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Stanisław Jemioło
Warsaw University of Technology
Institute of Structural Mechanics
and
Józef Joachim Telega
Polish Academy of Sciences
Institute of Fundamental Technological Research

REPRESENTATIONS OF TENSOR FUNCTIONS AND APPLICATIONS IN CONTINUUM MECHANICS

Abstract
This paper presents the theory of invariants and tensor functions in a unified manner suitable for applications in the continuum mechanics. Basic principles of the theory have been clarified. Important results on the determination of polynomial and nonpolynomial representation have been presented including higher-order tensor functions, spectral decomposition of tensors and regularity. Applications to the formulations of constitutive relationships are focussed on nonlinear elasticity, plasticity, locking materials, simple and anisotropic fluids.

Contents

Introduction....4
1. Basic notions....4
2. Basic theorems and principles....10
3. Polynomial representations....18
3. Non-polynomial representations....21
5. Invariants of tensor of order greater then two. Representation of tensor-valued functions of order greater then two....46
6. Selected applications to solid mechanics....57
7. Simple fluids and unimodular group....72
8. Applications of the tensor equation $\mathbf{AX} + \mathbf{XA} = \Phi(\mathbf{A}, \mathbf{H})$ to kinematics of continua....82
9. Spectral decomposition of Hooke's tensors....87
References....94

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Introduction

Generally speaking, there are two approaches to constitutive modelling. The first approach is more „intuitive" while the second one is based on the principles of rational mechanics. Obviously, both approaches often overlap. In the second approach the theory of invariants and representation of tensor functions is of vital importance. Materials symmetry are then naturally included into constitutive modelling.

Historical aspects of the invariant theory are sketched in the review papers by Fisher (1966), Rychlewski and Zhang (1991), Telega (1981), cf. also Caldonazzo (1932), Cisotti (1930a, 1930b, 1930c, 1930d), Dieudonné (1971), Gardner (1980), Pastori (1930a, 1930b, 1933), Racah (1933a, 1933b), Somigliana (1894). The formalism we shall use in this contribution is typical for the continuum mechanics. For a modern algebraic setting the reader should refer to Dieudonné and Carrell (1971), Processi (1976), Springer (1977).

Our comprehensive study is focussed on two topics, cf. also Jemioło and Telega (1995). First, we shall present all fundamental notions of the invariant theory and representations of isotropic and anisotropic tensor functions from the point of view of the continuum mechanics. Second, specific applications to constitutive modelling will be presented. Particularly we shall treat nonlinear elasticity, perfect plasticity, perfectly locking materials, as well as simple and anisotropic fluids.

Thorough and more detailed presentation including comprehensive approach to constitutive relationships, will be given in the future.

1. Basic notions

Prior to introducing the notion of a tensor-valued function, or simply a tensor function, we shall briefly discuss the so called concomitants (Kucharzewski and Kuczma, 1964; Telega, 1981).

Let there be given a geometric object ω transforming according to

(1.1) $$\overline{\omega} = f\left(\omega, T_{U_i \to \overline{U}_i}\right), \quad i = 1, \dots, I,$$

where $T_{U_i \to \overline{U}_i}$ stands for the transformation from one admissible coordinate system U_i to another one $\overline{U}_i$. In general, such a transformation is an element of the differential group of order s, denoted by L_s^n (n - dimensional case), which is a Lie group. For instance, if ω is a tensor then $L_1^n = GL(n)$, where $GL(n)$ is the full linear group.

Let now

(1.2) $$\Omega = H(\omega),$$

be a function of the object (1.1), which does not depend on the choice of admissible coordinate system. If Ω is a geometric object, then it is called the *algebraic* (or *geometric*) *concomitant*. Suppose that Ω is such a concomitant of the geometric object (1.1) and

(1.3) $$\overline{\Omega} = F\left(\Omega, T_{U_i \to \overline{U}_i}\right), \quad i = 1, \dots, I.$$

Because the function H cannot depend on the choice of admissible coordinate system, hence

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(1.4) $$\overline{\Omega} = H(\overline{\omega}),$$

From (1.1)-(1.4) we obtain

(1.5) $$H\left[f\left(\omega, T_{U_i \to \overline{U}_i}\right)\right] = F\left[H(\omega), T_{U_i \to \overline{U}_i}\right].$$

Let us pass now to differential concomitants. By Γ we denote the differential prolongation of order p of the object (1.1):

(1.6) $$\Gamma = \left(\omega, \partial_i \omega, \ldots, \partial_{i_1 \ldots i_p} \omega\right), \quad p \leq s,$$

where „∂" stands for the partial differentiation with respect to the coordinates of the system U_i.

By a *differential concomitants* of order p of the geometric object ω we mean every concomitant of Γ. Thus we see that differential concomitants can be treated as a particular case of algebraic concomitants.

Example. 1.1 Let $\Omega = f\left(b_{ij}, \partial_k b_{ij}\right)$, $\det\left[b_{(ij)}\right] \neq 0$, where $b_{(ij)} = \frac{1}{2}\left(b_{ij} + b_{ji}\right)$. It can be shown that

$$\Omega = f_1\left(b_{ij}, \nabla_k b_{[ij]}\right),$$

where $b_{[ij]} = \frac{1}{2}\left(b_{ij} - b_{ji}\right)$.

We note that differential concomitants are important in the study of invariant variational principles. In this paper however, we shall not develop further this interesting topic.

In the continuum mechanics we are not interested in the group $GL(n)$ (n=2 or n=3), but in its subgroups: the proper unimodular group U_n^0, the full orthogonal group O=$O(n)$ and, for anisotropic materials, in the so called material symmetry groups $S \subset O$.

Of main importance for our further developments are isotropic scalar- valued functions

(1.7) $$\omega = \Phi(\mathbf{X}),$$

where **X** denotes a set consisting of symmetric, and skew-symmetric second- order tensors and vectors. The function Φ is isotropic provided that

(1.8) $$\omega = \Phi(\mathbf{X}) = \omega' = \Phi(\mathbf{Q} \circ \mathbf{X}),$$

for each $\mathbf{Q} \in O$; obviously $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. The function Φ is anisotropic if (1.8) is satisfied for each $\mathbf{Q} \in S$.

Note on terminology. The isotropy is connected with invariance (or form invariance) under the full orthogonal group. The hemitropy is connected with the proper orthogonal

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group (Rychlewski 1970a, 1970b). In Montanaro and Pigozzi (1994) hemitropic tensors are called weakly isotropic. We observe that for tensors of even order the isotropy and hemitropy coincide.

Suppose now that Φ is a vector- or tensor-valued function:

(1.9) $$\mathbf{Y} = \underline{F}(\mathbf{X}).$$

$\underline{F}$ is isotropic if

(1.10) $$\forall \mathbf{Q} \in O \quad \mathbf{Q} * \mathbf{Y} = \underline{F}(\mathbf{Q} \circ \mathbf{X}),$$

and anisotropic provided that

(1.11) $$\forall \mathbf{Q} \in S \quad \mathbf{Q} * \mathbf{Y} = \mathbf{Q} * \underline{F}(\mathbf{X}) = \underline{F}(\mathbf{Q} \circ \mathbf{X}).$$

We shall see that representations of vector- and tensor-valued functions can be obtained from the corresponding representations of scalar functions. Moreover, representations of anisotropic functions can be reduced to the representations of isotropic functions by introducing the so called structural tensors (Lokhin, Sedov (1963), Sedov, Lokhin (1963), Boehler (1975, 1978, 1979, 1987a), Zhang and Rychlewski (1990a, 1990b), Rychlewski (1991a, 1991b), Zheng (1993b, 1993c,1994a), Xiao (1996a, 1996b)).

Let us briefly explain the notation $\mathbf{Q} \circ \mathbf{X}$ and $\mathbf{Q} * \mathbf{Y} = \mathbf{Q} * \underline{F}(\mathbf{X})$ by means of a simple example. Suppose that $\mathbf{X} = \{\mathbf{T}, \mathbf{v}\}$ where $\mathbf{T}$ is a second-order tensor and $\mathbf{v}$ a vector. Then $\mathbf{Q} \circ \mathbf{X} = \{\mathbf{Q}\mathbf{T}\mathbf{Q}^T, \mathbf{Q}\mathbf{v}\}$. Similarly, if $\underline{F}$ is a second-order tensor function, then $\mathbf{Q} * \mathbf{Y} = \mathbf{Q} * \underline{F}(\mathbf{X}) = \mathbf{Q}\underline{F}(\mathbf{X})\mathbf{Q}^T$.

Isotropic (anisotropic) tensor functions satisfying (1.10) ((1.11)) are called *form-invariant*. Comparing (1.10) and (1.11) with (1.5) we infer that tensor functions are algebraic concomitants, where $\mathbf{Q}$ replaces $T_{U_i \to \bar{U}_i}$.

Automorphisms of n-dimensional Euclidean vector space E^n can be represented by orthogonal tensors. We observe for n=1 the identity mapping is the only automorphism. Orthogonal tensors form a group:

$$O(n) \equiv \{\mathbf{Q} \in T_2 = E^n \otimes E^n: \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{I}\},$$

where $\mathbf{I}$ is the identity tensor. Since orthogonal transformations leave the length of a vector $\mathbf{a} \in E^n$ unchanged therefore we arrive at an equivalent definition

$$O(n) \equiv \{\mathbf{Q} \in T_2 = E^n \otimes E^n: \quad \|\mathbf{Q}\mathbf{a}\| = \|\mathbf{a}\|, \forall \mathbf{a} \in E^n\}.$$

The orthogonal transformations do not change the angle between two arbitrary vectors. Thus we may write

$$O(n) \equiv \{\mathbf{Q} \in T_2 = E^n \otimes E^n: \quad (\mathbf{Q}\mathbf{a}) \cdot (\mathbf{Q}\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \forall \mathbf{a}, \mathbf{b} \in E^n\}.$$

An orthogonal tensor $\mathbf{Q}$ with det$\mathbf{Q}$ =1 is called a rotation or a proper orthogonal tensor. Such tensors form a group:

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$$SO(n) \equiv \{\mathbf{Q} \in O(n): \quad \det \mathbf{Q} = 1\}.$$

Obviously, orthogonal tensor $\mathbf{Q}$ with det$\mathbf{Q}$ = -1 do not constitute a subgroup of $O(n)$. Such a tensor is a reflection or a composition of a rotation with reflection. For more details on rotations the reader should refer to Blinowski (1994a, 1994b), Guo (1981) and Zalewski (1987).

(i) *Automorphisms of* E^2.

This is the group of plane rotations and reflections $O(2)$. Its elements are orthogonal tensors

(i.1) $$\tilde{\mathbf{Q}} = \tilde{\mathbf{a}}_1 \otimes \tilde{\mathbf{b}}_1 + \tilde{\mathbf{a}}_2 \otimes \tilde{\mathbf{b}}_2,$$

where $\tilde{\mathbf{a}}_i \cdot \tilde{\mathbf{a}}_j = \delta_{ij}$, $\tilde{\mathbf{b}}_k \cdot \tilde{\mathbf{b}}_l = \delta_{kl}$ $(i, j, k, l = 1, 2)$. One can easily verify that (i.1) solves the equation $\tilde{\mathbf{Q}}^T \tilde{\mathbf{Q}} = \tilde{\mathbf{I}}$. Here $\tilde{\mathbf{I}}$ stands for the two-dimensional identity tensor. For $\tilde{\mathbf{Q}} \in O(2)$ with $\det \tilde{\mathbf{Q}} = 1$ we have

(i.2) $$\tilde{\mathbf{a}}_1 = \cos\phi\, \tilde{\mathbf{b}}_1 + \sin\phi\, \tilde{\mathbf{b}}_2, \quad \tilde{\mathbf{a}}_2 = -\sin\phi\, \tilde{\mathbf{b}}_1 + \cos\phi\, \tilde{\mathbf{b}}_2, \quad \phi \in [0, 2\pi).$$

Such orthogonal tensors form a subgroup of $O(2)$ denoted by $SO(2)$. Substitution of (i.2) into (i.1) provides a representation of $\tilde{\mathbf{Q}}$ with $\det \tilde{\mathbf{Q}} = 1$ in the orthonormal frame $\{\tilde{\mathbf{b}}_i\}$ $(i = 1, 2)$:

(i.3) $$[\tilde{Q}_{ij}] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

We note that

(i.4) $$\tilde{\mathbf{Q}}(\phi_1)\tilde{\mathbf{Q}}(\phi_2) = \tilde{\mathbf{Q}}(\phi_1 + \phi_2)$$

and

(i.5) $$\tilde{\mathbf{Q}}^{-1}(\phi) = \tilde{\mathbf{Q}}^T(\phi) = \tilde{\mathbf{Q}}(-\phi).$$

The eigenvalues of a rotation tensor $\tilde{\mathbf{Q}} \in SO(2)$ are complex numbers $(\cos\phi \pm i \sin\phi,\ i^2 = -1)$. It means that plane rotations have no eigenvectors.

For $\tilde{\mathbf{Q}} \in O(2)$ with $\det \tilde{\mathbf{Q}} = -1$ then in (i.1) we set

(i.6) $$\tilde{\mathbf{a}}_1 = \cos\varphi\, \tilde{\mathbf{b}}_1 + \sin\varphi\, \tilde{\mathbf{b}}_2, \quad \tilde{\mathbf{a}}_2 = \sin\varphi\, \tilde{\mathbf{b}}_1 - \cos\varphi\, \tilde{\mathbf{b}}_2, \quad \varphi \in [0, 2\pi).$$

Proceeding as previously, we get

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(i.7) $$\left[\tilde{Q}_{ij}\right]=\begin{bmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{bmatrix}.$$

in the orthonormal frame $\{\tilde{\mathbf{b}}_i\}\,(i=1,2)$. We note that a tensor $\tilde{\mathbf{Q}}$ with the representation (i.7) can be represented in the form

(i.8) $$\tilde{\mathbf{Q}}=2\tilde{\mathbf{m}}\otimes\tilde{\mathbf{m}}-\tilde{\mathbf{I}},$$

where

(i.9) $$\tilde{\mathbf{m}}=\cos\frac{\varphi}{2}\tilde{\mathbf{b}}_1+\sin\frac{\varphi}{2}\tilde{\mathbf{b}}_2.$$

The eigenvalues of a reflection tensor $\tilde{\mathbf{Q}}\in O(2)$ are the numbers 1 and (-1). Thus such a tensor has always eigenvectors. More precisely, (i.9) is the unit eigenvector corresponding to the eigenvalue 1 while eigentensors of (i.8) are tensors $\tilde{\mathbf{m}}\otimes\tilde{\mathbf{m}}$ and $\mathbf{I}-\tilde{\mathbf{m}}\otimes\tilde{\mathbf{m}}$.

A plane reflection tensor is a plane deviator with the unit norm; obviously it is a symmetric tensor. From two reflections one can compose a rotation. Obviously we have $O(2)=SO(2)\cup[O(2)\setminus SO(2)]$, $SO(2)\cap[O(2)\setminus SO(2)]=\varnothing$, where $O(2)\setminus SO(2)$ is a set of two-dimensional reflections.

(ii) *Automorphisms of* E^3
If plane rotation (i.3) takes place in the three-dimensional space, then we naturally have a rotation axis, which is orthogonal to the plane of rotation. The unit vector of the rotation axis is now the eigenvector corresponding to the unit eigenvalue of the rotation tensor, see the previous subsection and note that $\det\mathbf{Q}=1$. In this three-dimensional case a rotation tensor $\mathbf{Q}$ has the following representation in the right-handed basis $\{\tilde{\mathbf{b}}_i,\mathbf{n}\}\,(i=1,2)$

(ii.1) $$\left[Q_{ij}\right]=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Here $\mathbf{n}$ is the unit vector of the rotation axis.
Proceeding similarly, we obtain the representation of a tensor $\mathbf{Q}\in O(3)$ with $\det\mathbf{Q}=-1$, being a reflection tensor:

(ii.2) $$\left[Q_{ij}\right]=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ \sin\phi & -\cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}.$$

We recall that since the eigenvalues of a plane reflection tensor are 1 and (-1) therefore the third eigenvalue of a three-dimensional reflection tensor $\mathbf{Q}$ ($\det\mathbf{Q}=1$) is necessarily equal to 1. An orthogonal tensor represented by (ii.2) is often called a pure reflection in

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contrast to orthogonal tensors $\mathbf{Q}$ with $\det\mathbf{Q}=-1$. As we know, the last tensor may be a composition of a rotation and a reflection.
Summarizing we conclude that the rotation (ii.1) and reflection (ii.2) take place in a plane which is orthogonal to the eigenvector corresponding to the eigenvalue 1. Generalizing these considerations we see that orthogonal tensors can be written in the form which uses the knowledge of one of the eigenvalues (1 or -1) and of the eigentensor. For instance, rotations can be represented in the following way

(ii.3) $$\mathbf{Q}=\mathbf{n}\otimes\mathbf{n}+(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cos\phi-\boldsymbol{\varepsilon}\mathbf{n}\sin\phi,$$

where $\mathbf{I}$ is the identity tensor and $\boldsymbol{\varepsilon}$ the alternating symbol (a third-order hemitropic tensor). In an orthonormal basis we have $\boldsymbol{\varepsilon}=\epsilon_{ijk}\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$ with $\epsilon_{123}=\epsilon_{231}=\epsilon_{312}=1$, $\epsilon_{132}=\epsilon_{321}=\epsilon_{213}=-1$, $\epsilon_{ijk}=0$ unless i, j and k are all different. The vector $\mathbf{n}$ is the unit eigenvector of $\mathbf{Q}$, which can always be represented by (it is sufficient to use the spherical coordinate system)

(ii.4) $$\mathbf{n}=\sin\theta\cos\eta\,\mathbf{e}_1+\sin\theta\sin\eta\,\mathbf{e}_2+\cos\theta\,\mathbf{e}_3.$$

The relations (ii.3) and (ii.4) imply that $O=O(3)$ is characterized by three parameters, because similar statement is valid for orthogonal tensors with negative determinant. We recall that in the dynamics of rigid bodies the description using Euler's angles is very popular.

As we have shown, Eq. (ii.3) follows easily by a generalization of plane rotations. In contrast, the derivation of an invariant representation of arbitrary orthogonal tensor $\mathbf{Q}$ is more complicated, cf. Guo (1981). It can be shown that $\mathbf{Q}$ and $\mathbf{Q}^T$ have always the same real eigenvalue equal to $\det\mathbf{Q}$ and the same unit eigenvector $\mathbf{k}$. If $\mathbf{a}\perp\mathbf{k}$ then also $\mathbf{Qa}\perp\mathbf{k}$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{Qa}$ is an invariant of $\mathbf{Q}$:

(ii.5) $$\mathbf{a}\cdot(\mathbf{Qa})=\cos\alpha=\frac{1}{2}(tr\mathbf{Q}-\det\mathbf{Q}),\quad \alpha\in[0,\pi],$$

while the eigenvector is given by

(ii.6) $$\mathbf{k}=-\frac{\mathbf{E}\cdot\mathbf{Q}}{2\sin\alpha},\quad \alpha\in(0,\pi).$$

It is evident that for $\alpha=0$ and $\alpha=\pi$ the formula (ii.6) is not valid since $\mathbf{Q}$ is then a symmetric tensor and one can distinguish four cases: a) $\alpha=0$ and $\det\mathbf{Q}=1$, identity; b) $\alpha=\pi$ and $\det\mathbf{Q}=1$, a rotation about the the eigenvector (the angle of rotation is equal π); c) $\alpha=0$ and $\det\mathbf{Q}=-1$, a reflection in the plane orthogonal to the eigenvector; d) $\alpha=\pi$ and $\det\mathbf{Q}=-1$, inversion.
Finally, any orthogonal tensor can be uniquely represented in the following form:

(ii.7) $$\mathbf{Q}=(\det\mathbf{Q})\mathbf{k}\otimes\mathbf{k}+\cos\alpha(\mathbf{I}-\mathbf{k}\otimes\mathbf{k})-\sin\alpha(\boldsymbol{\varepsilon}\mathbf{k}).$$

If $\det\mathbf{Q}=1$ then (ii.7) represents a pure rotation about $\mathbf{k}$ while for $\det\mathbf{Q}=-1$ (ii.7) determines a rotation about $\mathbf{k}$ composed with a reflection in the plane orthogonal to $\mathbf{k}$.

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Introducing the skew-symmetric tensor **L** by

(ii.8) $$\mathbf{L} = -\varepsilon\mathbf{k},$$

we obtain an alternative form of Eq. (ii.7)

(ii.9) $$\mathbf{Q} = (\det\mathbf{Q})\mathbf{I} + \sin\alpha\mathbf{L} + (\det\mathbf{Q} - \cos\alpha)\mathbf{L}^2.$$

The skew-symmetric tensor (ii.8) can be found from the formula

(ii.10) $$\mathbf{L} = \frac{\mathbf{Q}-\mathbf{Q}^T}{2\sin\alpha},$$

where α is determined by (ii.5).

For more details on orthogonal tensors and their applications in the mechanics of rigid and deformable bodies the reader should refer to Guo (1981), Blinowski (1994a, 1994b) and Zalewski (1987).

2. Basic theorems and principles

The theory of representations of scalar, vector-, and tensor-valued functions as used in the continuum mechanics, exploits some results of the group representation theory and the principles which will now be expounded.

From that point of view of interest are the proper unimodular group U_n^0, the full orthogonal group O and its subgroups. Zheng and Boehler (1994) and Zheng (1994a) refer to a subgroup of the full orthogonal group as a *point group*.

Let us first introduce the notion of the *matrix representation*. Suppose that G is a group and e, a, b, c, ... its elements, where e is the identity element of the group. Let $\mathbf{D}(e)$, $\mathbf{D}(a)$, $\mathbf{D}(b)$, $\mathbf{D}(c)$,... denote a set of non-singular $n \times n$ matrices such that if $ab=c$, then

(2.1) $$\mathbf{D}(a)\mathbf{D}(b) = \mathbf{D}(c).$$

We then say that the matrices $\mathbf{D}(e)$, $\mathbf{D}(a)$, $\mathbf{D}(b)$, $\mathbf{D}(c)$,... form a *matrix representation* of dimension n of the group G, cf. Serre (1967). The formula (2.1) implies

(2.2) $$\mathbf{D}(e) = \mathbf{I} = [\delta_{ij}] \quad (i, j = 1,\ldots,n),$$

(2.3) $$\mathbf{D}(a^{-1}) = \mathbf{D}^{-1}(a).$$

Entries of the matrices $\mathbf{D}(e)$, $\mathbf{D}(a)$, ... are not necessarily real. A matrix **D** is said to be unitary if its adjoint $\mathbf{D}^*$ satisfies

(2.4) $$\mathbf{D}^* = \mathbf{D}^{-1},$$

where $D_{ij}^* = \overline{D}_{ij}$ and $\overline{D}_{ij}$ denotes the complex conjugate of D_{ij} .

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Let **T** be a second-order three-dimensional tensor whose components when refered to the reference frame $\{x_i\}$ are given by T_{ij} (i,j=1,2,3). Further, $G=\{\mathbf{Q}_1,\ldots,\mathbf{Q}_N\}=\{\mathbf{Q}_K\}$ denotes a group which describes the transformation properties of **T**. We set

(2.5) $$\mathbf{t}=[T_1,\ldots,T_9]^T=[T_\alpha]=[T_{11},T_{12},T_{13},T_{21},T_{22},T_{23},T_{31},T_{32},T_{33}].$$

Thus

(2.6) $$T_\alpha=C_{\alpha jk}T_{jk},\quad T_{jk}=D_{jk\alpha}T_\alpha,$$

where $i,j=1,2,3;\ \alpha=1,\ldots,9$ and

(2.7) $$C_{\alpha jk}D_{jk\beta}=\delta_{\alpha\beta},\quad D_{ij\alpha}C_{\alpha kl}=\delta_{ik}\delta_{jl}.$$

The matrices $[C_{\alpha jk}]$ and $[D_{jk\alpha}]$ may readily be obtained from (2.5). Let $\{x_i\}$ and $\{x'_i\}$ denote reference frames with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$, respectively; $\mathbf{e}'_i=Q_{ij}\mathbf{e}_j$, where $\mathbf{Q}\in G$, $\mathbf{Q}=[Q_{ij}]$. Then

(2.8) $$T'_{ij}=Q_{ik}Q_{jl}T_{kl},$$

and

(2.9) $$T'_\alpha=C_{\alpha jk}T'_{kl}.$$

On account of (2.6), (2.8) and (2.9) we obtain

(2.10) $$T'_\alpha=C_{\alpha jk}T'_{jk}=C_{\alpha jk}Q_{jp}Q_{kq}T_{pq}=D_{\alpha\beta}(\mathbf{Q})T_\beta,$$

where

(2.11) $$D_{\alpha\beta}(\mathbf{Q})=C_{\alpha jk}Q_{jp}Q_{kq}D_{pq\beta}.$$

Moreover we have

(2.12) $$\mathbf{D}(\mathbf{Q}_1)\mathbf{D}(\mathbf{Q}_2)=\mathbf{D}(\mathbf{Q}_1\mathbf{Q}_2),$$

or

(2.13) $$D_{\alpha\beta}(\mathbf{Q}_1)D_{\beta\gamma}(\mathbf{Q}_2)=D_{\alpha\gamma}(\mathbf{Q}_1\mathbf{Q}_2),$$

for each $\mathbf{Q}_1,\mathbf{Q}_2\in G$. Thus the set of matrices $\mathbf{D}(\mathbf{Q}_K)$ $(K=1,\ldots,N)$, describing the transformation properties of **t** under $G=\{\mathbf{Q}_K\}$, forms a matrix representation of G.

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An important role in the study of irreducible representations plays Schur's Lemma (see Schur and Grunsky, 1968). Let $\{\mathbf{D}_K\}$ and $\{\mathbf{R}_K\}$ denote n-dimensional and m-dimensional irreducible matrix representations of the finite group $G=\{\mathbf{Q}_K\}$. We may assume that the matrices $\mathbf{D}_K=\mathbf{D}(\mathbf{Q}_K)$ and $\mathbf{R}_K=\mathbf{R}(\mathbf{Q}_K)\,(K=1,\ldots,N)$ are unitary. Schur's Lemma solves the problem of determining the $n\times m$ matrix $\mathbf{C}$ such that

(2.14) $$\mathbf{D}_K\mathbf{C}=\mathbf{C}\mathbf{R}_K;\quad K=1,\ldots,N.$$

Schur's Lemma
Suppose that $\mathbf{C}$ is the matrix defined by (2.14). Then

(i) $\mathbf{C}=\mathbf{0}$ if $n\neq m$;
(ii) $\mathbf{C}=\mathbf{0}$ if the representations $\{\mathbf{D}_K\}$ and $\{\mathbf{R}_K\}$ are inequivalent;
(iii) $\mathbf{C}$ is non-singular if n=m and the representations $\{\mathbf{D}_K\}$ and $\{\mathbf{R}_K\}$ are equivalent;
(iv) if $\{\mathbf{D}_K\}$ = $\{\mathbf{R}_K\}$ in (2.14) so that $\mathbf{C}$ satisfies $\mathbf{D}_K\mathbf{C}=\mathbf{C}\mathbf{D}_K$, $(K=1,\ldots,N)$, for all $\mathbf{D}_K$ comprising an n-dimensional irreducible representation of $\mathbf{G}$, then $\mathbf{C}=\lambda\mathbf{I}$ where $\mathbf{I}=[\delta_{ij}]$ is the $n\times n$ identity matrix. ∇

The next important notion is the character of the representation $\{\mathbf{D}_K\}$ given by $(\xi_1,\ldots\xi_N)$, where

(2.15) $$\xi_K=tr\mathbf{D}_K=D^K_{ii}=D^K_{11}+\ldots+D^K_{nn}.$$

We observe that equivalent representations $\{\mathbf{D}_K\}$ and $\{\mathbf{S}\mathbf{D}_K\mathbf{S}^{-1}\}(\det\mathbf{S}\neq 0)$ have the same character since

(2.16) $$tr\mathbf{S}\mathbf{D}_K\mathbf{S}^{-1}=tr\mathbf{S}\mathbf{S}^{-1}\mathbf{D}_K=tr\mathbf{D}_K,$$

due to the fact that $tr\mathbf{ABC}=tr\mathbf{BCA}=tr\mathbf{CAB}$.

From the point of view of applications it is worth noting that all crystallographic groups are discrete groups. On the contrary, the symmetry properties of isotropic, hemitropic and transversely isotropic materials are defined by continuous groups. Then one has to define the so called group manifold and invariant integrals, cf. Smith (1994, Chap.II).

We shall now formulate the basic principles and theorems which are of fundamental importance for determining representations of scalar-, vector-, and tensor-valued function.

Neumann's Principle (Nye, 1957; Rychlewski, 1991a; Zheng, 1994a). The symmetry group of an investigated material must be included in the symmetry group of any tensor function in any constitutive law of the material. ∇

Hilbert (1893, see also 1970) formulated his celebrated theorem on the existence of a finite integrity basis. The contemporary version of this theorem may be given the following form, see Gurevich (1964), Boehler (1987a), Zheng (1994a).

Hilbert Theorem. For any finite number of vector and tensor agencies and relative to any compact point group, there exists a functional basis consisting of a finite number of

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basic invariants. Moreover, for any type of vector- and tensor-valued functions there exists a finite number of generators. ∇

We recall that the original Hilbert's theorems was formulated for integrity bases. The finiteness of a set generators is straightforward since the problem of determining polynomial representations of vector- and tensor-valued functions can be reduced to the scalar case (Rivlin, 1955; Smith, 1994; Spencer, 1971, 1984, 1987).

Non-polynomial representations are usually more concise (Wang, 1969, 1970, 1971; Smith, 1970, 1971; Boehler, 1977, 1978, 1979, 1987c; Korsgaard, 1990a, 1990b; Zheng, 1993a, 1993b, 1993c, 1994a, 1996), Rychlewski (1984c, 1984e). Here important is Wineman and Pipkin theorem (1964/65), see also Boehler (1987c), Zheng (1994a).

Wineman-Pipkin's Theorem. For any finite number of vector and tensor agencies and relative to any compact point group, a complete polynomial representation may be used as the complete non-polynomial representation. ∇

We note that though such a complete representation may be irreducible as the polynomial representation, some terms may be redundant for the non-polynomial irreducible representation, cf. Zheng (1994a).

Anisotropic properties of a material may be described by so called structural tensors (Boehler, 1978, 1987a; Liu, 1982; Rychlewski, Zhang, 1991; Sedov and Lokhin, 1963). Sometimes they are referred to as fabric tensors, cf. Oda (1972, 1993), Oda and Nakayama (1989), Kanatani (1984), Cowin (1985, 1986a, 1986b). Let G be a point group and $\mathbf{Q}$ any orthogonal tensor. Tensors $\underline{\xi}_1,\ldots,\underline{\xi}_s$ are said to be structural tensors if

$$\mathbf{Q}\circ\underline{\xi}_1=\underline{\xi}_1,\ldots,\mathbf{Q}\circ\underline{\xi}_s=\underline{\xi}_s, \tag{2.17}$$

is equivalent to $\mathbf{Q}\in G$. Recently, Zheng and Boehler (1994) have arrived at the following conclusion, called by them the

Structural Tensor Theorem:

Let G be a two-or three-dimensional point group. Then, (i) if G is compact, it can be described by a single tensor, which may even be irreducible; (ii) if G can be characterized by a finite number of tensors, then G is compact; (iii) if G is non-compact, it cannot be characterized by a finite number of tensors of finite orders. ∇

Structural tensors permit to reduce the problem of determining anisotropic representations to the corresponding problem of isotropic representations. Such study was initiated by Boehler (1978) and next developed further by Liu (1982), Rychlewski (1991a), Zhang and Rychlewski (1990a) and Zheng (1993b, 1994a), Svendsen (1994). We are now in a position to formulate

Isotropicization Theorem. Any anisotropic tensor function in the two-or three dimensional space of a finite number of tensor agencies, form-invariant under a compact point group can be expressed as an isotropic tensor function of the original tensor agencies and the structural tensors $\underline{\xi}_1,\ldots,\underline{\xi}_s$. ∇

Thus, if $F(\mathbf{S}_a)$ is an anisotropic tensor function of $\mathbf{S}_1,\ldots,\mathbf{S}_A$ and form-invariant under a compact point group G characterized by structural tensors $\underline{\xi}_1,\ldots,\underline{\xi}_s$, then there exists an isotropic tensor function $\overline{F}\left(\mathbf{S}_a,\underline{\xi}_1,\ldots,\underline{\xi}_s\right)$ such that

$$F(\mathbf{S}_a)=\overline{F}\left(\mathbf{S}_a,\underline{\xi}_1,\ldots,\underline{\xi}_s\right). \tag{2.18}$$

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Recently, Zheng and Boehler (1994) have formulated the following
Principle of Symmetry of Continuum. Compact point groups describe and classify all kind of real or ideal material symmetry and physical symmetry. The description of the symmetry of a continuous media or its any physical property by a non-compact point group is an unreality.
Polarization (Dieudonné and Carrell, 1971; Spencer, 1971). Of great importance in the search of invariants is the *polarization operator*, which we now introduce.

Let us consider a system of *m similar* tensors in the sense, that they are all of the same order, are defined in the same space and have the same symmetry with respect to a change of indices. We assume that each of those tensors has v *different* components. Those components, set up in a certain determined order, for an *r*th tensor (r=1,...,m) are denoted by $A_i^{(r)}\ (r=1,2,\dots,v)$. The polarization operator is defined by

$$D_{pq} = A_i^{(p)} \frac{\partial}{\partial A_i^{(p)}}.$$

This operator is used in Peano's theorem (sometimes called Pascal's theorems), which in turn allows to express invariants of an arbitrary (finite) number of tensors by means of invariants of $(v-1)$ tensors.

It should be noted that the choice of structural tensors is not uniquely determined, cf. Sedov and Lokhin (1963), Rychlewski and Zhang (1991), Zheng and Boehler (1994), Zheng and Spencer (1993) and Table 2.1. Tables 2.1 and 2.2 present the classification of compact point groups for all kinds of symmetries and the corresponding structural tensors. In Table 2.1 rotations are denoted by $\mathbf{R}(\theta)$ while in Table 2.2 by $\mathbf{R}(\theta\mathbf{n})$, where θ is the rotation angle and, in the 3D case, **n** stands for the unit vector of the rotation axis. The meaning of the set $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is obvious: these are orthonormal bases. Moreover, ε denotes the alternating symbol. The columns in both tables list the structural (or parametric) tensors.

Table 2.1. Two-dimensional compact point groups of all kinds and their structural tensors (Zheng, 1993b)

System	No	Group symbols	Group order	Group generators	Simple str.tensors	Single str.tensor
oblique	1 2	$\mathcal{C}_1$ $\mathcal{C}_2$	1 2	$\mathbf{R}(0)\equiv\mathbf{I}$ $\mathbf{R}(\pi)\equiv-\mathbf{I}$	$\mathbf{i},\mathbf{j}$ $\mathbf{P}_2,\varepsilon$	$(\mathbf{I}+\varepsilon)\otimes\mathbf{i}$ $\mathbf{P}_2+\varepsilon$
rectangular	3 4	$\mathcal{C}_{1v}$ $\mathcal{C}_{2v}$	2 4	$\mathbf{R}_j\equiv\mathbf{I}-2\mathbf{j}\otimes\mathbf{j}$ $\mathbf{R}(\pi),\mathbf{R}_j$	$\mathbf{i}$ $\mathbf{P}_2$	$\mathbf{i}$ $\mathbf{P}_2$
square (n=4) or n-gonal (n=4k)	5 6	$\mathcal{C}_n$ $\mathcal{C}_{nv}$	n 2n	$\mathbf{R}(2\pi/n)$ $\mathbf{R}(2\pi/n),\mathbf{R}_j$	$\mathbf{P}_n,\varepsilon$ $\mathbf{P}_n$	$\mathbf{P}_n+\varepsilon\otimes\mathbf{I}\otimes\ldots\otimes\mathbf{I}$ $\mathbf{P}_n$
trigonal (n=3) or n-gonal (n=2k+1)	7 8	$\mathcal{C}_n$ $\mathcal{C}_{nv}$	n 2n	$\mathbf{R}(2\pi/n)$ $\mathbf{R}(2\pi/n),\mathbf{R}_j$	$\mathbf{P}_n,\varepsilon$ $\mathbf{P}_n$	$(\mathbf{I}+\varepsilon)\otimes\mathbf{P}_n$ $\mathbf{P}_n$

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hexagonal (n=6) or n-gonal (n=4k+2)	9 10	$\mathcal{C}_n$ $\mathcal{C}_{nv}$	n 2n	$\mathbf{R}(2\pi/n)$ $\mathbf{R}(2\pi/n), \mathbf{R}_j$	$\mathbf{P}_n, \varepsilon$ $\mathbf{P}_n$	$\mathbf{P}_n + \varepsilon \otimes \mathbf{I} \otimes \ldots \otimes \mathbf{I}$ $\mathbf{P}_n$
circular $(0 \le \phi < 2\pi)$	11 12	$\mathcal{C}_\infty$ $\mathcal{C}_{\infty v}$	∞ ∞	$\mathbf{R}(\phi)$ $\mathbf{R}(\phi), \mathbf{R}_j$	ε $\mathbf{I}$	ε $\mathbf{I}$

k=integer>1

Table 2.2. Three-dimensional compact point groups of all kinds and their structural tensors (Zheng and Boehler, 1994)

System	No	Group symbols	Group order	Group generators	Simple structural tensors
triclinic	1 2	$\mathcal{C}_1$ $\mathcal{C}_i$	1 2	$\mathbf{I}$ $-\mathbf{I}$	$\mathbf{i}, \mathbf{j}, \mathbf{k}$ $\varepsilon\mathbf{i}, \varepsilon\mathbf{j}, \varepsilon\mathbf{k}$
monoclinic	3 4 5	$\mathcal{C}_2$ $\mathcal{C}_{1h}$ $\mathcal{C}_{2h}$	2 2 4	$\mathbf{R}(\pi\mathbf{k})$ $-\mathbf{R}(\pi\mathbf{k})$ $\mathbf{R}(\pi\mathbf{k}), -\mathbf{I}$	$\mathbf{P}_2, \varepsilon, \mathbf{k}$ $\mathbf{i}, \mathbf{j}$ $\mathbf{P}_2, \varepsilon\mathbf{k}$
orthorhombic	6 7 8	$\mathcal{D}_2$ $\mathcal{C}_{2v}$ $\mathcal{D}_{2h}$	4 4 8	$\mathbf{R}(\pi\mathbf{k}), -\mathbf{R}_i, -\mathbf{R}_j$ $-\mathbf{R}(\pi\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j$ $\mathbf{R}(\pi\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j, -\mathbf{I}$	$\mathbf{P}_2, \varepsilon$ $\mathbf{P}_2, \mathbf{k}$ $\mathbf{P}_2$
tetragonal (n=4) or (4k)-gonal (n=4k)	9 10 11 12 13 14 15	$\mathcal{C}_n$ $\mathcal{C}_{(n/2)i}$ $\mathcal{C}_{nh}$ $\mathcal{D}_n$ $\mathcal{C}_{nv}$ $\mathcal{D}_{(n/2)d}$ $\mathcal{D}_{nh}$	n n 2n 2n 2n 2n 4n	$\mathbf{R}(2\pi/n\mathbf{k})$ $-\mathbf{R}(2\pi/n\mathbf{k})$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{I}$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{R}_i, -\mathbf{R}_j$ $\mathbf{R}(2\pi/n\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j$ $-\mathbf{R}(2\pi/n\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j$ $\mathbf{R}(\pi\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j, -\mathbf{I}$	$\mathbf{P}_n, \mathbf{k}, \varepsilon$ $\mathbf{k} \otimes \mathbf{P}_{n/2}, \varepsilon\mathbf{k}$ $\mathbf{P}_n, \varepsilon\mathbf{k}$ $\mathbf{P}_n, \varepsilon$ $\mathbf{P}_n, \mathbf{k}$ $\mathbf{k} \otimes \mathbf{P}_{n/2}$ $\mathbf{P}_n$
trigonal (n=3) or (2k+1)-gonal (n=2k+1)	16 17 18 19 20	$\mathcal{C}_n$ $\mathcal{C}_{ni}$ $\mathcal{D}_n$ $\mathcal{C}_{nv}$ $\mathcal{D}_{nd}$	n 2n 2n 2n 4n	$\mathbf{R}(2\pi/n\mathbf{k})$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{I}$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{R}_i$ $\mathbf{R}(2\pi/n\mathbf{k}), \mathbf{R}_j$ $\mathbf{R}(\pi\mathbf{k}), \mathbf{R}_j, -\mathbf{I}$	$\mathbf{P}_n, \mathbf{k}, \varepsilon$ $\mathbf{k} \otimes \mathbf{P}_n, \varepsilon\mathbf{k}$ $\mathbf{P}_n, \varepsilon$ $\mathbf{P}_n, \mathbf{k}$ $\mathbf{k} \otimes \mathbf{P}_n$

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hexagonal (n=6) or (4k+2)-gonal (n=4k+2)	21 22 23 24 25 26 27	$\mathcal{C}_n$ $\mathcal{C}_{(n/2)h}$ $\mathcal{C}_{nh}$ $\mathcal{D}_n$ $\mathcal{C}_{nv}$ $\mathcal{D}_{(n/2)h}$ $\mathcal{D}_{nh}$	n n 2n 2n 2n 2n 4n	$\mathbf{R}(2\pi/n\mathbf{k})$ $-\mathbf{R}(2\pi/n\mathbf{k})$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{I}$ $\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{R}_i, -\mathbf{R}_j$ $\mathbf{R}(2\pi/n\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j$ $-\mathbf{R}(2\pi/n\mathbf{k}), -\mathbf{R}_i, \mathbf{R}_j$ $\mathbf{R}(\pi\mathbf{k}), \mathbf{R}_i, \mathbf{R}_j, -\mathbf{I}$	$\mathbf{P}_n, \mathbf{k}, \varepsilon$ $\mathbf{P}_{n/2}, \varepsilon\mathbf{k}$ $\mathbf{P}_n, \varepsilon\mathbf{k}$ $\mathbf{P}_n, \varepsilon$ $\mathbf{P}_n, \mathbf{k}$ $\mathbf{P}_{n/2}$ $\mathbf{P}_n$
cubic	28 29 30 31 32	$\mathcal{T}$ $\mathcal{T}_h$ $\mathcal{O}$ $\mathcal{T}_d$ $\mathcal{O}_h$	12 24 24 24 48	$\mathbf{R}(2\pi/3\mathbf{c}), -\mathbf{R}_i$ $\mathbf{R}(2\pi/3\mathbf{c}), \mathbf{R}_i, -\mathbf{I}$ $\mathbf{R}(2\pi/3\mathbf{c}), \mathbf{R}(\pi/2\mathbf{i}), -\mathbf{R}_j$ $\mathbf{R}(2\pi/3\mathbf{c}), -\mathbf{R}(\pi/2\mathbf{i}), -\mathbf{R}_j$ $\mathbf{R}(2\pi/3\mathbf{c}), \mathbf{R}(\pi/2\mathbf{i}), \mathbf{R}_j, -\mathbf{I}$	$\mathbf{T}_d, \varepsilon$ $\mathbf{T}_h$ $\mathbf{O}_h, \varepsilon$ $\mathbf{T}_d$ $\mathbf{O}_h$
icosahedral	33 34	$\mathcal{I}$ $\mathcal{I}_h$	60 120	$\mathbf{R}(2\pi/5\mathbf{k}), \mathbf{R}(2\pi/3\mathbf{i}), \mathbf{R}_j$ $\mathbf{R}(2\pi/5\mathbf{k}), \mathbf{R}(2\pi/3\mathbf{i}), \mathbf{R}_j, -\mathbf{I}$	$\mathbf{I}_h, \varepsilon$ $\mathbf{I}_h$
cylindrical $(0 \le \phi < 2\pi)$	35 36 37 38 39	$\mathcal{C}_\infty$ $\mathcal{C}_{\infty h}$ $\mathcal{D}_\infty$ $\mathcal{C}_{\infty v}$ $\mathcal{D}_{\infty h}$	∞ ∞ ∞ ∞ ∞	$\mathbf{R}(\phi\mathbf{k})$ $\mathbf{R}(\phi\mathbf{k}), -\mathbf{I}$ $\mathbf{R}(\phi\mathbf{k}), -\mathbf{R}_i$ $\mathbf{R}(\phi\mathbf{k}), \mathbf{R}_i$ $\mathbf{R}(\phi\mathbf{k}), \mathbf{R}_j, -\mathbf{I}$	$\mathbf{k}, \varepsilon$ $\varepsilon\mathbf{k}$ $\mathbf{k}\otimes\mathbf{k}, \varepsilon$ $\mathbf{k}$ $\mathbf{k}\otimes\mathbf{k}$
spherical $(0 \le \phi, \varphi < 2\pi)$	40 41	$\mathcal{K}$ $\mathcal{K}_h$	∞ ∞	$\mathbf{R}(\phi\mathbf{k}), \mathbf{R}(\varphi\mathbf{k})$ $\mathbf{R}(\phi\mathbf{k}), \mathbf{R}(\varphi\mathbf{k}), -\mathbf{I}$	ε $\mathbf{I}$

k=integer>1

The structural tensors $\mathbf{T}_d, \mathbf{T}_h, \mathbf{O}_h$ and $\mathbf{I}_h$ in Table 2.2. can be assumed in the following form:

$$\mathbf{T}_d = \mathbf{i}\otimes\mathbf{j}\otimes\mathbf{k} + \mathbf{j}\otimes\mathbf{k}\otimes\mathbf{i} + \mathbf{k}\otimes\mathbf{i}\otimes\mathbf{j} + \mathbf{k}\otimes\mathbf{j}\otimes\mathbf{i} + \mathbf{k}\otimes\mathbf{j}\otimes\mathbf{i} + \mathbf{j}\otimes\mathbf{i}\otimes\mathbf{k} + \mathbf{i}\otimes\mathbf{k}\otimes\mathbf{j},$$

$$\mathbf{T}_h = \mathbf{i}^2\otimes\mathbf{j}^2 - \mathbf{j}^2\otimes\mathbf{i}^2 + \mathbf{j}^2\otimes\mathbf{k}^2 - \mathbf{k}^2\otimes\mathbf{j}^2 + \mathbf{k}^2\otimes\mathbf{i}^2 - \mathbf{i}^2\otimes\mathbf{k}^2,$$

$$\mathbf{O}_h = \mathbf{i}^4 + \mathbf{j}^4 + \mathbf{k}^4,$$

$$\mathbf{I}_h = \mathbf{a}_1^6 + \mathbf{a}_2^6 + \ldots + \mathbf{a}_6^6,$$

where

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$$\mathbf{a}_6 = \mathbf{k}, \quad \sqrt{5}\mathbf{a}_l = \mathbf{k} + 2[\cos(2\pi l/5)\mathbf{i} + \sin(2\pi l/5)\mathbf{j}], \; l = 1,\ldots,5.$$

The abbreviation $\mathbf{w}^n$ for any real or complex vector $\mathbf{w}$ denotes the nth-order real or complex tensor

$$\mathbf{w}^n = \mathbf{w} \otimes \underbrace{\ldots}_{n} \otimes \mathbf{w}.$$

The nth-order *basic structural tensors* $\mathbf{P}_n\ (n = 1,\ldots,\mathbf{N})$ in Tables 2.1. and 2.2. are defined as follows

$$\mathbf{P}_n = \mathrm{Re}(\mathbf{i} + i\mathbf{j})^n, \quad i = \sqrt{-1},$$

where the prefix „*Re*" indicates real part. For example, we have

$$\mathbf{P}_1 = \mathbf{i},$$
$$\mathbf{P}_2 = \mathbf{i} \otimes \mathbf{i} - \mathbf{j} \otimes \mathbf{j},$$
$$\mathbf{P}_3 = \mathbf{i} \otimes \mathbf{i} \otimes \mathbf{i} - (\mathbf{i} \otimes \mathbf{j} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{i} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{j} \otimes \mathbf{i}).$$

The basic structural tensors $\mathbf{P}_n$ possess two important properties:
(i) with respect to the transformation

$$\mathbf{e}_1 = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \mathbf{e}_2 = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j},$$

it holds

$$\mathbf{P}_n = \mathrm{Re}\left[\exp(n\theta i)(\mathbf{e}_1 + i\mathbf{e}_2)^n\right].$$

(ii) $\mathbf{P}_n$ are *irreducible*, i.e., $\mathbf{P}_n$ are completely symmetric and traceless,

$$P^{(n)}_{ijk\ldots l} = P^{(n)}_{jik\ldots l} = P^{(n)}_{kj\ldots l} = \ldots = P^{(n)}_{ljk\ldots i}, \quad P^{(n)}_{mmk\ldots l} = O_{k\ldots l},$$

where the components $O_{k\ldots l}$ correspond to a zero-tensor. Each $\mathbf{P}_n$ may have in maximum two independent components $\cos(n\theta)$ and $\sin(n\theta)$.
These properties of $\mathbf{P}_n$ have significantly simplified the procedures of determining the representations of anisotropic tensor functions (Zheng, 1994a).

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3. Polynomial representations
Having presented the basic aspects of the theory of tensor functions representation, we shall now expound some more practical results from the point of view of the continuum mechanics.

Let $\mathbf{A}_k$, $\mathbf{W}_l$ and $\mathbf{v}_m$ (k=1,...,K; l=1,...,L; m=1,...,M) denote a set of second-order symmetric tensors, second-order skew-symmetric tensors and vectors respectively. Hilbert's theorem implies the existence of a finite integrity basis. The integrity basis is said to be irreducible if none of its elements is expressible as a polynomial in the remaining elements of the integrity basis. The problem of determining integrity bases for various symmetry groups was studied in many papers, cf. Adkins (1960a, 1960b), Betten and Helisch (1995), Boehler (1987a), Markov and Vakulenko (1981), Kiral and Eringen (1990), Kiral and Smith (1974), Smith (1960, 1965, 1994), Smith and Kiral (1978), Smith and Rivlin (1964), Smith et al. (1963), Spencer (1961, 1965, 1971), Spencer and Rivlin (1959a, 1959b, 1960, 1962), Rivlin (1955), Vakulenko (1972), Wesołowski (1964).

Having determined the integrity basis for $\mathbf{A}_k$, $\mathbf{W}_l$ and $\mathbf{v}_m$ we can pass to finding the representation of the vector- or tensor- valued function

$$\mathbf{T} = \underline{F}\left(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m\right), \tag{3.1}$$

which is form-invariant under a symmetry group S. In essence, the problem consists in determining the so called generators $\mathbf{G}_p$ (p=1,...,P), form- invariant under S. Then the function (3.1) has the following representation

$$\mathbf{T} = \alpha_p \mathbf{G}_p = \sum_{p=1}^{P} \alpha_p \mathbf{G}_p, \tag{3.2}$$

where α_p are arbitrary polynomial functions in the elements of the integrity basis.

There exists several methods of determining generators, cf. Spencer (1987), Betten (1987c). One of them was proposed by Pipkin and Rivlin (1959) in the case where $S = O$. To illustrate this method let **T**, **A** and **B** be symmetric, second- order tensors and

$$\mathbf{T} = \underline{F}(\mathbf{A}, \mathbf{B}). \tag{3.3}$$

The tensor function $\underline{F}$ is form-invariant provided that

$$\forall \mathbf{Q} \in O, \quad \mathbf{Q}\underline{F}(\mathbf{A}, \mathbf{B})\mathbf{Q}^T = \underline{F}\left(\mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}\mathbf{B}\mathbf{Q}^T\right). \tag{3.4}$$

In the known Cayley-Hamilton identity

$$\mathbf{A}^3 - (tr\mathbf{A})\mathbf{A}^2 + \frac{1}{2}\left(tr^2\mathbf{A} - tr\mathbf{A}^2\right)\mathbf{A} - (\det \mathbf{A})\mathbf{I} = \mathbf{0}, \tag{3.5}$$

where

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(3.6) $$3\det \mathbf{A} = tr\mathbf{A}^3 - \frac{3}{2}tr\mathbf{A}^2 tr\mathbf{A} + \frac{1}{2}tr^3\mathbf{A}, \quad tr\mathbf{A} = A_{ii},$$

we take $a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$ instead of $\mathbf{A}$ (a,b,c - real numbers, $\mathbf{C}$ - a symmetric second-order tensor). By equating to zero the term associated with abc we arrive at the identity for the first time derived by Rivlin (1955); cf. also Spencer (1971), Smith (1994),

(3.7) $$\begin{aligned}&\mathbf{ABC} + \mathbf{ACB} + \mathbf{BAC} + \mathbf{BCA} + \mathbf{CAB} + \mathbf{CBA} - (\mathbf{BC} + \mathbf{CB})tr\mathbf{A} - (\mathbf{CA} + \mathbf{AC})tr\mathbf{B} +\\ &-(\mathbf{AB} + \mathbf{BA})tr\mathbf{C} - \mathbf{A}(tr\mathbf{BC} - tr\mathbf{B}tr\mathbf{C}) - \mathbf{B}(tr\mathbf{CA} - tr\mathbf{C}tr\mathbf{A}) - \mathbf{C}(tr\mathbf{AB} - tr\mathbf{A}tr\mathbf{B}) +\\ &-\mathbf{I}(tr\mathbf{ABC} + tr\mathbf{CBA} - tr\mathbf{A}tr\mathbf{BC} - tr\mathbf{B}tr\mathbf{CA} - tr\mathbf{C}tr\mathbf{AB} + tr\mathbf{A}tr\mathbf{B}tr\mathbf{C}) = \mathbf{0}.\end{aligned}$$

Performing suitable substitutions in (3.7) and taking traces of expressions obtained, after simple though lengthy calculations we get the results summarized in Table 3.1.

Table 3.1 Integrity basis for three symmetric second-order tensors

Agencies	Invariants
$\mathbf{A}$	$tr\mathbf{A}, tr\mathbf{A}^2, tr\mathbf{A}^3$
$\mathbf{A}, \mathbf{B}$	$tr\mathbf{AB}, tr\mathbf{A}^2\mathbf{B}, tr\mathbf{AB}^2, tr\mathbf{A}^2\mathbf{B}^2$
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	$tr\mathbf{ABC}, tr\mathbf{A}^2\mathbf{BC}, tr\mathbf{AB}^2\mathbf{C}, tr\mathbf{ABC}^2, tr\mathbf{A}^2\mathbf{B}^2\mathbf{C}, tr\mathbf{AB}^2\mathbf{C}^2$

To determine the representation of the function (3.3) satisfying (3.4) we take a scalar-valued function φ given by

(3.8) $$\varphi = tr\mathbf{TC}.$$

The function φ depends linearly on a symmetric second-order tensor $\mathbf{C}$. Then

(3.9) $$\mathbf{T} = \frac{\partial\varphi}{\partial\mathbf{C}}.$$

By using Table 3.1 we readily obtain

(3.10) $$\begin{aligned}\varphi = \alpha_1 tr\mathbf{C} + \alpha_2 tr\mathbf{AC} + \alpha_3 tr\mathbf{BC} + \alpha_4 tr\mathbf{ABC} + \alpha_5 tr\mathbf{A}^2\mathbf{C} + \alpha_6 tr\mathbf{B}^2\mathbf{C} +\\ + \alpha_7 tr\mathbf{A}^2\mathbf{BC} + \alpha_8 tr\mathbf{AB}^2\mathbf{C} + \alpha_9 tr\mathbf{A}^2\mathbf{B}^2\mathbf{C},\end{aligned}$$

where

(3.11) $$\alpha_k = f_k\left(tr\mathbf{A}, tr\mathbf{A}^2, tr\mathbf{A}^3, tr\mathbf{B}, tr\mathbf{B}^2, tr\mathbf{B}^3, tr\mathbf{AB}, tr\mathbf{A}^2\mathbf{B}, tr\mathbf{AB}^2, tr\mathbf{A}^2\mathbf{B}^2\right), \; k = 1,\ldots,9.$$

With (3.9)-(3.11) and knowing that

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$$
\frac{\partial tr\mathbf{C}}{\partial \mathbf{C}} = \mathbf{I}, \quad \frac{\partial tr\mathbf{AC}}{\partial \mathbf{C}} = \mathbf{A}, \quad \frac{\partial tr\mathbf{BC}}{\partial \mathbf{C}} = \mathbf{B},
$$

$$
\frac{\partial tr\mathbf{A}^2\mathbf{C}}{\partial \mathbf{C}} = \mathbf{A}^2, \quad \frac{\partial tr\mathbf{B}^2\mathbf{C}}{\partial \mathbf{C}} = \mathbf{B}^2,
$$

(3.12)

$$
2\frac{\partial tr\mathbf{ABC}}{\partial \mathbf{C}} = \mathbf{AB} + \mathbf{BA}, \quad 2\frac{\partial tr\mathbf{A}^2\mathbf{BC}}{\partial \mathbf{C}} = \mathbf{A}^2\mathbf{B} + \mathbf{BA}^2,
$$

$$
2\frac{\partial tr\mathbf{AB}^2\mathbf{C}}{\partial \mathbf{C}} = \mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}, \quad 2\frac{\partial tr\mathbf{A}^2\mathbf{B}^2\mathbf{C}}{\partial \mathbf{C}} = \mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2,
$$

we find the following representation of the isotropic tensor function (3.3):

(3.13)

$$
\mathbf{T} = \alpha_1\mathbf{I} + \alpha_2\mathbf{A} + \alpha_3\mathbf{B} + \frac{1}{2}\alpha_4\left(\mathbf{AB} + \mathbf{BA}\right) + \alpha_5\mathbf{A}^2 + \alpha_6\mathbf{B}^2 +
$$

$$
+\frac{1}{2}\alpha_7\left(\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2\right) + \frac{1}{2}\alpha_8\left(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}\right) + \frac{1}{2}\alpha_9\left(\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2\right).
$$

Remark 3.1. Effort of researchers involved in the formulation of constitutive relationships by using scalar and tensor functions representations has evidently been focused on 2D and 3D problems. It seems to be the merit of just mechanicians that *nonpolynomial representations* have been developed, cf. the next section.

Analogous results for $n>3$ concerning representation of scalar and tensor functions are lacking. The available results are restricted to polynomial representations, general methods of constructions of scalar invariants and generators, finiteness theorems and estimates. Processi's (1976) paper provides a good account of a mathematician's approach to the theory of scalar (polynomial) invariants of $n \times n$ matrices as well as to the matrix-valued polynomial functions of vectors and matrices, cf. also Dieudonné and Carell (1971), Rasmyslov (1974), Springer (1977), Skwarczyński (1996). Similarly as in the case $n \leq 3$, the theorem of Cayley-Hamilton plays in important role in the study of relations among invariants and matrix-valued functions.

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4. Non-polynomial representations

In the previous Section we have recalled what is now referred to as Wineman- Pipkin Theorem, see also Pipkin and Wineman (1963). By employing this theorem, we conclude that in the case of non-polynomial representations, the coefficients α_p are *functions* and form the so called *functional basis*. Such a basis is complete but may contain redundant elements. If no such redundant elements exists, then the functional basis is irreducible. A set of invariants for given agencies (vectors and/or tensors) and symmetry group constitutes a functional basis if every other invariant of the same arguments is expressible as a function of the elements of the functional basis. A non-polynomial representation of a tensor function is said to be irreducible if in (3.2) the coefficients α_p are elements of the irreducible functional basis and none of the generators $\mathbf{G}_p$ is expressible as a linear combination of the remaining generators with coefficients being arbitrary scalar functions of the functional basis. Wang (1969, 1970, 1971), Smith (1970, 1971) and Boehler (1977, 1979) proved that, in general, a non-polynomial representation consists of a smaller number of elements of the integrity basis and generators in comparison with the corresponding polynomial representation, cf. also Korsgaard (1990a, 1990b), Zheng (1993a). Being more concise, non-polynomial representations are more appropriate for using in the formulation of constitutive relationships, cf. Boehler and Raclin (1977).

4.1. Representation of three-dimensional isotropic functions

We shall now provide well established results concerning representations of scalar-valued, vector-valued, symmetric tensor-valued and skew-symmetric tensor-valued isotropic functions:

(4.1) $$\phi = f(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m),$$

(4.2) $$\mathbf{h} = \mathrm{g}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m),$$

(4.3) $$\mathbf{S} = \underline{\mathrm{F}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m),$$

(4.4) $$\mathbf{V} = \underline{\mathrm{G}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m),$$

where $\mathbf{S} = \mathbf{S}^T$ and $\mathbf{V} = -\mathbf{V}^T$; $k = 1,\ldots,K$; $l = 1,\ldots,L$; $m = 1,\ldots,M$. The requirement of isotropy means that these functions should satisfy the following conditions, cf. Section 1:

(4.5) $$f(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m) = f\left(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m\right),$$

(4.6) $$\mathbf{Q}\mathrm{g}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m) = \mathrm{g}\left(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m\right),$$

(4.7) $$\mathbf{Q}\underline{\mathrm{F}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m)\mathbf{Q}^T = \underline{\mathrm{F}}\left(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m\right),$$

(4.8) $$\mathbf{Q}\underline{\mathrm{G}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m)\mathbf{Q}^T = \underline{\mathrm{G}}\left(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m\right),$$

for each $\mathbf{Q} \in O$.

The first important results due to Wang (1969) and Smith (1970) were not identical. Finally, the differences were clarified by Boehler (1977). The paper by Pennisi and Trovato (1987) justifies the results obtained by Smith and Boehler. Wang (1969, 1970, 1971) and Smith (1970, 1971) provided explicit forms of the functions f, $\mathbf{g}$, $\underline{\mathrm{F}}$ and $\underline{\mathrm{G}}$. Though their methods were different, yet now it is clear that their results coincide; such an identification was performed by Zheng (1993a). The results

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concerning non-polynomial representations of isotropic functions (4.1)-(4.4) are summarized in Tables 4.1-4.4.

In all these tables for representations of isotropic functions the following abbreviations are employed

(4.9) $$\mathbf{A}_1 = \mathbf{A}_i,\ \mathbf{A}_2 = \mathbf{A}_j,\ \mathbf{A}_3 = \mathbf{A}_k, \quad (i, j, k = 1, \ldots, K \ with\ i < j < k),$$

(4.10) $$\mathbf{W}_1 = \mathbf{W}_p,\ \mathbf{W}_2 = \mathbf{W}_q,\ \mathbf{W}_3 = \mathbf{W}_r, \quad (p, q, r = 1, \ldots, L \ with\ p < q < r),$$

(4.11) $$\mathbf{v}_1 = \mathbf{v}_m,\ \mathbf{v}_2 = \mathbf{v}_n,\ \mathbf{v}_3 = \mathbf{v}_t, \quad (m, n, t = 1, \ldots, M \ with\ m < n < t).$$

Table 4.1 Functional basis

Agencies	Basic invariants
$\mathbf{A}$	$tr\mathbf{A},\ tr\mathbf{A}^2,\ tr\mathbf{A}^3$
$\mathbf{v}$	$\mathbf{v}\cdot\mathbf{v}$
$\mathbf{W}$	$tr\mathbf{W}^2$
$\mathbf{A}_1, \mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2,\ tr\mathbf{A}_1^2\mathbf{A}_2,\ tr\mathbf{A}_1\mathbf{A}_2^2,\ tr\mathbf{A}_1^2\mathbf{A}_2^2$
$\mathbf{A}, \mathbf{v}$	$\mathbf{v}\cdot\mathbf{A}\mathbf{v},\ \mathbf{v}\cdot\mathbf{A}^2\mathbf{v}$
$\mathbf{A}, \mathbf{W}$	$tr\mathbf{A}\mathbf{W}^2,\ tr\mathbf{A}^2\mathbf{W}^2,\ tr\mathbf{A}^2\mathbf{W}^2\mathbf{A}\mathbf{W}$
$\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{v}_2$
$\mathbf{v}, \mathbf{W}$	$\mathbf{v}\cdot\mathbf{W}^2\mathbf{v}$
$\mathbf{W}_1, \mathbf{W}_2$	$tr\mathbf{W}_1\mathbf{W}_2$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$	$tr\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{v}$	$\mathbf{A}_1\mathbf{v}\cdot\mathbf{A}_2\mathbf{v}$
$\mathbf{A}, \mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{A}\mathbf{v}_2,\ \mathbf{v}_1\cdot\mathbf{A}^2\mathbf{v}_2$
$\mathbf{A}, \mathbf{W}_1, \mathbf{W}_2$	$tr\mathbf{A}\mathbf{W}_1\mathbf{W}_2,\ tr\mathbf{A}\mathbf{W}_1^2\mathbf{W}_2,\ tr\mathbf{A}\mathbf{W}_1\mathbf{W}_2^2$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{W}$	$tr\mathbf{A}_1\mathbf{A}_2\mathbf{W},\ tr\mathbf{A}_1^2\mathbf{A}_2\mathbf{W},\ tr\mathbf{A}_1\mathbf{A}_2^2\mathbf{W},\ tr\mathbf{A}_1\mathbf{W}^2\mathbf{A}_2\mathbf{W}$
$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$	$tr\mathbf{W}_1\mathbf{W}_2\mathbf{W}_3$
$\mathbf{v}_1, \mathbf{v}_2, \mathbf{W}$	$\mathbf{v}_1\cdot\mathbf{W}\mathbf{v}_2,\ \mathbf{v}_1\cdot\mathbf{W}^2\mathbf{v}_2$
$\mathbf{v}, \mathbf{W}_1, \mathbf{W}_2$	$\mathbf{W}_1\mathbf{v}\cdot\mathbf{W}_2\mathbf{v},\ \mathbf{W}_1^2\mathbf{v}\cdot\mathbf{W}_2\mathbf{v},\ \mathbf{W}_1\mathbf{v}\cdot\mathbf{W}_2^2\mathbf{v}$
$\mathbf{A}, \mathbf{v}, \mathbf{W}$	$\mathbf{A}\mathbf{v}\cdot\mathbf{W}\mathbf{v},\ \mathbf{A}^2\mathbf{v}\cdot\mathbf{W}\mathbf{v},\ \mathbf{A}\mathbf{W}\mathbf{v}\cdot\mathbf{W}^2\mathbf{v}$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{v}_1, \mathbf{v}_2$	$\mathbf{A}_1\mathbf{v}_1\cdot\mathbf{A}_2\mathbf{v}_2 - \mathbf{A}_1\mathbf{v}_2\cdot\mathbf{A}_2\mathbf{v}_1$
$\mathbf{A}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{W}$	$\mathbf{A}\mathbf{v}_1\cdot\mathbf{W}\mathbf{v}_2 - \mathbf{A}\mathbf{v}_2\cdot\mathbf{W}\mathbf{v}_1$
$\mathbf{W}_1, \mathbf{W}_2, \mathbf{v}_1, \mathbf{v}_2$	$\mathbf{W}_1\mathbf{v}_1\cdot\mathbf{W}_2\mathbf{v}_2 - \mathbf{W}_1\mathbf{v}_2\cdot\mathbf{W}_2\mathbf{v}_1$

Table 4.2 Generators for vector functions

Agencies	Generators
$\mathbf{v}$	$\mathbf{v}$
$\mathbf{v}, \mathbf{A}$	$\mathbf{A}\mathbf{v},\ \mathbf{A}^2\mathbf{v}$
$\mathbf{v}, \mathbf{W}$	$\mathbf{W}\mathbf{v},\ \mathbf{W}^2\mathbf{v}$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{v}$	$(\mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1)\mathbf{v}$
$\mathbf{v}, \mathbf{W}_1, \mathbf{W}_2$	$(\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1)\mathbf{v}$
$\mathbf{A}, \mathbf{v}, \mathbf{W}$	$(\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A})\mathbf{v}$

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Table 4.3 Generators for symmetric, tensor-valued functions of second-order

Agencies	Generators
-	$\mathbf{I}$
$\mathbf{A}$	$\mathbf{A},\ \mathbf{A}^2$
$\mathbf{v}$	$\mathbf{v}\otimes\mathbf{v}$
$\mathbf{W}$	$\mathbf{W}^2$
$\mathbf{A}_1, \mathbf{A}_2$	$\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1,\ \mathbf{A}_1^2\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1^2,\ \mathbf{A}_1\mathbf{A}_2^2+\mathbf{A}_2^2\mathbf{A}_1$
$\mathbf{A}, \mathbf{v}$	$\mathbf{v}\otimes\mathbf{A}\mathbf{v}+\mathbf{A}\mathbf{v}\otimes\mathbf{v},\ \mathbf{v}\otimes\mathbf{A}^2\mathbf{v}+\mathbf{A}^2\mathbf{v}\otimes\mathbf{v}$
$\mathbf{A}, \mathbf{W}$	$\mathbf{AW}-\mathbf{WA},\ \mathbf{WAW},\ \mathbf{A}^2\mathbf{W}-\mathbf{WA}^2,\ \mathbf{WA}^2\mathbf{W},\ \mathbf{WAW}^2-\mathbf{W}^2\mathbf{AW}$
$\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2+\mathbf{v}_2\otimes\mathbf{v}_1$
$\mathbf{v}, \mathbf{W}$	$\mathbf{Wv}\otimes\mathbf{Wv},\ \mathbf{v}\otimes\mathbf{Wv}+\mathbf{Wv}\otimes\mathbf{v},\ \mathbf{Wv}\otimes\mathbf{W}^2\mathbf{v}+\mathbf{W}^2\mathbf{v}\otimes\mathbf{Wv}$
$\mathbf{W}_1, \mathbf{W}_2$	$\mathbf{W}_1\mathbf{W}_2+\mathbf{W}_2\mathbf{W}_1,\ \mathbf{W}_1\mathbf{W}_2^2-\mathbf{W}_2^2\mathbf{W}_1,\ \mathbf{W}_1^2\mathbf{W}_2-\mathbf{W}_2\mathbf{W}_1^2$
$\mathbf{A}, \mathbf{v}_1, \mathbf{v}_2$	$(\mathbf{v}_1\otimes\mathbf{A}\mathbf{v}_2+\mathbf{A}\mathbf{v}_2\otimes\mathbf{v}_1)-(\mathbf{v}_2\otimes\mathbf{A}\mathbf{v}_1+\mathbf{A}\mathbf{v}_1\otimes\mathbf{v}_2)$
$\mathbf{W}, \mathbf{v}_1, \mathbf{v}_2$	$(\mathbf{v}_1\otimes\mathbf{W}\mathbf{v}_2+\mathbf{W}\mathbf{v}_2\otimes\mathbf{v}_1)-(\mathbf{v}_2\otimes\mathbf{W}\mathbf{v}_1+\mathbf{W}\mathbf{v}_1\otimes\mathbf{v}_2)$

Table 4.4 Generators for skew-symmetric, tensor-valued function of second-order

Agencies	Generators
-	$\mathbf{0}$
$\mathbf{W}$	$\mathbf{W}$
$\mathbf{A}_1, \mathbf{A}_2$	$\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1,\ \mathbf{A}_1^2\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1^2,\ \mathbf{A}_1\mathbf{A}_2^2-\mathbf{A}_2^2\mathbf{A}_1,$ $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_1^2-\mathbf{A}_1^2\mathbf{A}_2\mathbf{A}_1,\ \mathbf{A}_2\mathbf{A}_1\mathbf{A}_2^2-\mathbf{A}_2^2\mathbf{A}_1\mathbf{A}_2$
$\mathbf{A}, \mathbf{v}$	$\mathbf{v}\otimes\mathbf{A}\mathbf{v}-\mathbf{A}\mathbf{v}\otimes\mathbf{v},\ \mathbf{v}\otimes\mathbf{A}^2\mathbf{v}-\mathbf{A}^2\mathbf{v}\otimes\mathbf{v}$
$\mathbf{A}, \mathbf{W}$	$\mathbf{AW}+\mathbf{WA},\ \mathbf{AW}^2-\mathbf{W}^2\mathbf{A}$
$\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2-\mathbf{v}_2\otimes\mathbf{v}_1$
$\mathbf{v}, \mathbf{W}$	$\mathbf{v}\otimes\mathbf{Wv}-\mathbf{Wv}\otimes\mathbf{v},\ \mathbf{Wv}\otimes\mathbf{W}^2\mathbf{v}-\mathbf{W}^2\mathbf{v}\otimes\mathbf{Wv}$
$\mathbf{W}_1, \mathbf{W}_2$	$\mathbf{W}_1\mathbf{W}_2-\mathbf{W}_2\mathbf{W}_1$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$	$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3-\mathbf{A}_3\mathbf{A}_2\mathbf{A}_1+\mathbf{A}_2\mathbf{A}_3\mathbf{A}_1-\mathbf{A}_1\mathbf{A}_3\mathbf{A}_2+\mathbf{A}_3\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1\mathbf{A}_3$
$\mathbf{A}_1, \mathbf{A}_2, \mathbf{v}$	$\mathbf{A}_1\mathbf{v}\otimes\mathbf{A}_2\mathbf{v}-\mathbf{A}_2\mathbf{v}\otimes\mathbf{A}_1\mathbf{v}+\mathbf{v}(\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1)\otimes\mathbf{v}-(\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1)\mathbf{v}\otimes\mathbf{v}$
$\mathbf{A}, \mathbf{v}_1, \mathbf{v}_2$	$(\mathbf{v}_1\otimes\mathbf{A}\mathbf{v}_2-\mathbf{A}\mathbf{v}_2\otimes\mathbf{v}_1)+(\mathbf{v}_2\otimes\mathbf{A}\mathbf{v}_1-\mathbf{A}\mathbf{v}_1\otimes\mathbf{v}_2)$
$\mathbf{W}, \mathbf{v}_1, \mathbf{v}_2$	$(\mathbf{v}_1\otimes\mathbf{W}\mathbf{v}_2-\mathbf{W}\mathbf{v}_2\otimes\mathbf{v}_1)-(\mathbf{v}_2\otimes\mathbf{W}\mathbf{v}_1-\mathbf{W}\mathbf{v}_1\otimes\mathbf{v}_2)$

For instance, from Tables 4.1 and 4.3 it follows that in the case of nonpolynomial representation the isotropic tensor function (3.3) assumes the form (3.13), where $\alpha_9=0$ (since then the generator $\mathbf{A}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}^2$ is reducible), and the coefficients $\alpha_k\ (k=1,\ldots,8)$ are arbitrary functions of ten invariants appearing in (3.11).

We observe that when Table 4.1 is used then, from a given functional basis, one cannot directly determine a functional basis in which one agency appears in a linear manner. Such a case has to be solved separately from the very beginning; obviously, in the case of a polynomial representation, problems of this type do not arise. To illustrate this remark, in Table 4.5 are listed the invariants which are linear in a symmetric, second-

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order tensor **C**. For more details, the reader should refer to the paper by Korsgaard (1990b).

It should be added, however, that once invariants linear in **C** are available, then the general procedure of constructing generators as outlined in Section 3 still applies. It means that the approach used by the second author in (Telega, 1984) requires just such simple refinements.

Table 4.5 The invariants linear in **C** of the functional basis

Agencies	Invariants linear in C
$\mathbf{C}$	$tr\mathbf{C}$
$\mathbf{C}, \mathbf{A}_1$	$tr\mathbf{A}_1\mathbf{C}, tr\mathbf{A}_1^2\mathbf{C}$
$\mathbf{C}, \mathbf{A}_1, \mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2\mathbf{C}, tr\mathbf{A}_1^2\mathbf{A}_2\mathbf{C}, tr\mathbf{A}_1\mathbf{A}_2^2\mathbf{C}$
$\mathbf{C}, \mathbf{v}$	$\mathbf{v}\cdot\mathbf{C}\mathbf{v}$
$\mathbf{C}, \mathbf{W}$	$tr\mathbf{C}\mathbf{W}^2$
$\mathbf{C}, \mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{C}\mathbf{v}_2$
$\mathbf{C}, \mathbf{A}_1, \mathbf{v}$	$\mathbf{v}\cdot\mathbf{A}_1\mathbf{C}\mathbf{v}, \mathbf{v}\cdot\mathbf{A}_1^2\mathbf{C}\mathbf{v}$
$\mathbf{C}, \mathbf{W}_1, \mathbf{W}_2$	$tr\mathbf{C}\mathbf{W}_1\mathbf{W}_2, tr\mathbf{C}\mathbf{W}_1^2\mathbf{W}_2, tr\mathbf{C}\mathbf{W}_1\mathbf{W}_2^2$
$\mathbf{C}, \mathbf{A}_1, \mathbf{W}$	$tr\mathbf{A}_1\mathbf{C}\mathbf{W}, tr\mathbf{A}_1^2\mathbf{C}\mathbf{W}, tr\mathbf{W}\mathbf{A}_1\mathbf{W}\mathbf{C}, tr\mathbf{C}\mathbf{W}^2\mathbf{A}_1\mathbf{W}$
$\mathbf{C}, \mathbf{v}, \mathbf{W}$	$\mathbf{C}\mathbf{v}\cdot\mathbf{W}\mathbf{v}, \mathbf{C}\mathbf{W}\mathbf{v}\cdot\mathbf{W}\mathbf{v}, \mathbf{C}\mathbf{W}\mathbf{v}\cdot\mathbf{W}^2\mathbf{v}$
$\mathbf{C}, \mathbf{A}_1, \mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{A}_1\mathbf{C}\mathbf{v}_2 - \mathbf{v}_2\cdot\mathbf{A}_1\mathbf{C}\mathbf{v}_1$
$\mathbf{C}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{W}$	$\mathbf{C}\mathbf{v}_1\cdot\mathbf{W}\mathbf{v}_2 - \mathbf{C}\mathbf{v}_2\cdot\mathbf{W}\mathbf{v}_1$

4.2. Representation of two-dimensional isotropic functions

As is well known, two-dimensional problems are often studied in the continuum mechanics. Thus the problem of the representation of isotropic and anisotropic functions in the two-dimensional case is of interest in itself. However, such two-dimensional representations do not necessarily coincide with those derived directly from the corresponding three-dimensional cases.

Investigations of representation of two-dimensional isotropic functions have mainly been carried out in connection with representation of three-dimensional functions, cf. Boehler (1987c), Adkins (1960a, 1960b), Spencer (1971). The representations of these functions are deduced from the representation of the corresponding three-dimensional isotropic functions by reduction resulting from the Caley-Hamilton theorem for two-dimensional tensors. The nonpolynomial representation of scalar-valued, vector-valued, symmetric tensor-valued and skew-symmetric tensor-valued isotropic functions by direct methods (independent of the representation of three-dimensional isotropic functions) have been derived by Korsgaard (1990a). The method used by Korsgaard (1990a) is similar to the method used by Smith (1971). Zheng (1993a) proposed an alternative derivation procedure of determining the representations and obtained the same results as those given by Korsgaard. The results concerning non-polynomial representation of two-dimensional isotropic functions are summarized in Tables 4.6-4.9.

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Table 4.6 Functional basis in two-dimensional space

Agencies	Basic invariants
$\mathbf{A}$	$tr\mathbf{A}$, $tr\mathbf{A}^2$
$\mathbf{v}$	$\mathbf{v}\cdot\mathbf{v}$
$\mathbf{W}$	$tr\mathbf{W}^2$
$\mathbf{A}_1$, $\mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2$
$\mathbf{A}$, $\mathbf{v}$	$\mathbf{v}\cdot\mathbf{A}\mathbf{v}$
$\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{v}_2$
$\mathbf{W}_1$, $\mathbf{W}_2$	$tr\mathbf{W}_1\mathbf{W}_2$
$\mathbf{A}$, $\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{A}\mathbf{v}_2$
$\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{W}$	$tr\mathbf{A}_1\mathbf{A}_2\mathbf{W}$
$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{W}$	$\mathbf{v}_1\cdot\mathbf{W}\mathbf{v}_2$
$\mathbf{A}$, $\mathbf{v}$, $\mathbf{W}$	$\mathbf{A}\mathbf{v}\cdot\mathbf{W}\mathbf{v}$

Table 4.7 Generators for vector functions in two-dimensional space

Agencies	Generators
$\mathbf{v}$	$\mathbf{v}$
$\mathbf{v}$, $\mathbf{A}$	$\mathbf{A}\mathbf{v}$
$\mathbf{v}$, $\mathbf{W}$	$\mathbf{W}\mathbf{v}$

Table 4.8 Generators for symmetric, tensor-valued functions of second-order in two-dimensional space

Agencies	Generators
-	$\mathbf{I}$
$\mathbf{A}$	$\mathbf{A}$
$\mathbf{v}$	$\mathbf{v}\otimes\mathbf{v}$
$\mathbf{A}$, $\mathbf{W}$	$\mathbf{A}\mathbf{W}-\mathbf{W}\mathbf{A}$
$\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2+\mathbf{v}_2\otimes\mathbf{v}_1$
$\mathbf{v}$, $\mathbf{W}$	$\mathbf{v}\otimes\mathbf{W}\mathbf{v}+\mathbf{W}\mathbf{v}\otimes\mathbf{v}$

Table 4.9 Generators for skew-symmetric, tensor-valued function of second-order in two-dimensional space

Agencies	Generators
-	$\mathbf{0}$
$\mathbf{W}$	$\mathbf{W}$
$\mathbf{A}_1$, $\mathbf{A}_2$	$\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1$
$\mathbf{A}$, $\mathbf{v}$	$\mathbf{v}\otimes\mathbf{A}\mathbf{v}-\mathbf{A}\mathbf{v}\otimes\mathbf{v}$
$\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2-\mathbf{v}_2\otimes\mathbf{v}_1$

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Example 4.1. The right stretch tensor defined by

$$\mathbf{U} = \sqrt{\mathbf{C}},$$

is a smooth function of the right Cauchy-Green tensor $\mathbf{C}$ (Gurtin, 1981; Stephenson 1980) and is a nonpolynomial isotropic function, see Ting (1985). The representation of this function in the two-dimensional case is given by, see Hoger and Carlson (1984a, 1984b), Jemioło (1994a)

$$\mathbf{U} = \alpha\mathbf{I} + \beta\mathbf{C} = \frac{1}{\sqrt{I_C + 2\sqrt{II_C}}}\left(\sqrt{II_C}\,\mathbf{I} + \mathbf{C}\right),$$

where

$$I_C = tr\mathbf{C}, \quad II_C = \frac{1}{2}\left(tr^2\mathbf{C} - tr\mathbf{C}^2\right).$$

Example 4.2. The tensorial Hencky measure of strain defined by, cf. Marsden and Hughes (1983)

$$\mathbf{A} = \ln\mathbf{U} = \frac{1}{2}\ln\mathbf{C},$$

in the 2D case has the nonpolynomial representation in the form:

$$\mathbf{A} = a_0\mathbf{I} + a_1\mathbf{C} = \frac{1}{2(C_1 - C_2)}\left[\left(C_1\ln C_2 - C_2\ln C_1\right)\mathbf{I} + \left(\ln C_1 - \ln C_2\right)\mathbf{C}\right],$$

where

$$C_{1,2} = \frac{1}{2}\left(I_C \pm \sqrt{I_C^2 - 4II_C}\right).$$

More general isotropic tensor-valued functions of $\mathbf{C}$ in both two and three dimensions are derived in the papers by Sedov (1962), Hoger and Carlson (1984a, 1984b), Ting (1985), Morman (1986), Curnier and Rakotomanana (1991), Jemioło (1994a), Governatori et al. (1995).

4.3. Representation of two-dimensional orthotropic functions

The objective of this point is the determination of the general form of the functions (4.1)-(4.4) in the two-dimensional case.

In our 2D case, the orthotropy group S satisfies

(4.12) $$\forall\mathbf{Q}\in S, \quad \mathbf{Q}\mathbf{M}\mathbf{Q}^T = \mathbf{M},$$

where $\mathbf{M} = \mathbf{e}\otimes\mathbf{e}$ and the unit vector $\mathbf{e}$ characterizes orthotropy, see Boehler (1987c) p.51.

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For each $\forall \mathbf{Q} \in S$, the scalar-valued function (4.1), vector-valued function (4.2), symmetric tensor-valued function (4.3) and skew-symmetric tensor-valued function (4.4) satisfy the conditions (4.5)-(4.8). By applying Liu's (1982) theorem (see also Rychlewski (1991a)) and taking into account (4.12) the invariance requirements (4.5)-(4.8) may be written in the following way

$$f(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{M}) = f(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^T), \tag{4.13}$$

$$\mathbf{Q}\mathbf{g}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{M}) = \mathbf{g}(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^T), \tag{4.14}$$

$$\mathbf{Q}\underline{F}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^T = \underline{F}(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^T), \tag{4.15}$$

$$\mathbf{Q}\underline{G}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^T = \underline{G}(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{W}_l\mathbf{Q}^T, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^T), \tag{4.16}$$

for each $\forall \mathbf{Q} \in O(2)$, where $O(2)$ denotes the full orthogonal group. Now **M** plays the role of a parametric tensor, and the functions (4.1)-(4.4) depend explicitly on it. We observe that the approach leading to (4.13)-(4.16) has primarily been proposed by Boehler (1978, 1979).

In the sequel we shall derive the functional basis for the scalar function (4.13) and generators for the functions (4.14)-(4.16). Our method of the determination of the functional basis follows that used by Smith (1970, 1971) and Korsgaard (1990a, 1990b) for isotropic functions. Generators will be obtained similarly as in Jemioło, Kwieciński (1990), Jemioło (1994b), Jemioło, Telega (1996), following the idea proposed in the paper by the second author (Telega, 1984).

Determination of the orthotropic functional basis

Since the tensor **M** appearing in (4.13)-(4.16) is a parametric tensor, the determination of the functional basis is less complicated then in the case of isotropy examined by Korsgaard (1990a). Obviously, in the last case S=$O(2)$, because the invariance with respect to the full orthogonal group has been studied.

To find the functional basis for the orthotropic scalar function (4.13) it suffices to consider the following three cases, cf. Jemioło and Telega (1996).

Case 1

In the set of vectors $\{\mathbf{v}_m\}$ (m=1,...,M) there are vectors non-colinear with the direction of **e**.

Case 1.1

At least one vector from the set $\{\mathbf{v}_m\}$, say $\mathbf{v}_1$, is not colinear with **e** and $\mathbf{v}_m \neq \mathbf{0}$, $m = 1, \ldots, M$.

Case 1.2

Only one vector, say $\mathbf{v} \in \{\mathbf{v}_m\}$ is not colinear with **e**, whereas the remaining vectors are zero vectors.

Case 2

We assume that $\mathbf{v}_m = \mathbf{0}, m = 1, \ldots, M$. Since $\mathbf{M} = \mathbf{e} \otimes \mathbf{e} \neq \mathbf{0}$, hence the eigenvalues are: $M_1 = 1, M_2 = 0$.

Case 2.1

Among the tensors $\{\mathbf{A}_i\}$ there is none with non-zero off-diagonal components in the coordinate system $\{x_\alpha\}$, such that the axes of $\{x_\alpha\}$ coincide with the directions of the eigenvectors of **M**.

Case 2.2

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Let $\mathbf{B} \in \{\mathbf{A}_i\}$ denote a tensor with non-zero off-diagonal components. The positive direction of $0x_1$ is chosen in such a way that $B_{12} > 0$.

Case 3

All vectors $\{\mathbf{v}_m\}$ have the form $\mathbf{v}_m = c_m \mathbf{e}$.

Summarizing all the three cases: 1, 2 and 3 we obtain the orthotropic functional basis for the two-dimensional problem.

Table 4.10. Functional basis for the orthotropic scalar-valued function (4.13)

Agencies	Basic invariants
$\mathbf{A}$	$tr\mathbf{A}$, $tr\mathbf{AM}$, $tr\mathbf{A}^2$
$\mathbf{v}$	$\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{Mv}$
$\mathbf{W}$	$tr\mathbf{W}^2$
$\mathbf{A}$, $\mathbf{W}$	$tr\mathbf{MAW}$
$\mathbf{A}_1$, $\mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2$
$\mathbf{A}$, $\mathbf{v}$	$\mathbf{v}\cdot\mathbf{Av}$
$\mathbf{W}$, $\mathbf{v}$	$\mathbf{v}\cdot\mathbf{MWv}$
$\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{v}_2$, $\mathbf{v}_1\cdot\mathbf{Mv}_2$
$\mathbf{W}_1$, $\mathbf{W}_2$	$tr\mathbf{W}_1\mathbf{W}_2$
$\mathbf{A}$, $\mathbf{v}_1$, $\mathbf{v}_2$	$\mathbf{v}_1\cdot\mathbf{Av}_2$
$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{W}$	$\mathbf{v}_1\cdot\mathbf{Wv}_2$

The last table coincides with Zheng's (1993b) results, who has however used a different method.

Boehler (1978, 1979, 1987c) determined functional bases provided that functions appearing in (4.13) depend only on symmetric tensors. In the two-dimensional case, Boehler's results correspond to the first and fifth rows of our Table 4.10. This author approached the two-dimensional case through the three-dimensional one by using Cayley-Hamilton's theorem, cf. also Smith (1971). The method of the determination of a functional basis employed by Boehler and based on Cayley-Hamilton's theorem, proves that the functional basis is also the integrity basis.

Adkins (1960a, 1960b) determined integrity bases, in the two- and three-dimensional cases, for arbitrary second order tensors under the condition of linearity of invariants with respect to each argument. Consequently, two-dimensional reduction of the invariants in the case of transverse isotropy characterized by the parametric tensor $\mathbf{M}$ does not yield the invariants listed in the Table 4.10. It is worth noting that the tensor $\mathbf{M}$ describes only one of the five possible cases of 3D transverse isotropy, cf. Section 2.

Determination of generators of an orthotropic vector-valued function

In this section we shall derive the general form of the vector-valued function (4.14). To this end we consider the scalar function linear in $\mathbf{d}$, cf. Telega (1984), Jemioło (1993c), Jemioło and Telega (1996)

$$g = \mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m).\mathbf{d}. \tag{4.17}$$

Thus we may write

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(4.18) $$g(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_{:m}, \mathbf{d}) = \hat{g}(I_r, J_s) = \sum_{s=1}^{S} \varphi_s(I_r) J_s,$$

where I_r are the invariants listed in Table 4.10 while J_s are the following invariants, linear in $\mathbf{d}$:

(4.19) $$\mathbf{d}\cdot\mathbf{v}_{:m},\ \mathbf{d}\cdot\mathbf{M}\mathbf{v}_{:m},\ \mathbf{d}\cdot\mathbf{A}_i\mathbf{v}_{:m},\ \mathbf{d}\cdot\mathbf{W}_p\mathbf{v}_{:m}.$$

They are obtained by using the procedure outlined in the previous section. The canonical form of the vector-valued function (4.14) is given by

(4.20) $$\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_{:m}) = \frac{\partial\hat{g}}{\partial\mathbf{d}} = \sum_{s=1}^{S} \varphi_s(I_r)\frac{\partial J_s}{\partial\mathbf{d}} = \sum_{s=1}^{S} \varphi_s(I_r)\mathbf{g}_s.$$

The generators $\mathbf{g}_s$ are listed in Table 4.11 and coincide with the results due to Zheng (1993b).

Table 4.11. Generators of the orthotropic vector-valued function (4.14)

Agencies	Generators
v	**v**, **Mv**
v, **A**	**Av**
v, **W**	**Wv**

Determination of generators of the orthotropic symmetric tensor-valued function
Proceeding similarly as in the previous section we take, cf. Jemioło and Telega (1996)

(4.21) $$h = tr\underline{\mathrm{F}}\mathbf{C},$$

where $\mathbf{C}$ is a symmetric second-order tensor. The scalar-valued function h has now the form

(4.22) $$h(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_{:m}, \mathbf{d}) = \hat{h}(I_r, J_s) = \sum_{s=1}^{S} \eta_s(I_r) J_s,$$

where I_r are the invariants listed in Table 4.10 and J_s are linear in $\mathbf{C}$:

(4.23) $$tr\mathbf{C},\ tr\mathbf{MC},\ tr\mathbf{CA}_i,\ tr\mathbf{CMA}_i,\ \mathbf{v}_{:m}\cdot\mathbf{C}\mathbf{v}_{:m},\ \mathbf{v}_{:m}\cdot\mathbf{C}\mathbf{v}_{:n}.$$

The canonical form of the tensor-valued function (4.15) is given by

(4.24) $$\underline{\mathrm{F}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_{:m}) = \frac{1}{2}\left(\frac{\partial\hat{h}}{\partial\mathbf{C}} + \frac{\partial\hat{h}}{\partial\mathbf{C}^T}\right) = \frac{1}{2}\sum_{s=1}^{S} \eta_s(I_r)\left(\frac{\partial J_s}{\partial\mathbf{C}} + \frac{\partial J_s}{\partial\mathbf{C}^T}\right) = \sum_{s=1}^{S} \overline{\eta}_s(I_r)\underline{\mathrm{F}}_s.$$

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The results are summarized in Table 4.12. The generators $\underline{F}_s$ are the same as those obtained by Zheng (1993b). The case considered by Boehler (1978, 1987c) is covered by the first and second rows of Table 4.12.

Table 4.12. Generators of the orthotropic, symmetric tensor-valued function (4.15)

Agencies	Generators
-	$\mathbf{I}, \mathbf{M}$
$\mathbf{A}$	$\mathbf{A}$
$\mathbf{v}$	$\mathbf{v}\otimes\mathbf{v}$
$\mathbf{W}$	$\mathbf{MW}-\mathbf{WM}$
$\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2+\mathbf{v}_2\otimes\mathbf{v}_1$

Determination of generators of the orthotropic skew-symmetric tensor-valued function

We begin by constructing the scalar function, cf. Jemioło and Telega (1996)

(4.25)
$$k = tr\underline{G}\mathbf{X},$$

where $\mathbf{X}$ is a skew-symmetric tensor. Hence we may write

(4.26)
$$k(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{d}) = \hat{k}(I_r, J_s) = \sum_{s=1}^{S} \kappa_s(I_r) J_s,$$

where J_s are the invariants linear in $\mathbf{X}$:

(4.27)
$$tr\mathbf{MA}_i\mathbf{X},\ tr\mathbf{XW}_p,\ \mathbf{v}_m\cdot\mathbf{MXv}_m,\ \mathbf{v}_m\cdot\mathbf{Xv}_n.$$

The canonical form of the function $\underline{G}$ is given by

(4.24)
$$\underline{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) = \frac{1}{2}\left(\frac{\partial\hat{k}}{\partial\mathbf{X}} - \frac{\partial\hat{k}}{\partial\mathbf{X}^T}\right) = \frac{1}{2}\sum_{s=1}^{S}\kappa_s(I_r)\left(\frac{\partial J_s}{\partial\mathbf{X}} - \frac{\partial J_s}{\partial\mathbf{X}^T}\right) = \sum_{s=1}^{S}\overline{\kappa}_s(I_r)\underline{G}_s.$$

The generators of $\underline{G}_s$ are listed in Table 4.13. They coincide with those obtained by Zheng (1993b).

Table 4.13. Generators of the orthotropic, skew-symmetric tensor-valued function (4.16)

Agencies	Generators
-	$\mathbf{0}$
$\mathbf{W}$	$\mathbf{W}$
$\mathbf{A}$	$\mathbf{MA}-\mathbf{AM}$
$\mathbf{v}$	$\mathbf{v}\otimes\mathbf{Mv}-\mathbf{Mv}\otimes\mathbf{v}$
$\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{v}_1\otimes\mathbf{v}_2-\mathbf{v}_2\otimes\mathbf{v}_1$

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Equivalent functional bases and sets of generators

Zheng (1993c, 1994a) determined alternative form of the functional basis and generators in comparison with the results of his first paper (1993b). In Zhang's papers the representations of functions (4.1)-(4.4) corresponding to all anisotropy groups have been investigated. Then orthotropy group is the group $\mathcal{C}_{2v}$ (cf. also Smith (1994) and Table 2.1) and the parametric tensor **K** has the form

$$\mathbf{K} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 = \mathbf{P}_2. \tag{4.25}$$

Here $\mathbf{e}_\alpha$ $(\alpha = 1,2)$ are unit vectors specifying the directions of orthotropy. By setting $\mathbf{e}_1 = \mathbf{e}$, we readily obtain

$$\mathbf{K} = 2\mathbf{M} - \mathbf{I}. \tag{4.26}$$

This relation enables the passage from our results to those due to Zheng (1993c, 1994a) in the two-dimensional case of orthotropy.

The results obtained by Zheng and in (Jemioło and Telega, 1996) can be applied to the determination of representations of the following functions:

$$\begin{aligned}
&\tilde{s} = \tilde{f}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{H}),\\
&\hat{\mathbf{t}} = \tilde{\mathbf{f}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{H}),\\
&\tilde{\mathbf{S}} = \tilde{\underline{\mathrm{F}}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{H}), \quad \tilde{\mathbf{S}} = \tilde{\mathbf{S}}^T,\\
&\tilde{\mathbf{T}} = \tilde{\underline{\mathrm{G}}}(\mathbf{A}_k, \mathbf{W}_l, \mathbf{v}_m, \mathbf{H}), \quad \tilde{\mathbf{T}} = -\tilde{\mathbf{T}}^T,
\end{aligned} \tag{4.27}$$

where **H** is a symmetric, positive definite tensor. Its eigenvalues are denoted by H_1 and H_2, $H_1 > H_2$. Now we have

$$\mathbf{H} = H_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + H_2 \mathbf{e}_2 \otimes \mathbf{e}_2 = H_1 \mathbf{M} + H_2(\mathbf{I} - \mathbf{M}). \tag{4.28}$$

Consequently one can easily determine the representations of the functions appearing in (4.27).

The last case is important for applications if **H** plays the role of a fabric tensor, cf. Cowin (1985, 1986a, 1986b). This tensor is sometimes used to model the mechanical behaviour of materials as different as soils Boehler (1987a) and bones Cowin (1985, 1986a, 1986b).

In the case when $H_1 = H_2$, **H** is a spherical tensor and the representations of functions (4.27) coincide with those derived by Korsgaard (1990a); then the tensor **H** does not appear in these functions.

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4.4. Representation of three-dimensional orthotropic functions

Our aim is to determine the non-polynomial representations of the following functions

$$
(4.29)\qquad
\begin{aligned}
s &= f(\mathbf{A}_p;\mathbf{H}), \quad f: \underbrace{T_s \times \ldots \times T_s}_{(P+1)-times} \to R,\\
\mathbf{S} &= \underline{F}(\mathbf{A}_p;\mathbf{H}), \quad \underline{F}: \underbrace{T_s \times \ldots \times T_s}_{(P+1)-times} \to T_s,
\end{aligned}
$$

where $\mathbf{A}_p$ are symmetric second-order tensors and $\mathbf{H}$ is a symmetric, positive-definite tensor of the second order. The tensor $\mathbf{H}$ plays the role of a parametric tensor. The function f is a scalar-valued function while $\underline{F}$ is a symmetric, second order tensor function. Suppose that (4.29) are to be constitutive relationships. Then $\mathbf{A}_p$ are causes, $\mathbf{H}$ models the structure of a material while s and $\mathbf{S}$ are responses or effects. Within the framework of the classical continuum mechanics such relationships should be invariant with respect to the group of automorphisms of the space E^3, cf. Rychlewski (1991a). In other words, they have to satisfy the so called principle of isotropy of the physical space. Consequently the functions appearing in (4.29) fulfil the following conditions:

$$
(4.30)\qquad \forall \mathbf{Q} \in O: f(\mathbf{A}_p;\mathbf{H}) = f(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T;\mathbf{Q}\mathbf{H}\mathbf{Q}^T),\ \mathbf{Q}\underline{F}(\mathbf{A}_p;\mathbf{H})\mathbf{Q}^T = \underline{F}(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T;\mathbf{Q}\mathbf{H}\mathbf{Q}^T),
$$

where O denotes the full orthogonal group.

According to our assumption, the tensor $\mathbf{H}$ has three distinct eigenvalues, say H_i $(i = 1,2,3)$. Thus we may write

$$
(4.31)\qquad \mathbf{H} = H_1\mathbf{e}_1 \otimes \mathbf{e}_1 + H_2\mathbf{e}_2 \otimes \mathbf{e}_2 + H_3\mathbf{e}_3 \otimes \mathbf{e}_3, \quad H_1 \neq H_2 \neq H_3 \neq H_1,
$$

where $\mathbf{e}_i$ are unit eigenvectors of the tensor $\mathbf{H}$. We observe that the group of external symmetries of the tensor $\mathbf{H}$, given by

$$
(4.32)\qquad S \equiv \{\mathbf{Q} \in O: \quad \mathbf{Q}\mathbf{H}\mathbf{Q}^T = \mathbf{H}\},
$$

is the orthotropy group. Moreover, the eigenvectors of $\mathbf{H}$ determine the so called principal axes of orthotropy of a material. This statement becomes evident if we compare (4.31) and (4.32) with the corresponding definitions given in the papers by Boehler (1978, 1979) and Rychlewski (1991a).

Let

$$
(4.33)\qquad \mathbf{M}_i = \mathbf{e}_i \otimes \mathbf{e}_i \quad (\text{no summation on } i = 1,2,3),
$$

then we recover, by taking account of (4.31) and (4.33) in (4.29), provided that (4.30) is satisfied, the problem considered in Boehler's papers (1978, 1979).

From (4.30) and (3.32) it follows that

$$
(4.34)\qquad \forall \mathbf{Q} \in S: \quad f(\mathbf{A}_p;\mathbf{H}) = f(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T;\mathbf{H}), \quad \mathbf{Q}\underline{F}(\mathbf{A}_p;\mathbf{H})\mathbf{Q}^T = \underline{F}(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T;\mathbf{H}).
$$

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In other words the functions $f(\ldots;\mathbf{H})$, $\underline{F}(\ldots;\mathbf{H})$ are orthotropic functions of the tensors $\mathbf{A}_p$.

Determination of the orthotropic functional basis. Since the tensor **H** has three distinct eigenvalues therefore in order to determine the functional basis for the scalar function $(4.29)_1$ we may exploit the results obtained by Smith (1971), cf. Jemioło and Telega (1997a). To this end it is sufficient to consider the case (2ii) studied by Smith (1971, pp.905-907). The functional basis derived in this manner is presented in Table 4.14.

Table 4.14. Functional basis for the scalar function $(4.29)_1$

Arguments	Basic invariants
$\mathbf{A}_p$	$tr\mathbf{A}_p, tr\mathbf{A}_p^2, tr\mathbf{A}_p^3, tr\mathbf{H}\mathbf{A}_p, tr\mathbf{H}^2\mathbf{A}_p, tr\mathbf{H}\mathbf{A}_p^2, tr\mathbf{H}^2\mathbf{A}_p^2$
$\mathbf{A}_p, \mathbf{A}_q$	$tr\mathbf{A}_p\mathbf{A}_q, tr\mathbf{A}_p^2\mathbf{A}_q, tr\mathbf{A}_p\mathbf{A}_q^2, tr\mathbf{H}\mathbf{A}_p\mathbf{A}_q, tr\mathbf{H}^2\mathbf{A}_p\mathbf{A}_q$
$\mathbf{A}_p, \mathbf{A}_q, \mathbf{A}_r$	$tr\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r, \quad p,q,r=1,\ldots,P;\ p<q<r$

It can easily be proved that the representation of the scalar function $(4.29)_1$ depicted in Table 4.14 is equivalent to the results obtained by Boehler. Boehler's orthotropic functional basis is presented in Table 4.15.

Table 4.15. Orthotropic functional basis after Boehler (1979, 1987c)

Arguments	Basic invariants
$\mathbf{A}_p$	$tr\mathbf{M}_1\mathbf{A}_p, tr\mathbf{M}_1\mathbf{A}_p^2, tr\mathbf{A}_p^3, tr\mathbf{M}_2\mathbf{A}_p, tr\mathbf{M}_2\mathbf{A}_p^2, tr\mathbf{M}_3\mathbf{A}_p, tr\mathbf{M}_3\mathbf{A}_p^2$
$\mathbf{A}_p, \mathbf{A}_q$	$tr\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q, tr\mathbf{A}_p^2\mathbf{A}_q, tr\mathbf{A}_p\mathbf{A}_q^2, tr\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q, tr\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q$
$\mathbf{A}_p, \mathbf{A}_q, \mathbf{A}_r$	$tr\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r, \quad p,q,r=1,\ldots,P;\ p<q<r$

Both functional basis are equivalent because:

$$
\begin{aligned}
&tr\mathbf{A}_p = tr\mathbf{M}_1\mathbf{A}_p + tr\mathbf{M}_2\mathbf{A}_p + tr\mathbf{M}_3\mathbf{A}_p,\\
&tr\mathbf{A}_p^2 = tr\mathbf{M}_1\mathbf{A}_p^2 + tr\mathbf{M}_2\mathbf{A}_p^2 + tr\mathbf{M}_3\mathbf{A}_p^2,\\
(4.35)\qquad &tr\mathbf{H}^a\mathbf{A}_p^b = H_1^a tr\mathbf{M}_1\mathbf{A}_p^b + H_2^a tr\mathbf{M}_2\mathbf{A}_p^b + H_3^a tr\mathbf{M}_3\mathbf{A}_p^b,\\
&tr\mathbf{A}_p\mathbf{A}_q = tr\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q + tr\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q + tr\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q,\\
&tr\mathbf{H}^a\mathbf{A}_p\mathbf{A}_q = H_1^a tr\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q + H_2^a tr\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q + H_3^a tr\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q, \quad a,b=1,2.
\end{aligned}
$$

Determination of generators of an orthotropic tensor-valued function of the second order. In order to derive the representation of the function $(4.29)_2$ under the condition $(4.30)_2$, we shall apply the method similar to that used in the papers by Telega (1984), Jemioło and Kwieciński (1990). First, we construct a scalar function, say g, defined by, see Jemioło and Telega (1997a).

$$(4.36)\qquad g = tr\underline{F}\mathbf{C},$$

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linear with respect to the second argument or $\mathbf{C}$. Here $\mathbf{C}$ is a symmetric second order tensor while $\underline{\mathrm{F}}$ is the function $(4.29)_2$. The function g has the following form:

$$(4.37) \qquad g(\mathbf{A}_p, \mathbf{C}; \mathbf{H}) = \tilde{g}(I_t, J_s) = \sum_{s=1}^{S} \phi(I_t) J_s,$$

where I_t are invariants listed in Table 4.14, whereas J_s are invariants linear in $\mathbf{C}$, see Table 4.16 below.

Table 4.16. Invariants linear in **C**

Arguments	Invariants J_s
$\mathbf{C}$	$tr\mathbf{C}, tr\mathbf{HC}, tr\mathbf{H}^2\mathbf{C}$
$\mathbf{C}, \mathbf{A}_p$	$tr\mathbf{A}_p\mathbf{C}, tr\mathbf{A}_p^2\mathbf{C}, tr\mathbf{HA}_p\mathbf{C}, tr\mathbf{H}^2\mathbf{A}_p\mathbf{C}$
$\mathbf{C}, \mathbf{A}_p, \mathbf{A}_q$	$tr\mathbf{A}_p\mathbf{A}_q\mathbf{C}, \quad p,q = 1,\ldots,P;\ p<q$

The canonical form of the tensor-valued function $(4.29)_2$ is found from

$$(4.38) \qquad \underline{\mathrm{F}}(\mathbf{H}, \mathbf{A}_p) = \frac{1}{2}\left(\frac{\partial g}{\partial \mathbf{C}} + \frac{\partial g}{\partial \mathbf{C}^T}\right) = \frac{\partial g}{\partial \mathbf{C}} = \sum_{s=1}^{S} \phi_s(I_t)\frac{\partial J_s}{\partial \mathbf{C}} = \sum_{s=1}^{S} \tilde{\phi}_s(I_t)\mathbf{G}_s.$$

The results of calculations are summarized in Table 4.17, where the generators $\mathbf{G}_s$ are listed.

The generators obtained in this way are equivalent to those derived by Boehler (1979) and listed in Table 4.18. To corroborate this statement it is sufficient to exploit the following identities:

$$(4.39) \qquad \begin{aligned} &\mathbf{I} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3, \\ &\mathbf{H}^a = H_1^a\mathbf{M}_1 + H_2^a\mathbf{M}_2 + H_3^a\mathbf{M}_3, \quad a = 1,2, \\ &2\mathbf{A}_p = \mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p, \\ &\mathbf{H}^a\mathbf{A}_p + \mathbf{H}^a\mathbf{A}_p = H_1^a(\mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p) + H_2^a(\mathbf{M}_2\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p) + H_3^a(\mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p). \end{aligned}$$

Table 4.17. Generators of the function $(4.29)_2$

Arguments	Generators
	$\mathbf{I}, \quad \mathbf{H}, \quad \mathbf{H}^2$
$\mathbf{A}_p$	$\mathbf{A}_p, \quad \mathbf{A}_p^2, \quad \mathbf{HA}_p + \mathbf{A}_p\mathbf{H}, \quad \mathbf{H}^2\mathbf{A}_p + \mathbf{A}_p\mathbf{H}^2$
$\mathbf{A}_p, \mathbf{A}_q$	$\mathbf{A}_p\mathbf{A}_q + \mathbf{A}_q\mathbf{A}_p, \quad p,q = 1,\ldots,P;\ p<q$

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Table 4.18. Boehler's (1979, 1987c) generators of the orthotropic tensor function

Arguments	Generators
	$\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$
$\mathbf{A}_p$	$\mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p$, $\mathbf{M}_2\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p$, $\mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p$, $\mathbf{A}_p^2$
$\mathbf{A}_p, \mathbf{A}_q$	$\mathbf{A}_p\mathbf{A}_q + \mathbf{A}_q\mathbf{A}_p$, $p,q = 1,\ldots,P;\ p<q$

4.5. Comments on further results on anisotropic tensor functions of $\mathbf{A}_k$, $\mathbf{W}_l$ and $\mathbf{v}_m$

At present, in 2D case, complete and irreducible representation of anisotropic scalar-valued, vector-valued, symmetric second-order tensor-valued and skew- symmetric second-order tensor-valued functions of a finite number symmetric second-order tensors $\mathbf{A}_k$, skew- symmetric second-order tensors $\mathbf{W}_l$ and vectors $\mathbf{v}_m$ have been well established. The review paper by Zheng (1994a) summarizes the results achieved. The same author (Zheng, 1993c), by using certain properties of structural tensors listed in Table 2.1, significantly simplified the determination of representations of the functions just mentioned. By adopting the idea due to Pennisi and Trovato (1987) on isotropic representations, he was able to show irreducibility and completeness of the anisotropic representations derived.

In the 3D case, though parametric tensors are known for all kinds of anisotropy (see Table 2.2), yet the available anisotropic representations are not so complete. This unsatisfactory state-of-the-art is caused by appearance of tensors of order greater than two among structural tensors. However, complete and irreducible results are available for five types of transverse isotropy, three types of orthotropy and only certain kinds of the remaining types of anisotropy (see Zhang (1991b); Zheng, 1993b, 1994a; Pierce 1995). It seems that in the future new results will also be available. Recently Xiao (1996a, 1996b) has formulated several theorems which will probably be of importance in the search for representations of anisotropic tensor functions. Particularly, Xiao (1996a) proved that the problem of finding of a functional basis for both isotropic and anisotropic functions reduces to determining of invariants of at most four variables. He also showed which combinations of variables are to be analyzed. Similarly, in the case of second-order tensor functions it is sufficient to determine functions dependent on at most three variables. Next, Xiao's (1996b) analysis reveals that in the case of anisotropic tensor function of order not exceeding two, parametric tensors can be replaced by equivalent structural tensors of the order not greater than two. However, the last tensors are constructed from higher-order tensors. Xiao's (1996b) approach allows to determine complete functional bases and generators. In general, representations thus obtained are not irreducible.

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4.6. Regularity of representations

4.6.1. Reiner (1945) and Rivlin (1948b) derived the general form of the constitutive equation for isotropic, compressible non-Newtonian fluids:

$$(4.40) \qquad \mathbf{t} = -p\mathbf{I} + \underline{F}(\mathbf{D}) = (-p+\alpha)\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2.$$

Here **t** is the stress tensor, p the fluid pressure and $\alpha = 0$ when **D**=**0**. Obviously α, β and γ are functions of the principal invariants of **D**. This equation was derived by both Reiner (1945) and Rivlin (1948b) under the assumption that $\underline{F}(\mathbf{D})$ is *analytic*. The same representation was obtained by Serrin (1959) by using *algebraic* methods only. Fluids obeying Eq. (4.40) are called by Serrin (1959) „Stokesian fluids". We note that Truesdell (1952) uses the terminology „Stokesian fluid" in a more restricted sense.

For an incompressible Stokesian fluid the formula for stress becomes

$$(4.41) \qquad \mathbf{t} = -p\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2,$$

where β and γ are functions of the second and third invariants of **D**. Serrin's (1959) considerations take also into account the thermodynamical state, which from the point of view of tensor function representations plays the role of a parameter. More precisely, Serrin (1959) assumes that the tensor function $\underline{F}$ depends on **D** and ξ. This author writes: „ξ denotes the thermodynamical states". This quantity is a scalar; one may think of the temperature or entropy.

The subsequent development of tensor function representations has been based on algebraic methods. Now the time is ripe for asking the following question: what can be said about regularity of the existing nonpolynomial representations? Actually, we are at the very beginning of solving this difficult problem. In this section we shall summarize the available results. In the already mentioned paper Serrin (1959) proves that the coefficients α, β, γ inherit all the differentiability properties of the original relation $\underline{F}(\mathbf{D})$, provided that the principal values of **D** are different. This author gives also an example showing that α, β and γ might be discontinuous at a coalescence of the principal values of **D**, even though the original relation $\underline{F}(\mathbf{D})$ was differentiable. However, if $\underline{F}$ is of class C^3 in **D**, then the coefficients α, β and γ in (4.40) are continuous functions of the principal invariants of **D**. Serrin's regularity results were refined by Man (1994, 1995), see subsection 4.6.3 below.

4.6.2. In a remarkable paper by Ball (1984) regularity results for symmetric and isotropic scalar functions are presented. Those results are confined to scalar functions of vector or tensor argument.

Regularity of symmetric functions

Let us denote by S_n the group of permutations of n symbols. A function $f: R^n \to R$ is said to be invariant under S_n if

$$(4.42) \qquad f(x_{\mathsf{P}_1}, \ldots, x_{\mathsf{P}_n}) = f(x_1, \ldots, x_n),$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$, $\mathsf{P} \in \mathsf{S}_n$. It is well known that this holds if and only if

$$(4.43) \qquad f(\mathbf{x}) = F(\mathsf{S}(\mathbf{x})) \quad \textit{for all } \mathbf{x} \in R^n,$$

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for some F, where $\mathbf{S}(\mathbf{x}) = (S_1(\mathbf{x}), \ldots, S_n(\mathbf{x}))$, $\mathbf{S}: R^n \to R^n$, denotes the n-vector of *elementary symmetric functions*. More precisely, for $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$ the elementary symmetric functions are defined by

$$S_0(\mathbf{x}) = 1, \tag{4.44}$$
$$S_j(\mathbf{x}) = (-1)^j \sum_{1 \le i_1 < \ldots < i_j \le n} x_{i_1} \ldots x_{i_j}, \quad 1 \le j \le n.$$

We have

$$S_1(\mathbf{x}) = -(x_1 + \ldots + x_n), \quad S_2(\mathbf{x}) = x_1 x_2 + \ldots, \quad S_n(\mathbf{x}) = (-1) x_1 x_2 \ldots x_n.$$

Let $E \subset R^n$ be *open* and *symmetric*, i.e. $\mathsf{P}E = E$ for every permutation P of $(x_1, \ldots, x_n)$. Further, let K_n denote the open cone consisting of those points $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$ with $x_1 > x_2 > \ldots > x_n$. We set

$$\Omega_E = \mathbf{S}(E \cap K_n). \tag{4.45}$$

Basic properties of Ω_E:

(i) $\Omega_E = \operatorname{int} \mathbf{S}(E)$,

(ii) $\partial\Omega_E = \mathbf{S}(\partial E) \cup \mathbf{S}(E \cap \partial K_n)$,

(iii) $\overline{\Omega}_E = \mathbf{S}(\overline{E})$.

Obviously, the bar over a set denotes its closure.

Given any symmetric function $f: E \to R$ there exists a unique function $F: \mathbf{S}(E) \to R$ such that

$$f(\mathbf{x}) = F(\mathbf{S}(\mathbf{x})) \quad \textit{for all } \mathbf{x} \in E. \tag{4.46}$$

Ball (1984) assumes that E is convex, though this hypothesis can be weakened. The first results relating the differentiability properties of f and F is formulated as

Theorem 4.1. If $f \in C^{nr}(\overline{E})$, then $F \in C^r(\overline{\Omega}_E)$, $r = 0, 1, 2, \ldots$ ∇

Hence we have

Corollary 4.2. If $f \in C^\infty(\overline{E})$, then $F \in C^\infty(\overline{\Omega}_E)$.

The loss of derivatives given by Th.4.1 is optimal, thus in general $f \in C^{nr-1}(\overline{E})$ does not imply $F \in C^r(\overline{\Omega}_E)$, and $f \in C^{nr+1}(\overline{E})$ does not imply $F \in C^{r+1}(\overline{\Omega}_E)$ unless n=1. Consider the special case when

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \phi(x_i), \tag{4.47}$$

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where $\phi: I \to R$ and I is an open interval, possibly unbounded. Thus $f: E = I^n \to R$. Let $F = F_\phi$ be given by (4.43) and (4.47) Ball (1984) proves the following result.

Theorem 4.3. $\phi \in C^{nr}(\bar{I})$ if and only if $F \in C^r(\overline{\Omega}_E)$. ∇

These regularity results are also of interest in the study of isotropic functions. To make his paper rather self-contained, Ball (1984) includes also considerations on extension of differentiable functions, cf. also Stein (1970).

Regularity of isotropic functions

Let us denote by E_s^n the space of real, symmetric $n \times n$ matrices with inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i,j=1}^{n} A_{ij} B_{ij} .$$

Caution. Until the end of this section the summation convention is *not* used.

Obviously, in physical situations n does not exceed 3.

Let $E \subset E_s^n$ be open and invariant under $O_n^0 = SO(n)$; i.e., if $\mathbf{A} \in E$ and $\mathbf{Q} \in O_n^0$ then $\mathbf{QAQ}^T \in E$. Particularly the whole space E_s^n is invariant under the proper orthogonal group O_n^0. Further, by Γ_E we denote the set of diagonal matrices belonging to E. For such a matrix $\mathbf{A} \in E$ we write $\mathbf{A} = diag(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are diagonal element of this matrix. We recall that a function $h: E \to R$ is said to be isotropic if

$$h(\mathbf{QAQ}^T) = h(\mathbf{A}), \tag{4.48}$$

for all $\mathbf{A} \in E$, $\mathbf{Q} \in O_n^0$.

Let $v_i(\mathbf{A})\,(i = 1, \ldots, n)$ be eigenvalues of $\mathbf{A} \in E$. A standard result states that h is isotropic if and only if there exists a *symmetric* function $\tilde{h}: \Gamma_E \to R$ such that

$$h(\mathbf{A}) = \tilde{h}(v_1(\mathbf{A}), \ldots, v_n(\mathbf{A})), \tag{4.49}$$

for all $\mathbf{A} \in E$. Obviously, $\tilde{h}(a_1, \ldots, a_n) = h(diag(a_1, \ldots, a_n))$ for all $\mathbf{a} \in (a_1, \ldots, a_n) \in \Gamma_E$; more precisely we should write $diag\, \mathbf{a} = diag(a_1, \ldots, a_n) \in \Gamma_E$. The set Γ_E can be identified with a subset of R^n, cf. also Marques and Moreau (1982). Thus $h \in C^r(E)\,(resp.\; h \in C^{r,\alpha}(E))$ implies that $\tilde{h} \in C^r(\Gamma_E)\,(resp.\; \tilde{h} \in C^{r,\alpha}(\Gamma_E))$. Ball (1984) obtained also some results in the reverse direction.

The following two theorems summarize Ball's (1984) results on the regularity of isotropic functions.

Theorem 4.4. Let r=0,1 or 2. Then $h \in C^r(E)$ if and only if $\tilde{h} \in C^r(\Gamma_E)$. If $\mathbf{A} = \mathbf{Q} diag(a_1, \ldots, a_n) \mathbf{Q}^T \in E$ with $\mathbf{Q} \in O_n^0$ and $a_i \neq a_j$ for $i \neq j$ then

$$Dh(\mathbf{A})\mathbf{B} = \sum_{i=1}^{n} \tilde{h}_{,i}(\mathbf{a}) (\mathbf{Q}^T \mathbf{B} \mathbf{Q})_{ii}, \tag{4.50}$$

for all $\mathbf{B} \in E_s^n$ if $\tilde{h} \in C^1(\Gamma_E)$, and

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$$D^2h(\mathbf{A})(\mathbf{B},\mathbf{B}) =$$

(4.51)
$$= \sum_{i,j=1}^{n} \tilde{h}_{,ij}(\mathbf{a})(\mathbf{Q}^T\mathbf{B}\mathbf{Q})_{ii}(\mathbf{Q}^T\mathbf{B}\mathbf{Q})_{jj} + 2\sum_{i>j} \frac{\tilde{h}_{,i}(\mathbf{a}) - \tilde{h}_{,j}(\mathbf{a})}{a_i - a_j}(\mathbf{Q}^T\mathbf{B}\mathbf{Q})_{ij}(\mathbf{Q}^T\mathbf{B}\mathbf{Q})_{ij},$$

for all $\mathbf{B} \in E_s^n$ if $\tilde{h} \in C^2(\Gamma_E)$. ∇

It is evident that $\tilde{h}_{,i}(\mathbf{a}) = \partial\tilde{h}(\mathbf{a})/\partial a_i$, etc.

Remark 4.5. According to Ball (1984) he has not been able to prove that $h \in C^r(E)$ if and only if $\tilde{h} \in C^r(\Gamma_E)$ *for any r*. It is known that $h \in C^r(E)$ implies $\tilde{h} \in C^r(\Gamma_E)$ for any r. On the other hand, if $\tilde{h} \in C^r(\Gamma_E)$ then given any $\mathbf{A},\mathbf{B} \in E_s^n$ the map $t \to h(\mathbf{A}+t\mathbf{B})$ is C^r for sufficiently small $|t|$; this follows from the fact that the eigenvalues $v_i(t)$ of $\mathbf{A}+t\mathbf{B}$ can be ordered so as to be smooth in t and hence $t \to \tilde{h}(v_1(t),\ldots,v_n(t))$ is C^r. Unfortunately, the eigenvalues of a symmetric $\mathbf{A}$ cannot in general be ordered so as to be C^r in $\mathbf{A}$, even if n=2.

Remark 4.6. Let E_1 denote the open set consisting of those $\mathbf{A} \in E$ whose eigenvalues are all different. We observe that the eigenvalues $v_i(\mathbf{A})$ are smooth functions of $\mathbf{A}$ in E_1. The subset $E_2 = \{\mathbf{A} \in E_s^n \mid v_i(\mathbf{A}) = v_j(\mathbf{A}) \textit{ for some } i \neq j\}$ is closed and *sparse*, since it is the zero set of the discriminant

$$\left[\prod_{1\le i<j\le n}(v_i(\mathbf{A}) - v_j(\mathbf{A}))\right]^2,$$

which is a symmetric polynomial function of the v_i and is thus expressible as a polynomial in the entries of $\mathbf{A}$.

A set $K \subset R^n$ is *sparse* if given any $\mathbf{x} \in K$ and any nonzero $\xi \in R^n$ there exist sequences $\mathbf{x}^{(n)} \to \mathbf{x}$, $\xi^{(n)} \to \xi$ and a number $\varepsilon > 0$ such that for each n=1,2,..., the line segment $\{\mathbf{x}^{(n)} + t\xi^{(n)} \mid t \in [o,\varepsilon]\}$ intersects K at most countably often.

Obviously, similar definition can easily be formulated for a set $K \subset E_s^n$.

Let $\tilde{h} \in C^1(\Gamma_E)$. For $\mathbf{A} \in E_1$ we have

(4.52)
$$Dh(\mathbf{A}) = \sum_{i=1}^{n} \tilde{h}_{,i}(\mathbf{v}(\mathbf{A}))Dv_i(\mathbf{A}),$$

where $\mathbf{v}(\mathbf{A}) = (v_1(\mathbf{A}),\ldots,v_n(\mathbf{A}))$. Moreover, it can be shown that

(4.53)
$$Dv_i(\mathbf{A}) = P_i(\mathbf{A}),$$

for all $\mathbf{A} \in E_s^n \setminus E_1$ and $i = 1,\ldots,n$, where $P_i(\mathbf{A})$ stands for the projection onto the ith eigenspace of $\mathbf{A}$. This projection can be regarded as an element of E_s^n, so that $P_i(\mathbf{A})\mathbf{x} = (\mathbf{x},\mathbf{e}_i(\mathbf{A}))\mathbf{e}_i(\mathbf{A})$ for $\mathbf{x} \in R^n$, where $\mathbf{e}_i(\mathbf{A})$ denotes the ith unit eigenvector of $\mathbf{A}$ and (,) the inner product in R^n. Equivalently, $P_i(\mathbf{A}) = \mathbf{e}_i(\mathbf{A}) \otimes \mathbf{e}_i(\mathbf{A})$. $Dv_i(\mathbf{A})$ is the unique $n \times n$ symmetric matrix satisfying

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(4.54) $$\frac{d}{dt}v_i(\mathbf{A}+t\mathbf{B})\big|_{t=0}=\langle Dv_i(\mathbf{A}),\mathbf{B}\rangle.$$

Let $\tilde{h}\in C^2(\Gamma_E)$. For $\mathbf{A}\in E_1$ and $\mathbf{B}\in E_s^n$ we have

(4.55) $$\frac{d^2}{dt^2}h(\mathbf{A}+t\mathbf{B})\big|_{t=0}=\sum_{i=1}^{n}\tilde{h}_{,i}(\mathbf{v}(\mathbf{A}))\frac{d^2v_i}{dt^2}(\mathbf{A}+t\mathbf{B})\big|_{t=0}+ \\ +\sum_{i,j=1}^{n}\tilde{h}_{,ij}(\mathbf{v}(\mathbf{A}))\frac{dv_i}{dt}(\mathbf{A}+t\mathbf{B})\big|_{t=0}\frac{dv_j}{dt}(\mathbf{A}+t\mathbf{B})\big|_{t=0}.$$

It can be shown that if $\mathbf{A}=diag(a_1,\ldots,a_n)\in E_1$, $\mathbf{B}\in E_s^n$ then

(4.56) $$\frac{d^2v_i}{dt^2}(\mathbf{A}+t\mathbf{B})\big|_{t=0}=2\sum_{i\neq j}\frac{B_{ij}B_{ij}}{a_i-a_j}.$$ ∇

For $C^{r,\alpha}$ functions we have the following result.

Theorem 4.7. Let $0<\alpha<1, r=0,1,2,\ldots$. Then $h\in C^{r,\alpha}(E)$ if and only if $\tilde{h}\in C^{r,\alpha}(\Gamma_E)$. ∇

Now we are in a position to pass to applications to *nonlinear isotropic elasticity*.

Let E^n denote the set of real $n\times n$ matrices, and $E_+^n:=\{\mathbf{A}\in E^n \mid \det\mathbf{A}>0\}$, $E_{s+}^n:=\{\mathbf{A}\in E_s^n \mid \mathbf{A}\ \textit{is positive definite}\}$.

In the remaining part of this section we will be concerned with a homogeneous hyperelastic body possessing the density of the stored-energy function $W\colon D\to R$, where $D\in E_+^n$ is open and invariant under O_n^0. It means that $\mathbf{QF},\mathbf{FQ}\in D$ whenever $\mathbf{F}\in D$ and $\mathbf{Q}\in O_n^0$. We observe that *nonhomogeneous* materials can be treated similarly. More precisely, the function W is defined with respect to a fixed reference configuration in which the body occupies the closure of the bounded open set $\Omega\subset R^n$. For homogeneous materials W depends on the gradient of deformation, say $\mathbf{F}=\nabla\mathbf{x}(\mathbf{X})$; for nonhomegeneous materials W is a function of $\mathbf{X}\in\Omega$ and $\mathbf{F}$. We recall that *frame indifference* requires that, cf. Gurtin (1981),

(4.57) $$W(\mathbf{QF})=W(\mathbf{F})\quad \textit{for all}\ \mathbf{F}\in D, \mathbf{Q}\in O_n^0,$$

and W is isotropic if in addition

(4.58) $$W(\mathbf{F})=W(\mathbf{FQ})\quad \textit{for all}\ \mathbf{F}\in D, \mathbf{Q}\in O_n^0.$$

By using the polar decomposition theorem we write

(4.59) $$\mathbf{F}=\mathbf{RU}=\mathbf{VR},$$

where $\mathbf{F}\in D$, $\mathbf{R}\in O_n^0$ and $\mathbf{U},\mathbf{V}\in E=D\cap E_{s+}^n$.

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We set $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. By $0 < v_1 \le v_2 \le \ldots \le v_n$ we denote the principal stretches (or the singular values of $\mathbf{F}$), that is the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ (equivalently, of $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$). By virtue of (4.57), (4.59)

$$W(\mathbf{F}) = W(\mathbf{U}), \tag{4.60}$$

while if W is isotropic then

$$W(\mathbf{F}) = W(\mathbf{V}). \tag{4.61}$$

The last two relations follow by taking **Q**=**R**.
According to (4.49) if W is isotropic then there exists a symmetric function $\Phi: \Gamma_E \to E$ such that

$$W(\mathbf{F}) = \Phi(v_1, \ldots, v_n). \tag{4.62}$$

This relation is a characterization of isotropy. Obviously there are various other possible representations of an isotropic hyperelastic potential W. Below we are concerned with the following ones. Firstly, by (4.62) we can write

$$W(\mathbf{F}) = \theta(\mathbf{S}(\mathbf{v})), \tag{4.63}$$

where $\mathbf{S}(\mathbf{v})$ is the vector of *elementary* symmetric functions of $\mathbf{v} = (v_1, \ldots, v_n)$, cf. (4.44). Secondly, by (4.61) we have

$$W(\mathbf{F}) = h(\mathbf{B}), \tag{4.64}$$

and

$$W(\mathbf{F}) = H(b_1, \ldots, b_n), \tag{4.65}$$

where $b_i := v_i^2$ are the eigenvalues of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Thus we have the following representation

$$W(\mathbf{F}) = \tilde{H}(\mathbf{S}(\mathbf{b})), \tag{4.66}$$

where $\mathbf{b} = (b_i) = (b_1, \ldots, b_n)$.

The general results on the regularity of symmetric and isotropic functions will now be applied to relate the differentiability properties of W, Φ, θ, h, H and $\tilde{H}$.

Firstly, however, two technical lemmas will be formulated.

Lemma 4.8. The mapping $C \to C^{1/2}$ of E^n_{s+} to itself is C^∞. ∇

Lemma 4.9. The mapping $\mathbf{U}(\mathbf{F}) = \sqrt{\mathbf{F}^T\mathbf{F}}$ from E^n_+ to E^n_{s+} is C^∞. If $\mathbf{F} = diag(a_1, \ldots, a_n)$ with all $a_i > 0$ and if $\mathbf{G} \in E^n$ then the first and second derivatives with respect to t of $\mathbf{U}(t) := \mathbf{U}(\mathbf{F} + t\mathbf{G})$ at $t = 0$ are given by

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$$(4.67) \qquad \dot{U}_{ij}(0) = \frac{a_i G_{ij} + a_j G_{ji}}{a_i + a_j},$$

and

$$(4.68) \qquad \ddot{U}_{ij}(0) = \frac{2}{a_i + a_j} \sum_{k=1}^{n} \left[G_{ki} G_{kj} - \dot{U}_{ik}(0) \dot{U}_{jk}(0) \right].$$

∇

We recall that the summation convention does not apply to (4.67).

Theorem 4.10. Let $W: D \to R$ be isotropic, and Φ be given by (4.62).

(i) Let $r = 0,1,2$ or ∞ . Then $W \in C^r(D)$ if and only if $\Phi \in C^r(\Gamma_E)$.

(ii) Let $0 < \alpha < 1$, $r = 0,1,2,\ldots$. Then $W \in C^{r,\alpha}(D)$ if and only if $\Phi \in C^{r,\alpha}(\Gamma_E)$.

(iii) Let $\mathbf{F} = diag(v_1, \ldots, v_n) \in D$, where $\mathbf{v} = (v_1, \ldots, v_n)$ with all $v_i > 0$, and let $\mathbf{G} \in E^n$. Then if $\Phi \in C^1(\Gamma_E)$

$$(4.69) \qquad D_{\mathbf{F}} W(\mathbf{F})\mathbf{G} = \sum_{i=1}^{n} \Phi,_i(\mathbf{v}) G_{ii},$$

and if $\Phi \in C^2(\Gamma_E)$ then

$$(4.70) \qquad D_{\mathbf{F}}^2 W(\mathbf{F})(\mathbf{G},\mathbf{G}) = \sum_{i,j=1}^{n} \Phi,_{ij}(\mathbf{v}) G_{ii} G_{jj} + \sum_{i \neq j} \frac{v_i \Phi,_i(\mathbf{v}) - v_j \Phi,_j(\mathbf{v})}{v_i^2 - v_j^2} (G_{ij})^2 + \sum_{i \neq j} \frac{v_j \Phi,_i(\mathbf{v}) - v_i \Phi,_j(\mathbf{v})}{v_i^2 - v_j^2} G_{ij} G_{ji}.$$

Proof. By using Lemma 4.9, we conclude that $W \in C^r(E)$(resp. $W \in C^{r,\alpha}(E)$) if and only if $W \in C^r(D)$(resp. $W \in C^{r,\alpha}(D)$). Consequently (i) and (ii) follow from Theorems 4.4 and 4.7. We observe that the case $r = \infty$ in (i) is a consequence of (ii). Let now $\Phi \in C^1(\Gamma_E)$. Then by (4.50) and (4.67) we have

$$D_{\mathbf{F}} W(\mathbf{F})\mathbf{G} = \frac{d}{dt} W(\mathbf{F} + t\mathbf{G})\Big|_{t=0} = \sum_{i=1}^{n} \Phi,_i(\mathbf{v}) \dot{U}_{ii}(0) = \sum_{i=1}^{n} \Phi,_i(\mathbf{v}) G_{ii}.$$

Let now $\Phi \in C^2(\Gamma_E)$. Then by (4.51) and (4.68)

$$D_{\mathbf{F}}^2 W(\mathbf{F})(\mathbf{G},\mathbf{G}) = \frac{d^2}{dt^2} W(\mathbf{F} + t\mathbf{G})\Big|_{t=0} = \sum_{i=1}^{n} \Phi,_i(\mathbf{v}) \ddot{U}_{ii}(0) + D_{\mathbf{U}}^2 W(diag\mathbf{v})\left(\dot{\mathbf{U}}(0), \dot{\mathbf{U}}(0)\right) =$$

$$= \sum_{i \neq j} \frac{\Phi,_i(\mathbf{v})}{v_i} \left[(G_{ji})^2 - \left(\frac{v_i G_{ij} + v_j G_{ji}}{v_i + v_j} \right)^2 \right] + \sum_{i,j=1}^{n} \Phi,_{ij}(\mathbf{v}) G_{ii} G_{jj} + \sum_{i \neq j} \left(\frac{\Phi,_i(\mathbf{v}) - \Phi,_j(\mathbf{v})}{v_i - v_j} \right) \left(\frac{v_i G_{ij} + v_j G_{ji}}{v_i + v_j} \right)^2.$$

Hence (4.70) follows easily. ∇

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The next theorem provides an application of the formula (4.70) to the theory of constitutive inequalities.

Theorem 4.11. If $W \in C^2(D)$ is isotropic then W satisfies

$$D^2W(\mathbf{F})(\mathbf{G},\mathbf{G}) = \sum_{i,j,k,l} \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{kl}} G_{ij}G_{kl} > 0, \tag{4.71}$$

for all $\mathbf{F} \in D$ and nonzero $\mathbf{G} \in E^n$ if and only if Φ given by (4.62) satisfies

$$\sum_{i,j=1}^{n} \Phi,_{ij} \lambda_i \lambda_j > 0 \quad \textit{for all } \mathbf{v} \in \Gamma_E \textit{ and nonzero } \lambda \in R^n, \tag{4.72}$$

$$\frac{\Phi,_i(\mathbf{v}) - \Phi,_j(\mathbf{v})}{v_i - v_j} > 0 \quad \textit{for all } i \neq j \textit{ and all } \mathbf{v} = (v_1,\ldots,v_n) \in \Gamma_E \textit{ with } v_i \neq v_j, \tag{4.73}$$

and

$$\Phi,_i(\mathbf{v}) + \Phi,_j(\mathbf{v}) > 0 \quad \textit{for all } i \neq j \textit{ and all } \mathbf{v} \in \Gamma_E. \tag{4.74}$$

∇

Remark 4.12. A stored-energy function $W \in C^2(D)$ is said to be *strongly elliptic* if

$$\frac{d^2}{dt^2} W(\mathbf{F} + t\mathbf{a} \otimes \mathbf{b})\Big|_{t=o} = \sum_{i,j,k,l} \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij}\partial F_{kl}} a_i b_j a_k b_l > 0, \tag{4.75}$$

whenever $\mathbf{F} \in D$ and $\mathbf{a}$, $\mathbf{b} \in R^n$ are nonzero. Two consequences of strong ellipticity of an isotropic W follow immediately from (4.70). These are the strengthened tension-extension inequalities

$$\Phi,_{ii} > 0, \quad i = 1,\ldots,n, \tag{4.76}$$

and the Baker-Ericksen inequalities

$$\frac{v_i\Phi,_i(\mathbf{v}) - v_j\Phi,_j(\mathbf{v})}{v_i - v_j} > 0 \quad \textit{if } v_i \neq v_j. \tag{4.77}$$

∇

Now we pass to the representations (4.63)-(4.66). We set

$$D_+ = \{\mathbf{F}\mathbf{F}^T | \ \mathbf{F} \in D\} \subset E^n_{s+}.$$

Then $\Gamma_{D_+} = \{(v_1^2,\ldots,v_n^2) | \ \mathbf{v} = (v_1,\ldots,v_n) \in \Gamma_E\}$. We can formulate

Theorem 4.13. Let $W: D \to R$ be isotropic, and let h, H be given by (4.64), (4.65).

(i) Let $r = 0,1,2$ or ∞. Then $W \in C^r(D)$ if and only if $h \in C^r(D_+)$ and if and only if $H \in C^r(\Gamma_{D_+})$.

(ii) Let $0 < \alpha < 1$, $r = 0,1,2,\ldots$. Then $W \in C^{r,\alpha}(D)$ if and only if $h \in C^{r,\alpha}(D_+)$ and if and only if $H \in C^{r,\alpha}(\Gamma_{D_+})$.

Proof. This follows from Theorems 4.4, 4.7 and 4.10 and the fact that the map $(v_1,\ldots,v_n) \to (v_1^2,\ldots,v_n^2)$ from Γ_E to Γ_{D_+} is a smooth diffeomorphism. ∇

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The last results concerns smoothness of the function θ and $\tilde{H}$ given by (4.63) and (4.66) respectively. In contrast to Φ, h and H these functions are in general less differentiable than W.

Theorem 4.14. Let $W: D \to R$ be isotropic, and let θ and $\tilde{H}$ be given by (4.63) and (4.66) respectively. Let $r = 0,1,2,\ldots$. If $W \in C^{nr}(\overline{D})$ and Γ_E is convex then $\theta \in C^r(\overline{\Omega}_{\Gamma_E})$ and $\tilde{H} \in C^r(\overline{\Omega}_{\Gamma_{D_+}})$ (see the formula (4.45)).

Proof. If $W \in C^{nr}(\overline{D})$then $\Phi \in C^{nr}(\overline{\Gamma}_E)$, and hence $H \in C^{nr}(\overline{\Gamma}_{D_+})$. The result follows from Theorem 4.1. ∇

Remark 4.15. The last theorem is optimal. Indeed, let I be an open interval with $\bar{I} \subset (0,\infty)$ and let $D = \{\mathbf{F} \in E^n_+ | \text{ } each\ principal\ stretch\ v_i \in I\}$. Suppose that

$$\Phi(v_1,\ldots,v_n) = \sum_{i=1}^{n} \phi(v_i),$$

where $\phi: I \to R$. Then $\theta \in C^{r+1}(\overline{\Omega}_{\Gamma_E})$ (equivalently, $\tilde{H} \in C^{r+1}(\Omega_{\Gamma_{D_+}})$) if and only if $\phi \in C^{n(r+1)}(\bar{I})$ by virtue of Theorem 4.3. Consequently, $W \in C^{n(r+1)-1}(D)$ by Theorem 4.7 and Lemma 4.9.

Remark 4.16. Marques and Moreau (1982) proved the following theorem: „Let f be a symmetric function, $f: R^n \to R$ (or more generally $f: R^n \to \overline{R} = [-\infty,\infty]$) and set $g(\mathbf{A}) = f(a_1,\ldots,a_n)$, where a_i $(i = 1,\ldots,n)$ are the eigenvalues of $\mathbf{A} \in E^n_s$. Then $g: E^n_s \to R$ $(resp.\ \overline{R})$ is convex if and only if f is convex."

More restrictive theorem of this type has earlier been proven by Ball (1977, p.363 Th. 5.1.(i)). In the last paper a_i has to be nonnegative, i.e. $a_i \geq 0$ for $i = 1,\ldots,n$. Marques and Moreau (1982) suggest the application of their result to plasticity; then $\mathbf{A}$ is a symmetric stress tensor whose principal values are not necessarily nonnegative (think of compression and tension).

4.6.3. As we already known, Serrin (1959) asserts that the (nonpolynomial) functions α, β and γ are continuous provided that the tensor function $\underline{F}$ in (4.40) is of class C^3. Serrin's study was motivated by the formulation of a general class of what he calls „Stokesian fluids" (viscous, isotropic and in general compressible).

Let us consider once again the isotropic representation of a second-order symmetric tensor function $\underline{F}: D \to E^3_s$, where D is an open subset of E^3_s such that if $\mathbf{A} \in D$ then $\mathbf{QAQ}^T \in D$ for each orthogonal tensor $\mathbf{Q}$. Obviously, such a function has the following representation

$$\underline{F}(\mathbf{A}) = \alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2, \tag{4.78}$$

where α, β and γ are *symmetric functions* of the eigenvalues a_1, a_2 and a_3 of $\mathbf{A}$. Man (1994) proved that Serrin's sufficient condition on the smoothness of $\underline{F}$ can be weakened: the scalar coefficients α, β and γ in the representation (4.78) may be chosen to be continuous *symmetric functions of* $(a_i, i = 1,2,3)$ *if* $\underline{F}$ *is of class* C^2. In essence Man's (1994) approach consists in showing that

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(4.79)
$$\lim_{(a_1,a_2,a_3)\to(a,a,a)} \gamma(a_1,a_2,a_3)=\gamma_a,$$

where γ_a is given by his formula (24). In Eq. (4.79) (a_1,a_2,a_3) tends to a point of triple coalescence. In order to prove (4.79) Man (1994) uses the classical Taylor's theorem and supplies the estimate of $|\gamma-\gamma_a|$. Next he shows that the coefficients α and β as defined by his Eqs (2) and (3) may likewise be extended by continuity at points of the form (a,a,a). Thus we may assume that α, β and γ are continuous functions whose domain has been extended to the entire

$$\mathsf{E}=\left\{(a_1,a_2,a_3)\in R^3 \,\middle|\, a_1,a_2,a_3 \textit{ are the repeated eigenvalues of some } \mathbf{A}\in D\right\}.$$

Let $\mathsf{S}: R^3 \to R^3$ be a 3-vector of elementary functions. Man (1994) omits the sign minus and takes, cf. (4.44)

(4.80)
$$\mathsf{S}(a_1,a_2,a_3)=(a_1+a_2+a_3,\, a_1a_2+a_2a_3+a_3a_1,\, a_1a_2a_3).$$

Since the functions α, β and γ are symmetric, there exist unique functions $\hat\alpha, \hat\beta$ and $\hat\gamma$ defined on $\mathsf{S}(\mathsf{E})$ such that

(4.81)
$$\alpha=\hat\alpha\circ\mathsf{S}, \quad \beta=\hat\beta\circ\mathsf{S}, \quad \gamma=\hat\gamma\circ\mathsf{S}.$$

According to Ball (1984, Th. 3.2; see also subsection 4.6.2) the functions $\hat\alpha, \hat\beta$ and $\hat\gamma$ are continuous because α, β and γ are continuous.
All in all, if $\underline{\mathsf{F}}$ is of class C^2, then there exist unique continuous functions $\hat\alpha, \hat\beta$ and $\hat\gamma$ such that

(4.82)
$$\underline{\mathsf{F}}(\mathbf{A})=\hat\alpha(I,II,III)\mathbf{I}+\hat\beta(I,II,III)\mathbf{A}+\hat\gamma(I,II,III)\mathbf{A}^2,$$

where *I*, *II*, *III* are the principal invariants of **A**.

Similarly, for the *two-dimensional case* an isotropic, tensor-valued function $\underline{\mathsf{G}}=\mathsf{D}\to E_s^2$ has a representation of the form

(4.83)
$$\underline{\mathsf{G}}(\mathbf{A})=\alpha\mathbf{I}+\beta\mathbf{A},$$

where α and β are symmetric functions of the eigenvalues a_1, a_2 of **A**. By applying the same approach as in the three-dimensional case, Man (1994) asserts that if $\underline{\mathsf{G}}$ is of class C^1, then there exist unique continuous functions $\hat\alpha$ and $\hat\beta$ such that

(4.84)
$$\underline{\mathsf{G}}(\mathbf{A})=\hat\alpha(I,II)\mathbf{I}+\hat\beta(I,II)\mathbf{A},$$

where *I* and *II* are the principal invariants of **A**.

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In a later paper Man (1995) proved two general theorems on the smoothness of the scalar coefficients in the Reiner-Rivlin-Ericksen-Serrin representation formula, cf. (4.78).

As previously, $D \subset E_s^3$ is a set such that if $\mathbf{A} \in D$ then $\mathbf{QAQ}^T \in D$ for each orthogonal tensor $\mathbf{Q}$. The first result due to Man (1995) is formulated as

Theorem 4.17. If the isotropic tensor function $\underline{\mathrm{F}}: D \to E_s^3$ is of class C^{r+2} $(r = 0,1,2,\dots)$, then there exists a unique set of symmetric C^r functions $\alpha, \beta, \gamma : \mathrm{E} \to R$ such that representation formula

$$\underline{\mathrm{F}}(\mathbf{A}) = \alpha(a_1, a_2, a_3)\mathbf{I} + \beta(a_1, a_2, a_3)\mathbf{A} + \gamma(a_1, a_2, a_3)\mathbf{A}^2, \tag{4.85}$$

holds for each $\mathbf{A} \in D$. ∇

From practical point of view it is important to express the scalar coefficients as functions of the principal invariants of $\mathbf{A}$. Man's (1995) second result solves the smoothness problem in this case.

Theorem 4.18. Let $\mathbf{S}(a_1, a_2, a_3)$ be given by (4.80). If the isotropic tensor function $\underline{\mathrm{F}}: D \to E_s^3$ is of class C^{3r+2} $(r = 0,1,2,\dots)$, then there exists a unique set of symmetric C^r functions $\hat{\alpha}, \hat{\beta}, \hat{\gamma} : \mathbf{S}(\mathrm{E}) \to R$ such that (4.82) takes place. ∇

In essence the proofs of Theorems 4.17 and 4.18 are based on the differentiability of the remainder term in Taylor's formula (Whitney, 1943) combined with Ball's (1984) results. The two-dimensional case has also been considered by Man (1995).

It seems that Man's (1994, 1995) approach can be generalized to the case of nonpolynomial, isotropic representations of symmetric second-order tensor functions of more than one tensor argument.

5. Invariants of tensors of order greater then two. Representation of tensor-valued functions of order greater then two

Results concerning representations of tensor-valued functions of order greater then two, depending upon tensors of arbitrary order are not numerous. The same may be said about complete results concerning scalar-, vector- and tensor-valued functions of second-order tensor functions depending upon arbitrary tensors. Though there exist general theorems enabling the determination of integrity bases in the case of invariance under arbitrary compact point groups, yet algebraic difficulties are often such that actually it seems impossible to obtain complete and satisfactory results, cf. (Rychlewski and Zhang, 1991; Zheng, 1994a). The same concerns functional bases and generators.

The simplest to determine is a representation of a tensor-valued function of the order greater then two provided that the order of tensor agencies is not greater then two. In this case both the integrity and functional bases are known. Let us pass now to providing illustrative examples. Moreover, we shall propose a method of determining tensor-valued functions of fourth-order depending upon symmetric, second-order tensors in the two- and three dimensional cases.

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Representations of third- and fourth-order tensor functions depending on tensors of order not greater than two were examined by Pennisi (1992) and Zheng (1994a, 1994b, 1996), cf. also Silber (1988a, 1988b, 1990), Telega (1984). The main idea consists in a decomposition of the spaces of third- or fourth-order tensors into direct sums of subspaces so as to essentially reduce the number of independent components which belong to the corresponding subspace. For instance, Pennisi (1992) determines several types of third-order tensor functions. Let

(5.1) $$\underline{\Phi}:\ \mathcal{U} = \underbrace{T_1 \times \ldots \times T_1}_{I\ \text{times}} \times \underbrace{T_2^s \times \ldots \times T_2^s}_{J\ \text{times}} \times \underbrace{T_2^a \times \ldots \times T_2^a}_{K\ \text{times}} \to T_3,$$

be a third-order tensor function which is form-invariant under the isotropy group. Here $T_3 = E^3 \otimes E^3 \otimes E^3$ and T_1, T_2^s, T_2^a are the spaces of 3-dimensional vectors, symmetric second-order tensors and skew-symmetric second-order tensors, respectively. It is convenient to decompose the space of three-dimensional third-order tensors T_3 in the following way:

(5.2) $$T_3 = T_3^{is} \oplus T_3^{ia} \quad (\textit{no summation on } i),$$

where

(5.3) $$T_3^{is} = \{\underline{\Psi} \in T_3 |\quad 2\underline{\Psi} = \sigma_i \underline{\Psi} + \sigma_{6-i}\underline{\Psi}\},$$

(5.4) $$T_3^{ia} = \{\underline{\Psi} \in T_3 |\quad 2\underline{\Psi} = \sigma_i \underline{\Psi} - \sigma_{6-i}\underline{\Psi}\}.$$

Here $\sigma_1 = (1,2,3)$, $\sigma_2 = (2,3,1)$, $\sigma_3 = (3,1,2)$, $\sigma_4 = (1,3,2)$, $\sigma_5 = (3,2,1)$, and $\sigma_6 = (2,1,3)$ are permutations acting on a third-order tensor; for instance if $\underline{\Psi} = [\Psi_{ijk}]$ in an orthonormal basis then $\sigma_6 \Psi_{ijk} = \Psi_{jik}$. Thus T_3^{is} (T_3^{ia}) may be referred to as the space of symmetric (skew-symmetric) third-order tensors.

Next, one determines generators of non-polynomial tensor functions

(5.5) $$\underline{\Phi}^{is}:\ \mathcal{U} \to T_3^{is}, \quad \underline{\Phi}^{ia}:\ \mathcal{U} \to T_3^{ia},$$

and this allows us to find the canonical form of the tensor function (5.1):

(5.6) $$\underline{\Phi} = \underline{\Phi}^{1s} + \underline{\Phi}^{1a} + \underline{\Phi}^{2s} + \underline{\Phi}^{2a} + \underline{\Phi}^{3s} + \underline{\Phi}^{3a},$$

Pennisi (1992) also determined the so called fully symmetric $\underline{\Phi}^{fs}$ and fully skew-symmetric $\underline{\Phi}^{fa}$ third-order tensor functions:

(5.7) $$3\underline{\Phi}^{fs} = \underline{\Phi}^{1s} + \underline{\Phi}^{2s} + \underline{\Phi}^{3s},$$

(5.8) $$3\underline{\Phi}^{fa} = \underline{\Phi}^{1a} + \underline{\Phi}^{2a} + \underline{\Phi}^{3a}.$$

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This author did not give the procedure leading to the determination of the generators of the tensor functions appearing in (5.5), but presented only the final results. Moreover, he proved that his set of generators is complete and irreducible. We observe that Pennisi and Trovato (1987) devised a procedure permitting to prove the irreducibility of functional bases and sets of generators.

Recently Zheng (1994b, 1996) have derived representations of scalar, vector and second-order, symmetric and skew-symmetric tensor functions for all types of anisotropy listed in Table 1.1. This author considered also third-order tensor functions. All these functions depend on a finite set of vectors, symmetric and skew-symmetric second-order tensors as well as on third-order tensors. To determine representations of the functions mentioned Zheng (1994b, 1996) applied methods elaborated in his earlier papers, see Zheng (1993a, 1993c). In the case of tensor functions depending on third-order tensors, Zheng (1996) introduced a decomposition of an arbitrary third-order tensor into a sum of three vectors and one tensor, called *irreducible third-order tensor* with two independent components. Moreover, completeness and irreducibility of the representations obtained in this manner have been proved by exploiting the ideas due to Pennisi and Trovato (1987). We observe that an elementary method of the decomposition of arbitrary tensors into a sum of irreducible tensors was already proposed by Spencer (1970). We recall that a tensor of the order of two or greater than two is said to be irreducible if it is *completely symmetric* and *traceless*, i.e. if $\mathbf{H} = \left(H_{ijk\ldots l}\right)$ is an n-order tensor in an orthonormal basis then

$$H_{ijk\ldots l} = H_{jik\ldots l} = H_{kji\ldots l} = \ldots = H_{ljk\ldots i}, \quad H_{mmk\ldots l} = 0_{k\ldots l},$$

where $0_{k\ldots l}$ are components of the (n-2)-order zero tensor. It is evident that the notion of an irreducible tensor is a generalization of the deviator of a second-order symmetric tensor.

In the relevant literature irreducible tensors are usually called „*harmonic tensors*", cf. Backus (1970). An isomorphism between irreducible tensors and homogeneous polynomials of an appropriate degree satisfying Laplace's equation justifies such a terminology, cf. Backus (1970) and Forte and Vianello (1996) as well as the references cited therein.

More attention was devoted to the determination of invariants of a single fourth-order tensor, cf. Telega (1981), Rychlewski and Zhang (1991), Zheng (1994a). It seems, however, that until now the problem of the determination of complete and irreducible integrity and functional bases, for this tensor, remains open, see also Smith (1994). Having in mind the elasticity tensor, damage tensor, etc., most papers deal with the so called symmetric fourth-order tensor $\mathbf{C} \in T_4^{\Sigma}$, $T_4 = \mathrm{E}^3 \otimes \mathrm{E}^3 \otimes \mathrm{E}^3 \otimes \mathrm{E}^3$, $\dim T_4^{\Sigma} = 21$ and Σ is a permutation group consisting of

$$(5.9) \qquad \tau_1 = (1,2,3,4), \quad \tau_2 = (2,1,3,4), \quad \tau_3 = (1,2,4,3), \quad \tau_4 = (3,4,1,2).$$

For further results on invariants and representations of higher order tensors, the reader should refer to (Betten, 1982, 1986, 1987a, 1987b; Silber, 1990; Rychlewski and Zhang, 1991, pp.83-84; Zheng, 1994a, pp.571-579).

Though the tensor **C** posses 21 independent components, yet its complete functional basis should contain more then 18 independent invariants, which are still unknown (Rychlewski and Zhang, 1991). Boehler et al. (1994) imbedded the 18-dimensional manifold of the distinct orbits of elasticity moduli into the 37-dimensional euclidean

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space. Next, the space T_4^Σ is decomposed into the direct sum of spaces of dimensions 1,1,5,5,9 and the integrity basis for **C** consisting of 39 basic polynomial invariants is determined.

Let us pass now to providing two examples.

Example 5.1

Let $T_4 = E^2 \otimes E^2 \otimes E^2 \otimes E^2$, where E^2 is two-dimensional Euclidean space. In order to determine a non-polynomial representation of the symmetric fourth-order tensor-valued function we employ the usual procedure, which now consists in taking

(5.10)
$$\mathrm{G}:\; \underbrace{T_2^S \times \ldots \times T_2^S}_{(k+1)\,times} \to T_2^S,$$

such that

(5.11)
$$\mathbf{G} = \mathbb{F}(\mathbf{A}_k)\cdot\mathbf{X} \quad or \quad G_{\alpha\beta} = \mathbb{F}_{\alpha\beta\lambda\kappa}(\mathbf{A}_k)X_{\lambda\kappa},$$

where $\mathbf{X} = \mathbf{X}^T$ and $\alpha,\beta,\lambda,\kappa = 1,2;\ k = 1,\ldots,K$. Obviously, $\mathbf{A}_k$ are symmetric, second-order tensors, $\mathbf{A}_k \in T_2^s$. The set of invariants and generators for the function **G** consists of those for the set of tensors $\{\mathbf{A}_k\}$ and, additionally of the invariants linear in **X**

(5.12)
$$tr\mathbf{X},\quad tr\mathbf{A}_k\mathbf{X},$$

as well as the generator **X**. The tensor function $\mathbb{F}$ is given by

(5.13)
$$\mathbb{F} = \frac{\partial\mathbf{G}}{\partial\mathbf{X}} = \alpha_n\mathbb{F}_n,$$

where α_n are arbitrary scalar functions of the elements of the functional basis for the tensors $\mathbf{A}_k$, and $\mathbb{F}_n \in T_4^\Sigma$ are generators. The final results are summarized in Table 5.1.

Table 5.1 Representation of two-dimensional symmetric fourth-order tensor-valued function of symmetric second-order tensors

Agency	Functional basis	Generators
$\mathbf{A}_k$	$tr\mathbf{A}_k,\ tr\mathbf{A}_k^2$	$\mathbf{I}\otimes\mathbf{I},\ \mathbb{I},\ \mathbf{A}_k\otimes\mathbf{A}_k,\ \mathbf{I}\otimes\mathbf{A}_k+\mathbf{A}_k\otimes\mathbf{I}$
$\mathbf{A}_k,\ \mathbf{A}_l$ $k,l = 1,\ldots,K;\ l<k$	$tr\mathbf{A}_k\mathbf{A}_l$	$\mathbf{A}_l\otimes\mathbf{A}_k+\mathbf{A}_k\otimes\mathbf{A}_l$

Notation: $\mathbb{I}$ -the unit fourth-order tensor, **I**-the unit second-order tensor

The results obtained in this way coincide with those given by Zheng (1994a).

Let us consider more closely the case where the set $\{\mathbf{A}_k\}$ reduces to a symmetric tensor **A**. Then

(5.14)
$$\mathbf{G} = \alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{X},$$

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where

$$\alpha = \alpha_1 tr\mathbf{X} + \alpha_2 tr\mathbf{AX}, \quad \beta = \beta_1 tr\mathbf{X} + \beta_2 tr\mathbf{AX},$$

and $\alpha_1, \ldots, \beta_2, \gamma$ are arbitrary functions of $tr\mathbf{A}$ and $tr\mathbf{A}^2$. Now (5.13) takes the form

$$\mathbb{F}(\mathbf{A}) = \delta_1 \mathbf{I} \otimes \mathbf{I} + \delta_2 \mathbb{I} + \delta_3 (\mathbf{A} \otimes \mathbf{I} + \mathbf{A} \otimes \mathbf{I}) + \delta_4 \mathbf{A} \otimes \mathbf{A}, \tag{5.15}$$

where $\delta_j = f_j(tr\mathbf{A}, tr\mathbf{A}^2)$, $j = 1,2,3,4$; moreover

$$\delta_1 = \alpha_1, \delta_2 = \gamma, 2\delta_3 = \alpha_2 + \beta_1, \delta_4 = \beta_2.$$

Example 5.2
Similar procedure applies to the three-dimensional case. For instance, the counterpart of (5.15) is

$$\begin{aligned} \mathbb{F}(\mathbf{A}) &= \gamma_1 \mathbf{I} \otimes \mathbf{I} + \gamma_2 \mathbb{I} + \gamma_3 (\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}) + \gamma_4 (\mathbf{A} \Diamond \mathbf{I} + \mathbf{I} \Diamond \mathbf{A}) + \gamma_5 \mathbf{A} \otimes \mathbf{A} + \\ &+ \gamma_6 (\mathbf{A}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}^2) + \gamma_7 (\mathbf{A}^2 \Diamond \mathbf{I} + \mathbf{I} \Diamond \mathbf{A}^2) + \gamma_8 \mathbf{A}^2 \otimes \mathbf{A}^2 + \gamma_9 (\mathbf{A}^2 \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{A}^2), \end{aligned} \tag{5.16}$$

where $\gamma_j = f_j(tr\mathbf{A}, tr\mathbf{A}^2, tr\mathbf{A}^3)$ $(j = 1, \ldots, 9)$ and in an orthonormal basis we have $(\mathbf{A} \Diamond \mathbf{B})_{ijkl} = (A_{ik} B_{jl} + A_{il} B_{jk})/2$. In such a basis (5.16) coincides with results obtained by Telega (1984).

From the point of view of the theory of tensor function representations, Examples 5.1 and 5.2 provide constructions of fourth-order tensor functions which are not difficult to carry out. This simplicity is due to the fact that they are functions of second-order symmetric tensors only. The same procedure applies to fourth-order tensor functions of any finite number of vectors, second-order symmetric and skew-symmetric tensors. As usual, from the computational point of view, 2D case is simpler than the corresponding 3D problems.

It seems that a general and efficient method for finding complete and irreducible isotropic and anisotropic representations of functions of higher order tensors is still lacking.

Below, we shall discuss some of the isotropic and orthotropic representations of functions of fourth-order tensors due to Zheng (1994a, 1994b). In essence, Zheng's method relies on the decomposition of fourth-order tensors which is now described.

Decompositions of fourth-order tensors and some fourth-order tensor-valued functions

Let $\mathbb{A}, \mathbb{W}, \alpha$ and ω be fourth-order tensors. We assume that these tensors possess the following symmetries, cf. Zheng (1994a)

$$\begin{aligned} &\textit{double-symmetric} && \mathbb{A}_{ijkl} = \mathbb{A}_{jikl} = \mathbb{A}_{ijlk} = \mathbb{A}_{klij}, \\ &\textit{antisymm-symmetric} && \mathbb{W}_{ijkl} = \mathbb{W}_{jikl} = \mathbb{W}_{ijlk} = -\mathbb{W}_{klij}, \\ &\textit{symm-antisymmetric} && \alpha_{ijkl} = -\alpha_{jikl} = -\alpha_{ijlk} = \alpha_{klij}, \\ &\textit{double-antisymmetric} && \omega_{ijkl} = -\omega_{jikl} = -\omega_{ijlk} = -\omega_{klij}. \end{aligned} \tag{5.17}$$

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In three dimensions the number of non-zero independent components of $\mathbb{A}$, $\mathbb{W}$, α and ω is equal to 21, 15, 6 and 3, respectively, while in the two-dimensional case this number amounts to 6, 3, 1 and zero, respectively. In general, any fourth-order tensor $\mathbb{K}$ can be decomposed as follows

$$\mathbb{K} = \mathbb{A} + \mathbb{W} + \alpha + \omega + \mathbb{K}^{-} + \mathbb{K}^{+}, \tag{5.18}$$

where the tensors $\mathbb{K}^{-}$ and $\mathbb{K}^{+}$ possess the following symmetries:

$$\mathbb{K}^{-}_{ijkl} = \mathbb{K}^{-}_{jikl} = -\mathbb{K}^{-}_{ijlk}, \quad \mathbb{K}^{+}_{ijkl} = -\mathbb{K}^{+}_{jikl} = \mathbb{K}^{+}_{ijlk}. \tag{5.19}$$

Thus in the 3D case the number of non-zero independent components of both $\mathbb{K}^{-}$ and $\mathbb{K}^{+}$ is equal to 18 and 3 in two dimensions. At this point, we introduce the notation:

$$\begin{aligned} &\mathbb{K}\mathbb{L} = \mathbb{K}:\mathbb{L} = \left(\mathbb{K}_{ijmn}\mathbb{L}_{mnkl}\right)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \\ &\mathbb{K}^2 = \mathbb{K}\mathbb{K}, \ldots, \mathbb{K}^n = \mathbb{K}\mathbb{K}^{n-1}, \quad Tr\mathbb{K} = \mathbb{K}_{klkl}, \\ &\mathbb{K}\mathbf{s} = \mathbb{K}\cdot\mathbf{s} = \left(\mathbb{K}_{ijkl}s_{kl}\right)\mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{s}:\mathbf{t} = s_{kl}t_{kl}. \end{aligned} \tag{5.20}$$

The irreducible isotropic function basis of a single two-dimensional fourth-order double-symmetric tensor was derived by Zheng (1994a, 1996); it includes five invariants (see also Table 5.2):

$$I_1 = Tr\mathbb{A}, I_2 = Tr\mathbb{A}^2, I_3 = Tr\mathbb{A}^3, L_1 = \mathbf{I}:\mathbb{A}\mathbf{I}, L_2 = \mathbf{I}:\mathbb{A}^2\mathbf{I}, \tag{5.21}$$

or

$$\begin{aligned} &I_1 = \mathbb{A}_{ijij}, \quad I_2 = \mathbb{A}_{ijkl}\mathbb{A}_{klij}, \quad I_3 = \mathbb{A}_{ijkl}\mathbb{A}_{klmn}\mathbb{A}_{mnij}, \\ &L_1 = \mathbb{A}_{iijj}, \quad L_2 = \mathbb{A}_{iikl}\mathbb{A}_{kljj}. \end{aligned} \tag{5.22}$$

An equivalent set of invariants was proposed by Betten (1986). It arises from the so-called *extended characteristic polynomial*:

$$\det\left(\lambda\mathbf{1} + \mu\mathbf{I}\otimes\mathbf{I} - \mathbb{A}\right) = \lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 + 2\mu\left(\lambda^2 + K_1\lambda + K_2\right), \tag{5.23}$$

where

$$\begin{aligned} &J_1 = I_1, \quad 2J_2 = J_1I_1 - I_2, \quad 3J_3 = J_2I_1 - J_1I_2 + I_3, \\ &2K_1 = 2J_1 - L_1, \quad 2K_2 = 2J_2 - J_1L_1 + L_2. \end{aligned} \tag{5.24}$$

For μ=0, (5.23) reduces to the usual characteristic polynomial.
In the 3D case one can easily write the explicit form of the extended characteristic polynomial, cf. Betten (1987a), Zheng (1994a), and more generally even in n-dimensional case. Unfortunately, for $n \geq 3$ this method does not lead to a complete set of

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invariants of $\mathbb{A}$. It means that such a set constitutes neither the integrity nor functional basis. In reality, in 3D case we obtain the following isotropic invariants of $\mathbb{A}$:

$$(5.25) \qquad Tr\mathbb{A}^i, \quad \mathbf{I}:\mathbb{A}^j\mathbf{I}, \quad i=1,\ldots,6;\ j=1,\ldots,5.$$

Hence we conclude that in the 3D case the problem of the determination of the integrity and functional bases remains open.

Similarly as in the case of functions involving third-order tensors, the results obtained by Zheng (1994a, 1994b) and Zheng and Betten (1994) concern mainly two-dimensional problems. The same can be said about functions of a fourth-order tensor. Complete and irreducible representations due to Zheng (1994b) and Zheng and Betten (1994) cover: isotropic and orthotropic scalar functions, second-order and fourth-order tensor-valued functions of any finite number of fourth-order double-symmetric tensors $\mathbb{A}_\alpha\ (\alpha=1,\ldots,A)$, antisymm-symmetric tensors $\mathbb{W}_\beta\ (\beta=1,\ldots,B)$, symm-antisymmetric tensors $\alpha_\gamma\ (\gamma=1,\ldots,G)$, double-antisymmetric tensors $\omega_\delta\ (\delta=1,\ldots,D)$, second-order symmetric tensors $\mathbf{A}_i\ (i=1,\ldots,I)$ and skew-symmetric second-order tensors $\mathbf{W}_p\ (p=1,\ldots,P)$.

In the 3D case the available results concern isotropic and hemitropic functions of $\alpha_\gamma, \omega_\delta$ and $\mathbf{W}_p$. For a more thorough account the reader should refer to Zheng (1994a).

Having in mind applications, for instance in the continuum damage mechanics, we shall now present selected representations of isotropic and orthotropic functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$. Some of the available results are summarized in Tables 5.2-5.5.

Two-dimensional isotropic fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

From Tables 5.2 and 5.4 it follows that the isotropic representation of a double-symmetric, fourth-order tensor function of $\mathbb{A}$ and $\mathbf{A}$ has the form

$$(5.26) \qquad \begin{aligned} \mathbb{F}(\mathbb{A},\mathbf{A}) &= \alpha_1\mathbf{I}\otimes\mathbf{I} + \alpha_2\mathbb{I} + \alpha_3(\mathbf{I}\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{I}) + \alpha_4\mathbf{A}\otimes\mathbf{A} + \alpha_5\mathbb{A} + \alpha_6(\mathbf{I}\otimes\mathbb{A}\mathbf{I}+\mathbb{A}\mathbf{I}\otimes\mathbf{I}) \\ &+\alpha_7\mathbb{A}^2 + \alpha_8(\mathbf{I}\otimes\mathbb{A}^2\mathbf{I}+\mathbb{A}^2\mathbf{I}\otimes\mathbf{I}) + \alpha_9\left[\mathbb{A}(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I}) - (\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})\mathbb{A}\right], \end{aligned}$$

where

$$(5.27) \qquad \alpha_i = f_i\left(tr\mathbf{A}, tr\mathbf{A}^2, Tr\mathbb{A}, Tr\mathbb{A}^2, Tr\mathbb{A}^3, \mathbf{I}:\mathbb{A}\mathbf{I}, \mathbf{I}:\mathbb{A}^2\mathbf{I}, \mathbf{I}:\mathbb{A}\mathbf{A}, \mathbf{I}:\mathbb{A}^2\mathbf{A}, \mathbf{A}:\mathbb{A}\mathbf{A}\right).$$

We observe that if $\mathbb{F}$ does not depend on $\mathbb{A}$ then (5.26) and (5.27) yield the representation (5.15).

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Table 5.2 Functional basis in two-dimensional space of isotropic scalar-valued function of $\mathbb{A}_\alpha, \mathbb{W}_\beta, \mathbf{A}_i$

Agencies	Functional basic
$\mathbb{A}$	$Tr\mathbb{A},\ Tr\mathbb{A}^2,\ Tr\mathbb{A}^3,\ \mathbf{I}:\mathbb{A}\mathbf{I},\ \mathbf{I}:\mathbb{A}^2\mathbf{I}$
$\mathbb{A}_1, \mathbb{A}_2$	$Tr\mathbb{A}_1\mathbb{A}_2,\ Tr\mathbb{A}_1^2\mathbb{A}_2,\ Tr\mathbb{A}_1\mathbb{A}_2^2,\quad \mathbf{I}:\mathbb{A}_1\mathbb{A}_2\mathbf{I}$
$\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$	$Tr\mathbb{A}_1\mathbb{A}_2\mathbb{A}_3$
$\mathbb{W}$	$Tr\mathbb{W}^2,\ \mathbf{I}:\mathbb{W}\mathbf{I}$
$\mathbb{A}, \mathbb{W}$	$Tr\mathbb{A}\mathbb{W}^2,\ Tr\mathbb{A}^2\mathbb{W}^2,\ Tr\mathbb{A}^2\mathbb{W}^2\mathbb{A}\mathbb{W},\ \mathbf{I}:\mathbb{A}\mathbb{W}\mathbf{I},\ \mathbf{I}:\mathbb{A}^2\mathbb{W}\mathbf{I},\ \mathbf{I}:\mathbb{W}\mathbb{A}\mathbb{W}^2\mathbf{I}$
$\mathbb{A}_1, \mathbb{A}_2, \mathbb{W}$	$Tr\mathbb{A}_1\mathbb{A}_2\mathbb{W},\ Tr\mathbb{A}_1^2\mathbb{A}_2\mathbb{W},\ Tr\mathbb{A}_1\mathbb{A}_2^2\mathbb{W},\ Tr\mathbb{A}_1\mathbb{W}^2\mathbb{A}_2\mathbb{W}$
$\mathbb{W}_1, \mathbb{W}_2$	$Tr\mathbb{W}_1\mathbb{W}_2,\ \mathbf{I}:\mathbb{W}_1\mathbb{W}_2\mathbf{I},\ \mathbf{I}:\mathbb{W}_1^2\mathbb{W}_2\mathbf{I},\ \mathbf{I}:\mathbb{W}_1\mathbb{W}_2^2\mathbf{I}$
$\mathbb{A}, \mathbb{W}_1, \mathbb{W}_2$	$Tr\mathbb{A}\mathbb{W}_1\mathbb{W}_2,\ Tr\mathbb{A}\mathbb{W}_1^2\mathbb{W}_2,\ Tr\mathbb{A}\mathbb{W}_1\mathbb{W}_2^2$
$\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$	$Tr\mathbb{W}_1\mathbb{W}_2\mathbb{W}_3$
$\mathbf{A}$	$tr\mathbf{A},\ tr\mathbf{A}^2$
$\mathbb{A}, \mathbf{A}$	$\mathbf{I}:\mathbb{A}\mathbf{A},\ \mathbf{I}:\mathbb{A}^2\mathbf{A},\ \mathbf{A}:\mathbb{A}\mathbf{A}$
$\mathbb{A}_1, \mathbb{A}_2, \mathbf{A}$	$\mathbf{I}:(\mathbb{A}_1\mathbb{A}_2 - \mathbb{A}_2\mathbb{A}_1)\mathbf{A}$
$\mathbb{W}, \mathbf{A}$	$\mathbf{I}:\mathbb{W}\mathbf{A},\ \mathbf{I}:\mathbb{W}^2\mathbf{A}$
$\mathbb{A}, \mathbb{W}, \mathbf{A}$	$\mathbf{I}:(\mathbb{A}\mathbb{W} + \mathbb{W}\mathbb{A})\mathbf{A},\ \mathbf{A}:\mathbb{A}\mathbb{W}\mathbf{A}$
$\mathbb{W}_1, \mathbb{W}_2, \mathbf{A}$	$\mathbf{I}:(\mathbb{W}_1\mathbb{W}_2 - \mathbb{W}_2\mathbb{W}_1)\mathbf{A}$
$\mathbf{A}_1, \mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2$
$\mathbb{A}, \mathbf{A}_1, \mathbf{A}_2$	$\mathbf{A}_1:\mathbb{A}\mathbf{A}_2$
$\mathbb{W}, \mathbf{A}_1, \mathbf{A}_2$	$\mathbf{A}_1:\mathbb{W}\mathbf{A}_2$

Table 5.3 Generators for isotropic symmetric, second-order tensor-valued functions of $\mathbb{A}_\alpha, \mathbb{W}_\beta, \mathbf{A}_i$

Agencies	Generators
-	$\mathbf{I}$
$\mathbb{A}$	$\mathbb{A}\mathbf{I},\ \mathbb{A}^2\mathbf{I}$
$\mathbb{A}_1, \mathbb{A}_2$	$(\mathbb{A}_1\mathbb{A}_2 - \mathbb{A}_2\mathbb{A}_1)\mathbf{I}$
$\mathbb{W}$	$\mathbb{W}\mathbf{I},\ \mathbb{W}^2\mathbf{I}$
$\mathbb{A}, \mathbb{W}$	$(\mathbb{A}\mathbb{W} + \mathbb{W}\mathbb{A})\mathbf{I}$
$\mathbb{W}_1, \mathbb{W}_2$	$(\mathbb{W}_1\mathbb{W}_2 - \mathbb{W}_2\mathbb{W}_1)\mathbf{I}$
$\mathbf{A}$	$\mathbf{A}$
$\mathbb{A}, \mathbf{A}$	$\mathbb{A}\mathbf{A}$
$\mathbb{W}, \mathbf{A}$	$\mathbb{W}\mathbf{A}$

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Table 5.4 Generators for isotropic double-symmetric, fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Generators
-	$\mathbf{I}$, $\mathbf{I}\otimes\mathbf{I}$
$\mathbb{A}$	$\mathbb{A}$, $\mathbb{A}^2$, $\mathbf{I}\otimes\mathbb{A}\mathbf{I}+\mathbb{A}\mathbf{I}\otimes\mathbf{I}$, $\mathbf{I}\otimes\mathbb{A}^2\mathbf{I}+\mathbb{A}^2\mathbf{I}\otimes\mathbf{I}$
$\mathbb{A}_1$, $\mathbb{A}_2$	$\mathbb{A}_1\mathbb{A}_2+\mathbb{A}_2\mathbb{A}_1$
$\mathbb{W}$	$\mathbb{W}^2$, $\mathbf{I}\otimes\mathbb{W}\mathbf{I}+\mathbb{W}\mathbf{I}\otimes\mathbf{I}$, $\mathbb{W}\mathbf{I}\otimes\mathbf{I}\mathbb{W}$, $\mathbb{W}\mathbf{I}\otimes\mathbb{W}^2\mathbf{I}+\mathbb{W}^2\mathbf{I}\otimes\mathbb{W}\mathbf{I}$
$\mathbb{A}$, $\mathbb{W}$	$\mathbb{A}\mathbb{W}-\mathbb{W}\mathbb{A}$, $\mathbb{W}\mathbb{A}\mathbb{W}^2-\mathbb{W}^2\mathbb{A}\mathbb{W}$, $\mathbb{A}^2\mathbb{W}-\mathbb{W}\mathbb{A}^2$
$\mathbb{W}_1$, $\mathbb{W}_2$	$\mathbb{W}_1\mathbb{W}_2+\mathbb{W}_2\mathbb{W}_1$, $\mathbb{W}_1^2\mathbb{W}_2-\mathbb{W}_2\mathbb{W}_1^2$, $\mathbb{W}_1\mathbb{W}_2^2-\mathbb{W}_2^2\mathbb{W}_1$
$\mathbf{A}$	$\mathbf{A}\otimes\mathbf{A}$, $\mathbf{I}\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{I}$
$\mathbb{A}$, $\mathbf{A}$	$\mathbb{A}(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})-(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})\mathbb{A}$
$\mathbb{W}$, $\mathbf{A}$	$\mathbb{W}(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})+(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})\mathbb{W}$, $\mathbf{A}\otimes\mathbb{W}\mathbf{A}+\mathbb{W}\mathbf{A}\otimes\mathbf{A}$
$\mathbf{A}_1$, $\mathbf{A}_2$	$\mathbf{A}_1\otimes\mathbf{A}_2+\mathbf{A}_2\otimes\mathbf{A}_1$

Table 5.5 Generators for isotropic antisymmetric-symmetric, fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Generators
$\mathbb{A}$	$\mathbf{I}\otimes\mathbb{A}\mathbf{I}-\mathbb{A}\mathbf{I}\otimes\mathbf{I}$, $\mathbf{I}\otimes\mathbb{A}^2\mathbf{I}-\mathbb{A}^2\mathbf{I}\otimes\mathbf{I}$, $\mathbb{A}\mathbf{I}\otimes\mathbb{A}^2\mathbf{I}-\mathbb{A}^2\mathbf{I}\otimes\mathbb{A}\mathbf{I}$
$\mathbb{A}_1$, $\mathbb{A}_2$	$\mathbb{A}_1\mathbb{A}_2-\mathbb{A}_2\mathbb{A}_1$, $\mathbb{A}_1^2\mathbb{A}_2-\mathbb{A}_2\mathbb{A}_1^2$, $\mathbb{A}_1\mathbb{A}_2^2-\mathbb{A}_2^2\mathbb{A}_1$, $\mathbb{A}_1\mathbf{I}\otimes\mathbb{A}_2\mathbf{I}-\mathbb{A}_2\mathbf{I}\otimes\mathbb{A}_1\mathbf{I}+\mathbf{I}\otimes(\mathbb{A}_1\mathbb{A}_2-\mathbb{A}_2\mathbb{A}_1)\mathbf{I}-(\mathbb{A}_1\mathbb{A}_2-\mathbb{A}_2\mathbb{A}_1)\mathbf{I}\otimes\mathbf{I}$
$\mathbb{W}$	$\mathbb{W}$, $\mathbf{I}\otimes\mathbb{W}\mathbf{I}-\mathbb{W}\mathbf{I}\otimes\mathbf{I}$, $\mathbb{W}\mathbf{I}\otimes\mathbf{I}\mathbb{W}$, $\mathbb{W}\mathbf{I}\otimes\mathbb{W}^2\mathbf{I}-\mathbb{W}^2\mathbf{I}\otimes\mathbb{W}\mathbf{I}$
$\mathbb{A}$, $\mathbb{W}$	$\mathbb{A}\mathbb{W}+\mathbb{W}\mathbb{A}$, $\mathbb{A}\mathbb{W}^2-\mathbb{W}^2\mathbb{A}$
$\mathbb{W}_1$, $\mathbb{W}_2$	$\mathbb{W}_1\mathbb{W}_2-\mathbb{W}_2\mathbb{W}_{11}$
$\mathbf{A}$	$\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I}$
$\mathbb{A}$, $\mathbf{A}$	$\mathbb{A}(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})+(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})\mathbb{A}$, $\mathbf{A}\otimes\mathbb{A}\mathbf{A}-\mathbb{A}\mathbf{A}\otimes\mathbf{A}$
$\mathbb{W}$, $\mathbf{A}$	$\mathbb{W}(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})-(\mathbf{I}\otimes\mathbf{A}-\mathbf{A}\otimes\mathbf{I})\mathbb{W}$
$\mathbf{A}_1$, $\mathbf{A}_2$	$\mathbf{A}_1\otimes\mathbf{A}_2-\mathbf{A}_2\otimes\mathbf{A}_1$

Two-dimensional orthotropic fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Since the two-dimensional orthotropy can be characterized by $\mathbf{M}_1=\mathbf{e}_1\otimes\mathbf{e}_1$ and $\mathbf{M}_2=\mathbf{e}_2\otimes\mathbf{e}_2$, therefore we can establish complete and irreducible representations for isotropic tensor functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$, $\mathbf{M}_1$, $\mathbf{M}_2$ equivalent to orthotropic tensor functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$, cf. Zheng and Betten (1994). Such complete and irreducible representations of orthotropic functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$ and $\mathbf{A}_i$ are summarized in Tables 5.6-5.9.

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Table 5.6 Functional basis in two-dimensional space of orthotropic scalar-valued function of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Functional basic
$\mathbb{A}$	$Tr\mathbb{A}$, $\mathbf{M}_1:\mathbb{A}\mathbf{M}_1$, $\mathbf{M}_2:\mathbb{A}\mathbf{M}_2$, $\mathbf{M}_1:\mathbb{A}\mathbf{M}_2$, $\mathbf{M}_1:\mathbb{A}^2\mathbf{M}_1$, $\mathbf{M}_2:\mathbb{A}^2\mathbf{M}_2$, $\mathbf{M}_1:\mathbb{A}^2\mathbf{M}_2$
$\mathbb{A}_1$, $\mathbb{A}_2$	$Tr\mathbb{A}_1\mathbb{A}_2$, $\mathbf{M}_1:(\mathbb{A}_1\mathbb{A}_2-\mathbb{A}_2\mathbb{A}_1)\mathbf{M}_2$
$\mathbb{W}$	$\mathbf{M}_1:\mathbb{W}\mathbf{M}_2$, $\mathbf{M}_1:\mathbb{W}^2\mathbf{M}_1$, $\mathbf{M}_2:\mathbb{W}^2\mathbf{M}_2$, $\mathbf{M}_1:\mathbb{W}^2\mathbf{M}_2$
$\mathbb{A}$, $\mathbb{W}$	$Tr\mathbb{S}\mathbb{A}\mathbb{W}$, $\mathbf{M}_1:(\mathbb{A}\mathbb{W}+\mathbb{W}\mathbb{A})\mathbf{M}_2$
$\mathbb{W}_1$, $\mathbb{W}_2$	$Tr\mathbb{W}_1\mathbb{W}_2$, $\mathbf{M}_1:(\mathbb{W}_1\mathbb{W}_2-\mathbb{W}_2\mathbb{W}_1)\mathbf{M}_2$
$\mathbf{A}$	$tr\mathbf{M}_1\mathbf{A}$, $tr\mathbf{M}_2\mathbf{A}$, $tr\mathbf{A}^2$
$\mathbb{A}$, $\mathbf{A}$	$\mathbf{M}_1:\mathbb{A}\mathbf{A}$, $\mathbf{M}_2:\mathbb{A}\mathbf{A}$
$\mathbb{W}$, $\mathbf{A}$	$\mathbf{M}_1:\mathbb{W}\mathbf{A}$, $\mathbf{M}_2:\mathbb{W}\mathbf{A}$
$\mathbf{A}_1$, $\mathbf{A}_2$	$tr\mathbf{A}_1\mathbf{A}_2$

$\mathbb{S}=\mathbf{M}_1\otimes\mathbf{M}_1+\mathbf{M}_2\otimes\mathbf{M}_2$

Table 5.7 Generators for orthotropic symmetric, second-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Generators
-	$\mathbf{M}_1$, $\mathbf{M}_2$
$\mathbb{A}$	$\mathbb{A}\mathbf{M}_1$, $\mathbb{A}\mathbf{M}_2$
$\mathbb{W}$	$\mathbb{W}\mathbf{M}_1$, $\mathbb{W}\mathbf{M}_2$
$\mathbf{A}$	$\mathbf{A}$

Table 5.8 Generators for orthotropic double-symmetric, fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Generators
-	$\mathbf{M}_1\otimes\mathbf{M}_1$, $\mathbf{M}_2\otimes\mathbf{M}_2$, $\mathbf{M}_1\otimes\mathbf{M}_2+\mathbf{M}_2\otimes\mathbf{M}_1$
$\mathbb{A}$	$\mathbb{A}$, $\mathbb{A}\mathbb{Q}-\mathbb{A}\mathbb{Q}$
$\mathbb{W}$	$\mathbb{W}\mathbb{Q}+\mathbb{W}\mathbb{Q}$, $\mathbb{W}\mathbb{S}-\mathbb{S}\mathbb{W}$
$\mathbf{A}$	$\mathbf{M}_1\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{M}_1$, $\mathbf{M}_2\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{M}_2$

$\mathbb{Q}=\mathbf{M}_1\otimes\mathbf{M}_2-\mathbf{M}_2\otimes\mathbf{M}_1$, $\quad\mathbb{S}=\mathbf{M}_1\otimes\mathbf{M}_1+\mathbf{M}_2\otimes\mathbf{M}_2$

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Table 5.9 Generators for orthotropic antisymmetric-symmetric, fourth-order tensor-valued functions of $\mathbb{A}_\alpha$, $\mathbb{W}_\beta$, $\mathbf{A}_i$

Agencies	Generators
-	$\mathbb{Q} = \mathbf{M}_1 \otimes \mathbf{M}_2 - \mathbf{M}_2 \otimes \mathbf{M}_1$
$\mathbb{A}$	$\mathbb{A}\mathbb{S} - \mathbb{S}\mathbb{A},\ \mathbb{A}\mathbb{Q} + \mathbb{A}\mathbb{Q}$
$\mathbb{W}$	$\mathbb{W},\ \mathbb{W}\mathbb{Q} - \mathbb{W}\mathbb{Q}$
$\mathbf{A}$	$\mathbf{M}_1 \otimes \mathbf{A} - \mathbf{A} \otimes \mathbf{M}_1,\ \mathbf{M}_2 \otimes \mathbf{A} - \mathbf{A} \otimes \mathbf{M}_2$

For instance, from Tables 5.6 and 5.7 it follows that the representation of a symmetric, second-order tensor function of $\mathbb{A}$ and $\mathbf{A}$ has the following form

$$(5.28)\qquad \mathbb{F}_{ort}(\mathbb{A}, \mathbf{A}) = \tilde{\mathbb{F}}(\mathbb{A}, \mathbf{A}, \mathbf{M}_1, \mathbf{M}_2) = \alpha_1 \mathbf{M}_1 + \alpha_2 \mathbf{M}_2 + \alpha_3 \mathbf{A} + \alpha_4 \mathbb{A}\mathbf{M}_1 + \alpha_5 \mathbb{A}\mathbf{M}_2,$$

where

$$(5.29)\qquad \begin{aligned} \alpha_i = f_i\big(&tr\mathbf{M}_1\mathbf{A},\ tr\mathbf{M}_2\mathbf{A},\ tr\mathbf{A}^2,\ Tr\mathbb{A},\ \mathbf{M}_1{:}\mathbb{A}\mathbf{M}_1,\ \mathbf{M}_2{:}\mathbb{A}\mathbf{M}_2,\\ &\mathbf{M}_1{:}\mathbb{A}\mathbf{M}_2,\ \mathbf{M}_1{:}\mathbb{A}^2\mathbf{M}_1,\ \mathbf{M}_2{:}\mathbb{A}^2\mathbf{M}_2,\ \mathbf{M}_1{:}\mathbb{A}^2\mathbf{M}_2,\ \mathbf{M}_2{:}\mathbb{A}\mathbf{A},\ \mathbf{M}_1{:}\mathbb{A}\mathbf{A}\big), \quad i = 1,\ldots,5. \end{aligned}$$

Similarly, Tables 5.2 and 5.3 yield the isotropic representation of a symmetric second-order tensor function of $\mathbb{A}$ and $\mathbf{A}$:

$$(5.30)\qquad \mathbb{F}_{is}(\mathbb{A}, \mathbf{A}) = \beta_1 \mathbf{I} + \beta_2 \mathbf{A} + \beta_3 \mathbb{A}\mathbf{I} + \beta_4 \mathbb{A}^2\mathbf{I} + \beta_5 \mathbb{A}\mathbf{A},$$

where

$$(5.31)\qquad \beta_i = f_i\big(tr\mathbf{A},\ tr\mathbf{A}^2,\ Tr\mathbb{A},\ Tr\mathbb{A}^2,\ Tr\mathbb{A}^3,\ \mathbf{I}{:}\mathbb{A}\mathbf{I},\ \mathbf{I}{:}\mathbb{A}^2\mathbf{I},\ \mathbf{I}{:}\mathbb{A}\mathbf{A},\ \mathbf{I}{:}\mathbb{A}^2\mathbf{A},\ \mathbf{A}{:}\mathbb{A}\mathbf{A}\big), \quad i = 1,\ldots,5.$$

From Sections 2 and 4 of the present paper we already know that the choice of parametric tensors is not uniquely defined. Consequently the parametric tensors $\mathbf{M}_1$ and $\mathbf{M}_2$ can be replaced by $\mathbf{H} = H_1\mathbf{M}_1 + H_2\mathbf{M}_2$ or its deviator $\mathbf{H}_d = \mathbf{H} - \frac{1}{2}(tr\mathbf{H})\mathbf{I}$ or just by one of the tensors $\mathbf{M}_1$ or $\mathbf{M}_2$ since $\mathbf{M}_1 + \mathbf{M}_2 = \mathbf{I}$. Here H_1 and H_2 are eigenvalues of $\mathbf{H}$ with $H_1 \neq H_2$. Then one can easily construct alternative orthotropic functional basis and generators to the ones listed in Tables 5.6-5.8. For instance, the equivalent form of the representation of the function (5.28) is given by

$$(5.32)\qquad \mathbb{F}_{ort}(\mathbb{A}, \mathbf{A}) = \tilde{\mathbb{F}}(\mathbb{A}, \mathbf{A}, \mathbf{M}) = \gamma_1 \mathbf{I} + \gamma_2 \mathbf{M} + \gamma_3 \mathbf{A} + \gamma_4 \mathbb{A}\mathbf{I} + \gamma_5 \mathbb{A}\mathbf{M},$$

where

$$(5.33)\qquad \begin{aligned} \gamma_i = f_i\big(&tr\mathbf{A},\ tr\mathbf{M}\mathbf{A},\ tr\mathbf{A}^2,\ Tr\mathbb{A},\ \mathbf{I}{:}\mathbb{A}\mathbf{I},\ \mathbf{M}{:}\mathbb{A}\mathbf{M},\\ &\mathbf{M}{:}\mathbb{A}\mathbf{I},\ \mathbf{I}{:}\mathbb{A}^2\mathbf{I},\ \mathbf{M}{:}\mathbb{A}^2\mathbf{M},\ \mathbf{I}{:}\mathbb{A}^2\mathbf{M},\ \mathbf{I}{:}\mathbb{A}\mathbf{A},\ \mathbf{M}{:}\mathbb{A}\mathbf{A}\big), \quad i = 1,\ldots,5. \end{aligned}$$

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6. Selected applications to solid mechanics

This Section is concerned with some applications of tensor functions to the formulation of the constitutive relationships describing a class of solids. In a future contribution we hope to study the problem more thoroughly and completely. Nevertheless, the role of nonpolynomial representations will become evident.

6.1 Nonlinear elasticity

In the theory of hyperelasticity one assumes that the free energy of a body depends only on its actual deformation and is referred to as the stored energy function. It is also assumed that there exists a neutral state of the body at which this function vanishes. Since it cannot depend on rigid rotations of a material point, hence we may write, cf. Gurtin (1981):

$$W = \tilde{W}(\mathbf{U}), \quad \tilde{W}(\mathbf{I}) = 0, \tag{6.1}$$

where $\mathbf{U}$ denotes the right stretch tensor which is a symmetric, positive definite second-order tensor appering in the polar decomposition of the deformation tensor: $\mathbf{F} = \mathbf{R}\mathbf{U}$, $\mathbf{R} \in O$, $\det \mathbf{R} = 1$. For nonhomogeneous materials $\tilde{W}$ depends additionally on a space variable determining the position of a material point. For isotropic materials $\tilde{W}$ is an isotropic scalar-valued functions of $\mathbf{U}$. Otherwise, for anisotropic materials, according to Isotropicization Theorem (Section 2) the stored energy function may be written as follows

$$W = \overline{W}(\mathbf{U}, \mathbf{P}_1, \ldots, \mathbf{P}_M), \tag{6.2}$$

where $\mathbf{P}_1, \ldots, \mathbf{P}_M$ are structural tensors characterizing an anisotropy group S, i.e.:

$$\forall \mathbf{Q} \in S \subset O, \quad \overline{W}(\mathbf{U}, \mathbf{P}_1, \ldots, \mathbf{P}_M) = \overline{W}\left(\mathbf{Q}\mathbf{U}\mathbf{Q}^T, \mathbf{P}_1, \ldots, \mathbf{P}_M\right), \tag{6.3}$$

where

$$S = S_1 \cap S_2 \cap \ldots \cap S_M \quad \textit{and} \quad S_m \equiv \left\{\mathbf{Q} \in O \middle| \quad \mathbf{Q} \circ \mathbf{P}_m = \mathbf{P}_m\right\}; \quad m = 1, \ldots, M.$$

Consequently, the function defined by (6.2) depends on *anisotropic invariants* $I_n \ (n = 1, \ldots, N)$ of the tensor $\mathbf{U}$:

$$W = \hat{W}(I_n). \tag{6.4}$$

The constitutive relationship for hyperelastic materials has the form:
(i) in the reference configuration

$$\mathbf{T} = \mathbf{R}\frac{\partial \hat{W}}{\partial \mathbf{U}}, \tag{6.5}$$

where $\mathbf{T}$ stands for the first Piola-Kirchhoff stress tensor, which is an unsymmetric, second-order tensor.

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(ii) In the deformed configuration

(6.6) $$\mathbf{t} = \frac{1}{\det \mathbf{U}} \mathbf{R} \frac{\partial \hat{W}}{\partial \mathbf{U}} \mathbf{R}^T,$$

where **t** is the Cauchy (symmetric) stress tensor.

In order to determine the general, canonical form of the constitutive relationship by using (6.5) (or (6.6)) we write

(6.7) $$\frac{\partial \hat{W}}{\partial \mathbf{U}} = \alpha_n \mathbf{G}_n, \quad \alpha_n = \frac{\partial \hat{W}}{\partial I_n}, \quad \mathbf{G}_n = \frac{\partial I_n}{\partial \mathbf{U}}.$$

The scalar coefficients α_n satisfy the obvious condition

(6.8) $$\frac{\partial \alpha_n}{\partial I_p} = \frac{\partial \alpha_p}{\partial I_n}; \quad n, p = 1, \ldots, N.$$

The symmetric, second Piola-Kirchhof stress tensor is given by $\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}$. Hence the equivalent form of (6.5) is

(6.9) $$\mathbf{S} = \mathbf{U}^{-1} \frac{\partial \hat{W}}{\partial \mathbf{U}}.$$

Constitutive relationships (6.5), (6.6) and (6.9) may be used in experimental verification of an a priori postulated function (6.4) or its identification on the basis of standard tests, cf. Ogden (1972a, 1972b) for relevant discussion in the case of isotropic materials. For uniaxial tests, $U=l/L$, where L stands for the initial length and l is the final length of a sample.

Let $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ and $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$ denote the right Cauchy-Green tensor and Green strain tensor, respectively. Then the constitutive relationships, equivalent to (6.5), (6.6) and (6.9) are expressible in the following form

(6.10) $$\mathbf{S} = 2\frac{\partial \bar{W}}{\partial \mathbf{C}} \quad or \quad \mathbf{S} = \frac{\partial \breve{W}}{\partial \mathbf{E}},$$

where

(6.11) $$W = \bar{W}(J_n) \quad or \quad W = \breve{W}(K_n)$$

and J_n, K_n $(n = 1, \ldots, N)$ are anisotropic invariants of the tensors **C** and **E**, respectively. We observe that the linearized form of $(6.10)_2$ describes so called Saint Venant-Kirchhoff materials. Such constitutive equation is formally the same as the classical Hooke's law, cf. Ciarlet (1988).

For hyperelastic *incompressible* materials $\det \mathbf{F} = \det \mathbf{U} = \det \mathbf{C} = 1$. Then it is convenient to consider the following constitutive equation

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(6.12) $$\mathbf{S} = -p(\det \mathbf{C})\mathbf{C} + \overline{\mathbf{S}}, \quad tr\overline{\mathbf{S}}\mathbf{C} = 0,$$

(6.13) $$p = -2\det \mathbf{C} \frac{\partial \dot{W}}{\partial \det \mathbf{C}},$$

(6.14) $$\overline{\mathbf{S}} = \frac{2}{\det \mathbf{C}}\left[\frac{\partial \dot{W}}{\partial \overline{\mathbf{C}}} - \frac{1}{3}\left(\frac{\partial \dot{W}}{\partial \overline{\mathbf{C}}} \cdot \overline{\mathbf{C}}\right)\overline{\mathbf{C}}^{-1}\right], \quad \frac{\partial \dot{W}}{\partial \overline{\mathbf{C}}} \cdot \overline{\mathbf{C}} = \frac{\partial \dot{W}}{\partial \overline{C}_{IJ}} \cdot \overline{C}_{IJ} \quad (I, J = 1, 2, 3)$$

and

(6.15) $$W = \dot{W}\left(\det \mathbf{C}, \overline{\mathbf{C}}, \mathbf{P}_1, \ldots, \mathbf{P}_M\right), \quad \overline{\mathbf{C}} = \frac{1}{\det \mathbf{C}}\mathbf{C}.$$

Here $\overline{\mathbf{C}}$ is a measure of shear deformations and was introduced by Rubin (1988). The incompressibility condition det $\mathbf{C}$=1 can easily be taken into account in Eq. (6.12) and (6.14). We observe that Rubin (1988) examined only isotropic materials.

Now we shall provide some examples of applications of the invariant theory and tensor functions to the formulation of constitutive equations modelling the behaviour of hyperelastic materials.

Example 6.1.1 (isotropic materials)
The set of structural tensors reduces now to the unit tensor since for each $\mathbf{Q} \in O, \quad \mathbf{QIQ}^T = \mathbf{I}$, and the set of isotropic invariants of the tensor $\mathbf{U}$ has the form

(6.16) $$\{I_n\} = \{tr\mathbf{U}, tr\mathbf{U}^2, tr\mathbf{U}^3\}, \quad n = 1, 2, 3.$$

Consequently, according to (6.7) the set of generators is given by

(6.17) $$\{\mathbf{G}_n\} = \{\mathbf{I}, 2\mathbf{U}, 3\mathbf{U}^2\},$$

while the scalar functions α_n are defined by (6.7) . Usually, instead of (6.16) one takes the following basic invariants

(6.18) $$\{\tilde{I}_n\} = \{I_U, II_U, III_U\},$$

where

$$I_U = tr\mathbf{U}, \quad II_U = \frac{1}{2}\left(tr^2\mathbf{U} - tr\mathbf{U}^2\right), \quad III_U = \det \mathbf{U}.$$

Hence

(6.19) $$\{\tilde{\mathbf{G}}_n\} = \{\mathbf{I}, I_U\mathbf{I} - \mathbf{U}, II_U\mathbf{I} - I_U\mathbf{U} + \mathbf{U}^2\}.$$

By applying Cayley-Hamilton's theorem, from (6.6) we obtain

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(6.20) $$\mathbf{t}=\frac{\partial\hat{W}}{\partial III_U}\mathbf{I}+\frac{1}{III_U}\left[\left(\frac{\partial\hat{W}}{\partial I_U}+I_U\frac{\partial\hat{W}}{\partial II_U}\right)\mathbf{V}-\frac{\partial\hat{W}}{\partial II_U}\mathbf{V}^2\right],$$

where $W=\hat{W}(I_U, II_U,\ III_U)$ and $\mathbf{V}=\mathbf{RUR}^T$ is the left stretch tensor appearing in the polar decomposition of the deformation tensor $\mathbf{F}$; $\mathbf{F}=\mathbf{VR}^T$.

The stored energy function for the Saint-Venant Kirchhoff material is defined by

(6.21) $$W=\frac{1}{2}\left(\lambda tr^2\mathbf{E}+2\mu tr\mathbf{E}^2\right),$$

where λ and μ are the so-called Lamé constants as introduced in the classical linear elasticity; hence

(6.22) $$\mathbf{S}=\lambda(tr\mathbf{E})\mathbf{I}+2\mu\mathbf{E}.$$

Because of their simplicity, Saint-Venant Kirchhoff materials are often used in computations. Unfortunately, such materials can reach infinite compression rates with finite energy and do not satisfy the polyconvexity assumption used in the existence theory (cf. Ciarlet, 1988; Raoult, 1986). An explicit expression for the quasiconvex envelope of the Saint-Venant-Kirchhoff stored energy function in terms of singular values was derived by Le Dret and Raoult (1994), see also Benaouda and Telega (1997).
For more details on specific forms of the stored energy functions and their experimental verification the reader should refer to the papers by Aron (1991), Aron and Creasy (1989), Ball (1977, 1984), Beatty (1987), Billinngton (1986a, 1986b, 1986c and 1986d), Blinowski (1980, 1982), Bolzon (1993), Bolzon and Vitaliani (1993), Bowen (1989), Caroll (1988), Green and Zerna (1968), Harren (1993), Jiang and Knowles (1991), Ogden (1972a, 1972b, 1984), Rajagopal and Wineman (1987), Rivlin and Ericksen (1955), Rubin (1988), Trumel and Dragon (1994).

(i) *Classical incompressible rubberlike solids*
Treloar (1944) constructed the so-called *incompressible neo-Hookean* form of the strain energy function (it is the simplest example of neo-Hookean material):

$$\hat{W}(\mathbf{C})=\frac{1}{2}\mu(tr\mathbf{C}-3),$$

where μ is the shear modulus. If we add a linear term in $tr(cof\mathbf{C})=\frac{1}{2}\left[(tr\mathbf{C})^2-tr\mathbf{C}^2\right]$ to this function, we get the well-known Mooney-Rivlin materials (Mooney, 1940; Rivlin, 1948a) whose energy is given by

$$\hat{W}(\mathbf{C})=A_{10}\left[tr\mathbf{C}-3\right]+A_{01}\left[tr(cof\mathbf{C})-3\right].$$

For rubbers, the numerical values of the above constants are typically equal to:

$$A_{10}=0.183\ MPa,\quad A_{01}=0.0034\ MPa.$$

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The Mooney-Rivlin form of the strain energy function gives a marginaly better fit to the experimental data than the neo-Hookean form.
The strain energy function defined by

$$\tilde{W}(\mathbf{U}) = \mu(tr\mathbf{U} - 3),$$

was proposed by Varga (1966) as a first approximation to the behaviour of rubberlike solids.
Rivlin and Saunders (1951) suggested a strain energy function in the form

$$\hat{W}(\mathbf{C}) = \frac{1}{2}\mu[tr\mathbf{C} - 3] + f(tr(cof\mathbf{C}) - 3).$$

Another possibility is, cf. Ogden (1972a):

$$\hat{W}(\mathbf{C}) = \sum_{m,n=0}^{\infty} A_{mn}(tr\mathbf{C} - 3)^m (tr(cof\mathbf{C}) - 3)^n, \quad A_{00} = 0,$$

where A_{mn} are constants. This last expression is the generalization of the Mooney-Rivlin material.
Hart-Smith (1966) proposed the following strain energy function

$$\hat{W}(\mathbf{C}) = \mu_1 \int \exp\left[\nu_1 (I_C - 3)^2\right] dI_C + \mu_2 \ln\left(\frac{II_C}{3}\right),$$

which gives good correlations with the data for small and moderate strains of the vulcanized natural rubber. Alexander (1968) found that this function was not suitable for the synthetic rubber at moderate strains. According to this author better results gives the function

$$\hat{W}(\mathbf{C}) = \mu_1 \int \exp\left[\nu_1 (I_C - 3)^2\right] dI_C + \mu_2 \ln\left(\frac{II_C - 3 + \nu_2}{\nu_2}\right) + \mu_3 (II_C - 3),$$

where μ_i $(i = 1,2,3)$, ν_α $(\alpha = 1,2)$ are constants.
We observe that constitutive equations for nonlinear isotropic thermoelastic solids were extensively studied by Haddow and Ogden (1990) and Ogden (1987, 1992), cf. also Green and Adkins (1970).

(ii) *Ogden's materials* (Ogden, 1972a and 1972b)
They are described by

$$\hat{W}(\mathbf{F}) = \sum_{i=1}^{M} a_i tr\mathbf{C}^{\gamma_i/2} + \sum_{j=1}^{N} b_j tr(cof\mathbf{C})^{\delta_j/2} + \Gamma(\det \mathbf{F}),$$

where

$$\det \mathbf{F} = \det \mathbf{U}, \quad tr(cof\mathbf{C}) = \frac{1}{2}\left[(tr\mathbf{C})^2 - tr\mathbf{C}^2\right].$$

Equivalently we write

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$$\hat{W}(\mathbf{U})=\sum_{i=1}^{M}a_i\left(U_1^{\gamma_i}+U_2^{\gamma_i}+U_3^{\gamma_i}\right)+\sum_{j=1}^{N}b_j\left[\left(U_1U_2\right)^{\delta_j}+\left(U_1U_3\right)^{\delta_j}+\left(U_2U_3\right)^{\delta_j}\right]+\Gamma\left(U_1U_2U_3\right),$$

where $U_i\ (i=1,2,3)$ are the eigenvalues of the tensor $\mathbf{U}\left(\mathbf{U}=\sqrt{\mathbf{C}}, U_i=\sqrt{C_i}\right)$,

$$a_i>0, \gamma_i\geq 1, b_j>0, \delta_j\geq 0,$$

and

$$\Gamma:[0,+\infty]\rightarrow R$$

is a convex function satisfying $\Gamma(\delta)\rightarrow +\infty$ as $\delta\rightarrow 0^+$ and subjected to suitable growth conditions as $\delta\rightarrow +\infty$. Notice that in the literature, the normalizing constant $3=tr\mathbf{I}$ is often introduced, as in

$$\hat{W}(\mathbf{F})=\sum_{i=1}^{M}a_i\left[tr\mathbf{C}^{\gamma_i/2}-3\right]+\sum_{j=1}^{N}b_j\left[tr\left(cof\mathbf{C}\right)^{\delta_j/2}-3\right]+\Gamma\left(\det\mathbf{F}\right),$$

in order that the first terms vanish when $\mathbf{F}^T\mathbf{F}=\mathbf{I}$.

(iii) *Compressible neo-Hookean materials* are characterized by (Blatz, 1971)

$$\hat{W}(\mathbf{F})=a\|\mathbf{F}\|^2+\Gamma\left(\det\mathbf{F}\right),$$

$$\|\mathbf{F}\|^2=tr\mathbf{C},\quad a>0.$$

Particularly we have

$$\hat{W}(\mathbf{F})=\frac{1}{2}a\|\mathbf{F}\|^2+\frac{1}{\sigma}\left(\det\mathbf{F}\right)^{-\sigma},\quad \sigma>0.$$

(iv) *Compressible Mooney-Rivlin materials* behave according to the following potential

$$\hat{W}(\mathbf{F})=a\|\mathbf{F}\|^2+b\|cof\mathbf{F}\|^2+\Gamma\left(\det\mathbf{F}\right),$$

$$a>0, b>0,\quad \Gamma(\delta)=c\delta^2-d\log\delta, c>0, d>0.$$

(v) *Hadamard-Green materials* are characterized by

$$\hat{W}(\mathbf{F})=\frac{\alpha}{2}\|\mathbf{F}\|^2+\frac{\beta}{4}\left[\|\mathbf{F}\|^2-\|\mathbf{F}\mathbf{F}^T\|^2\right]+\Gamma\left(\det\mathbf{F}\right),\quad \alpha>0,\beta>0.$$

Example 6.1.2 (transverse isotropy)

There exist five types of transverse isotropy (Spencer, 1987) characterized by the following structural tensors (Zhang and Rychlewski, 1990a and 1990b):

(i) $\quad \mathcal{C}_{\infty},\quad \{\mathbf{P}_m\}=\{\mathbf{e}_3,\mathbf{e}\},$

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(ii) $\mathcal{C}_{\infty v}$, $\{\mathbf{P}_m\} = \{\mathbf{e}_3\}$,

(iii) $\mathcal{C}_{\infty h}$, $\{\mathbf{P}_m\} = \{\mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1\}$,

(iv) $\mathcal{D}_{\infty}$, $\{\mathbf{P}_m\} = \{\mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}\}$,

(v) $\mathcal{D}_{\infty h}$, $\{\mathbf{P}_m\} = \{\mathbf{e}_3 \otimes \mathbf{e}_3\}$,

where $\mathbf{e} = \in_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ $(i, j, k = 1,2,3)$ and $\in_{ijk}$ stands for the permutation symbol. Obviously, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal frame and $\mathbf{e}_3$ is a privileged direction. An alternative set of structural tensors has been recently proposed by Zheng (1993b, 1994a).

Since the parametric tensors (6.22) appear in the scalar-valued function (6.2), which must be positive, therefore the cases (ii), (iv) and (v) lead to identical sets of basic invariants and generators:

$$\begin{aligned} &\{I_n\} = \{tr\mathbf{U}, tr\mathbf{U}^2, tr\mathbf{U}^3, \mathbf{e}_3 \cdot \mathbf{U}\mathbf{e}_3, \mathbf{e}_3 \cdot \mathbf{U}^2\mathbf{e}_3\}, \\ &\{\mathbf{G}_n\} = \{\mathbf{I}, 2\mathbf{U}, 3\mathbf{U}^2, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_3 \otimes \mathbf{U}\mathbf{e}_3 + \mathbf{U}\mathbf{e}_3 \otimes \mathbf{e}_3\}. \end{aligned} \tag{6.23}$$

Similarly, for (i) and (iii) we obtain

$$\begin{aligned} &\{I_n\} = \{tr\mathbf{U}, tr\mathbf{U}^2, tr\mathbf{U}^3, tr\mathbf{U}\mathbf{N}^2, tr\mathbf{U}^2\mathbf{N}^2, tr\mathbf{U}^2\mathbf{N}^2\mathbf{U}\mathbf{N}\}, \\ &\{\mathbf{G}_n\} = \{\mathbf{I}, 2\mathbf{U}, 3\mathbf{U}^2, \mathbf{N}^2, \mathbf{U}\mathbf{N} + \mathbf{N}\mathbf{U}, \mathbf{N}\mathbf{U}^2\mathbf{N}^2 + \mathbf{N}^2\mathbf{U}^2\mathbf{N}\}, \end{aligned} \tag{6.24}$$

where $\mathbf{N} = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$.

In $(6.23)_1$ and $(6.24)_1$, the first three invariants may be replaced by the invariants (6.18) what yields, after substitution into (6.23), the physical relations considered by Spencer (1972, 1980, 1987). This autor investigated particular cases of physical relations for: a) incompressible materials where $\det \mathbf{C} = 1$, b) materials which are incompressible in a privileged direction, $\mathbf{e}_3 \cdot \mathbf{C}\mathbf{e}_3 = 1$ as well as both these cases for so called strongly anisotropic materials (for instance, a metal matrix reinforced with fibres).

Example 6.1.3 : Porous or cellular solids characterized by a symmetric second-order structural tensor (fabric tensor)

In the papers by Cowin (1985, 1986a) the so called *fabric tensor* $\mathbf{M}$ is used, cf. also Harrigan and Mann (1984), Zysset and Curnier (1995). This is a positive-definite, symmetric, second-order tensor, which characterizes porosity. This tensor plays the role of a structural tensor. For the form invariant constitutive relationship

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \mathbf{M}),$$

the set of basic invariants is given by

$$\{K_n\} = \{tr\mathbf{E}, tr\mathbf{E}^2, tr\mathbf{E}^3, tr\mathbf{M}\mathbf{E}, tr\mathbf{M}^2\mathbf{E}, tr\mathbf{M}\mathbf{E}^2, tr\mathbf{M}^2\mathbf{E}^2\} \tag{6.25}$$

whilst the set of generators has the following form

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(6.26) $$\{\mathbf{G}_n\} = \{\mathbf{I},\, 2\mathbf{E},\, 3\mathbf{E}^2,\, \mathbf{M},\, \mathbf{M}^2,\, \mathbf{ME}+\mathbf{EM},\, \mathbf{M}^2\mathbf{E}+\mathbf{M}^2\mathbf{E}\},$$

where **E** is the strain tensor. By taking account of (6.25) and (6.26) in (6.10) we obtain the following constitutive equation

(6.27) $$\mathbf{S} = \alpha_1\mathbf{I} + 2\alpha_2\mathbf{E} + 3\alpha_3\mathbf{E}^2 + \alpha_4\mathbf{M} + \alpha_5\mathbf{M}^2 + \alpha_6(\mathbf{ME}+\mathbf{EM}) + \alpha_7(\mathbf{M}^2\mathbf{E}+\mathbf{EM}^2),$$

where

(6.28) $$\alpha_k = \frac{\partial \breve{W}}{\partial K_k}, \quad k = 1,\ldots,7.$$

The simplest is the linear dependence of **S** on **E** and **M**; then

(6.29) $$\alpha_1 = \lambda tr\mathbf{E} + \beta tr\mathbf{ME}, \quad \alpha_2 = 2\mu, \quad \alpha_3 = \alpha_5 = \alpha_7 = 0, \quad \alpha_4 = \beta tr\mathbf{E}, \quad \alpha_6 = 2\gamma.$$

By comparing Eq.(6.27), into which (6.29) is substituted, with the classical Hooke's law for an orthotropic material in a matricial form

(6.30) $$\underline{\sigma} = \underline{C}\varepsilon,$$

where

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T, \quad \underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]^T,$$

$$\underline{C} = \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0 \\ f_3 & e_2 & f_1 & 0 & 0 & 0 \\ f_2 & f_1 & e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_1 \end{bmatrix},$$

we obtain the relations between 9 classical elastic coefficients and $\lambda, \mu, \beta, \gamma$ and M_ρ cf. also Boehler (1987d)

$$\begin{aligned} e_i &= \lambda + 2\mu + 2(2\gamma + \beta)M_i, \\ f_i &= \lambda + \beta(M_j + M_k), \\ g_i &= 2[\mu + \gamma(M_j + M_k)]. \end{aligned}$$

Here M_i are the eigenvalues of **M** and (i,j,k) = (1,2,3), (2,3,1), (3,1,2).
We observe that Gurtin (1974) provided a simple proof of the representation theorem for isotropic, linear elasticity, cf. also Knowles (1995), Martins and Podio-Guidugli (1978), Pericack-Spector and Spector (1995). Some relations between the moduli for anisotropic

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elastic materials were derived by Hayes (1972). Isotropic tensors of orders up to eight were examined by Kearsley and Fong (1975).

6.2 Perfect plasticity, perfectly locking behaviour

From the formal point of view the general mathematical framework for perfectly plastic and perfectly locking materials is the same, cf. Jemioło and Telega (1992, 1994 and 1997b).

Let us consider a second-order form-invariant tensor-valued function $\hat{\mathbf{A}}$ of a second-order symmetric tensor $\mathbf{B}$ and structural tensors $\mathbf{P}_m \quad (m = 1, \ldots, M)$

(6.31) $$\mathbf{A} = \hat{\mathbf{A}}(\mathbf{B}, \mathbf{P}_m),$$

subject to

(6.32) $$\frac{\partial \hat{\mathbf{A}}}{\partial \mathbf{B}} \cdot \mathbf{B} = \mathbf{0} \quad or \quad \left(\frac{\partial \hat{\mathbf{A}}}{\partial \mathbf{B}} \cdot \mathbf{B} \right)_{ij} = \frac{\partial \hat{A}_{ij}}{\partial B_{kl}} B_{kl}, \quad if \quad \frac{\partial \hat{\mathbf{A}}}{\partial \mathbf{B}} \neq \overline{\mathbf{0}}.$$

Here $\mathbf{A}, \mathbf{B} \in T_s$ and $\mathbf{0}, \overline{\mathbf{0}}$ are zero second- and fourth-order tensors, respectively. The requirement of form-invariance expresses as follows

(6.33) $$\forall \mathbf{Q} \in O, \quad \mathbf{QAQ}^T = \hat{\mathbf{A}}\left(\mathbf{QBQ}^T, \mathbf{Q} \circ \mathbf{P}_m\right).$$

Physically motivated is the condition

(6.34) $$tr\mathbf{AB} > 0 \quad if \quad \mathbf{B} \neq \mathbf{0},$$

which imposes restrictions on material coefficients.

Condition (6.32) means that the tensor function $\hat{\mathbf{A}}$ is a homogeneous function of degree zero with respect to the first argument. Hence $\det\left(\partial \hat{\mathbf{A}} / \partial \mathbf{B}\right) = 0$ and consequently there exists a scalar relation $f(\mathbf{A}, \mathbf{P}_m) = 0$.

We observe that if

(6.35) $$\frac{\partial \hat{A}_{ij}}{\partial B_{kl}} = \frac{\partial \hat{A}_{kl}}{\partial B_{ij}},$$

then (6.31) is a potential (associated) law, cf. (Telega, 1974; Jemioło and Telega, 1992, 1994, 1997b). Then under condition (6.32), the potential $P(\mathbf{B}, \mathbf{P}_m)$ results from (6.34).

Relationship (6.31), satisfying (6.32)-(6.34) models:

(i) perfectly plastic materials provided that $\mathbf{A} = \mathbf{t}$, $\mathbf{B} = \mathbf{d}$, where $\mathbf{t}$ is the Cauchy stress tensor and $\mathbf{d}$ denotes the strain rate tensor, cf. Sawczuk and Stutz (1968), Sawczuk (1982).

(ii) Perfectly locking materials, if $\mathbf{A} = \underline{\varepsilon}$, $\mathbf{B} = \underline{\dot{\sigma}}$, where $\underline{\varepsilon}$ stands for the (small) strain tensor and $\underline{\dot{\sigma}}$ is the stress rate tensor, cf. Jemioło and Telega (1992, 1997b).

Relation (6.33) yields the following form of the representation of the tensor function (6.31)

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where $\beta_n = f_n(I_k)\,(k = 1,\dots,K)$ and I_k are anisotropic basic invariants of $\mathbf{B}$. We recall that the anisotropy group is characterized by the structural tensors $\{\mathbf{P}_m\}$. $\mathbf{G}_n$ are anisotropic generators of $\mathbf{B}$; they are symmetric, second-order tensors. With (6.36) we obtain

(6.37) $$\frac{\partial\hat{\mathbf{A}}}{\partial\mathbf{B}}\cdot\mathbf{B} = \left(\mathbf{G}_n \otimes \frac{\partial\beta_n}{\partial I_k}\frac{\partial I_k}{\partial\mathbf{B}}\right)\cdot\mathbf{B} + \beta_n\frac{\partial\mathbf{G}_n}{\partial\mathbf{B}}\cdot\mathbf{B} = \mathbf{0}.$$

Further we have

(6.38) $$\frac{\partial I_k}{\partial\mathbf{B}}\cdot\mathbf{B} = I_k;\quad k = 1,\dots,N,$$

and

(6.39) $$\frac{\partial\mathbf{G}_n}{\partial\mathbf{B}}\cdot\mathbf{B} = p_{(n)}\mathbf{G}_n \quad (\textit{no summation on } n),$$

where $p_{(n)} > 0$ is a natural number. Taking account of (6.38) and (6.39), Eq. (6.37) takes the form

(6.40) $$I_k\frac{\partial\beta_n}{\partial I_k}\mathbf{G}_n + p_{(n)}\beta_n\mathbf{G}_n = \mathbf{0} \quad (\textit{the summation convention is used}).$$

Because $\mathbf{G}_n \neq \mathbf{0}$, Eq.(6.40) readily yields

(6.41) $$I_k\frac{\partial\beta_n}{\partial I_k} + p_{(n)}\beta_n = 0 \quad (\textit{no summation on } n).$$

It means that β_n is a homogeneous function of degree $(-p_{(n)})$ with respect to $\mathbf{B}$. From the set $\{I_k\}$ we take an invariant, say I such that $I > 0$ for $\mathbf{B} \neq \mathbf{0}$. A general solution of Eq. (6.41) is then given by

(6.42) $$\beta_n = \frac{1}{I^{p_{(n)}}}A_n\left(\frac{I_l}{I}\right);\quad l = 1,\dots,N-1,$$

where $\{I_l\} = \{I_k\}\setminus\{I\}$. Now the functions A_n are *homogeneous of degree zero* with respect to $\mathbf{B}$. Hence (6.36) may be written in the following way

(6.42) $$\mathbf{A} = \hat{\mathbf{A}}(\mathbf{B},\mathbf{P}_m) = \frac{1}{I^{p_{(n)}}}A_n\left(\frac{I_l}{I}\right)\mathbf{G}_n.$$

Next we construct anisotropic invariants $J_k\ (k = 1,\dots,K)$

(6.43) $$J_k = f_k(\mathbf{A},\mathbf{P}_m).$$

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By taking account of (6.43) in (6.44) we may write

$$J_k = \tilde{f}_k\left(\frac{I_l}{I}\right); \quad k = 1,\ldots,K; \quad l = 1,\ldots,N-1. \tag{6.44}$$

Suppose that (N-1) non-dimensional parameters I_l / I can be eliminated from the system (6.45). Then one scalar relations between the invariants J_k exists, say

$$f(J_k) = 0. \tag{6.45}$$

The last relation represents: (i) a plastic yield locus if **A** = **t** and (ii) a locking condition if $\mathbf{A} = \underline{\varepsilon}$.

"Inverse" to (6.43) is the equation

$$\frac{\mathbf{B}}{I} = \lambda\left(\frac{I_l}{I}\right)\gamma_n \mathbf{H}_n, \tag{6.46}$$

where $\gamma_n = f_n(J_k)$ and $\mathbf{H}_n$ are anisotropic generators of **A**. For **B** = **d**, Eq. (6.47) is refered to as the flow law, while for $\mathbf{B} = \dot{\underline{\sigma}}$ we have the locking law. We note that semi-invertibility for isotropic functions was earlier studied by Truesdell and Moon (1975).

If N = K and

$$\gamma_k \mathbf{H}_k = \frac{\partial f}{\partial \mathbf{A}} = \frac{\partial f}{\partial J_k}\frac{\partial J_k}{\partial \mathbf{A}}, \qquad \gamma_k = \frac{\partial f}{\partial J_k}, \quad \mathbf{H}_k = \frac{\partial J_k}{\partial \mathbf{A}}, \tag{6.48}$$

then (6.41) is called the associated flow law $(\mathbf{B} = \mathbf{d})$ or associated locking law $(\mathbf{B} = \dot{\underline{\sigma}})$. Otherwise the laws are referred to as non-associated laws.

Within such a fromework, Sawczuk and Stutz (1968) investigated isotropic materials, cf. also Sawczuk and Telega (1975), Telega (1974, 1978), Jemioło (1991a, 1993a and 1993b), Lanier and Zitouni (1990). We observe that Sawczuk and Stutz (1968) developed an idea due to Thomas (1954). Anisotropic materials were studied by Boehler (1978, 1987e), Aravas (1992), Jemioło (1991b, 1991c, 1994c) Jemioło et al. (1990a, 1990b, 1993), Basista (1985a and 1985b), Zhang (1991a). Geometrically nonlinear effects were examined by Murakami and Sawczuk (1979, 1981), cf. also Atluri (1984), Backhaus (1988), Dashner (1986a, 1986b, 1986c), Duszek (1980), Duszek and Perzyna (1988, 1991), Freudenthal and Gou (1969), John and Bergander (1994), Lehmann (1982, 1985), Lehmann et al. (1985), Loret (1983), Lubarda (1991), Naghdi (1990), Ning and Aifantis (1994), Petryk (1991), Raniecki and Samanta (1989), Raniecki and Bruhns (1991) Sidoroff (1973, 1975) Stumpf and Badur (1990).

Example 6.2.1. (Invariant formulation of Hill's yield condition)
In 1948 Hill proposed the following yield condition for orthotropic, incompressible materials:

$$\begin{aligned} 2f(\sigma) = {} & F_0(\sigma_{22}-\sigma_{33})^2 + G_0(\sigma_{33}-\sigma_{11})^2 + H_0(\sigma_{11}-\sigma_{22})^2 + \\ & + 2L_0\sigma_{23}^2 + 2M_0\sigma_{31}^2 + 2N_0\sigma_{12}^2 = 1. \end{aligned}$$

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Here $\sigma = (\sigma_{ij})(i, j = 1,2,3)$ is the stress tensor in the so called principal axes of orthotropy of the material and the material coefficients $F_0, \ldots, N_0$ are determined in standard tests of tension or compression in each of the principal axis of orthotropy and pure shear in the planes 1-2, 2-3 and 1-3. Substitution of those tests data into Hill's criterion gives:

$$F_0 = \frac{1}{2}\left(\frac{1}{Y_{t2}^2} + \frac{1}{Y_{t3}^2} - \frac{1}{Y_{t1}^2}\right), \quad G_0 = \frac{1}{2}\left(\frac{1}{Y_{t3}^2} + \frac{1}{Y_{t1}^2} - \frac{1}{Y_{t2}^2}\right), \quad H_0 = \frac{1}{2}\left(\frac{1}{Y_{t1}^2} + \frac{1}{Y_{t2}^2} - \frac{1}{Y_{t3}^2}\right),$$

$$L_0 = \frac{1}{2k_{23}^2}, \quad M_0 = \frac{1}{2k_{13}^2}, \quad N_0 = \frac{1}{2k_{12}^2},$$

where Y_{ti} are yield limit in tension in the principal axes of orthotropy $\mathbf{m}_i$ while k_{ij} are yield limit in shear in the orthotropy planes *i-j* with $i \neq j$. By using orthotropic invariants of the stress deviator s, Hill's criterion takes the form:

$$2f(\sigma) - 1 = a_1 tr\mathbf{s}^2 + a_2 tr\mathbf{M}_1\mathbf{s}^2 + a_3 tr\mathbf{M}_2\mathbf{s}^2 + \\ + a_4(tr\mathbf{M}_1\mathbf{s})^2 + a_5(tr\mathbf{M}_2\mathbf{s})^2 + a_6 tr\mathbf{M}_1\mathbf{s}\, tr\mathbf{M}_2\mathbf{s} - 1 = 0,$$

where

$$a_1 = L_0 + M_0 - N_0, \quad a_2 = 2N_0 - 2L_0, \quad a_3 = 2N_0 - 2M_0,$$
$$a_4 = H_0 + 4G_0 + F_0 - 2M_0, \quad a_5 = H_0 + G_0 + 4F_0 - 2L_0,$$
$$a_6 = 2(2G_0 - H_0 + 2F_0 - L_0 - M_0 + N_0).$$

It is evident that according to this criterion the material is insensitive to a hydrostatic pressure.

For transversely isotropic materials Hill's criterion reduces to:

$$2f(\sigma) - 1 = atr\mathbf{s}^2 + btr\mathbf{M}_1\mathbf{s}^2 + c(tr\mathbf{M}_1\mathbf{s})^2 - 1 = 0,$$

where

$$a = -\frac{1}{2Y_{t1}^2} + \frac{2}{Y^2}, \quad b = \frac{1}{k^2} - \frac{4}{Y^2} + \frac{1}{Y_{t1}^2}, \quad c = -\frac{1}{k^2} + \frac{1}{Y^2} + \frac{2}{Y_{t1}^2}.$$

Here k and Y are material coefficients determined in tests performed in the plane of isotropy of the material.

Example 6.2.2. Hoffman (1967) generalized Hill's criterion by adding linear terms in σ_{11}, σ_{22} and σ_{33}:

$$2f(\sigma) = C_1(\sigma_{22} - \sigma_{33})^2 + C_2(\sigma_{33} - \sigma_{11})^2 + C_3(\sigma_{11} - \sigma_{22})^2 + \\ + C_4\sigma_{11} + C_5\sigma_{22} + C_6\sigma_{33} + C_7\sigma_{23}^2 + C_8\sigma_{31}^2 + C_9\sigma_{12}^2 = 1.$$

This author proposed the last condition as a brittle failure criterion. He showed that

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$$C_1=\frac{1}{2}\left(\frac{1}{Y_{t2}Y_{c2}}+\frac{1}{Y_{t3}Y_{c3}}-\frac{1}{Y_{t1}Y_{c1}}\right),\quad C_2=\frac{1}{2}\left(\frac{1}{Y_{t3}Y_{c3}}+\frac{1}{Y_{t1}Y_{c1}}-\frac{1}{Y_{t2}Y_{c2}}\right),$$
$$C_3=\frac{1}{2}\left(\frac{1}{Y_{t1}Y_{c1}}+\frac{1}{Y_{t2}Y_{c2}}-\frac{1}{Y_{t3}Y_{c3}}\right),\quad C_4=\frac{1}{Y_{t1}}-\frac{1}{Y_{c1}},\quad C_5=\frac{1}{Y_{t2}}-\frac{1}{Y_{c2}},\quad C_6=\frac{1}{Y_{t3}}-\frac{1}{Y_{c3}},$$
$$C_7=\frac{1}{2k_{23}^2},\quad C_8=\frac{1}{2k_{13}^2},\quad C_9=\frac{1}{2k_{12}^2},$$

where Y_{ti}, Y_{ci} are the strength limits in tension and compression in the direction $\mathbf{m}_i$ $(i=1,2,3)$ respectively, while k_{ij} denote the strength limits in the planes *i-j* with $i\neq j$. The invariant form of Hoffman's criterion is expressed by:

$$2f(\sigma)-1=a_1tr\mathbf{s}^2+a_2tr\mathbf{M}_1\mathbf{s}^2+a_3tr\mathbf{M}_2\mathbf{s}^2+a_4\left(tr\mathbf{M}_1\mathbf{s}\right)^2+a_5\left(tr\mathbf{M}_2\mathbf{s}\right)^2+$$
$$+a_6tr\mathbf{M}_1\mathbf{s}\,tr\mathbf{M}_2\mathbf{s}+b_1tr\mathbf{M}_1\mathbf{s}+b_2tr\mathbf{M}_2\mathbf{s}+b_3tr\sigma-1=0,$$

where

$$a_1=\frac{1}{2}(C_7+C_8-C_9),\quad a_2=C_9-C_7,\quad a_3=C_9-C_8,$$
$$a_4=C_1+4C_2-C_8+C_3,\quad a_5=C_2-C_7+4C_1+C_3,\quad a_6=C_9+4C_2+4C_1-C_7-C_8-2C_3,$$
$$b_1=C_4-C_6,\quad b_2=C_5-C_6,\quad b_3=\frac{1}{3}(C_4+C_5+C_6).$$

In the case of transverse isotropy Hoffman's criterion reduces to:

$$2f(\sigma)-1=atr\mathbf{s}^2+btr\mathbf{M}_1\mathbf{s}^2+c\left(tr\mathbf{M}_1\mathbf{s}\right)^2+dtr\mathbf{M}_1\mathbf{s}+etr\sigma-1=0,$$

where

$$a=\frac{2}{Y_tY_c}-\frac{1}{2Y_{t1}Y_{c1}},\quad b=\frac{1}{k^2}-\frac{4}{Y_tY_c}+\frac{1}{Y_{t1}Y_{c1}},\quad c=-\frac{1}{k^2}-\frac{1}{Y_tY_c}+\frac{2}{Y_{t1}Y_{c1}},$$
$$d=\frac{1}{Y_{t1}}-\frac{1}{Y_{c1}}-\frac{1}{Y_t}+\frac{1}{Y_c},\quad e=\frac{1}{3Y_{t1}}-\frac{1}{3Y_{c1}}+\frac{2}{3Y_t}-\frac{2}{3Y_c}.$$

For materials with cubic symmetry and different properties in tension and compression we have

$$Y_{t1}=Y_{t2}=Y_{t3}=Y_t,\quad Y_{c1}=Y_{c2}=Y_{c3}=Y_c,\quad k_{12}=k_{13}=k_{23}=k.$$

Consequently the nine constants in Hoffman's criterion reduce to three only:

$$C_1=C_2=C_3=\frac{1}{2Y_tY_c},\quad C_4=C_5=C_6=\frac{1}{Y_t}-\frac{1}{Y_c},\quad C_7=C_8=C_9=\frac{1}{k^2}.$$

The criterion itself takes then the following form

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$$2f(\sigma)-1=\frac{1}{2k^2}tr\mathbf{s}^2+\left(\frac{3}{Y_tY_c}-\frac{1}{k^2}\right)\left[(tr\mathbf{M}_1\mathbf{s})^2+(tr\mathbf{M}_2\mathbf{s})^2+tr\mathbf{M}_1\mathbf{s}\,tr\mathbf{M}_2\mathbf{s}\right]+$$
$$+\left(\frac{1}{Y_t}-\frac{1}{Y_c}\right)tr\sigma-1=0.$$

We conclude that Hoffman's criterion reduces to the Hill condition provided that material properties in tension and compression coincide (in the sense of absolute values). Yield criteria including linear terms and hardening were also discussed by Telega (1984), cf. also Baltov and Sawczuk (1965), Bassani (1977), Bergandner et al. (1992), Betten (1976, 1987d), Billinngton (1984), Boehler (1985), Boehler and Sawczuk (1970, 1974, 1976), Darrieulat et al (1992, 1996), Ferron et al (1994), Horz et al (1994), Jemioło (1996), Kurtyka (1985), Litewka (1977), Olesiak and Węgrowska (1985), Pipkin and Rakotomanana et al (1991), Rivlin (1965), Shrivastava et al (1973a, 1973b), Szczepiński (1993), Tanaka and Miyagawa (1975).

6.3. Further bibliographical comments on applications

Invariants and tensor functions are indispensable for rational formulation of constitutive relations to which fluids and solid obey, cf. Boehler (1987b). We have already mentioned many applications in the solid mechanics; fluids are dealt with in the next section, cf. also Skwarczyński (1996).

Materials with memory are described by form invariant functionals, cf. Green and Rivlin (1957), Spencer (1971).

A simple continuum model of rigid-perfectly plastic fibre-reinforced materials was formulated by Mulhern et al. (1967). Further developments of this theory were given by Spencer (1972). An elastic-perfectly plastic continuum model of fibre-reinforced materials was proposed by Mulhern et al. (1969). Shaw and Spencer (1978) used a strain-hardening rigid-plastic theory for fibre-reinforced plates, and Spencer (1993) has briefly discussed elastic-plastic strain-hardening fibre-reinforced materials.

Ostrowska-Maciejewska and Harris (1990) proposed the constitutive equation for a granular material that is a tensor function relating the strain-rate tensor to the Cauchy stress tensor and to the co-rotational rate of the Cauchy stress, cf. also Harris (1992, 1993).

Yield functions for anisotropic materials with hardening were examined by Dafalias (1979), Mróz and Jemioło (1991), Mróz and Rodzik (1996), Rees (1981, 1982, 1983a, 1983b, 1993), Schreyer and Zuo (1995), cf. also Backhaus (1988), Karafillis and Boyce (1993).

Perforated plastic materials were studied by Litewka and Sawczuk (1982) and Markov (1993).

Arminjon et al. (1994) proposed a fourth-order plastic potential, being a fourth-order polynomial in strain-rate, from which a constitutive equation for anisotropic metals can be derived, cf. also Gotoh (1977).

Some creep problems are discussed in Anisimowicz et al. (1982), Betten and Waniewski (1989), Jakowluk (1993), Litewka (1989), Murakami and Ohno (1980).

Sobotka (1975, 1976, 1984, 1992, 1993) applied tensor functions to the formulation of viscoelastic and viscoplastic constitutive equations, cf. also Perzyna (1966, 1978)

The papers by Betten (1986, 1987b, 1992) offer a good account of possibilities of application of invariants and tensor functions to damage mechanics, cf. also Basista (1984), He and Curnier (1995), Lam and Zhang (1995), Litewka (1985), Litewka and Moszyńska (1985, 1987), Schreyer (1995).

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Failure criterions for composites were reviewed and compared by Theocaris (1992), cf. also Betten (1993), Jemioło (1996), Theocaris (1987, 1994), Theocaris and Philippidis (1989), Tsai and Wu (1971).

Porous media were investigated by Kubik (1982), Kubik and Mielniczuk (1985) and Wilmański (1996), cf. also De Boer (1996).

The monograph by Kiral and Eringen (1990) deals with the material symmetry regulations arising from the crystallographic symmetry of magnetic crystals. Unfortunately, the authors apply componentwise notation which renders the constitutive equations lengthy and not always easy for handling.

Zheng and He (1997) have proved that there are only fifteen symmetry classes for piezoelectric tensors, and not sixteen as believed earlier, cf. also Dieulesaint and Royer (1974).

Micropolar media were studied by Kafadar and Eringen (1971).

The purpose of Hoger's (1991, 1993a, 1993b, 1993c, 1996) papers was to propose a method for determining the dependence of the elasticity tensor on residual stress for an elastic material with known material symmetry, cf. also Johnson and Hoger (1993). *Residual stress* is the stress present in a body in an unloaded equilibrium configuration; so the residual stress is symmetric, satisfies the equilibrium equations with zero body force, and the associated traction on the boundary of the body vanishes. This zero traction condition causes the residual stress to be dependent on body geometry. Since the residual stress field is necessarily inhomogeneous, the response of a body that supports a residual stress is also inhomogeneous, cf. Hoger (1991), Johnson and Hoger (1993). Thus, although the constitutive equation of a residually stressed body may have the same form at every point in the body, the residual stress must be a function of position and the material parameters may vary with position as well. The stress constitutive equation for the finite elastic behaviour was obtained from the derivative of the strain energy function. The strain energy function was expressed as a function of the set of basic polynomial invariants of the strain measure and the residual stress appropriate to the given material symmetry. This set of invariants forms an integrity basis, and therefore a functional basis, and thus ensures that the representation of the constitutive equation provides the canonical form. Specifically, the general form for the constitutive equation appropriate for a hyperelastic transversely isotropic material was derived, cf. Hoger (1993c, 1996). The approach used by Hoger (1993c, 1996) can also be applied for any other symmetry, where an integrity basis is known.

In the papers by He and Curnier (1993) and Zmitrowicz (1989, 1993) invariants and tensor functions were consequently used in the formulation of anisotropic friction laws, cf. also Telega (1988). We observe that He and Curnier (1993) applied the formalism of Sawczuk and Stutz (1968) to constitutive modelling of the friction phenomenon.

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7. Simple fluids and the unimodular group

Let $U(n)$ be the full real unimodular group consisting of all linear transformations $\mathbf{H} \in GL(n)$ such that $|\det \mathbf{H}| = 1$. Brauer (1965) and Noll (1965) established that the full orthogonal group $O_n = O(n)$ is a maximal subgroup of $U_n = U(n)$. Similarly the proper orthogonal group O_n^o is a maximal subgroup of the proper unimodular group U_n^o consisting of the elements of O_n and U_n respectively of determinant +1. In the case n=3, the subscript n will be omitted, thus $U^o = U_3^o$ and $O^o = O_3^o$, etc.

7.1 The orthogonal group as the maximal subgroup of the unimodular grup

In this subsection we shall present both Noll's (1965) and Brauer's (1965) approaches to the proof of the following

Theorem 7.1. The group O_n is a maximal subgroup of U_n.

(i) Noll's proof exploits the polar decomposition theorem: every invertible linear transformation is the product of an orthogonal transformation and one that is symmetric and positive definite, cf. Gurtin (1981). Consequently every group G containing O_n is generated by its positive definite and symmetric elements. Hence we deduce that it is sufficient to prove the following

Proposition 7.2. Assume that $\mathbf{S} \in U_n$ is positive definite and symmetric and has at least two distinct eigenvalues s and t with $s>t$. Then every positive definite and symmetric $\mathbf{H} \in U_n$ belongs to the group G generated by O_n and $\mathbf{S}$.

The proof of this statement is based on

Lemma 7.3. For every ξ such that

$$\left(\frac{t}{s}\right)^2 \le \xi \le \left(\frac{s}{t}\right)^2, \tag{7.1}$$

there exists a $\mathbf{Q} \in O_n$ such that the symmetric transformation

$$\mathbf{T} = \mathbf{S}^{-1}\mathbf{Q}\mathbf{S}^2\mathbf{Q}^{-1}\mathbf{S}^{-1}, \tag{7.2}$$

has the eigenvalues $\left(\xi, \xi^{-1}, 1, \ldots, 1\right)$. ∇

By using this lemma we are now in a position to prove Prop.7.2.

Let $\{\mathbf{e}_i\}_{1\le i\le n}$ be an orthonormal basis of proper vectors of $\mathbf{H}$, with corresponding eigenvalues h_i $(1 \le i \le n)$. Let $\mathbf{H}_k$ be the symmetric tensor that has the same eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ as $\mathbf{H}$, but with the corresponding eigenvalues $1, \ldots, 1, \eta_k, \eta_k^{-1}, 1, \ldots, 1$, where η_k corresponds to $\mathbf{e}_k$ and η_k^{-1} to $\mathbf{e}_{k+1}$. Suppose that η_k are chosen in the following way:

$\eta_1 = h_1$, $\eta_k = \eta_{k-1} h_k$, $\quad k = 1, 2, \ldots, n-1$.

By noting that $h_1 h_2 \ldots h_n = 1$, we conclude

$$\mathbf{H} = \mathbf{H}_1 \mathbf{H}_2 \ldots \mathbf{H}_{n-1}. \tag{7.3}$$

Let us now fix k and choose m possibly large enough such that $\xi = \sqrt[m]{\eta_k}$ satisfies (7.1). We then determine $\mathbf{T}$ according to (7.2). Let $\overline{\mathbf{g}}$ and $\overline{\mathbf{f}}$ be unit proper vectors of $\mathbf{T}$ which correspond to the eigenvalues ξ and ξ^{-1}, respectively. We can find an orthogonal

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transformation $\tilde{\mathbf{Q}}$ which maps the basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ onto an orthonormal basis of proper vectors of $\mathbf{T}$ in such a way that $\tilde{\mathbf{Q}}\mathbf{e}_k = \bar{\mathbf{g}}$, $\tilde{\mathbf{Q}}\mathbf{e}_{k+1} = \bar{\mathbf{f}}$.
With ξ, $\mathbf{T}$ and $\tilde{\mathbf{Q}} \in O_n$ chosen in this way, we have

$$\text{(7.4)} \qquad \mathbf{H}_k = \tilde{\mathbf{Q}}^{-1}\mathbf{T}^m\tilde{\mathbf{Q}}.$$

It is evident that $\mathbf{T}$, $\tilde{\mathbf{Q}}$ and m depend on k.
By substituting (7.2) into (7.4) and the result into (7.3), and doing so for each of the $\mathbf{H}_k$, we conclude that $\mathbf{H}$ is generated by $\mathbf{S}$ and orthogonal transformations.
(ii) Brauer (1965) proof is different. This author first shows that $SO_n = O_n^o$ is a maximal subgroup of U_n^o. The proof is based on Schur's Lemma and induction on dimension n. The final step in the proof of Th.7.1 is then rather easy. Let G be a subgroup of the full unimodular group U_n which includes O_n. Then $G^o = G \cap U_n^o$ is a subgroup of U_n^o which includes SO_n. Thus we have

$$G^o = SO_n \quad or \quad G^o = U_n^o.$$

Now O_n contains elements $\mathbf{Q}$ of determinant (-1) and any such element jointly with U_n^o generates U_n. If $G = U_n^o$, it follows from $\mathbf{Q} \in G$ that $G = U_n$. If $G = SO_n$, let $\mathbf{Q}_1$ be any element of G not in G^o. Then $\mathbf{Q}_1\mathbf{Q}^{-1}$ has determinant 1, and hence $\mathbf{Q}_1\mathbf{Q}^{-1} \in G \cap U_n^o = SO_n$. It follows that $\mathbf{Q}_1 \in O_n$. Thus $G = O_n$.

7.2. Simple fluids

Those fluids constitute a subclass of simple materials, cf. Noll (1972), Truesdell and Noll (1965). We shall now study this subclass more closely, cf. Fahy and Smith (1980). Let $\underline{\sigma}(X,t)$ denote the stress tensor at a particle identified with a point X at time t. A simple material is a material for which $\underline{\sigma}(X,t)$ is a functional of the history of the deformation gradient from time $\tau = 0$ until $\tau = t$ measured with respect to a fixed reference frame x:

$$\text{(7.5)} \qquad \underline{\sigma}(X,t) = \underline{F}(\mathbf{F}(X,t)), \quad F_{iA}(X,t) = \frac{\partial x_i(X_B,\tau)}{\partial X_A}.$$

Here $x_i(X_B,\tau)$ are the coordinates of a particle at time τ in the rectangular Cartesian coordinate system x. The constitutive relationship (7.5) must satisfy the requirement of invariance under a superposed rotation, hence

$$\text{(7.6)} \qquad \mathbf{Q}(t)\underline{F}(\mathbf{F}(X,\tau))\mathbf{Q}^T(t) = \underline{F}(\mathbf{Q}(\tau)\mathbf{F}(X,\tau)),$$

holds for all time-dependent matrices $\mathbf{Q}(\tau) \in O^o$, where $\mathbf{Q}(0) = \mathbf{I}$.

A *simple fluid* is a simple material for which the functional $\underline{F}(\mathbf{F}(X,\tau))$ satisfies the condition

$$\text{(7.7)} \qquad \underline{F}(\mathbf{F}(X,\tau)) = \underline{F}(\mathbf{F}(X,\tau)\mathbf{H}),$$

for each $\mathbf{H} \in U^o$. We recall that for $\mathbf{H} \in U^o$, $\det \mathbf{H} = 1$.

A functional $\underline{F}(\mathbf{F}(X,\tau))$ which satisfies (7.6) and (7.7) is given by

$$\text{(7.8)} \; \underline{F}(\mathbf{F}(X,\tau)) = \mathbf{F}(X,t)\underline{P}[\mathbf{C}(X,\tau)]\mathbf{F}^T(X,t) = \mathbf{F}(X,t)\mathbf{H}\underline{P}[\mathbf{H}^T\mathbf{C}(X,\tau)\mathbf{H}]\mathbf{H}^T\mathbf{F}^T(X,t).$$

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The last relationship must obviously hold for all $\mathbf{H} \in U^o$. It is well known that $\det \mathbf{F}(X,t) > 0$; consequently the functional $\underline{P}[\mathbf{C}(X,\tau)]$ must satisfy

$$\underline{P}[\mathbf{C}(X,\tau)] = \mathbf{H}\underline{P}[\mathbf{H}^T\mathbf{C}(X,\tau)\mathbf{H}]\mathbf{H}^T, \tag{7.9}$$

for all $\mathbf{H} \in U^o$. Such a functional is referred to as a *unimodular functional*. We observe that a symmetric matrix-valued function $\mathbf{P}(\mathbf{A}_1,\ldots,\mathbf{A}_N)$ of the symmetric 3×3 matrices $\mathbf{A}_1,\ldots,\mathbf{A}_N$ is said to be form-invariant under U^o if

$$\mathbf{P}(\mathbf{A}_1,\ldots,\mathbf{A}_N) = \mathbf{H}\mathbf{P}(\mathbf{H}^T\mathbf{A}_1\mathbf{H},\ldots,\mathbf{H}^T\mathbf{A}_N\mathbf{H})\mathbf{H}^T, \tag{7.10}$$

holds for each $\mathbf{H} \in U^o$ and is referred to as a *unimodular function*.

Before proceeding to the study of the functional $\underline{P}[\mathbf{C}(X,\tau)]$ which satisfies (7.9) we shall recall some results from the theory of invariants in the case of the proper unimodular group U^o, see Fahy and Smith (1980) and the references cited therein.

A scalar-valued function $\Theta(\mathbf{A}_1,\ldots,\mathbf{A}_N)$ of the symmetric 3×3 matrices $\mathbf{A}_1,\ldots,\mathbf{A}_N$ is said to be invariant under the three dimensions proper unimodular group U^o if

$$\Theta(\mathbf{A}_1,\ldots,\mathbf{A}_N) = \Theta(\mathbf{H}^T\mathbf{A}_1\mathbf{H},\ldots,\mathbf{H}^T\mathbf{A}_N\mathbf{H}), \tag{7.11}$$

for all $\mathbf{H} \in U^o$. Prior to listing the integrity basis for polynomial functions of five symmetric matrices $\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_5$, which are invariant under U^o, we introduce the following notation:

$$(\mathbf{A}\times\mathbf{B})_{ij} = \in_{ipq}\in_{jrs} A_{pr}B_{qs}, \tag{7.12}$$

where $\in_{ijk}$ denotes the alternating symbol: $\in_{123}=\in_{231}=\in_{312}=1$, $\in_{132}=\in_{321}=\in_{213}=-1$, $\in_{ipq}=0$ unless i, p and q are all different.

Integrity basis for unimodular invariants of $\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_5$

(7.13)

$$tr\mathbf{A}_i(\mathbf{A}_j\times\mathbf{A}_k), \quad (i\le j\le k);$$

$$tr(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_j(\mathbf{A}_k\times\mathbf{A}_k)\mathbf{A}_j, \quad (i<j<k);$$

$$tr(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_j(\mathbf{A}_k\times\mathbf{A}_k)\mathbf{A}_l, \quad (i,j,k,l \text{ are all different}; i<k, j<l);$$

$$tr(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_l(\mathbf{A}_j\times\mathbf{A}_k)\mathbf{A}_m, \quad tr(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_k(\mathbf{A}_j\times\mathbf{A}_l)\mathbf{A}_m,$$
$$(i,j,k,l \text{ are all different}; j<k<l<m);$$

$$tr(\mathbf{A}_i\times\mathbf{A}_j)\mathbf{A}_l(\mathbf{A}_j\times\mathbf{A}_j)\mathbf{A}_m(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_k, \quad (i,j,k,l,m \text{ are all different}; i<j; k<l<m);$$

$$tr(\mathbf{A}_i\times\mathbf{A}_i)\mathbf{A}_m(\mathbf{A}_k\times\mathbf{A}_k)\mathbf{A}_i(\mathbf{A}_j\times\mathbf{A}_j)\mathbf{A}_l, \quad (i,j,k,l,m \text{ are all different}; j<k; l<m);$$

$$tr(\mathbf{A}_j\times\mathbf{A}_j)\mathbf{A}_i(\mathbf{A}_k\times\mathbf{A}_k)\mathbf{A}_m(\mathbf{A}_l\times\mathbf{A}_l)\mathbf{A}_i,$$

$$tr(\mathbf{A}_k\times\mathbf{A}_k)\mathbf{A}_i(\mathbf{A}_j\times\mathbf{A}_j)\mathbf{A}_m(\mathbf{A}_l\times\mathbf{A}_l)\mathbf{A}_i, \quad (i<j<k<l<m).$$

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The subscripts i,j,k,l,m each independently take on the values 1,2,3,4,5 subject to the restrictions listed above. There is no summation on the repeated indices.

By using Peano's theorem (Grace and Young, 1903, p.358), an integrity basis for polynomial functions of N symmetric matrices $\mathbf{A}_1,\ldots,\mathbf{A}_N$ which are invariant under U^o, is formed by the invariants obtained by complete polarization of the listed unimodular invariants, together with the invariants obtained by complete polarization of the invariant

$$(7.14)\quad J(\mathbf{A}_1,\ldots,\mathbf{A}_6)=\left(\in_{i_1i_2i_3}\in_{j_1i_4i_5}\in_{j_2j_4i_6}\in_{j_3j_5j_6}-\in_{i_1i_2i_4}\in_{j_1i_3i_5}\in_{j_2j_3i_6}\in_{j_4j_5j_6}\right)A^{(1)}_{i_1j_1}A^{(2)}_{i_2j_2}A^{(3)}_{i_3j_3}A^{(4)}_{i_4j_4}A^{(5)}_{i_5j_5}A^{(6)}_{i_6j_6}.$$

For instance, we note that the set of invariants given by

$$(\mathbf{A}_i\partial/\partial\mathbf{A}_1)(\mathbf{A}_j\partial/\partial\mathbf{A}_2)(\mathbf{A}_k\partial/\partial\mathbf{A}_3)tr\mathbf{A}_1(\mathbf{A}_2\times\mathbf{A}_3),$$

where i,j and k each independently take on the values $1,\ldots,N$ is referred to as the set of invariants obtained from $tr\mathbf{A}_1(\mathbf{A}_2\times\mathbf{A}_3)$ by the process of complete polarization.

Let us consider the specific case of (7.11), where **P** is a polynomial function of three symmetric matrices $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3$; then **P** satisfies

$$(7.15)\qquad \mathbf{P}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)=\mathbf{HP}(\mathbf{H}^T\mathbf{A}_1\mathbf{H},\mathbf{H}^T\mathbf{A}_2\mathbf{H},\mathbf{H}^T\mathbf{A}_3\mathbf{H})\mathbf{H}^T,$$

for all $\mathbf{H}\in U^o$. The general expression for **P** has the form

$$(7.16)\qquad \mathbf{P}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)=\sum_{i=1}^{9}\alpha_i(\mathbf{L}_i+\mathbf{L}_i^T),$$

where

$$(7.17)\qquad \begin{aligned}\mathbf{L}_1,\ldots,\mathbf{L}_9=\ &\mathbf{A}_1\times\mathbf{A}_1,\ \mathbf{A}_1\times\mathbf{A}_2,\ \mathbf{A}_1\times\mathbf{A}_3,\ \mathbf{A}_2\times\mathbf{A}_2,\ \mathbf{A}_2\times\mathbf{A}_3,\ \mathbf{A}_3\times\mathbf{A}_3,\\ &(\mathbf{A}_1\times\mathbf{A}_1)\mathbf{A}_2(\mathbf{A}_3\times\mathbf{A}_3),(\mathbf{A}_1\times\mathbf{A}_1)\mathbf{A}_3(\mathbf{A}_2\times\mathbf{A}_2),(\mathbf{A}_2\times\mathbf{A}_2)\mathbf{A}_1(\mathbf{A}_3\times\mathbf{A}_3),\end{aligned}$$

and where the $\alpha_i\ (i=1,\ldots,9)$ are polynomial functions of the unimodular invariants, cf. (7.13).

$$(7.18)\quad tr\mathbf{A}_i(\mathbf{A}_j\times\mathbf{A}_k),\quad (i,j,k=1,2,3;\ i\le j\le k);\quad tr(\mathbf{A}_1\times\mathbf{A}_1)\mathbf{A}_2(\mathbf{A}_3\times\mathbf{A}_3)\mathbf{A}_2.$$

We recall the well known formula

$$tr\mathbf{A}_i(\mathbf{A}_i\times\mathbf{A}_i)=6\det\mathbf{A}_i.$$

Now we are in a position to consider the special case where the unimodular functional $\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]$ is approximated by a unimodular function $\mathbf{P}[\mathbf{C}(X,\tau)]$. Suppose that τ is close to t. Then the functional $\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]$ may be approximated by a unimodular function $\mathbf{P}(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}})$, where

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$$\dot{\mathbf{C}}(X,\tau)=\frac{\partial\mathbf{C}(X,\tau)}{\partial\tau},\quad \ddot{\mathbf{C}}(X,\tau)=\frac{\partial^2\mathbf{C}(X,\tau)}{\partial\tau^2},$$
$$\mathbf{C}=\mathbf{C}(X,\tau),\quad \dot{\mathbf{C}}=\dot{\mathbf{C}}(X,\tau),\quad \ddot{\mathbf{C}}=\ddot{\mathbf{C}}(X,\tau).$$

On account of (7.8) and (7.17)-(7.18) we have

$$\text{(7.19)}\qquad \mathbf{FP}(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}})\mathbf{F}^T=\sum_{i=1}^{9}\alpha_i\mathbf{F}(\mathbf{L}_i+\mathbf{L}_i^T)\mathbf{F}^T,\quad \mathbf{F}=\mathbf{F}(X,\tau),$$

where $\mathbf{L}_i=\mathbf{L}_i(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}})$ are given by

$$\text{(7.20)}\qquad \begin{aligned}\mathbf{L}_1,\ldots,\mathbf{L}_9=\;&\mathbf{C}\times\mathbf{C},\mathbf{C}\times\dot{\mathbf{C}},\mathbf{C}\times\ddot{\mathbf{C}},\dot{\mathbf{C}}\times\dot{\mathbf{C}},\dot{\mathbf{C}}\times\ddot{\mathbf{C}},\ddot{\mathbf{C}}\times\ddot{\mathbf{C}},\\&(\mathbf{C}\times\mathbf{C})\dot{\mathbf{C}}(\ddot{\mathbf{C}}\times\ddot{\mathbf{C}}),(\mathbf{C}\times\mathbf{C})\ddot{\mathbf{C}}(\dot{\mathbf{C}}\times\dot{\mathbf{C}}),(\dot{\mathbf{C}}\times\dot{\mathbf{C}})\mathbf{C}(\ddot{\mathbf{C}}\times\ddot{\mathbf{C}}).\end{aligned}$$

Here the $\alpha_i\ (i=1,\ldots,9)$ are polynomial functions of the unimodular invariants $I_j(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}})\ (j=1,\ldots,11)$ defined by

$$\begin{aligned}&I_1(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}}),\ldots,I_{11}(\mathbf{C},\dot{\mathbf{C}},\ddot{\mathbf{C}})=\det\mathbf{C},\det\dot{\mathbf{C}},\det\ddot{\mathbf{C}},tr\mathbf{C}(\mathbf{C}\times\dot{\mathbf{C}}),tr\mathbf{C}(\mathbf{C}\times\ddot{\mathbf{C}}),\\&tr\mathbf{C}(\dot{\mathbf{C}}\times\dot{\mathbf{C}}),tr\mathbf{C}(\dot{\mathbf{C}}\times\ddot{\mathbf{C}}),tr\mathbf{C}(\ddot{\mathbf{C}}\times\ddot{\mathbf{C}}),tr\dot{\mathbf{C}}(\dot{\mathbf{C}}\times\ddot{\mathbf{C}}),tr\dot{\mathbf{C}}(\ddot{\mathbf{C}}\times\ddot{\mathbf{C}}),tr(\mathbf{C}\times\mathbf{C})\dot{\mathbf{C}}(\ddot{\mathbf{C}}\times\ddot{\mathbf{C}})\dot{\mathbf{C}}.\end{aligned}$$

Fahy and Smith (1980) derived also the Eulerian form of (7.20). Moreover, by employing the procedure outlined by Wineman and Pipkin (1964) (see also Spencer (1971)) the general expression for the functional $\mathbf{F}\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]\mathbf{F}^T$ was proposed which satisfies

$$\text{(7.21)}\qquad \mathbf{F}\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]\mathbf{F}^T=\mathbf{FH}\underline{\mathrm{P}}[\mathbf{H}^T\mathbf{C}(X,\tau)\mathbf{H}]\mathbf{H}^T\mathbf{F}^T,\quad \mathbf{F}=\mathbf{F}(X,\tau),$$

for all $\mathbf{H}\in U^o$. The first step is similar as in the case of the determination of tensor functions: both sides of (7.21) are multiplied by an arbitrary symmetric matrix $\mathbf{M}$; by taking the trace we get

$$tr\mathbf{K}\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]=tr\mathbf{H}^T\mathbf{KH}\,\underline{\mathrm{P}}[\mathbf{H}^T\mathbf{C}(X,\tau)\mathbf{H}],\quad \mathbf{K}=\mathbf{F}^T\mathbf{MF}.$$

Thus $tr\mathbf{K}\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]$ is a functional of the history $\mathbf{C}(X,\tau)$ and $\mathbf{K}$ which is linear in $\mathbf{K}$ and invariant under the proper unimodular group. Finally, one arrives at the following representation

$$\mathbf{F}\underline{\mathrm{P}}[\mathbf{C}(X,\tau)]\mathbf{F}^T=\sum_{\beta=1}^{6}\underline{\mathrm{P}}^{\beta}\left(\mathbf{F}(\mathbf{N}_\beta+\mathbf{N}_\beta^T)\mathbf{F}^T,I_1,I_2,I_3,I_4\right),$$

where $\underline{\mathrm{P}}^{\beta}$ are functionals linear in $\mathbf{F}(\mathbf{N}_\beta+\mathbf{N}_\beta^T)\mathbf{F}^T$, and $I_1,\ldots,I_4$ are basic unimodular invariants. Moreover $\mathbf{N}_\beta\ (\beta=1,\ldots,6)$ are given by

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$$\mathbf{N}_1 = \mathbf{C}_1 \times \mathbf{C}_2, \quad \mathbf{N}_2 = (\mathbf{C}_1 \times \mathbf{C}_2)\mathbf{C}_3(\mathbf{C}_4 \times \mathbf{C}_5), \quad \mathbf{N}_3 = (\mathbf{C}_1(\mathbf{C}_2 \times \mathbf{C}_3)\mathbf{C}_4) \times \mathbf{C}_5,$$

$$\mathbf{N}_4 = \left[\left(\in_{ii_1i_2} \in_{ji_3i_4} \in_{j_1j_3i_5} \in_{j_2j_4j_5} - \in_{ii_1i_3} \in_{ji_2i_4} \in_{j_1j_2i_5} \in_{j_3j_4j_5}\right) C^{(1)}_{i_1j_1} C^{(2)}_{i_2j_2} C^{(3)}_{i_3j_3} C^{(4)}_{i_4j_4} C^{(5)}_{i_5j_5}\right],$$

$$\mathbf{N}_5 = (\mathbf{C}_1 \times \mathbf{C}_2)\mathbf{C}_3(\mathbf{C}_4 \times \mathbf{C}_5)\mathbf{C}_6(\mathbf{C}_7 \times \mathbf{C}_8), \quad \mathbf{N}_6 = (\mathbf{C}_1(\mathbf{C}_2 \times \mathbf{C}_3)\mathbf{C}_4(\mathbf{C}_5 \times \mathbf{C}_6)\mathbf{C}_7) \times \mathbf{C}_8.$$

Here the quantities $C^{(\alpha)}_{ij}$ appearing in the expression defining $\mathbf{N}_4$ are the components of the matrix $\mathbf{C}_{(\alpha)} = \mathbf{C}(X, \tau_\alpha)$.

7.3. Anisotropic fluids

Serrin (1959) derived a general form of the constitutive relations for isotropic fluids, cf. Eq. (4.41) in Sec.4.6. Formally, this representation is the same both for polynomial and nonpolynomial representations. Recall that such a representation has also been used by Sawczuk and Stutz (1968) for studying the general form of constitutive relations for isotropic perfectly plastic materials, cf. also Section 6.2 of the present paper. The same representation has been used by Jemioło and Telega (1992 and 1994) in the study of perfectly locking materials, see also Sec.6.2.

The theory of tensor functions suggests how to derive constitutive relations for anisotropic fluids: it is sufficient to introduce structural tensors, see Sec.2. Thus one can consider constitutive relations of the following form

$$\mathbf{t} = \hat{\mathbf{t}}(\mathbf{D}, \underline{\xi}_1, \ldots, \underline{\xi}_s), \tag{7.22}$$

where $\mathbf{t}$ is the stress tensor, $\mathbf{D}$ is the strain rate tensor and $\underline{\xi}_1, \ldots, \underline{\xi}_s$ are structural tensors, cf. also Remark 7.1 below. Particular forms of (7.22) can then be derived by applying the available representations of the tensor function $\hat{\mathbf{t}}$.

Liquid crystals are rather simple anisotropic fluids having a single preferred direction at each point, cf. Rymarz (1993).

7.3.1. Ericksen (1960a 1960b) introduced a simple properly invariant theory of transversely isotropic fluids. In the simplest theory this author assumes that the stress tensor is a function of the velocity gradient of the fluid at time t and also of a vector $\mathbf{n}$. Suppose that the fluid is incompressible and let $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$, $\mathbf{L} = grad\mathbf{v}$, $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$, where $\mathbf{v}$ stands for the velocity vector. By $\mathbf{t} = \mathbf{s} + p\mathbf{I}$ we denote the extra stress referred to fixed rectangular Cartesian coordinates. Using established invariance principles (balance of mass, etc.) it is shown that

$$\mathbf{s} = -p\mathbf{I} + \alpha_1\mathbf{N} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2 + \alpha_4(\mathbf{ND} + \mathbf{DN}) + \alpha_5(\mathbf{ND}^2 + \mathbf{D}^2\mathbf{N}). \tag{7.23}$$

Obviously, p is a scalar function of coordinates and $\alpha_1, \ldots, \alpha_5$ are polynomials in the invariants

$$tr\mathbf{N}, tr\mathbf{ND}, tr\mathbf{ND}^2, tr\mathbf{D}, tr\mathbf{D}^3. \tag{7.24}$$

Both Ericksen (1960a,1960b) and Green (1964a) consider only polynomial representations. It is now evident that non-polynomial representations can also be used.

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In order to complete his theory Ericksen (1960a, 1960b) assumes that the time derivative $\dot{\mathbf{n}}$ of $\mathbf{n}$ depends on $\mathbf{n}$ and $\mathbf{L}$. Then, by applying standard invariance principles one obtains

$$(7.25)\quad \dot{n}_i - w_{ij}n_j = \beta_1\left(D_{ij}n_j - D_{km}n_k n_m n_i\right) + \beta_2\left(D_{ik}D_{kj}n_j - D_{km}D_{mp}n_k n_p n_i\right),$$

provided that $\|\mathbf{n}\| = n_i n_i = 1$. Here $\mathbf{w}$ denotes the vorticity and β_1, β_2 are polynomials in the invariants (7.24). Under the assumption of linearity in $\mathbf{D}$, Eqs (7.23),(7.25) reduce to

$$(7.26)\quad \begin{aligned} &\mathbf{s} = -p\mathbf{I} + 2\mu\mathbf{D} + \left[\mu_1 + \mu_2 tr(\mathbf{DN})\right]\mathbf{N} + 2\mu_3(\mathbf{ND} + \mathbf{DN}), \\ &\dot{n}_i - w_{ij}n_j = \lambda\left(D_{ij}n_j - D_{km}n_k n_m n_i\right), \end{aligned}$$

where μ, μ_1, μ_2, μ_3 and λ are constants.

According to Green (1964a), Ericksen's postulate (7.25) does not seem to lie within the framework of established ideas in continuum mechanics. Green claims that this postulate is nevertheless quite reasonable. In his paper, Green (1964a) proposed a theory of anisotropic fluids which do not introduces additional assumptions of type (7.25). Green's approach is based on the paper by Noll (1955). We shall now briefly present Green's results.

Referred to rectangular Cartesian coordinates the position of a typical particle of fluid at time t is denoted by x_i $(or\ x^i)$ where

$$(7.27)\quad \mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t).$$

It is clear that $X_i(=X^i)$ is the position of the particle at a given time t. Green requires a more general reference position which is defined by a general curvilinear system of coordinates θ_i $(or\ \theta^i)$ where

$$(7.28)\quad \theta_i = \theta_i(X_1, X_2, X_3).$$

We assume that

$$(7.29)\quad \det\left[\frac{\partial x_i}{\partial X_j}\right] > 0, \quad \det\left[\frac{\partial X^i}{\partial \theta^j}\right] > 0.$$

The velocity of the fluid at time t is

$$(7.30)\quad v_i = \dot{x}_i,$$

where $\dot{x}_i$ is the material derivative. The tensor or matrix of velocity gradients is denoted by $grad\mathbf{v} = \left[\frac{\partial v_i}{\partial x_j}\right]$. By virtue of (7.29) we have

$$(7.31)\quad \det\left[\frac{\partial x^i}{\partial \theta^j}\right] > 0.$$

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Therefore, by using the polar decomposition theorem we write

$$\left[\frac{\partial x^i}{\partial \theta^j}\right] = \mathbf{RU}, \tag{7.32}$$

where $\mathbf{RR}^T = \mathbf{I}$, $\det \mathbf{R} = 1$ and $\mathbf{U}$ is a positive definite symmetric matrix. As previously, the symmetric stress tensor or matrix at the position x_i at time t is denoted by $\mathbf{s} = \left[s_{ij}\right]$. Confining the investigation to incompressible fluids and assuming that the extra stress $\mathbf{s} + p\mathbf{I}$ is a tensor function of $\mathrm{grad}\mathbf{v}$ and the rotation matrix $\mathbf{R}$ we have

$$\mathbf{s} + p\mathbf{I} = \Phi(grad\mathbf{v}, \mathbf{R}). \tag{7.33}$$

The principle of isotropy of space (or the principle of material indifferençe) reduces the last tensorial constitutive equation to

$$\mathbf{s} + p\mathbf{I} = \Psi(\mathbf{D}, \mathbf{R}). \tag{7.34}$$

As we already know, the tensor function Ψ must be form invariant, i.e.:

$$\mathbf{Q}\Psi(\mathbf{D}, \mathbf{R})\mathbf{Q}^T = \Psi\left(\mathbf{QDQ}^T, \mathbf{QR}\right), \tag{7.35}$$

for any proper orthogonal matrix $\mathbf{Q}$. For $\mathbf{Q} = \mathbf{R}^T$ the last relation takes the following form

$$\mathbf{s} + p\mathbf{I} = \mathbf{R}\Psi\left(\mathbf{R}^T\mathbf{DR}, \mathbf{I}\right)\mathbf{R}^T = \mathbf{R}\Theta\left(\mathbf{R}^T\mathbf{DR}\right)\mathbf{R}^T, \tag{7.36}$$

where Θ is a tensor function. It can easily be verified that (7.35) is satisfied for any proper orthogonal matrix $\mathbf{Q}$. Recall that we are considering constitutive equations for a particular particle (X_i) so that we may choose $\mathbf{Q} = \mathbf{R}^T$, where $\mathbf{R}$ depends on (X_i), even though $\mathbf{Q}$ is only a function of time t.

For fluids which are transversely isotropic with respect to the θ_1 coordinate direction (7.36) can be represented in the following polynomial form

$$\mathbf{s} + p\mathbf{I} = \alpha_1\mathbf{N} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2 + \alpha_4(\mathbf{ND} + \mathbf{DN}) + \alpha_5\left(\mathbf{ND}^2 + \mathbf{D}^2\mathbf{N}\right), \tag{7.37}$$

where

$$\mathbf{N} = \left[N_{ij}\right], \quad N_{ij} = R_{i1}R_{j1}. \tag{7.38}$$

Here $\alpha_1, \ldots, \alpha_5$ are polynomials in the invariants (7.24). Comparing Ericksen's equation (7.23) with Eq. (7.37) we infer that they are formally identical. However, $\mathbf{N}$ is now defined by (7.38) entirely in terms of the fluid motion and some initial structure of the fluid at a specified time, instead of being expressed in terms of a vector $\mathbf{n} = (n_i)$ which

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should satisfy Eq. (7.25). We observe that if the fluid is isotropic in the preferred state then (7.36) takes the form

$$\mathbf{s} + p\mathbf{I} = \Theta_1(\mathbf{D}), \tag{7.39}$$

where Θ_1 is an isotropic tensor function.

If $\mathbf{D} = \mathbf{0}$ then (7.37) reduces to

$$\mathbf{s} + p\mathbf{I} = \alpha_1 \mathbf{N}, \tag{7.40}$$

where α_1 is a polynomial in $tr\mathbf{N}$. This means that when the fluid is at rest, or moving as a rigid body, there is a stress which is not a hydrostatic pressure. The fluid at rest therefore sustains shearing stresses across arbitrary planes at any point and this may contradict an intuitive feeling that a fluid at rest should only exert normal pressure across any plane. This situation can be avoided if α_1 in (7.40) is zero, i.e. if α_1 in (7.37) as a polynomial in the invariants (7.24) contains no terms independent of $\mathbf{D}$.

We observe that for anisotropic plastic solids a coupling between hydrostatic pressure and shearing stresses is well known, cf. Boehler and Sawczuk (1976, 1977). Green (1964a) expressed also the result (7.36) in terms of a general fixed curvilinear coordinate system in which the position of the particle (x^i) at time t is denoted by (ξ^i) and

$$x^i = x^i(\xi^k), \quad \det\left[\frac{\partial x^i}{\partial \xi^j}\right] > 0. \tag{7.41}$$

The polar decomposition is then written in the form

$$\left[\frac{\partial x^i}{\partial \xi^j}\right] = \mathbf{rh}, \tag{7.42}$$

where

$$\mathbf{rr}^T = \mathbf{I}, \quad \det \mathbf{r} = 1, \quad \mathbf{h} = \mathbf{h}^T, \tag{7.43}$$

and $\mathbf{h}$ is positive definite.

Remark 7.1. In now more familiar terminology, the tensor $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$ appearing in Eq. (7.23) plays the role a structural tensor while Eq. (7.25) is the evolution equation for the vector $\mathbf{n}$. Thus Green's criticism does not seem to be justified. In Green's approach such a quite natural evolution equation is replaced by the polar decomposition (7.32) at each time t. The general constitutive relation (7.22) has obviously to be completed by evolution laws of the structural tensors.

For more information on application of the invariant theory to the formulation of not necessarily linear constitutive equations describing liquid crystals the reader should refer to Eringen (1978, 1993) and Leslie (1992) as well as to the references cited therein.

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7.3.2. In Section 7.2 we have studied constitutive relations for simple fluids from the point of view of tensor functions representations. Prior to passing to anisotropic simple fluids let us recall some fundamental results of the rational mechanics, cf. Noll (1955), Coleman and Noll (1959).

By $\hat{\varepsilon}$ we denote the energy density (per unit mass) of the deformation gradient **F** and the entropy η. The unimodular transformations **H** for which the following relation is satisfied

$$\hat{\varepsilon}(\mathbf{F}, \eta) = \hat{\varepsilon}(\mathbf{FH}, \eta), \tag{7.44}$$

form a group called *the isotropy group* G *of* $\hat{\varepsilon}$ *or of the material defined by* $\hat{\varepsilon}$. This group depends, in general, on the choice of the local reference configuration, but it can be shown that the groups corresponding to two different local configurations are always conjugate and hence isomorphic.

The energy function $\hat{\varepsilon}$ is said to define a simple fluid if its *isotropy group* G is the full unimodular group U.

We say that a material point is an *isotropic material point* if the isotropy group of its energy function $\hat{\varepsilon}$, relative to some local reference configuration, *contains* the orthogonal group O. Those local reference configurations of the material point for which G contains O are said *undistorted*. Thus a simple fluid is isotropic, and all of its local configurations are undistorted.

Further, the energy function $\hat{\varepsilon}$ is said to define a *simple solid* if its isotropy group is *contained* as a group in the orthogonal group O. Obviously, for an isotropic simple solids, the isotropy group G coincides with the orthogonal group O.

Green (1964b) defines a *simple anisotropic fluid* as one for which the stress tensor at a particular particle at time t is dependent on the whole history of the displacement gradients measured with respect to the current configuration at time t, the whole history of the rotation tensor measured with respect to the θ_i curvilinear coordinates, and the density $\rho(t)$. Thus we write

$$\mathbf{s} = \mathop{\mathcal{L}}_{-\infty}^{t}\left[\mathbf{F}_t(\tau), \mathbf{R}(\tau); \rho(t)\right]. \tag{7.45}$$

We recall that such a fluid is a special case of a simple material. Green (1964b) develops a general theory of such fluids and considers next specific cases allowing for application of tensor functions representations.

It seems that an alternative approach to the formulation of constitutive relations for anisotropic simple fluids would consists in a generalization of the results due to Fahy and Smith (1980) (cf. also Sec.7.2) by including structural tensors into the constitutive relation (7.5) completed with their evolution equations. However, this is a subject for a separate study.

For more information on viscoelastic fluids, including their classification, the reader should refer to the book by Zahorski (1981).

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8. Applications of the tensor equation $\mathbf{AX} + \mathbf{XA} = \Phi(\mathbf{A}, \mathbf{H})$ to kinematics of continua

Some problems in continuum mechanics require the knowledge of a solution $\mathbf{X}$ of a linear algebraic equation of the following form, cf. Scheidler (1994)

$$\mathbf{AX} + \mathbf{XA} = \Phi(\mathbf{A}, \mathbf{H}), \tag{8.1}$$

where $\mathbf{A}$, $\mathbf{X}$ and $\mathbf{H}$ are second-order 2D or 3D tensors, while $\Phi(\mathbf{A}, \mathbf{H})$ is an isotropic tensor function of $\mathbf{A}$ and $\mathbf{H}$ which is linear in $\mathbf{H}$. The special cases of the equations (8.1) have been studied by various authors: Sidoroff (1978), Guo (1984, 1992), Guo et al. (1994), Hoger and Carlson (1984a), Carlson and Hoger (1986b), Mehrabadi and Nemat-Nasser (1987). Particular cases of (8.1) were considered with

$$\mathbf{A} = \mathbf{B}, \mathbf{B}^{-1}, \mathbf{V}, \mathbf{U} \; or \; \mathbf{U}^{-1}, etc., \tag{8.2}$$

where $\mathbf{B}$ is the *left Cauchy-Green tensor*, $\mathbf{V}$ and $\mathbf{U}$ are the *left* and *right stretch tensors* respectively; for definitions of these tensors see for example: Gurtin (1981), Ostrowska-Maciejewska (1995). $\Phi(\mathbf{A}, \mathbf{H})$ is usually taken in one of the following forms:

$$\mathbf{H}, \mathbf{AH} - \mathbf{HA}, \mathbf{HA} - \mathbf{AH}^T, \mathbf{AHA}, \mathbf{A}^2\mathbf{H} - \mathbf{HA}^2, etc. \tag{8.3}$$

It is well known that $\mathbf{A}$ given by (8.2) is symmetric and positive-definite. Then a solution $\mathbf{X}$ exists and is unique, cf. Scheidler (1994). Observe that $\mathbf{X}$ is symmetric (skew-symmetric) iff $\Phi(\mathbf{A}, \mathbf{H})$ is symmetric (skew-symmetric).

Let us mention most important fields where Eq. (8.1) intervenes:

(i) direct formulas for the derivatives of the stretch and rotation tensors with respect to the deformation gradients.

(ii) Direct formulas for a work-conjugate stress tensor in terms of another.

(iii) The kinematics and dynamics of rigid bodies and pseudo-rigid bodies.

(iv) Traction boundary value problems in finite elasticity.

(v) Stability analysis of system of ordinary differential equations.

Below our study is confined to possible applications of the algebraic equation (8.1) to the kinematics of continuous media. More precisely, we shall show how to determine material derivatives of the stretch and rotation tensors. We recall the well known fact that derivatives of these tensors are applied to the formulation of objective derivatives of fundamental quantities used in the continuum mechanics, cf. Bowen and Wang (1971), Carlson and Hoger (1986a, 1986b), Casey (1992), Chu (1986), Dubey (1985), Eringen (1980), Giesekus (1984), Gurtin (1981), Gurtin and Spear (1983), Haupt and Tsakmakis (1996), Hoger (1986), Lehman and Liang Haoyun (1993), Mac Millan (1992), Metzger and Dubey (1986), Sansour (1994), Scheidler (1991), Sidoroff (1973), Wang and Duan (1992), Wheeler (1990), Youzhi Ma and Desai (1990).

We assume that the deformation tensor $\mathbf{F}$ is a continuously differentiable function of time. Using the formula for the material derivative of the deformation gradient, see Gurtin (1981)

$$\dot{\mathbf{F}} = \mathbf{LF}, \tag{8.4}$$

one can easily find the rates of $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{V}^2 = \mathbf{FF}^T$. We have

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$$\dot{\mathbf{C}} = \mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F} = 2\mathbf{F}^T\mathbf{D}\mathbf{F} = 2\mathbf{U}\overset{\bullet}{\mathbf{D}}\mathbf{U} = \dot{\mathbf{C}}^T, \tag{8.5}$$
$$\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T = \mathbf{LB} + \mathbf{BL}^T = \mathbf{DB} + \mathbf{BD} + \mathbf{WB} - \mathbf{BW} = \dot{\mathbf{B}}^T.$$

They are expressed in terms of deformation measures $\mathbf{F}$ (or $\mathbf{U}$), $\mathbf{B}$ and kinematic quantities $\mathbf{D}$, $\mathbf{L}$ (or $\overset{\bullet}{\mathbf{D}}$, $\mathbf{D}$, $\mathbf{W}$). Here, the velocity gradient $\mathbf{L}$ has as its symmetric part the stretching tensor: $\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right)$, and as its skew part the spin tensor: $\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)$.

In Eq. (8.5) the intermediate material rate $\overset{\bullet}{\mathbf{D}} = \mathbf{R}^T\mathbf{D}\mathbf{R}$ is the rotated rate of deformation. Making use of the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ we get expressions for the rate of rotation $\dot{\mathbf{R}}$, the right stretch rate $\dot{\mathbf{U}}$ and the left stretch rate $\dot{\mathbf{V}}$,

$$\dot{\mathbf{F}} = \dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}} = \dot{\mathbf{V}}\mathbf{R} + \mathbf{V}\dot{\mathbf{R}}. \tag{8.6}$$

The above formulas provide additional insight into the structure of the mixed velocity gradient $\dot{\mathbf{F}}$, see Guo (1984), Curnier and Rokotomanana (1991). From Eq. (8.6) we get

$$\dot{\mathbf{F}} = \mathbf{R}\left(\dot{\mathbf{U}} + \Theta\mathbf{U}\right) = \left(\dot{\mathbf{V}}\mathbf{R} + \mathbf{V}\Omega\right)\mathbf{R}, \tag{8.7}$$

where

$$\Theta = \mathbf{R}^T\dot{\mathbf{R}} = -\Theta^T, \quad \Omega = \dot{\mathbf{R}}\mathbf{R}^T = -\Omega^T. \tag{8.8}$$

Here Θ and Ω are the material relative spin and the material stretch spin, respectively. Rewriting Eq. (8.6) in the form

$$\dot{\mathbf{R}} = \mathbf{LR} - \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{V}^{-1}\left(\mathbf{LV} - \dot{\mathbf{V}}\right)\mathbf{R}, \tag{8.9}$$

and using Eq. (8.8) we get expressions for $\dot{\mathbf{R}}$, Θ and Ω in terms of the rotation, the velocity gradient, the stretch tensors and the stretch rate tensors. Differentiation of Eq. $\mathbf{U}^2 = \mathbf{C}$ yields

$$\dot{\mathbf{C}} = \mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}. \tag{8.10}$$

Consequently we conclude that the problem of the determination of the material time derivatives $\dot{\mathbf{R}}$, $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$ is reduced to solving Eq. (8.10), cf. Guo (1984), Hoger and Carlson (1984a).

Now we are going to formulate three lemmas indispensable for the determination of solutions of Eq. (8.10). Both 2D- and 3D-cases will be investigated. Our approach is a modified version of the methods used by Sidoroff (1978), Guo (1984) and Hoger and Carlson (1984a), Carlson and Hoger (1986b).

In the three lemmas below $\mathbf{S}$ is an arbitrary symmetric positive definite tensor.

Lemma 8.1. The homogeneous tensor equation

$$\mathbf{SX} + \mathbf{XS} = \mathbf{0}, \tag{8.11}$$

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has only the trivial solution **X=0**.

Proof. The proof of this lemma is very simple, see Gurtin (1981), Guo (1984). ∇

Proofs of the remaining two lemmas are more complicated.

Lemma 8.2. In the 2D case the solution of the tensor equation

$$\mathbf{SX} + \mathbf{XS} = \mathbf{A}, \tag{8.12}$$

has the following form:

$$\mathbf{X} = (2I_S II_S)^{-1}\left\{\left[I_A\left(I_S^2 - II_S\right) - I_S I_{SA}\right]\mathbf{I} + (I_{SA} - I_S I_A)\mathbf{S} + 2II_S\mathbf{A}\right\}, \tag{8.13}$$

where $I_S = tr\mathbf{S}$, $II_S = \det\mathbf{S} = \frac{1}{2}\left[(tr\mathbf{S})^2 - tr\mathbf{S}^2\right]$, $I_A = tr\mathbf{A}$, $I_{SA} = tr\mathbf{SA}$, and **A** is an arbitrary second-order tensor. Particularly the same formula holds for symmetric tensors. For skew symmetric tensors Eq. (8.13) simplifies to

$$\mathbf{X} = (I_S)^{-1}\mathbf{A}. \tag{8.14}$$

Proof. Applying the extended 2D Cayley-Hamilton theorem to Eq. (8.12), we get

$$\mathbf{SX} + \mathbf{XS} = I_S\mathbf{X} + I_X\mathbf{S} + (I_{SX} - I_S I_X)\mathbf{I} = \mathbf{A}, \tag{8.15}$$

and

$$\mathbf{X} = (I_S)^{-1}\left[(I_S I_X - I_{SX})\mathbf{I} - I_X\mathbf{S} + \mathbf{A}\right]. \tag{8.16}$$

Taking the trace of Eq. (8.12), we obtain

$$I_{SX} = \frac{1}{2}I_A. \tag{8.17}$$

By multiplying Eq. (8.12) on the right by **S**, on the left by **S**, and adding the equations thus obtained and using the extended 2D Cayley-Hamilton theorem, we get

$$\mathbf{S}^2\mathbf{X} + \mathbf{XS}^2 + 2\mathbf{SXS} = \mathbf{AS} + \mathbf{SA} = I_A\mathbf{S} + I_S\mathbf{A} - (I_A I_S - I_{AS})\mathbf{I}. \tag{8.18}$$

Next, the 2D Cayley-Hamilton theorem yields

$$\mathbf{S}^2\mathbf{X} + \mathbf{XS}^2 + 2\mathbf{SXS} = I_S(\mathbf{XS} + \mathbf{SX}) + 2I_X\mathbf{S}^2 - 2(I_S I_X - I_{SX})\mathbf{S}. \tag{8.19}$$

Combining Eqs (8.18), (8.19) and taking the trace, we obtain

$$I_X = (2II_S)^{-1}(I_S I_A - I_{SA}). \tag{8.20}$$

Substituting Eqs (8.17) and (8.20) into (8.16) we complete the proof for an arbitrary **A**.

For $\mathbf{A} = \mathbf{A}^T$, from Eq. (8.12) we obtain

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(8.21) $$\mathbf{S}(\mathbf{X}-\mathbf{X}^T)+(\mathbf{X}-\mathbf{X}^T)\mathbf{S}=\mathbf{0}.$$

Lemma 8.1 implies

(8.22) $$\mathbf{X}=\mathbf{X}^T.$$

Consequently (8.13) is also the solution of Eq. (8.12) for this particular case.

If $\mathbf{A}=-\mathbf{A}^T$ then adding to Eq. (8.12) its transpose, we get

(8.23) $$\mathbf{S}(\mathbf{X}+\mathbf{X}^T)+(\mathbf{X}+\mathbf{X}^T)\mathbf{S}=\mathbf{0}.$$

By virtue of Lemma 8.1

(8.24) $$\mathbf{X}=-\mathbf{X}^T.$$

Then (8.13) yields immediately the solution (8.14). ∇

Lemma 8.3. For the 3D case the solution of the tensor equation (8.12) is specified by

(8.25) $$\mathbf{X}=a_0^{-1}\left[-(\mathbf{S}^2\mathbf{A}+\mathbf{A}\mathbf{S}^2)+a_1\mathbf{A}+a_2\mathbf{S}^2+a_3\mathbf{S}+a_4\mathbf{I}\right],$$

where

(8.26) $$\begin{aligned}
a_0&=I_S II_S-III_S=\frac{1}{3}\left[(tr\mathbf{S})^3-tr\mathbf{S}^3\right],\\
a_1&=I_S^2-II_S=\frac{1}{2}\left[(tr\mathbf{S})^2+tr\mathbf{S}^2\right],\\
2a_2&=I_A+K=I_A+I_S III_S^{-1}\left(I_{S^2A}-I_S I_{SA}+I_A II_S\right)=\\
&=tr\mathbf{A}+tr\mathbf{S}(\det\mathbf{S})^{-1}\left[tr\mathbf{S}^2\mathbf{A}-tr\mathbf{S}\mathbf{A}tr\mathbf{S}+\frac{1}{2}tr\mathbf{A}\left((tr\mathbf{S})^2-tr\mathbf{S}^2\right)\right],\\
2a_3&=I_{SA}+I_S K=tr\mathbf{S}\mathbf{A}+Ktr\mathbf{S},\\
2a_4&=I_{S^2A}-I_A\left(I_S^2-II_S\right)+II_S K=\\
&=tr\mathbf{S}^2\mathbf{A}-\frac{1}{2}tr\mathbf{A}\left[(tr\mathbf{S})^2+tr\mathbf{S}^2\right]+\frac{1}{2}\left[(tr\mathbf{S})^2-tr\mathbf{S}^2\right]K,
\end{aligned}$$

where in turn:

$$I_S=tr\mathbf{S},\ II_S=\frac{1}{2}\left[(tr\mathbf{S})^2-tr\mathbf{S}^2\right],\ III_S=\det\mathbf{S}=\frac{1}{6}(tr\mathbf{S})^3-\frac{1}{2}tr\mathbf{S}tr\mathbf{S}^2+\frac{1}{3}tr\mathbf{S}^3,$$

$$I_A=tr\mathbf{A},\ I_{SA}=tr\mathbf{S}\mathbf{A},\ I_{S^2A}=tr\mathbf{S}^2\mathbf{A}.$$

Here $\mathbf{A}$ is an arbitrary second-order tensor. Specifically, the same formula holds for symmetric tensors.

If $\mathbf{A}=-\mathbf{A}^T$, then

(8.27) $$\mathbf{X}=a_0^{-1}\left[-(\mathbf{S}^2\mathbf{A}+\mathbf{A}\mathbf{S}^2)+a_1\mathbf{A}\right].$$

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Proof. By multiplying Eq. (8.12) on the right by $\mathbf{S}^2$, on the left by $\mathbf{S}^2$ and adding the obtained expressions, we get

$$\mathbf{S}^3\mathbf{X} + \mathbf{X}\mathbf{S}^3 + \mathbf{S}^2\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X}\mathbf{S}^2 = \mathbf{A}\mathbf{S}^2 + \mathbf{S}^2\mathbf{A}. \tag{8.28}$$

Eq. (8.28) is a keynote of the proof.

Taking now the trace of Eq. (8.12) after multiplying by S, we have

$$I_{S^2X} = \frac{1}{2} I_{SA}. \tag{8.29}$$

From the extended 3D Cayley-Hamilton theorem

$$\begin{aligned} &\mathbf{S}^2\mathbf{X} + \mathbf{S}\mathbf{X}\mathbf{S} + \mathbf{X}\mathbf{S}^2 = \\ &= I_X\mathbf{S}^2 + I_S(\mathbf{S}\mathbf{X} + \mathbf{X}\mathbf{S}) + (I_{SX} - I_S I_X)\mathbf{S} - II_S\mathbf{X} + (I_{S^2X} + I_{SX}I_S + I_X II_S)\mathbf{I}, \end{aligned} \tag{8.30}$$

we obtain

$$\begin{aligned} &\mathbf{S}^2\mathbf{X} + \mathbf{S}\mathbf{X}\mathbf{S} + \mathbf{X}\mathbf{S}^2 = \\ &= I_X\mathbf{S}^2 + I_S\mathbf{A} + \left(\frac{1}{2}I_A - I_S I_X\right)\mathbf{S} - II_S\mathbf{X} + \left[II_S I_X + \frac{1}{2}(I_{SA}I_S - I_S I_A)\right]\mathbf{I}. \end{aligned} \tag{8.31}$$

Rewriting Eq. (8.12) in the form

$$\mathbf{S}\mathbf{X}\mathbf{S} = -\frac{1}{2}\left(\mathbf{S}^2\mathbf{X} + \mathbf{X}\mathbf{S}^2\right) + \frac{1}{2}(\mathbf{A}\mathbf{S} + \mathbf{S}\mathbf{A}), \tag{8.32}$$

and substituting Eq. (8.32) into (8.31) we get an expression for $\mathbf{S}^2\mathbf{X} + \mathbf{X}\mathbf{S}^2$. By multiplying Eq. (8.12) on the left by $\mathbf{S}^{-1}$, we obtain

$$\mathbf{X} + \mathbf{S}^{-1}\mathbf{X}\mathbf{S} = \mathbf{S}^{-1}\mathbf{A}. \tag{8.33}$$

Next, after taking the trace of Eq. (8.33) we have

$$I_X = \frac{1}{2} I_{AS^{-1}} = \frac{1}{2} tr\mathbf{A}\mathbf{S}^{-1}. \tag{8.34}$$

Similarly, from the 3D Cayley-Hamilton theorem we get

$$\mathbf{S}^2\mathbf{A} = I_S\mathbf{S}\mathbf{A} - II_S\mathbf{A} + III_S\mathbf{S}^{-1}\mathbf{A}, \tag{8.35}$$

and

$$\mathbf{S}^3\mathbf{X} + \mathbf{X}\mathbf{S}^3 = I_S\left(\mathbf{S}^2\mathbf{X} + \mathbf{X}\mathbf{S}^2\right) - II_S\mathbf{A} + 2III_S\mathbf{X}. \tag{8.36}$$

Thus we arrive at

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$$I_{AS^{-1}} = III_S^{-1}\left[I_{AS^2} - I_S I_{SA} + I_A II_S\right], \tag{8.37}$$

and

$$I_X = \frac{1}{2} III_S^{-1}\left[I_{AS^2} - I_S I_{SA} + I_A II_S\right]. \tag{8.38}$$

Substituting Eqs (8.31), (8.32) and the invariants (8.17), (8.29) and (8.38) into Eq. (8.36), we obtain an expression for $\mathbf{S}^3\mathbf{X} + \mathbf{X}\mathbf{S}^3$. Then Eqs (8.12) and (8.30) yield

$$\begin{aligned} &\mathbf{S}^2\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X}\mathbf{S}^2 = \mathbf{S}\mathbf{A}\mathbf{S} = -\left(\mathbf{S}^2\mathbf{A} + \mathbf{A}\mathbf{S}^2\right) + I_A\mathbf{S}^2 + \\ &+ I_S(\mathbf{A}\mathbf{S} + \mathbf{S}\mathbf{A}) + (I_{AS} - I_A I_S)\mathbf{S} - II_S\mathbf{A} + \left(I_A II_S + I_{S^2A} - I_S I_{AS}\right)\mathbf{I}. \end{aligned} \tag{8.39}$$

Substituting now Eq. (8.39) and the obtained identity for $\mathbf{S}^3\mathbf{X} + \mathbf{X}\mathbf{S}^3$ into Eq. (8.28) and next solving for $\mathbf{X}$, we finally arrive at (8.25). Using Lemma 8.1 similarly as it was done in Lemma 8.2, we complete the proof.

9. Spectral decomposition of Hooke's tensors

Double-symmetric fourth-order tensors (5.17) can be considered as a symmetric linear operator mapping the space T_s of symmetric second-order tensors into itself. We observe that Hooke's tensors possess all properties of symmetric second-order tensors belonging to the symmetric part of tensor product of six-dimensional Euclidean spaces (Rychlewski, 1983, 1984a, 1995). In addition, fourth-order tensors appearing in the Hooke law or in the von Mises (1928) yield condition are positively definite, though positively semi-definite tensors can also be considered, cf. Arnold and Falk (1987).

In the sequel of this section we shall formulate the spectral decomposition theorem for Hooke's tensors and give several illustrative examples.

Theorem. 9.1. (Rychlewski, 1984a, 1995) For each Hooke's tensor $\mathbb{H}$ there exists exactly one orthogonal decomposition of the space T_s

$$T_s = P_1 \oplus \ldots \oplus P_r, \quad r \leq 6,$$

where $P_i \perp P_j$ *for* $i \neq j$, and exactly one sequence of moduli $\lambda_1 < \ldots < \lambda_r$, such that

$$\mathbb{H} = \lambda_1 \mathbb{P}_1 + \ldots + \lambda_r \mathbb{P}_r,$$

where $\mathbb{P}_1, \ldots, \mathbb{P}_r$ are orthogonal projectors which map the space T_s onto subspaces $P_1, \ldots, P_r$, respectively. The numbers $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\mathbb{H}$.

The operators $\mathbb{P}_1, \ldots, \mathbb{P}_r$ constitute proper orthogonal decomposition of the unit operator $\mathbb{I}$:

$$\mathbb{I} = \mathbb{P}_1 + \ldots + \mathbb{P}_r, \quad \mathbb{P}_i : \mathbb{P}_j = \begin{cases} \mathbb{P}_i & if \quad i = j, \\ \mathbb{O} & if \quad i \neq j. \end{cases}$$

The dimension of the subspace $P_i \left(i = 1, \ldots, r\right)$,

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$$q_i = \dim P_i = Tr\mathbb{P}_i, \quad Tr\mathbb{P}_i = tr_{(1,3)}tr_{(2,4)}\mathbb{P}_i,$$

is called the multiplicity of the modulus λ_i. ∇

9.1. Consider now Hooke's law

(9.1) $$\sigma = \mathbb{C}\cdot\varepsilon, \quad \varepsilon = \mathbb{D}\cdot\sigma, \quad \mathbb{C}:\mathbb{D} = \mathbb{D}:\mathbb{C} = \mathbb{1},$$

where σ is the stress tensor, ε is the strain tensor, and the properties of the material are described by the *stiffness tensor* $\mathbb{C}$ or by the *compliance tensor* $\mathbb{D}$. The parameter λ is called the *stiffness modulus* of the *elastic material* if there exists such a symmetric second order tensor ω that

(9.2) $$\mathbb{C}\cdot\omega = \lambda\omega .$$

The tensor ω is called the *proper elastic state* of the material corresponding to the stiffness modulus λ.

The spectral decomposition of the stiffness tensor and the compliance tensor are expressed by

(9.3) $$\mathbb{C} = \lambda_1\mathbb{P}_1 + \ldots + \lambda_r\mathbb{P}_r, \quad \mathbb{D} = \frac{1}{\lambda_1}\mathbb{P}_1 + \ldots + \frac{1}{\lambda_r}\mathbb{P}_r .$$

From the spectral theorem we see that subspace $P_i\,(i=1,\ldots,r)$ contains all *proper elastic states*, (Rychlewski, 1983, 1984a, 1984d) corresponding to stiffness modulus λ_i. Decomposing the stress and strain spaces into the proper subspaces we can write the Hooke's law in an equivalent form of $r \le 6$ proportionalities of parts of stress and strain (*ut tensio sic vis*)

(9.4) $$\sigma_i = \lambda_i\varepsilon_i, \quad \varepsilon_i = \frac{1}{\lambda_i}\sigma_i, (no\ summation\ on\ i),$$

where

(9.5) $$\sigma_i = \mathbb{P}_i\cdot\sigma, \quad \varepsilon_i = \mathbb{P}_i\cdot\varepsilon.$$

The stiffness moduli λ_i are called *Kelvin moduli* (Rychlewski, 1984a).

Of course, we can also write the spectral decomposition in the classical form

(9.6) $$\mathbb{C} = \lambda_I\omega_I \otimes \omega_I + \ldots + \lambda_{VI}\omega_{VI} \otimes \omega_{VI},$$

where $\omega_K\,(K = I,\ldots,VI)$ is the orthonormal basis in the stress-strain space, $\omega_K\cdot\omega_L = \delta_{KL}$. For a completely anisotropic material the stiffness tensor can be described by 21 parameters which constitute the following independent groups: 6 Kelvin moduli $\lambda_K\,(\lambda_K \ge 0)$ defining the stiffnesses, 12 invariants describing the orthogonal basis $\omega_K\,(K = I,\ldots,VI)$ which Rychlewski (1983) proposed to call the *elasticity distributors*, and 3 *orientation angles* placing the tensor $\mathbb{C}$ against the laboratory. The Kelvin moduli

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and elasticity distributors constitute a *local functional basis* of 18 invariants, cf. also Khatkevich (1962). From the uniqueness of the spectral decomposition follows the *theorem on elastic symmetry*: The group of rotational symmetry of the stiffness tensor $\mathbb{C}$ is a symmetry group of orthogonal decomposition of stress-strain space $T_s = P_1 \oplus \ldots \oplus P_r$, which corresponds to $\mathbb{C}$ (Table 9.1. Rychlewski, 1995).

Table 9.1.

Elastic symmetry	Spectral decomposition of strain-stress space	Kelvin moduli	elasticity distributors	orientation angles
isotropy	1+5	2	0	0
cubic	1+2+3	3	0	3
transverse isotropy	(1+1)+2+2	4	1	2
4-fold symmetry axis	(1+1)+1+1+2	5	1	3
3-fold symmetry axis	(1+1)+(2+2)	4	2	3
orthotropy	(1+1+1)+1+1+1	6	3	3
one symmetry plane	(1+1+1+1)+(1+1)	6	7	3
full anisotropy	1+1+1+1+1+1	6	12	3

Example 9.1.1.
The spectral decomposition of the stiffness and compliance tensors of an isotropic material has the following form

(9.7)
$$\mathbb{C} = \lambda_1 \mathbb{P}_1 + \lambda_2 \mathbb{P}_2, \quad \mathbb{D} = \frac{1}{\lambda_1}\mathbb{P}_1 + \frac{1}{\lambda_2}\mathbb{P}_2,$$

where the projectors are given by

(9.8)
$$\mathbb{P}_1 = \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \quad \mathbb{P}_2 = \mathbf{1} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \quad Tr\mathbb{P}_1 = 1, Tr\mathbb{P}_2 = 5.$$

The Kelvin's moduli are equal to: $\lambda_1 = 3K = 2\mu + 3\lambda$, $\lambda_2 = 2\mu$, where λ, μ are Lame's constants.

Example 9.1.2.
The spectral decomposition of the stiffness and compliance tensors of a material with the cubic symmetry has the form (Ostrowska-Maciejewska and Rychlewski, 1988)

(9.9)
$$\mathbb{C} = \lambda_1 \mathbb{P}_1 + \lambda_2 \mathbb{P}_2 + \lambda_3 \mathbb{P}_3, \quad \mathbb{D} = \frac{1}{\lambda_1}\mathbb{P}_1 + \frac{1}{\lambda_2}\mathbb{P}_2 + \frac{1}{\lambda_3}\mathbb{P}_3,$$

where the projectors are specified by

(9.10)
$$\mathbb{P}_1 = \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \quad Tr\mathbb{P}_1 = 1,$$
$$\mathbb{P}_2 = \mathbf{M}_1\otimes\mathbf{M}_1 + \mathbf{M}_2\otimes\mathbf{M}_2 + \mathbf{M}_3\otimes\mathbf{M}_3 - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \quad Tr\mathbb{P}_2 = 2,$$
$$\mathbb{P}_3 = \mathbf{1} - \left(\mathbf{M}_1\otimes\mathbf{M}_1 + \mathbf{M}_2\otimes\mathbf{M}_2 + \mathbf{M}_3\otimes\mathbf{M}_3\right), \quad Tr\mathbb{P}_3 = 3.$$

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Here $\mathbf{M}_i = \mathbf{m}_i \otimes \mathbf{m}_i$ and $\mathbf{m}_i$ are unit vectors of the principal axes of anisotropy. Kelvin's moduli are then equal to:

$$\lambda_1 = C_{11} + 2C_{12}, \quad \lambda_2 = C_{11} - C_{12}, \quad \lambda_3 = 2C_{44},$$

(9.11)

$$C_{11} = \frac{D_{11} + D_{12}}{(D_{11} - D_{12})(D_{11} + 2D_{12})}, \quad C_{12} = -\frac{D_{12}}{(D_{11} - D_{12})(D_{11} + 2D_{12})}, \quad C_{44} = \frac{1}{D_{44}},$$

where C_{11}, C_{12}, C_{44} (D_{11}, D_{12}, D_{44}) denote stiffnesses (compliances) in a more traditional notation, cf. Nye (1957).

The *bulk modulus K*, *Young's modulus* $E(\mathbf{n})$ in a direction $\mathbf{n}$, *Poisson ratio* $\nu(\mathbf{k},\mathbf{n})$ in a direction $\mathbf{k}$ under stretch in the direction $\mathbf{n}$, *shear modulus* $G(\mathbf{k},\mathbf{n})$ in the plane defined by $\mathbf{k}$ and $\mathbf{n}$ are as follows (Hayes, 1972; Rychlewski, 1983):

$$\frac{1}{K} = \mathbf{I} : \mathbb{D} : \mathbf{I} = \frac{3}{\lambda_1},$$

$$\frac{1}{E(\mathbf{n})} = (\mathbf{n} \otimes \mathbf{n}) : \mathbb{D} : (\mathbf{n} \otimes \mathbf{n}) = \frac{1}{3\lambda_1} + \frac{2}{3\lambda_2} - 2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right)\left(n_1^2 n_1^2 + n_2^2 n_2^2 + n_3^2 n_3^2\right),$$

(9.12)

$$\frac{1}{4G(\mathbf{k},\mathbf{n})} = (\mathbf{k} \otimes \mathbf{n}) : \mathbb{D} : (\mathbf{k} \otimes \mathbf{n}) = \frac{1}{\lambda_3} + \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right)\left(n_1^2 k_1^2 + n_2^2 k_2^2 + n_3^2 k_3^2\right),$$

$$-\frac{\nu(\mathbf{k},\mathbf{n})}{E(\mathbf{n})} = (\mathbf{k} \otimes \mathbf{k}) : \mathbb{D} : (\mathbf{n} \otimes \mathbf{n}) = \frac{1}{3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right)\left(n_1^2 k_1^2 + n_2^2 k_2^2 + n_3^2 k_3^2\right),$$

where $\mathbf{k} = k_i \mathbf{m}_i$, $\mathbf{n} = n_i \mathbf{m}_i$, $\mathbf{k} \perp \mathbf{n}$.

It is worth noting that in Eq. (9.6) the eigenvalues are not ordered, in general. Since the elasticity tensor $\mathbb{C}$ must be positive definite, hence only Kelvin's moduli are positive. It means that in the considered case of the cubic symmetry there exist six possible orderings of those moduli. From Eqs $(9.10)_1$ and $(9.12)_1$ it follows that for materials with the cubic symmetry the spherical state is an elastic eigenstate, similarly as for isotropic materials. The bulk modulus K is then a constant. From the formulae $(9.12)_2 - (9.12)_4$ we infer that particularly important is ordering of the second and third of Kelvin's moduli. For metals, as a rule, the double shear modulus λ_2 is smaller than the triple shear modulus λ_3, cf. Table 9.2, Eq. (9.12) and the data included in the monograph by Schulze (1982). Exceptions are: the tungsten which is elastically isotropic, the chromium, molybdenum, niobium and vanadium for which $\lambda_2 > \lambda_3$. Table 9.2 presents only illustrative values of Kelvin's moduli and the elasticity coefficients for selected crystals with cubic symmetry. The elasticity coefficients are cited after Hearmon (1961), Nye (1957) and Schulze (1982).

For materials with cubic symmetry the invariant form of the density of the elastic energy is given by

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$$
\begin{aligned}
(9.13)\qquad & 2W(\varepsilon,\mathbb{P}_1,\mathbb{P}_2,\mathbb{P}_3)=\varepsilon\cdot\mathbb{C}\cdot\varepsilon=\lambda_1\varepsilon\cdot\mathbb{P}_1\cdot\varepsilon+\lambda_2\varepsilon\cdot\mathbb{P}_2\cdot\varepsilon+\lambda_3\varepsilon\cdot\mathbb{P}_3\cdot\varepsilon_3=\\
& \frac{1}{3}\lambda_1 tr^2\varepsilon+\lambda_2\left(tr^2\mathbf{M}_{11}\mathbf{e}+tr^2\mathbf{M}_{22}\mathbf{e}+tr^2\mathbf{M}_{33}\mathbf{e}\right)+\\
& +\lambda_3\left[tr\mathbf{e}^2-\left(tr^2\mathbf{M}_{11}\mathbf{e}+tr^2\mathbf{M}_{22}\mathbf{e}+tr^2\mathbf{M}_{33}\mathbf{e}\right)\right],
\end{aligned}
$$

where $\mathbf{e}=\varepsilon-\frac{1}{3}(tr\varepsilon)\mathbf{I}$. Similarly, the density of the complementary energy is expressed by

$$
\begin{aligned}
(9.14)\qquad & 2\Omega(\sigma,\mathbb{P}_1,\mathbb{P}_2,\mathbb{P}_3)=\sigma\cdot\mathbb{D}\cdot\sigma=\frac{1}{\lambda_1}\sigma\cdot\mathbb{P}_1\cdot\sigma+\frac{1}{\lambda_2}\sigma\cdot\mathbb{P}_2\cdot\sigma+\frac{1}{\lambda_3}\sigma\cdot\mathbb{P}_3\cdot\sigma=\\
& \frac{1}{3\lambda_1}tr^2\sigma+\frac{1}{\lambda_2}\left(tr^2\mathbf{M}_{11}\mathbf{s}+tr^2\mathbf{M}_{22}\mathbf{s}+tr^2\mathbf{M}_{33}\mathbf{s}\right)+\\
& +\frac{1}{\lambda_3}\left[tr\mathbf{s}^2-\left(tr^2\mathbf{M}_{11}\mathbf{s}+tr^2\mathbf{M}_{22}\mathbf{s}+tr^2\mathbf{M}_{33}\mathbf{s}\right)\right],
\end{aligned}
$$

where $\mathbf{s}=\sigma-\frac{1}{3}(tr\sigma)\mathbf{I}$.

Table 9.2.

Material	D_{11}	D_{12}	D_{44}	λ_1	λ_2	λ_3
Sodium chloride	2.21	-0.45	7.83	0.763359	0.37594	0.255428
Sodium chlorate	2.20	-0.60	8.60	1.0	0.357143	0.232558
Tungsten	0.257	-0.073	0.66	9.00901	3.0303	3.0303
Aluminium	1.59	-0.58	3.52	2.32558	0.460829	0.568182
Cooper	1.49	-0.63	1.33	4.34783	0.471698	1.50376
Nickel	0.799	-0.312	0.844	5.71429	0.90009	2.36967
Diamond	0.113	-0.023	0.212	14.9254	7.35294	9.43396

$S_{11}, S_{12}; S_{44}\left[10^{-11}\frac{m^2}{N}\right]$; $\lambda_i\left[10^{11}\frac{N}{m^2}\right]$

9.2. Theoretical results on the spectral decomposition sketched in the previous subsection can likewise be applied to the von Mises (1928) yield condition given by

$$
(9.15)\qquad \sigma\cdot\mathbb{M}\cdot\sigma=1.
$$

Here $\mathbb{M}$ is a fourth-order positive definite tensor with the usual symmetries, i.e. $\mathbb{M}_{ijkl}=\mathbb{M}_{jikl}=\mathbb{M}_{klij}$. By using Th. 9.1, the quadratic form (9.15) is written in the form (Rychlewski, 1984b)

$$
(9.16)\qquad \sigma\cdot\mathbb{M}\cdot\sigma=\frac{1}{\kappa_1^2}\sigma\cdot\mathbb{P}_1\cdot\sigma+\ldots+\frac{1}{\kappa_r^2}\sigma\cdot\mathbb{P}_r\cdot\sigma=1,\quad r\le 6.
$$

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where $\kappa_i^2\ (i=1,\dots,r)$ are *plastic* moduli taken from tests allowing for the determination of stress states $\mathbb{P}_i \cdot \sigma$. We see that the criterion (9.16) is a sum of partial energies which depend on orthogonal stress states. Thus (9.16) may be viewed as a generalization of Beltrami's yield condition proposed already in 1885, cf. Rychlewski (1984b).
It is now clear that Olszak and Urbanowski's (1956) problem on the decomposition of the elastic energy of an anisotropic material into spherical and deviatoric parts was solved by Rychlewski (1983, 1984b), cf. also Olszak and Ostrowska-Maciejewska (1985). Consequently one can correctly formulate the generalized hypothesis of Maxwell-Huber-von Mises-Hencky (MHMH) for an arbitrary anisotropic material. When plastic yielding of a material depends on a critical value of the distortion energy, then in (9.15) $\mathbb{M}$ should be replaced by a modified tensor $\tilde{\mathbb{M}}$, cf. Rychlewski (1984b). Consequently we get

(9.17)
$$\sigma \cdot \tilde{\mathbb{M}} \cdot \sigma = \sigma \cdot \left(\mathbb{M} - \frac{(\mathbf{I} \cdot \mathbb{M}) \otimes (\mathbb{M} \cdot \mathbf{I})}{\mathbf{I} \cdot \mathbb{M} \cdot \mathbf{I}} \right) \cdot \sigma = 1.$$

Purely formal generalization of Hill's (1948) criterion to any anisotropic material has the form

(9.18)
$$\mathbf{s} \cdot \mathbb{H} \cdot \mathbf{s} = \frac{1}{\eta_1^2} \mathbf{s} \cdot \mathbb{R}_1 \cdot \mathbf{s} + \ldots + \frac{1}{\eta_s^2} \mathbf{s} \cdot \mathbb{R}_s \cdot \mathbf{s} = 1, \quad s \le 5,$$

where $\mathbf{s} = \sigma - \frac{1}{3}(tr\sigma)\mathbf{I}$ while $\mathbb{R}_i\ (i=1,\dots,s)$ are orthogonal projectors of the tensor $\mathbb{H}$; $\mathbb{H}$ belongs to the 15-dimensional space. Obviously we have

(9.19)
$$\mathbb{R}_1 + \ldots + \mathbb{R}_s = \mathbf{1} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I}.$$

The coefficients $\eta_i^2\ (i=1,\dots,s)$ appearing in the yield condition are plastic moduli (η_i are yield limits) determined in the tests $\mathbb{R}_i \cdot \mathbf{s}$.
It should be noted that only for isotropic materials and materials with the cubic symmetry the yield conditions (9.17) and (9.18) are equivalent.
We shall provide now an example of the application of the theoretical results just presented to materials with the cubic symmetry.
Example 9.2.1. For materials with the cubic symmetry, the yield condition (9.16) takes the following form (Ostrowska-Maciejewska and Rychlewski, 1988):

(9.20)
$$\frac{1}{\kappa_1^2} \sigma \cdot \mathbb{P}_1 \cdot \sigma + \frac{1}{\kappa_2^2} \sigma \cdot \mathbb{P}_2 \cdot \sigma + \frac{1}{\kappa_3^2} \sigma \cdot \mathbb{P}_3 \cdot \sigma = 1,$$

where the projection operators $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are defined by (9.10) while $1/\kappa_i^2\ (i=1,2,3)$ are the eigenvalues of the tensor $\mathbb{M}$ with multiplicities 1, 2 and 3 respectively. A physical interpretation, in the sense of energy, of the complementary energy (9.14) follows by comparing it with (9.20). Particularly for $\kappa_2 = \kappa_3$, the already mentioned Beltrami's

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condition for isotropic materials is obtained. Spherical stress state $\mathbb{P}_1 \cdot \sigma$ is a safe state if $\kappa_1^2 \to \infty$. Then the condition (9.20) is associated with the critical distortion energy, where κ_2^2 and κ_3^2 are determined by performing two independent shear tests, namely $\mathbb{P}_2 \cdot \sigma$ and $\mathbb{P}_3 \cdot \sigma$. We observe that in the case of the cubic symmetry Hill's criterion has the following form:

$$(9.21) \qquad \frac{1}{\eta_1^2} \mathbf{s} \cdot \mathbb{P}_1 \cdot \mathbf{s} + \frac{1}{\eta_2^2} \mathbf{s} \cdot \mathbb{P}_2 \cdot \mathbf{s} = 1,$$

where the projection generator $\mathbb{P}_1$ and $\mathbb{P}_2$ are given by

$$(9.22) \qquad \begin{aligned} &\mathbb{P}_1 = \mathbf{M}_1 \otimes \mathbf{M}_1 + \mathbf{M}_2 \otimes \mathbf{M}_2 + \mathbf{M}_3 \otimes \mathbf{M}_3 - \frac{1}{3} \mathbf{I} \otimes \mathbf{I}, \quad Tr\mathbb{P}_1 = 2, \\ &\mathbb{P}_2 = \mathbf{1} - \left(\mathbf{M}_1 \otimes \mathbf{M}_1 + \mathbf{M}_2 \otimes \mathbf{M}_2 + \mathbf{M}_3 \otimes \mathbf{M}_3 \right), \quad Tr\mathbb{P}_2 = 3. \end{aligned}$$

Hence we conclude that $\mathbb{P}_2 = \mathbb{P}_1$ and $\mathbb{P}_3 = \mathbb{P}_2$ and if for the determination of η_1^2 and η_2^2 tests $\mathbb{P}_1 \cdot \mathbf{s}$ and $\mathbb{P}_2 \cdot \mathbf{s}$ are applied respectively, then (9.22) coincides with the condition (9.20) for $\kappa_1^2 = \infty$, $\kappa_2^2 = \eta_1^2$, $\kappa_3^2 = \eta_2^2$. Usually, for materials with regular symmetry obeying Hill's condition two standard strength tests are carried out: tension in one of the directions $\mathbf{m}_i$ $(i = 1, 2, 3)$ and shear in one of the plane determined, for instance by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$. Applying these tests to the determination of η_1^2 and η_2^2, the condition (9.22) can be written in the form

$$(9.23) \qquad tr\mathbf{s}^2 + \left(\frac{3k^2}{Y^2} - 1 \right) \left(tr^2 \mathbf{M}_1 \mathbf{s} + tr^2 \mathbf{M}_2 \mathbf{s} + tr^2 \mathbf{M}_3 \mathbf{s} \right) = 2k^2,$$

or

$$(9.24) \qquad tr\mathbf{s}^2 + 2 \left(\frac{3k^2}{Y^2} - 1 \right) \left(tr^2 \mathbf{M}_1 \mathbf{s} + tr^2 \mathbf{M}_2 \mathbf{s} + tr\mathbf{M}_1 \mathbf{s} tr\mathbf{M}_2 \mathbf{s} \right) = 2k^2.$$

We recall that Y is the yield limit in the in tension while k in torsion. It can easily be shown that (9.23) follows from orthotropic Hill's (1948, 1950) condition. In the case of plastic materials with regular symmetry and undergoing plastic yielding according to a criterion based on the distortion energy, it makes sense to distinguish two classes. These classes depend on the ordering of the eigenvalues $1/\eta_1^2$ and $1/\eta_2^2$. As we have already shown, to the first (second) class belong materials with the yield limit in tension grater than $\sqrt{3}k$ (smaller than $\sqrt{3}k$).

Tracing back the application of the spectral decomposition theorem one finds that Miller (1981) applied it to a hexagonal structure.

The symmetry and properties of elasticity tensors were also studied by other authors, cf. Blinowski (1984), Blinowski and Ostrowska-Maciejewska (1996), Blinowski, Ostrowska-Maciejewska and Rychlewski (1996), Chernykh (1988), Cowin (1989, 1992a, 1992b, 1994, 1995), Cowin and Mehrabadi (1987, 1992, 1995), Forte and Vianello (1996), Huo and Del Piero (1991), Mehrabadi and Cowin (1990), Khatkevich (1962), Litvin (1982), Musgrave (1990, 1992), Norris (1989), Ostrowska-Maciejewska and Rychlewski (1988), Pratz (1983), Rathkjen (1980), Rychlewski (1984d, 1995), Rychlewski and Xiao (1991),

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Rychlewski and Zhang (1991), Sutcliffe (1992), Surrel (1993), Ting (1987,1988), Vianello (1997), Wooster (1973), Xiao (1995), cf. also Aberth (1967), Beatty (1987).

Miehe (1993) proposed algorithms for the computation of 2D and 3D isotropic tensor-valued tensor functions and their derivatives for symmetric positive-definite tensor arguments. The formulation is based on a spectral decomposition.

The paper by Huber (1904) was one of earlier attempts to formulate, saying it in modern terminology, an invariant form of the deviatoric energy strength criterion, cf. also Burzyński (1928). We observe that already in 1856 Maxwell in a letter to Lord Kelvin suggested that strength criterion should be expressed by „*the distortion work*".

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