{\boldsymbol{\gamma}}_2] = 0$ or $\langle\boldsymbol{\gamma}_1|\boldsymbol{\gamma}_2\rangle = 0$. The first condition leads to one-dimensional unsteady flows ([6]), while the other, taking into account the form of $\boldsymbol{\gamma}_c$, gives:

$$\mathring{\mathbf{e}}_1\cdot\mathring{\mathbf{e}}_2 = -\frac{\varkappa-1}{2}; \tag{3.17}$$

hence

$$\cos\alpha_\varkappa = \cos(\mathring{\mathbf{e}}_1, \mathring{\mathbf{e}}_2) = -\frac{\varkappa-1}{2} \tag{3.17'}$$

which gives the bound $-1 \leqslant \varkappa \leqslant 3$. The solution in this case represents two simple waves which intersect under the angle $\alpha_\varkappa$ [8]. We observe that for such waves the principle of linear superposition is valid (they are not scattered by each other), since the corresponding Pfaff forms are separable. Integration of the Pfaff system

$$dR^a = \eta^a\lambda^a, \quad a = 1, 2,$$

where

$$\boldsymbol{\lambda}^1 = (\langle\boldsymbol{\gamma}_1|\boldsymbol{\gamma}_1\rangle R^1,\ -\mathring{\boldsymbol{\gamma}}_1), \quad \boldsymbol{\lambda}^2 = (\langle\boldsymbol{\gamma}_2|\boldsymbol{\gamma}_2\rangle R^2,\ -\mathring{\boldsymbol{\gamma}}_2),$$

([6]) In fact, if $[\mathring{\boldsymbol{\gamma}}_1, \mathring{\boldsymbol{\gamma}}_2] = 0$, then we may assume $\boldsymbol{\gamma}_1 = \left(\frac{\varkappa-1}{2}, 1, 0, 0\right)$ and $\boldsymbol{\gamma}_2 = \left(\frac{\varkappa-1}{2}, -1, 0, 0\right)$ (in an appropriate coordinate system) and the Eqs. (3.1) will reduce to equations of one-dimensional unsteady flows.

gives the solution

$$R^a = \varphi^a(\lambda_i^a x^i) \tag{3.18}$$

with two arbitrary functions φ^1, φ^2.

c) *Triple waves* $C_1 - C_1 - C_1$. We assume the expression

$$\mathbf{h} = \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + \boldsymbol{\gamma}_3 R^3 + \mathbf{h}_0, \quad \boldsymbol{\gamma}_i \in C$$

as a hodograph of the triple waves under consideration. The conditions of involution are reduced to the following:

$$\dot{\Delta}_2^1 = 0, \quad \dot{\Delta}_3^1 = 0, \quad \dot{\Delta}_3^2 = 0,$$

since in this case $\dot{\Delta}_b^a = -\dot{\Delta}_a^b$.

Analogously as for double waves, the condition of involution will be satisfied if we put:

$$\langle \boldsymbol{\gamma}_b | \boldsymbol{\gamma}_c \rangle = 0, \quad b, c = 1, 2, 3; \quad b \neq c. \tag{3.19}$$

The solution has the form:

$$R^a = \varphi^a(\lambda_i^a x^i) \tag{3.20}$$

and represents three simple waves linearly interacting without being mutually scattered. The solution is valid for adiabatic exponent $\varkappa < 2$, since we have

$$\cos \alpha_\varkappa = \cos(\dot{\mathbf{e}}_b, \dot{\mathbf{e}}_c) = -\frac{\varkappa - 1}{2},$$

and only for $\alpha_\varkappa \leqslant \frac{2}{3}\pi \left(\cos \alpha_\varkappa \geqslant -\frac{1}{2}\right)$ there may exist in three-dimensional space three such vectors $\dot{\mathbf{e}}_c$ that the angles which they form are equal.

3.4.2. The solutions of type $C_1 - P_2$. The hodograph of the solution is taken in the form:

$$\mathbf{h} = \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + \mathbf{h}_0,$$

where $\boldsymbol{\gamma}_1 \in C$ and $\boldsymbol{\gamma}_2 \in P$. The conditions of involution take the form:

$$\begin{aligned} &\dot{\Delta}_2^1 = 0, \quad \dot{\Delta}_{\tilde{2}}^{1} = 0, \\ &\Delta_1^2 = 0, \quad \Delta_{\tilde{1}}^{\dot{2}} = 0. \end{aligned} \tag{3.21}$$

These conditions may be satisfied only by the vectors $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ for which

$$\langle \boldsymbol{\gamma}_1 | \boldsymbol{\gamma}_2 \rangle = 0 \quad \text{and} \quad \dot{\boldsymbol{\gamma}}_1 \cdot \dot{\boldsymbol{\gamma}}_2 = 0. \tag{3.22}$$

Performing rotation of the coordinate system, we may reduce $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ to the following form:

$$\boldsymbol{\gamma}_1 = \left(\frac{\varkappa - 1}{2}, 1, 0, 0\right), \quad \boldsymbol{\gamma}_2 = (0, 0, 1, 0).$$

As the vectors $\mathring{\mathbf{m}}$ and $\dot{\mathbf{m}}$ we may take:

$$\mathring{\mathbf{m}} = \mathring{\boldsymbol{\gamma}}_1, \quad \dot{\mathbf{m}} = [\mathring{\boldsymbol{\gamma}}_1, \mathring{\boldsymbol{\gamma}}_2].$$

The vectors $\boldsymbol{\lambda}$ knotted to $\boldsymbol{\gamma}$ then have the form:

$$\begin{aligned} \boldsymbol{\lambda}^1 &= (\langle \boldsymbol{\gamma}_1 | \boldsymbol{\gamma}_1 \rangle R^1 + \langle \boldsymbol{\gamma}_1 | \mathbf{h}_0 \rangle, -\mathring{\boldsymbol{\gamma}}_1), \\ \boldsymbol{\lambda}^2 &= (0, 0, 0, -1), \\ \tilde{\boldsymbol{\lambda}}^2 &= (-R^1, -1, 0, 0), \end{aligned}$$

and the corresponding Pfaff system defining R^1 and R^2 is given by:

$$\text{(3.23)} \qquad \begin{aligned} dR^1 &= \eta^1 \left(\left(\frac{\varkappa+1}{2} R^1 + c_0 \right) dt - dx \right), \\ dR^2 &= -\eta^2 dz + \tilde{\eta}^2 (R^1 dt + dx). \end{aligned}$$

Integration of the first equation yields:

$$\text{(3.24)} \qquad R^1 = \varphi^1 \left(\left(\frac{\varkappa+1}{2} R^1 + c_0 \right) t - x \right),$$

where φ^1 is an arbitrary function. The second equation indicates that $\partial R^2 / \partial y = 0$, and that dependence on z is arbitrary, while the dependence on t and x is determined by a linear equation

$$\text{(3.25)} \qquad \frac{\partial R^2}{\partial t} - R^1(t, x) \frac{\partial R^2}{\partial x} = 0,$$

whose solution depends on an arbitrary function $\varphi^2(I)$ of the general integral $I(t, x)$ of the ordinary differential equation:

$$\text{(3.26)} \qquad \frac{dx}{dt} = -R^1(t, x).$$

Taking into account the form of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, we have:

$$\mathbf{h} = (c, u, v, w) = \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + c_0 = \left(\frac{2}{\varkappa-1} R^1 + c_0, R^1, R^2, 0 \right),$$

and hence

$$\text{(3.27)} \qquad c = \frac{\varkappa-1}{2} R^1 + c_0, \quad u = R^1, \quad v = R^2, \quad w = 0;$$

therefore,

$$\text{(3.28)} \qquad \begin{aligned} c &= \frac{\varkappa-1}{2} \varphi^1 \left(\left(\frac{\varkappa+1}{2} R^1 + c_0 \right) t - x \right) + c_0; \\ u &= \varphi^1 \left(\frac{\varkappa+1}{2} R^1 t - x \right), \end{aligned}$$

and $v = v(t, x, z)$ is the solution of the equation

$$\text{(3.29)} \qquad \frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x}, \quad v_{,y} = 0, \quad w = 0,$$

which is equivalent to

$$\frac{dv}{dt} = 0, \quad \text{where} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla .$$

Thus we can conclude that in this case the flow depends on one function of a single variable, — e.g., $u(0, x)$ — and on one function of two variables, for instance $v(0, x, z)$.

3.4.3. Solution of type $C-C-P$. a) *The case* $C_1 - C_2 - P_2$. Let the hodograph have the form

$$\mathbf{h} = \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + \boldsymbol{\gamma}_0 R^0 + \mathbf{h}_0$$

and let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ be vectors taken from the cone C and $\boldsymbol{\gamma}_0$ from the hyperplane P. The conditions of involution

$$\begin{gathered} \dot{\Delta}_2^1 = 0, \quad \dot{\Delta}_0^1 = 0, \quad \dot{\Delta}_{0}^{\dot{1}} = 0, \quad \dot{\Delta}_1^2 = 0, \quad \dot{\Delta}_0^2 = 0, \quad \dot{\Delta}_{\tilde{0}}^2 = 0, \\ \Delta_1^0 = 0, \quad \Delta_2^0 = 0, \quad \Delta_1^{\tilde{0}} = 0, \quad \Delta_2^{\tilde{0}} = 0 \end{gathered} \tag{3.30}$$

are satisfied if either

1. $\langle \boldsymbol{\gamma}_1 | \boldsymbol{\gamma}_2 \rangle = 0, \quad \langle \boldsymbol{\gamma}_0 | \boldsymbol{\gamma}_c \rangle = 0 \ (c = 1, 2)$ — that is, if $\boldsymbol{\gamma}_1 \cdot \dot{\boldsymbol{\gamma}}_2 = -(\varkappa - 1)/2$ and $\dot{\boldsymbol{\gamma}}_0 \| [\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2]$, or if

2. $[\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2] = 0$ and $\langle \boldsymbol{\gamma}_0 | \boldsymbol{\gamma}_c \rangle = 0, \quad c = 1, 2.$

The first of these conditions may be satisfied by letting $\boldsymbol{\gamma}_1 = \left(\frac{\varkappa - 1}{2}, \dot{\mathbf{e}}\right)$ and $\boldsymbol{\gamma}_2 = \left(\frac{\varkappa - 1}{2}, -\dot{\mathbf{e}}\right)$.

In the case 1, by a rotation of coordinates we may impart to the vectors γ the form:

$$\begin{aligned} \gamma_1 &= \left(\frac{\varkappa - 1}{2}, 1, 0, 0\right), \\ \gamma_2 &= \left(\frac{\varkappa - 1}{2}, \cos\alpha, \sin\alpha, 0\right), \\ \gamma_0 &= (0, 0, 0, 1), \end{aligned}$$

where $\cos\alpha = -\frac{\varkappa - 1}{2}$ and $\sin\alpha = \frac{1}{\varkappa - 1}\sqrt{(1+\varkappa)(3-\varkappa)}$. As the vectors $\dot{\bar{\mathbf{m}}}$ and $\dot{\mathbf{m}}$, we take $\dot{\mathbf{m}} = \dot{\boldsymbol{\gamma}}_1$, $\dot{\bar{\mathbf{m}}} = \dot{\boldsymbol{\gamma}}_2$. The knotted $\boldsymbol{\lambda}$ vectors are then:

$$\begin{aligned} \boldsymbol{\lambda}^1 &= (\langle \boldsymbol{\gamma}_1 | \boldsymbol{\gamma}_1 \rangle R^1 + c_0, -\dot{\boldsymbol{\gamma}}_1) = \left(\frac{\varkappa + 1}{2} R^1 + c_0, -1, 0, 0\right), \\ \boldsymbol{\lambda}^2 &= (\langle \boldsymbol{\gamma}_2 | \boldsymbol{\gamma}_2 \rangle R^2 + c_0, -\dot{\boldsymbol{\gamma}}_2) = \left(\frac{\varkappa + 1}{2} R^2 + c_0, \cos\alpha, -\sin\alpha, 0\right), \\ \boldsymbol{\lambda}^0 &= ([\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_0, \dot{\boldsymbol{\gamma}}_2] R^2, -[\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_0]) = (-R^2 \sin\alpha, 0, 1, 0), \\ \tilde{\boldsymbol{\lambda}}^0 &= ([\dot{\boldsymbol{\gamma}}_2, \dot{\boldsymbol{\gamma}}_0, \dot{\boldsymbol{\gamma}}_1] R^1, -[\dot{\boldsymbol{\gamma}}_2, \dot{\boldsymbol{\gamma}}_0]) = (R^1 \sin\alpha, -\sin\alpha, \cos\alpha, 0). \end{aligned}$$

The corresponding Pfaff system has the form:

$$
(3.31)\qquad
\begin{aligned}
dR^1 &= \eta^1\left(\left(\frac{\varkappa+1}{2}R^1+c_0\right)dt-dx\right),\\
dR^2 &= \eta^2\left(\left(\frac{\varkappa+1}{2}R^2+c_0\right)dt-\cos\alpha\,dx-\sin\alpha\,dy\right),\\
dR^0 &= \sin\alpha(\tilde{\eta}^0R^1-\eta^0R^2)dt-\tilde{\eta}^0\sin\alpha\,dx+(\eta+\tilde{\eta}^0\cos\alpha)\,dy.
\end{aligned}
$$

From the first and second equations, we obtain:

$$
(3.32)\qquad
\begin{aligned}
R^1 &= \varphi^1\left(\left(\frac{\varkappa+1}{2}R^1+c_0\right)t-x\right),\\
R^2 &= \varphi^2\left(\left(\frac{\varkappa+1}{2}R^2+c_0\right)t-x\cos\alpha-y\sin\alpha\right)
\end{aligned}
$$

with arbitrary functions φ^1 and φ^2. From $(3.31)_3$, we observe that $\partial R^0/\partial z = 0$ and that (R^0_t, R^0_x, R^0_y) lies in a plane spanned by $(\lambda^0_t, \lambda^0_x, \lambda^0_y)$ and $(\tilde{\lambda}^0_t, \tilde{\lambda}^0_x, \tilde{\lambda}^0_y)$. Thus the scalar product of (R^0_t, R^0_x, R^0_y) with a vector orthogonal to this plane is equal to zero. Thus we arrive at an equation of the first order:

$$
(3.32')\qquad \frac{\partial R^0}{\partial t}+\left(R^1-\frac{\varkappa-1}{2}R^2\right)\frac{\partial R^0}{\partial x}+\frac{1}{\varkappa-1}\sqrt{(1+\varkappa)(3-\varkappa)}R^2\frac{\partial R^0}{\partial y} = 0.
$$

The flow parameters are given by the formulae:

$$
(3.33)\qquad
\begin{aligned}
c &= \frac{\varkappa-1}{2}(R^1+R^2)+c_0,\\
u &= R^1-\frac{\varkappa-1}{2}R^2,\\
v &= \frac{1}{\varkappa-1}\sqrt{(1+\varkappa)(3-\varkappa)}R^2,\\
w &= R^0,
\end{aligned}
$$

and they depend on two arbitrary functions of one argument and one arbitrary function of two arguments.

In the variables c, u, v, w, the Eq. (3.32′) means that

$$
(3.34)\qquad \frac{dw}{dt} = 0,\quad w_{,z} = 0.
$$

In case 2, the vectors $\boldsymbol{\gamma}$, after corresponding rotation of coordinates, take the form:

$$
\boldsymbol{\gamma}_1 = \left(\frac{\varkappa-1}{2}, 1, 0, 0\right),\quad \boldsymbol{\gamma}_2 = \left(\frac{\varkappa-1}{2}, -1, 0, 0\right),\quad \boldsymbol{\gamma}_3 = (0, 0, 1, 0),
$$

which implies that the flow is plane. Flow parameters are expressed by Riemann invariants as follows:

$$
(3.35)\qquad c = \frac{\varkappa-1}{2}(R^1+R^2)+c_0,\quad u = R^1-R^2,\quad v = R^0,\quad w = 0.
$$

Taking $\dot{\mathbf{m}} = (1, 0, 0)$ and $\dot{\mathbf{m}} = (0, 0, 1)$, we obtain as $\boldsymbol{\lambda}$ vectors

$$\boldsymbol{\lambda}^1 = \left(\frac{\varkappa+1}{2}R^1 + \frac{3-\varkappa}{2}R^2 + c_0, -1, 0, 0\right),$$

$$\boldsymbol{\lambda}^2 = \left(\frac{\varkappa+1}{2}R^2 + \frac{3-\varkappa}{2}R^1 + c_0, 1, 0, 0\right),$$

$$\boldsymbol{\lambda}^0 = (0, 0, 0, 1),$$

$$\tilde{\boldsymbol{\lambda}}^0 = (R^2 - R^1, 1, 0, 0),$$

and the corresponding Pfaff system leads to a hyperbolic system

$$\text{(3.36)} \qquad \begin{aligned} &\frac{\partial R^1}{\partial t} + \left(\frac{\varkappa+1}{2}R^1 + \frac{3-\varkappa}{2}R^2 + c_0\right)\frac{\partial R^1}{\partial x} = 0, \\ &\frac{\partial R^2}{\partial t} - \left(\frac{\varkappa+1}{2}R^2 + \frac{3-\varkappa}{2}R^1 + c_0\right)\frac{\partial R^2}{\partial x} = 0, \end{aligned}$$

for unknown $R^1 = R^1(t, x)$, $R^2 = R^2(t, x)$, and to the equation

$$\text{(3.36)}_3 \qquad \frac{\partial R^0}{\partial t} + (R^1 - R^2)\frac{\partial R^0}{\partial x} = 0$$

for the Riemann invariant $R^0 = R^0(t, x, z)$ in which the dependence on z is parametrical.

In the flow variables c, u, v, w, these equations take the form:

$$\text{(3.37)} \qquad \begin{aligned} &\frac{dc}{dt} + \frac{\varkappa-1}{2}c\frac{\partial u}{\partial x} = 0, \\ &\frac{du}{dt} + \frac{2c}{\varkappa-1}\frac{\partial c}{\partial x} = 0, \\ &\frac{dv}{dt} = 0, \\ &w = 0, \quad c_{,y} = u_{,y} = v_{,y} = 0. \end{aligned}$$

If at the initial moment, the flow parameters are independent of z coordinate, the class of flows obtained above degenerates into a certain class of double waves (the solution depends only on time t and x coordinate). If the initial conditions imply $R^0 = 0$, then the solution will describe ordinary one-dimensional unsteady flows.

3.4.4. The solution of type $C-P-P$. Constructing these solutions, we take one of the vectors $\boldsymbol{\gamma}$ from the cone C (say $\boldsymbol{\gamma}_0$) and two vectors, denoted $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ from the hyperplane P. We have two cases in which the conditions of involution can be satisfied:

$$\text{a) } C_1 - P_1 - P_2 \quad \text{and} \quad \text{b) } C_1 - P_1 - P_1.$$

In both cases, we have to deal with the hodograph:

$$\mathbf{h} = \boldsymbol{\gamma}_0 R_0 + \boldsymbol{\gamma}_1 R^1 + \boldsymbol{\gamma}_2 R^2 + \mathbf{h}_0.$$

Case a). The conditions of involution take the form:

$$\dot{\Delta}_1^0 = 0, \quad \dot{\Delta}_2^0 = 0, \quad \dot{\Delta}_{\tilde{2}}^0 = 0, \quad \dot{\Delta}_0^1 = 0, \quad \dot{\Delta}_2^1 = 0, \quad \dot{\Delta}_{\tilde{2}}^1 = 0,$$
$$\Delta_0^2 = 0, \quad \Delta_1^2 = 0, \quad \Delta_0^{\tilde{2}} = 0, \quad \Delta_1^{\tilde{2}} = 0.$$

These conditions will be satisfied if we take $\langle \boldsymbol{\gamma}_0 | \boldsymbol{\gamma}_p \rangle = 0$ $p = 1, 2$ (which means that $\dot{\boldsymbol{\gamma}}_0 \perp \dot{\boldsymbol{\gamma}}_p$) and if $\dot{\mathbf{m}}^1 = \dot{\boldsymbol{\gamma}}_2$. The vectors $\dot{\mathbf{m}}^2$ and $\dot{\tilde{\mathbf{m}}}^2$ independent of $\dot{\boldsymbol{\gamma}}_2$ are arbitrary. Without loss of generality, the vectors $\boldsymbol{\gamma}_i$ may be chosen as follows:

$$\boldsymbol{\gamma}_0 = \left(\frac{\varkappa-1}{2}, 1, 0, 0\right), \quad \boldsymbol{\gamma}_1 = (0, 0, 1, 0), \quad \boldsymbol{\gamma}_2 = (0, 0, 0, 1).$$

When we take $\dot{\tilde{\mathbf{m}}}^2 = (0, 1, 0)$, $\dot{\mathbf{m}}^2 = (1, 0, 0)$, we have

$$\boldsymbol{\lambda}^0 = \left(\frac{\varkappa-1}{2} R^0 + c_0, -1, 0, 0\right),$$
$$\boldsymbol{\lambda}^1 = (-R^0, 1, 0, 0),$$
$$\boldsymbol{\lambda}^2 = (R^0, -1, 0, 0),$$
$$\tilde{\boldsymbol{\lambda}}^2 = (-R^1, 0, 1, 0),$$

as the corresponding $\boldsymbol{\lambda}$ vectors.

The form of $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ assumed leads to the following Pfaff system:

$$\begin{aligned} dR^0 &= \eta^0 \left(\left(\frac{\varkappa+1}{2} R^0 + c_0\right) dt - dx\right), \\ dR^1 &= \eta^1(-R^0 dt + dx), \\ dR^2 &= \eta^2(R^0 dt - dx) + \tilde{\eta}^2(-R^1 dt - dy). \end{aligned} \tag{3.38}$$

Thus we have

$$c = \frac{\varkappa-1}{2} R^0 + c_0, \quad u = R^0, \quad v = R^1, \quad w = R^2$$

for the velocity components.

The first Eq. (3.38) easily integrates, giving:

$$R^0 = \varphi\left(\frac{\varkappa+1}{2} R^0 t + c_0 t - x\right), \tag{3.39}$$

while the remaining Eqs. (3.38) lead to the following Eqs. for R^1:

$$\frac{\partial R^1}{\partial t} + R^0 \frac{\partial R^1}{\partial x} = 0, \quad R^1_{,y} = R^1_{,z}\, 0, \tag{3.40$_1$}$$

and for R^2:

$$\frac{\partial R^2}{\partial t} + R^0 \frac{\partial R^2}{\partial x} + R^1 \frac{\partial R^2}{\partial y} = 0, \quad R^2_{,z} = 0, \tag{3.40$_2$}$$

which may be integrated successively.

The Eqs. $(3.40)_1$ and $(3.40)_2$ are equivalent to

$$\frac{dv}{dt} = 0, \quad \frac{dw}{dt} = 0,$$

$$v_{,y} = v_{,z} = w_{,z} = 0.$$

Case b). The conditions of involution

$$\overset{\bullet}{\Delta}{}^0_1 = 0, \quad \overset{\bullet}{\Delta}{}^0_2 = 0, \quad \overset{\bullet}{\Delta}{}^1_0 = 0, \quad \overset{\bullet}{\Delta}{}^1_2 = 0, \quad \overset{\bullet}{\Delta}{}^2_0 = 0,$$

$$\overset{\bullet}{\Delta}{}^2_1 = 0, \quad (\overset{\bullet}{\Delta}{}^1_2 = -\overset{\bullet}{\Delta}{}^2_1)$$

will be satisfied if we take the same configuration of $\boldsymbol{\gamma}$ vectors as in case a):

$$\boldsymbol{\gamma}_0 = \left(\frac{\varkappa-1}{2}, 1, 0, 0\right), \quad \boldsymbol{\gamma}_1 = (0, 0, 1, 0), \quad \boldsymbol{\gamma}_2 = (0, 0, 0, 1),$$

and $\overset{\bullet}{\mathbf{m}}{}^1 = \overset{\bullet}{\boldsymbol{\gamma}}_2$, $\overset{\bullet}{\mathbf{m}}{}^2 = \overset{\bullet}{\boldsymbol{\gamma}}_1$.

Thus,

$$\boldsymbol{\lambda}^0 = \left(\frac{\varkappa+1}{2} R^0 + c_0, -1, 0, 0\right),$$

$$\boldsymbol{\lambda}^1 = (-R, 1, 0, 0),$$

$$\boldsymbol{\lambda}^2 = (R^0, -1, 0, 0),$$

are corresponding $\boldsymbol{\lambda}$ vectors. The velocity components are given by the expressions:

$$c = \frac{\varkappa-1}{2} R^0 + c_0, \quad u = R^0, \quad v = R^1, \quad w = R^2.$$

The corresponding Pfaff system is:

$$\begin{aligned} dR^0 &= \eta^0 \left(\left(\frac{\varkappa+1}{2} R^0 + c_0 \right) dt - dx \right), \\ dR^1 &= \eta^1 (-R^0 dt + dx), \\ dR^2 &= \eta^2 (R^0 dt - dx). \end{aligned} \tag{3.41}$$

From the first equation we have

$$R^0 = \varphi \left(\left(\frac{\varkappa+1}{2} R^0 + c_0 \right) t - x^0 \right). \tag{3.42}$$

The last two equations in (3.41) indicate that dR^1 and dR^2 are linearly dependent. Thus both functions R^1 and R^2 are functionally dependent and are given as the solutions of

$$\frac{\partial R}{\partial t} + R^0 \frac{\partial R}{\partial x} = 0, \tag{3.43}$$

or

$$\frac{dR}{dt} = 0, \quad R_{,y} = R_{,z} = 0.$$

The hodograph of the flow is two-dimensional, although we have used three independent vectors $\boldsymbol{\gamma}$. The shape of this two-dimensional surface lying in the hyperplane spanned by $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ depends on the initial conditions for the flow.

3.4.4'. Solution of type $C-C-P-P$. Solution of this type may be obtained only in the case $C_1-C_1-P_1-P_1$. We assume

$$\mathbf{h} = \boldsymbol{\gamma}_1 R^1+\boldsymbol{\gamma}_2 R^2+\boldsymbol{\gamma}_3 R^3+\boldsymbol{\gamma}_4 R^4+\mathbf{h}_0$$

as the hodograph, where $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ denote the C-type vectors and $\boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4$ — the P-type vectors. The conditions of involution

$$\begin{aligned}
&\dot{\Delta}^1_2 = 0, \quad \dot{\Delta}^1_3 = 0, \quad \dot{\Delta}^1_4 = 0, \quad \dot{\Delta}^2_1 = 0, \quad \Delta^2_3 = 0,\\
&\dot{\Delta}^2_4 = 0, \quad \dot{\Delta}^1_2 = -\dot{\Delta}^2_1,\\
&\dot{\Delta}^3_1 = 0, \quad \dot{\Delta}^3_2 = 0, \quad \dot{\Delta}^3_4 = 0, \quad \dot{\Delta}^4_1 = 0, \quad \dot{\Delta}^4_2 = 0,\\
&\dot{\Delta}^4_3 = 0, \quad \dot{\Delta}^3_4 = -\dot{\Delta}^4_3,
\end{aligned}$$

may be satisfied if we take $\boldsymbol{\gamma}^0_1 = \boldsymbol{\gamma}^0_2$ and $\boldsymbol{\gamma}^*_1 = -\boldsymbol{\gamma}^*_2$ and $\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_p\rangle = 0$, $p = 3, 4$ and $\dot{\mathbf{m}}^p$ assumed as $\dot{\mathbf{m}}^3 = \dot{\boldsymbol{\gamma}}_4$ and $\dot{\mathbf{m}}^4 = \dot{\boldsymbol{\gamma}}_3$. By a rotation of the coordinate system, we may impart to the $\boldsymbol{\gamma}$ vectors the form:

$$\boldsymbol{\gamma}_1 = \left(\frac{\varkappa-1}{2}, 1, 0, 0\right), \quad \boldsymbol{\gamma}_2 = \left(\frac{\varkappa-1}{2}, -1, 0, 0\right), \quad \boldsymbol{\gamma}_3 = (0, 0, 1, 0),$$

$$\boldsymbol{\gamma}_4 = (0, 0, a, b).$$

The corresponding $\boldsymbol{\lambda}$ vectors are then

$$\begin{aligned}
\boldsymbol{\lambda}^1 &= \left(\frac{\varkappa+1}{2} R^1+\frac{3-\varkappa}{2} R^2+c, -1, 0, 0\right),\\
\boldsymbol{\lambda}^2 &= \left(\frac{3-\varkappa}{2} R^1+\frac{\varkappa+1}{2} R^2+c, 1, 0, 0\right),\\
\boldsymbol{\lambda}^3 &= b(R^2-R^1, 1, 0, 0),\\
\boldsymbol{\lambda}^4 &= -b(R^2-R^1, 1, 0, 0).
\end{aligned}$$

For the velocity components, we have

$$c = \frac{\varkappa-1}{2}(R^1+R^2)+c_0, \quad u = R^1-R^2, \quad v = R^3+aR^4, \quad w = R^4.$$

Since the vectors $\boldsymbol{\lambda}^i$ $(i = 1, 2, 3, 4)$ span a two-dimensional plane, the solution will describe a class of flows with two-dimensional hodographs and will depend on four arbitrary functions of one argument. The functions R^i, $i = 1, 2, 3, 4$, will be dependent on t and x only. The Pfaff system for R^1 and R^2 leads to the following system of partial differential equations:

$$(3.44)_1 \qquad \begin{aligned} &\frac{\partial R^1}{\partial t}+\left(\frac{\varkappa+1}{2}R^1+\frac{3-\varkappa}{2}R^2+c_0\right)\frac{\partial R^1}{\partial x}=0, \quad R^1_{,y}=R^1_{,z}=0,\\ &\frac{\partial R^2}{\partial t}-\left(\frac{\varkappa+1}{2}R^2+\frac{3-\varkappa}{2}R^1+c_0\right)\frac{\partial R^2}{\partial x}=0, \quad R^2_{,y}=R^2_{,z}=0, \end{aligned}$$

and for R^3 and R^4 to two equations

$$(3.44)_2 \qquad \begin{aligned} &\frac{\partial R^3}{\partial t}+(R^1-R^2\ \frac{\partial R^3}{\partial x}=0, \quad R^3_{,y}=R^3_{,z}=0,\\ &\frac{\partial R^4}{\partial t}+(R^1-R^2)\frac{\partial R^4}{\partial x}=0, \quad R^4_{,y}=R^4_{,z}=0. \end{aligned}$$

Using the flow parameters c, u, v, w, the system of Eqs. (3.44) may be transformed as follows:

$$(3.45) \qquad \begin{aligned} &\frac{dc}{dt}+\frac{\varkappa-1}{2}c\frac{\partial u}{\partial x}=0, \quad c_{,y}=c_{,z}=0,\\ &\frac{2c}{\varkappa-1}\frac{\partial c}{\partial x}+\frac{du}{dt}=0, \quad u_{,y}=u_{,z}=0,\\ &\frac{dv}{dt}=0, \quad v_{,y}=v_{,z}=0,\\ &\frac{dw}{dt}=0, \quad w_{,y}=w_{,z}=0. \end{aligned}$$

The first two Eqs. (3.45) are the equations of one-dimensional nonstationary flow, whereas the next two govern a perpendicular rotational flow.

In all cases of the mixed type considered (i.e., when we have taken the $\boldsymbol{\gamma}$ vectors of the type C and P), it can be seen that a certain rotational flow is superimposed on a potential flow. The rotational flow is perpendicular to the potential flow; it depends on the same variables as the potential flow and satisfies the equation

$$\frac{d}{dt}\dot{\mathbf{h}}=0.$$

3.4.5. Pure P-type solutions. It may be verified that all P_1-P_1 and P_1-P_2 solutions may be obtained from $C-P_1-P_1$ and $C-P_1-P_2$ solutions by making the amplitude of C-wave equal to zero (in other words by appropriate degeneration of the above solutions). The conditions of involution cannot be satisfied in the case P_2-P_2. Further, let us observe that for $\boldsymbol{\gamma}_i \in P$ also any linear combination of $\boldsymbol{\gamma}_i$ belongs to P. This fact makes possible to superpose an arbitrary number of simple waves.

Indeed, in the case of $P_1-P_1-\ldots-P_1$ solutions, we have

$$\mathbf{h}=\boldsymbol{\gamma}_1R^1+\boldsymbol{\gamma}_2R^2+\ldots+\boldsymbol{\gamma}_kR^k, \quad \boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\ldots,\boldsymbol{\gamma}_k\in P,$$

for which the condition of involution takes the form

$$\Delta^\alpha_\beta=[\dot{\mathbf{m}}^\alpha,\dot{\boldsymbol{\gamma}}_\alpha,\dot{\boldsymbol{\gamma}}_\beta]\,[[\dot{\mathbf{m}}^\alpha,\dot{\boldsymbol{\gamma}}_\alpha],[\dot{\mathbf{m}}^\beta,\dot{\boldsymbol{\gamma}}_\beta]]=0$$

and may be satisfied if $\dim\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_k\} = 2$. In this case, we can take

$$\dot{\boldsymbol{\lambda}}^\alpha = [\dot{\mathbf{m}}^\alpha, \dot{\boldsymbol{\gamma}}_\alpha] = \dot{\boldsymbol{\lambda}} \perp \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_k\}.$$

In such a case, the resulting solution is given by

$$R^i = \varphi^i(\dot{\lambda}_\nu x^\nu)$$

and again describes a simple wave. Physically, this means that the result of such interaction of the simple waves

$$\mathbf{h} = \boldsymbol{\gamma}_1 \varphi^1(\dot{\lambda}_\nu x^\nu),$$
$$\mathbf{h} = \boldsymbol{\gamma}_2 \varphi^2(\dot{\lambda}_\nu x^\nu),$$
$$\dots\dots\dots\dots\dots\dots$$
$$\mathbf{h} = \boldsymbol{\gamma}_k \varphi^k(\dot{\lambda}_\nu x^\nu)$$

is again a simple wave.

Table 1

k	1	2	3	4
Types of γ vectors	C_1 simple waves	C_1-C_1 $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = \frac{1-\varkappa}{2}$ $1 \leqslant \varkappa \leqslant 3$	$C_1-C_1-C_1$ $\cos(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}}_b) = \frac{1-\varkappa}{2}$ $1 \leqslant \varkappa \leqslant 2$	$C_1-C_1-C_1-C_1$ *
		C_1-C_1 $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = -1$	$C_1-C_1-P_2$ $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = \frac{1-\varkappa}{2}$ $\cos(\dot{\boldsymbol{\gamma}}_a, \dot{\boldsymbol{\gamma}}_0) = 0$	$C_1-C_1-C_1-P_{1,2}$ *
		C_1-P_2 $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = 0$ (C_1-P_1) (2)	$C_1-C_1-P_2$ $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = -1$ $\cos(\dot{\boldsymbol{\gamma}}_a, \dot{\boldsymbol{\gamma}}_0) = 0$	
	P_2 simple waves non-planar	P_1-P_2 simple waves	$C_1-P_1-P_2$ $\cos(\dot{\boldsymbol{\gamma}}_0, \dot{\boldsymbol{\gamma}}_p) = 0$ $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = 0$ $(C_1-P_1-P_1)$ (2)	$C_1-C_1-P_1-P_1$ $\cos(\dot{\boldsymbol{\gamma}}_1, \dot{\boldsymbol{\gamma}}_2) = -1$ $\cos(\dot{\boldsymbol{\gamma}}_0, \dot{\boldsymbol{\gamma}}_p) = a$ (1)
				$C_1-P_1-P_1-P_1$ *
	P_1	P_1-P_1	$P_1-P_1-P_1$	$P_1-P_1-P_1-P_1$

k — the number of Riemann invariants considered (i.e., the number of interacting waves).

(1) These solutions with four indepedent γ vectors are only double waves with four arbitrary functions and not the plane hodograph. The corresponding flows depend only on t and x.

(2) Each of these solutions must have the same configuration of γ vectors as the solution placed above it in the Table.

The discussion presented in this Section may be summarized in the Table 1, where the arrows indicate the possible kinds of flow degeneration (7) and the asterisk * shows that configuration of $\boldsymbol{\gamma}$ vectors cannot satisfy the conditions of involution under which all the solutions considered up till now were constructed. In other case, it needs the prolongations of the Pfaff system in a way analogous to that given in Sec. 4 in agreement with the Cartan algorithm outlined in Sec. 2.

4. Investigation of a case in which prolongations are needed

4.1. Formulation of the problem

The flows considered in this Section are governed by the system:

$$
\begin{aligned}
&\chi(c_t+uc_x+vc_y)+c(u_x+v_y) = 0,\\
(4.1)\qquad &u_t+uu_x+vu_y+\chi cc_x = 0,\\
&v_t+uv_x+vv_y+\chi cc_y = 0,
\end{aligned}
$$

where, as in Sec. 3, c denotes speed of sound and u and v are velocity components with respect to the x and y axis, respectively, and $\chi = 2/(\varkappa-1)$.

We shall be concerned with *nondegenerated* solutions defined by the linear Riemann invariants:

$$(4.2)\qquad c = \gamma_a^1 R^a, \quad u = \gamma_a^2 R^a, \quad v = \gamma_a^3 R,^a$$

where characteristic vectors $\boldsymbol{\gamma}_a = (\gamma_a^1, \gamma_a^2, \gamma_a^3)$ are taken from the cone $\chi^2(\gamma^1)^2 = (\gamma^2)^2 + (\gamma^3)^2$. The fact that the solutions under consideration should be nondegenerated means that $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ vectors form a basis in the space H. The corresponding knotted $\boldsymbol{\lambda}^a$ vectors are given as

$$(4.3)\qquad \boldsymbol{\lambda}^a = (\chi c\gamma_a^1+u\gamma_a^2+v\gamma_a^3, -\gamma_a^2, -\gamma_a^3),$$

or on substitution of (4.2), as

$$(4.3')\qquad \boldsymbol{\lambda}^a = (\Lambda_{ab}R^b, -\gamma_a^2, -\gamma_a^3),$$

where coefficients Λ_{ab} symmetric in low indices are given by

$$(4.4)\qquad \Lambda_{ab} = \chi\gamma_a^1\gamma_b^1+\gamma_a^2\gamma_b^2+\gamma_a^3\gamma_b^3.$$

Pfaff's system defining Riemann invariants has the form:

$$(4.5)\qquad \theta^a \equiv dR^a-\eta^a\lambda^a = 0, \quad a = 1, 2, 3,$$

where the linear differential forms λ^a are given by

$$(4.6)\qquad \lambda^a = \Lambda_{ab}R^b dt-\gamma_a^2 dz-\gamma_a^3 dy.$$

These forms may be taken as the forms independent on the solutions of (4.5), since $\det[\lambda^1, \lambda^2, \lambda^3] = \chi c \operatorname{Det}(\gamma_1, \gamma_2, \gamma_3)$. In particular, in terms of these forms we shall express:

$$(4.7)\qquad dt = \frac{A_a\lambda^a}{\chi\Delta c},$$

(7) Such degeneration may arise as the result of too narrow initial conditions.

where

$$\Delta = \mathrm{Det}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3), \quad A_1 = \gamma^2_{[2}\gamma^3_{3]}, \quad A_2 = \gamma^2_{[3}\gamma^3_{1]}, \quad A_3 = \gamma^2_{[1}\gamma^3_{2]}.$$

By exterior differentiation of (4.5), we obtain:

$$d\theta^a = \lambda^a \wedge d\eta^a - \eta^a d\lambda^a, \tag{4.8}$$

where $d\lambda^a = \Lambda_{ab} dR^b \wedge dt$ and by the formula (4.7)

$$d\lambda^a = \frac{\Lambda_{ab} A_c \eta^b}{\chi \Delta c} \lambda^b \wedge \lambda^c \quad \text{[on solution of (4.5)]}, \tag{4.9}$$

or more explicitly

$$d\lambda^a = \frac{1}{\chi \Delta c}[(\Lambda_{a1}\eta^1 A_2 - \Lambda_{a2}\eta^2 A_1)\lambda^1 \wedge \lambda^2 + (\Lambda_{a1}\eta^1 A_3 - \Lambda_{a3}\eta^3 A_1)\lambda^1 \wedge \lambda^3 + (\Lambda_{a2}\eta^2 A_3 - \Lambda_{a3}\eta^3 A_2)\lambda^2 \wedge \lambda^3]. \tag{4.10}$$

The equations closing the system (4.5) thus reduce to the

$$d\theta^a = \lambda^a \wedge d\eta^a - \eta^a d\lambda^a = 0 \quad (\mathrm{mod}\,\theta^a = 0). \tag{4.11}$$

On the integral manifolds on which forms λ^a are independent differentials, $d\eta^a$ should be decomposed as follows

$$d\eta^a = l^a_b \lambda^b, \quad a = 1, 2, 3. \tag{4.12}$$

The new free parameters l^a_b appearing in this decomposition are determined on substitution of (4.12) into (4.11) and by the requirement that the coefficients of $\lambda^1 \wedge \lambda^2$, $\lambda^1 \wedge \lambda^3$ and $\lambda^2 \wedge \lambda^3$ should vanish.

In this way, we arrive at the conclusion that l^a_b are free for $a = b$, l^a_b are given functions of η^1, η^2, η^3 for $a \neq b$, and that the coefficients on the diagonal in the decomposition (4.10) should vanish. Otherwise,

$$\begin{aligned} KA_3\eta^2 - LA_2\eta^3 &= 0, \\ -KA_3\eta^1 + MA_1\eta^3 &= 0, \\ LA_2\eta^1 - MA_1\eta^2 &= 0, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} K = \Lambda_{12} &= \chi\gamma^1_1\gamma^1_2 + \gamma^2_1\gamma^2_2 + \gamma^3_1\gamma^3_2, \\ L = \Lambda_{31} &= \chi\gamma^1_1\gamma^1_3 + \gamma^2_1\gamma^2_3 + \gamma^3_1\gamma^3_3, \\ M = \Lambda_{32} &= \chi\gamma^1_2\gamma^1_3 + \gamma^2_2\gamma^2_3 + \gamma^3_2\gamma^3_3. \end{aligned} \tag{4.14}$$

The formula (4.13) provides a constraint on η^1, η^2, η^3. It cannot be eliminated, otherwise the solutions will be degenerated. Since only two of the equations in (4.13) are independent, we write

$$\frac{\eta^1}{MA_1} = \frac{\eta^2}{LA_2} = \frac{\eta^3}{KA_3} = \xi,$$

where ξ is a new free parameter.

Thus we see that our problem differs from those considered in Sec. 3, where by means of strong conditions of involution the constraints on free parameters in an original Pfaff system are eliminated. In this Section, we are forced to begin our problem again with one free parameter ξ, otherwise it would be a contradiction.

4.2. A new free Pfaff system and the first prolongation

Now, we start our considerations with the Pfaff system

$$\begin{aligned}\theta^1 &= dR^1 - MA_1\xi\lambda^1 = 0,\\ \theta^2 &= dR^2 - LA_2\xi\lambda^2 = 0,\\ \theta^3 &= dR^3 - KA_3\xi\lambda^3 = 0.\end{aligned} \tag{4.15}$$

By exterior differentiation, we obtain

$$\begin{aligned}d\theta^1 &= MA_1(\lambda^1\wedge d\xi - \xi d\lambda^1),\\ d\theta^2 &= LA_2(\lambda^2\wedge d\xi - \xi d\lambda^2),\\ d\theta^3 &= KA_3(\lambda^3\wedge d\xi - \xi d\lambda^3),\end{aligned} \tag{4.16}$$

where on the solutions of (4.15)

$$\begin{aligned}d\lambda^a = \frac{\xi}{\chi\Delta c}\,[A_1A_2(\Lambda_{a1}M - \Lambda_{a2}L)\lambda^1\wedge\lambda^2 + A_1A_3(\Lambda_{a1}M - \Lambda_{a3}K)\lambda^1\wedge\lambda^3\\ + A_2A_3(\Lambda_{a2}L - \Lambda_{a3}K)\lambda^2\wedge\lambda^3].\end{aligned} \tag{4.17}$$

If we close the system (4.15) and substitute in those equations

$$d\xi = l_1\lambda^1 + l_2\lambda^2 + l_3\lambda^3, \tag{4.18}$$

we arrive at six conditions for new free parameters. These algebraic conditions are consistent if, and only if,

$$\begin{aligned}A_2(\Lambda_{22}L - KM) &= A_3(\Lambda_{33}K - LM),\\ A_1(\Lambda_{11}M - KL) &= A_3(\Lambda_{33}K - LM),\end{aligned} \tag{4.19}$$

and the corresponding values of l_a are as follows:

$$\begin{aligned}l_1 &= \frac{\xi^2}{\chi\Delta c}A_1A_2(\Lambda_{22}L - KM),\\ l_2 &= \frac{\xi^2}{\chi\Delta c}A_1A_3(\Lambda_{33}K - ML),\\ l_3 &= \frac{\xi^2}{\chi\Delta c}A_1A_2(\Lambda_{11}M - KL).\end{aligned} \tag{4.20}$$

With these values of l_a, the Eq. (4.18) should be joined to the original system. Thus we arrive at the first prolongation of (4.15), in which there are no free parameters:

$$\begin{aligned}\theta^1 &= dR^1 - A_1M\xi\lambda^1 = 0,\\ \theta^2 &= dR^2 - A_2L\xi\lambda^2 = 0,\end{aligned} \tag{4.21}$$

$$\theta^3 = dR^3 - A_3 K\xi\lambda^3 = 0,$$

(4.21) [cont.]
$$\theta^4 = d\left(\frac{1}{\xi}\right) + c^{-1}(L_1\lambda^1 + L_2\lambda^2 + L_3\lambda^3) = 0,$$

where

(4.22)
$$L_1 = \frac{1}{\chi\Delta} A_1 A_2(\Lambda_{22} L - KM) = \frac{1}{\chi\Delta} A_1 A_3(\Lambda_{33} K - LM),$$
$$L_2 = \frac{1}{\chi\Delta} A_2 A_3(\Lambda_{33} K - ML) = \frac{1}{\chi\Delta} A_1 A_2(\Lambda_{11} M - KL),$$
$$L_3 = \frac{1}{\chi\Delta} A_1 A_3(\Lambda_{11} M - KL) = \frac{1}{\chi\Delta} A_2 A_3(\Lambda_{22} L - MK)$$

are constant coefficients. Let us observe that integrability of (4.21) is finally controlled only by choice of the constants $\gamma_a^1, \gamma_a^2, \gamma_a^3$, $a = 1, 2, 3$ and that its general solution may depend only on arbitrary constants.

4.3. Integrability conditions

By virtue of (4.21) and (4.22), we have $d\theta^1 = 0$, $d\theta^2 = 0$, $d\theta^3 = 0$ on the solutions of (4.21). Thus we are left with the condition

(4.23)
$$d\theta^4 = 0 \quad (\mathrm{mod}\,\theta^1 = 0,\ \theta^2 = 0,\ \theta^3 = 0,\ \theta^4 = 0).$$

However,

$$d\theta^4 = -c^{-2} dc \wedge (L_1\lambda^1 + L_2\lambda^2 + L_3\lambda^3) + c^{-1}(L_1 d\lambda^1 + L_2 d\lambda^2 + L_3 d\lambda^3),$$

and

$$dc = \gamma_a^1 dR^a = \xi(\gamma_1^1 A_1 M\lambda^1 + \gamma_2^1 A_2 L\lambda^2 + \gamma_3^1 A_3 K\lambda^3) \quad (\mathrm{mod}\,\theta^a = 0,\ a = 1, 2, 3).$$

From (4.17), making use of (4.22), we can write the following expressions:

(4.24)
$$d\lambda^1 = \xi c^{-1}(L_2\lambda^1 \wedge \lambda^2 + L_3\lambda^1 \wedge \lambda^3),$$
$$d\lambda^2 = \xi c^{-1}(-L_1\lambda^1 \wedge \lambda^2 + L_3\lambda^2 \wedge \lambda^3),$$
$$d\lambda^3 = \xi c^{-1}(-L_1\lambda^1 \wedge \lambda^3 - L_2\lambda^2 \wedge \lambda^3)$$

for differentials of λ^a forms, taken on solution of (4.21). The conditions (4.23) may be finally expressed as follows:

(4.25)
$$(\gamma_1^1 A_1 M\lambda^1 + \gamma_2^1 A_2 L\lambda^2 + \gamma_3^1 A_3 K\lambda^3) \wedge (L_1\lambda^1 + L_2\lambda^2 + L_3\lambda^3) = \\ = L_1 d\lambda^1 + L_2 d\lambda^2 + L_3 d\lambda^3.$$

It may easily be verified that, by virtue of (4.24), the right-hand side of (4.25) vanishes identically. The corresponding coefficients of $\lambda^1 \wedge \lambda^2$, $\lambda^1 \wedge \lambda^3$, $\lambda^2 \wedge \lambda^3$ on the left-hand side must be subsequently equal to zero:

(4.26)
$$\gamma_1^1 A_1 ML_2 - \gamma_2^1 A_2 LL_1 = 0,$$
$$\gamma_1^1 A_1 ML_3 - \gamma_3^1 A_3 KL_1 = 0,$$
$$\gamma_2^1 A_2 LL_3 - \gamma_3^1 A_3 KL_2 = 0.$$

After certain manipulations, making use of (4.22), these conditions may be reduced to the form

$$\frac{\gamma_1^1}{L} = \frac{\gamma_2^1}{M}, \quad \frac{\gamma_1^1}{K} = \frac{\gamma_3^1}{M}, \quad \frac{\gamma_2^1}{K} = \frac{\gamma_3^1}{L}. \tag{4.27}$$

Among these conditions only two are independent, since we have

$$\frac{\gamma_1^1}{KL} = \frac{\gamma_2^1}{KM} = \frac{\gamma_3^1}{LM}. \tag{4.28}$$

In the process of formation of integrability conditions, the conditions (4.27) enable us to give a much simpler form for the conditions (4.19). Performing the corresponding calculations, we arrive at the following expressions for (4.19):

$$\frac{A_1C_3}{\gamma_1^1\gamma_1^3} = \frac{A_3C_1}{\gamma_3^1\gamma_1^3}, \quad \frac{A_2C_3}{\gamma_3^1\gamma_3^3} = \frac{A_3C_2}{\gamma_3^1\gamma_2^3}, \tag{4.29}$$

where the meaning of C_1, C_2, C_3 and A_1, A_2, A_3 is clear from the expansions of $\mathrm{Det}(\gamma_1, \gamma_2, \gamma_3) = \Delta$

$$\Delta = \begin{vmatrix} \gamma_1^1 & \gamma_2^1 & \gamma_3^1 \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \\ \gamma_1^3 & \gamma_2^3 & \gamma_3^3 \end{vmatrix} = \begin{cases} \gamma_1^1 A_1 + \gamma_2^1 A_2 + \gamma_3^1 A_3, \\ \gamma_1^2 B_1 + \gamma_2^2 B_2 + \gamma_3^2 B_3, \\ \gamma_1^3 C_1 + \gamma_2^3 C_2 + \gamma_3^3 C_3. \end{cases} \tag{4.30}$$

Let us observe, at this opportunity, that

$$W = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \Delta^2. \tag{4.31}$$

4.4. Geometrical meaning of integrability conditions

The vectors $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ taken from the cone C: $\chi^2(\gamma_a^1)^2 = (\gamma_a^2)^2 + (\gamma_a^3)^2$ may be characterized by two parameters; their coordinate γ_a^1 along c-axis and by the angle φ_a. Thus we have

$$\boldsymbol{\gamma}_a = \gamma_a^1(1, \chi\cos\varphi_a, \chi\sin\varphi_a), \tag{4.32}$$

and consequently

$$\begin{aligned} A_1 &= \chi^2\gamma_2^1\gamma_3^1\sin(\varphi_3-\varphi_2), \\ A_2 &= \chi^2\gamma_1^1\gamma_3^1\sin(\varphi_1-\varphi_3), \\ A_3 &= \chi^2\gamma_1^1\gamma_2^1\sin(\varphi_2-\varphi_1); \\ B_1 &= \chi\gamma_2^1\gamma_3^1(\sin\varphi_3-\sin\varphi_2), \\ B_2 &= \chi\gamma_1^1\gamma_3^1(\sin\varphi_1-\sin\varphi_3), \\ B_3 &= \chi\gamma_1^1\gamma_2^1(\sin\varphi_2-\sin\varphi_1); \\ C_1 &= \chi\gamma_2^1\gamma_3^1(\cos\varphi_2-\cos\varphi_3), \\ C_2 &= \chi\gamma_1^1\gamma_3^1(\cos\varphi_3-\cos\varphi_1), \\ C_3 &= \chi\gamma_1^1\gamma_2^1(\cos\varphi_1-\cos\varphi_2). \end{aligned} \tag{4.33}$$

Writing (4.27) in a more conscise form

$$\gamma_1^2 C_1 = \gamma_1^3 B_1, \quad \gamma_2^2 C_2 = \gamma_2^3 B_2, \tag{4.34}$$

and taking into account (4.32) and (4.33), we arrive at a set of two trigonometrical equations with unknown angles $\varphi_1, \varphi_2, \varphi_3$

$$\begin{aligned} \cos\varphi_1(\cos\varphi_3 - \cos\varphi_2) &= \sin\varphi_1(\sin\varphi_2 - \sin\varphi_3), \\ \cos\varphi_2(\cos\varphi_1 - \cos\varphi_3) &= \sin\varphi_2(\sin\varphi_3 - \sin\varphi_1), \end{aligned} \tag{4.35}$$

and with the solutions

$$\varphi_1 = \varphi \text{ (arbitrary)}, \quad \varphi_2 = \varphi + \frac{2}{3}\pi, \quad \varphi_3 = \varphi + \frac{4}{3}\pi. \tag{4.36}$$

Effecting the same substitution in (4.29), we obtain

$$\begin{aligned} \frac{\sin(\varphi_3 - \varphi_2)(\cos\varphi_2 - \cos\varphi_1)}{\sin\varphi_3} &= \frac{\sin(\varphi_2 - \varphi_1)(\cos\varphi_3 - \cos\varphi_2)}{\sin\varphi_1}, \\ \frac{\sin(\varphi_1 - \varphi_3)(\cos\varphi_3 - \cos\varphi_2)}{\sin\varphi_1} &= \frac{\sin(\varphi_3 - \varphi_2)(\cos\varphi_1 - \mathrm{cis}\,\varphi_3)}{\sin\varphi_2}. \end{aligned} \tag{4.37}$$

These conditions may by means of (3.35) be reduced to the form

$$\begin{aligned} \cos(\varphi_1 - \varphi_2) &= \cos(\varphi_3 - \varphi_2), \\ \cos(\varphi_3 - \varphi_2) &= \cos(\varphi_3 - \varphi_1), \end{aligned} \tag{4.38}$$

which is obviously satisfied by the solutions (4.36) of (4.35).

Thus we see that the configuration of a hodograph characteristic basis $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ in a hodograph space H for which linear Riemann invariants exist is given by the conditions (4.36). There are four arbitrary constants γ_a^1, $a = 1, 2, 3$ and φ which may be taken arbitrarily in choosing such basis.

4.5. Integration of Pfaff's system

Since the integrability conditions (4.36) and therefore (4.27) are satisfied, we can integrate Pfaff system (4.21). Expressing λ^1, λ^2 and λ^3 from the first three equations, we obtain on substitution in fourth equation:

$$\theta^4 = d\ln\frac{1}{\xi} + c^{-1}\left(\frac{L_1}{A_1 H} dR^1 + \frac{L_2}{A_2 L} dR^2 + \frac{L_3}{A_3 K} dR^3\right). \tag{4.39}$$

By virtue of (4.29) and (4.27), this equation may be reduced to the form

$$d\ln\frac{1}{\xi} = \frac{E}{\chi\Delta c}(\gamma_1^1 dR^1 + \gamma_2^1 dR^2 + \gamma_3^1 dR^3) = \frac{E}{\chi\Delta}\frac{dc}{c}, \tag{4.40}$$

where E denotes the common value of the quotiens:

$$\frac{A_1 A_2 C_3}{\gamma_1^1 \gamma_2^1 \gamma_3^3} = \frac{A_1 A_3 C_2}{\gamma_1^1 \gamma_3^1 \gamma_2^3} = \frac{A_2 A_3 C_1}{\gamma_2^1 \gamma_3^1 \gamma_1^3} = E. \tag{4.41}$$

(see 4.29).

The Eq. (4.41) may easily be integrated with the result

$$\xi = \xi_0 c^{-\frac{E}{\chi\Delta}}. \tag{4.42}$$

Now, we return to the remaining equations of Pfaff's system (4.21). First, we resolve these equations with respect to dt, dx and dy. On substitution of (4.42) and again using integrability conditions, we may observe that the right-hand sides of the expressions obtained are exact differentials. Performing integration, we finally obtain

$$\begin{aligned} t &= \tilde{C}_1(\gamma_1^1 R^1+\gamma_2^1 R^2+\gamma_3^1 R^3)^{-\chi/2}+\tilde{C}_2, \\ x &= \tilde{C}_1(\gamma_1^2 R^1+\gamma_2^2 R^2+\gamma_3^2 R^3)(\gamma_1^1 R^1+\gamma_2^1 R^2+\gamma_3^1 R^3)^{-\chi/2}+\tilde{C}_3, \\ y &= \tilde{C}_1(\gamma_1^3 R^1+\gamma_2^3 R^2+\gamma_3^3 R^3)(\gamma_1^1 R^1+\gamma_2^1 R^2+\gamma_3^1 R^3)^{-\chi/2}+\tilde{C}_0, \end{aligned} \tag{4.43}$$

where $\tilde{C}_1, \ldots, \tilde{C}_0$ are arbitrary constants. The exponent $-\chi/2$ in these formulas is related to the exponent $-E/\chi\Delta$ in (4.42) by the equality

$$\frac{E}{\chi\Delta} = -\frac{\chi}{2}, \tag{4.44}$$

which may be proved to be true by means of (4.40) and the identity

$$\frac{\gamma_1^2}{A_{[2}C_{3]}} = \frac{\gamma_2^2}{A_{[3}C_{1]}} = \frac{\gamma_3^2}{A_{[2}C_{1]}} = \frac{\Delta}{W} = \frac{1}{\Delta} \qquad \text{(cf 4.31)}, \tag{4.45}$$

and by substitution of (4.33) taken for angles (4.36).

The formulas (4.43), together with those for c, u, v given by (4.2), represent the solutions of the Eqs. (4.1). Let us observe that as a result of the integrability conditions and the prolongation procedure, functional freedom is eliminated from such superposition of simple waves. In conclusion, let us also observe that Riemann invariants may be subsequently eliminated with the results:

$$c = (\tilde{C}_1 t+\tilde{C}_2)^{-\chi/2}, \quad u = \frac{\tilde{C}_1 x+\tilde{C}_3}{\tilde{C}_1 t+\tilde{C}_2}, \quad v = \frac{\tilde{C}_1 y+\tilde{C}^4}{\tilde{C}_1 t+\tilde{C}_2.}. \tag{4.46}$$

At the same time, also, the constant parameter φ is shown to be inessential in the solution.

This form of solution suggest an analogous form in the general case of the system (3.1) with four independent variables. In fact, as may be verified, the solution has the form:

$$c = (\tilde{C}_1 t+\tilde{C}_2)^{-\chi/2}, \quad u = \frac{\tilde{C}_1 x+\tilde{C}_3}{\tilde{C}_1 t+\tilde{C}_2}, \quad v = \frac{\tilde{C}_1 y+\tilde{C}_4}{\tilde{C}_1 t+\tilde{C}_2}, \quad w = \frac{\tilde{C}_1 z+\tilde{C}_5}{\tilde{C}_1 t+\tilde{C}_2} \tag{4.47}$$

and may be obtained as the solution of the $C_1-C_1-C_1-C_1$ type with symmetrical configuration of constant vectors $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4$ in a manner similar to (4.46).

The solutions obtained are not of great mechanical interest. We present them to show how the "mechanism of prolongation" works, and we expect that such an example, completely analysed, will be of some value in further investigations of case intermediate as regards those from Sec. 3, where there is no maximal functional freedom of solutions of Pfaff's system for Riemann invariants. The case presented in this Section represents a second extreme case — no functional freedom at all.

References

1. M. BURNAT, *Riemann invariants*, Fluid Dynamics Transactions, **4**, 17–27, 1969.
2. M. BURNAT, *The method of Riemann invariants for multidimensional nonelliptic system*, Bull. Acad. Polon. Sci., Série Sci. Techn., **17**, 11, 12, 97–104, 1969.
3. Z. PERADZYŃSKI, *On algebraic aspects of the generalized Riemann invariants method*, Bull. Acad. Polon. Sci., Série Sci. Techn., **17**, 4, 37–42, 1970.
4. E. CARTAN, *Sur la structure des groupes infinis de transformations* (Chapitre I, *Les systèmes différentiells en involution*), Ann. Ec. Norm., **21**, 3, 153–308 (159–170), Oeuvres Complètes, Partie II, 2, 571–714 (577–589), Gauthiers-Villars, Paris 1953.
5. S. P. FINIKOV, *The method of Cartan's exterior forms in differential geometry* [in Russian], Moscow-Leningrad 1948.
6. Z. PERADZYŃSKI, *Nonlinear plane k-waves and Riemann invariants*, Bull. Acad. Polon. Sci., Série Sci, Techn., **19**, 9, 59–65, 1971.
7. Z. PERADZYŃSKI, *Riemann invariants for non-planar k-waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 10, 67–74, 1971.
8. Z. PERADZYŃSKI, *On certain classes of exact solutions for gas-dynamics equations*, Arch. Appl. Mech., **24**, 2, 287–303, 1972.

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