1.1.9 – modelowanie i symulacja

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A TOLERANCE APPROACH TO VISUAL SHAPE PERCEPTION AND IMAGE MATCHING ANALYSIS FOR MODELING OF MICRO-HETEROGENEOUS MATERIALS

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A TOLERANCE APPROACH TO VISUAL SHAPE PERCEPTION AND IMAGE MATCHING ANALYSIS FOR MODELING OF MICRO-HETEROGENEOUS MATERIALS

Summary

The aim of this paper is twofold. First, we propose a mathematical model of how the visible shapes of illuminated material objects are perceived. The leading assumption is that the visual shape perception depends not only on the received 2D-images but also on a certain "higher level knowledge" about the viewed objects ("we perceive what we expect to perceive"). The model obtained is represented by the image tolerance relation between the expected and received monochromatic images. On this basis the perceptible shapes of objects can be determined. The second aim of this paper is to show how the proposed image tolerance analysis can be applied to the formation of the representative distributions of constituents for certain micro-heterogeneous material structures.

Contents

1. Introduction

In this paper we deal with two problems: (1) how to recover the shapes of 3-D objects on the basis of the recorded 2-D monochromatic images and a certain given a priori knowledge about the objects, and (2) how to determine the representative elements of micro-heterogeneous material structures on the basis of received images and certain extra information about the distribution of heterogeneities. Problem (1) belongs to the computational vision which is a subfield of artificial intelligence, [9], and problem (2) is significant in the modeling of non-periodic composite structures, [2,6,13], i.e. it belongs to the continuum mechanics. The objective of this paper is to propose a special mathematical model of the visual shape perception which makes it possible to obtain the exact solutions to both aforementioned problems. This model is based on the tolerance image matching analysis and on the concept of convex constraints, constituting the mathematical background of the theory.

The image matching analysis plays an important role in the computational vision and has been investigated in a number of papers (for list of references cf. [6,9,10]). However, below we are going to investigate the problems of vision on a higher level of abstraction leading to what is called *a mathematical model of the visual shape perception*. To this purpose all vision concepts, experimental facts and heuristic hypotheses will be presented from the very beginning in the mathematical form. We shall also keep mathematics and the physical background of the theory separate. The proposed approach to the concept of vision is slightly similar to that of Zeeman, [14], but is aimed at the shape recovering of the objects viewed rather than at the human visual perception (how the brain works). The main mathematical concept is that of the tolerance systems, which constitutes a certain generalization of Zeeman's tolerance space, [14], and is a basis for the matching analysis proposed in the paper and hence for the model of visual shape perception. The characteristic feature of this model is that it can be treated as an algorithm specifying how the received images and certain extra information about objects viewed as the input data produce an output representation in the form of reconstructed (perceived) shapes of the visible parts of these objects. For the two-dimensional vision (where the viewed objects are plane) the output representations are given in the form of new images which can be interpreted as the reconstruction of the received (recorded) images on the basis of the given a priori knowledge about the shapes of the viewed (plane) objects.

The contents of this paper can be outlined as follows. We start in Sect.2 with the primary concepts and relations which describe an exact mathematical model of the known phenomenological process of image formation, cf. [5,9,10]. The global and local aspects of the image matching analysis based on the concepts of tolerance system and convex constraints, cf. [12], are detailed in Sect.3 and constitute the main mathematical tool for the proposed approach. In Sect.4 we give a mathematical explanation of how the received images combined with the known a priori information about the viewed objects produce what can be called the "perceived image" of the objects under consideration. *The perception paradigm is that the visual shape perception depends not only on the received images of the illuminated objects, but also on our knowledge about the presumed shapes of the viewed objects* (roughly speaking "we perceive what we expect to perceive"). The general result given in Sect.4 has the form of the global relation between the expected shapes of viewed objects and monochromatic images received by the viewer. In the case of 2D vision this result reduces to the relation between the expected and received images of certain illuminated objects; the monochromatic images that satisfy this relation can be referred to as the reconstructed images. The aforementioned relation is applied in Sect.5 to the continuum mechanics in order to formulate what are called *the representative macro-elements of micro-heterogeneous material structures* [6]. The macro-elements describe the characteristic distribution of constituents in composites and constitute the starting point for the derivation of the effective properties of composite materials, [2,6,13]. We conclude the paper with a critical look at the advantages and drawbacks of the proposed approach. In the Appendix we outline the concept of tolerance system which is the main mathematical tool for the analysis.

From a physical viewpoint all considerations are carried out at a gross phenomena level. By the viewer we mean a human being (then the received image coincides with the retinal image) and/or an artificial device (e.g. the computing system) where the received image is provided by a system of imaging sensors. We restrict ourselves to the monochromatic (gray-scale) images and assume that both light sources and viewers are situated far from the objects viewed relative to the size of these objects (for the perspective images cf.[1]). We also restrict the analysis to the stationary (time independent) problems and to situations in which the mutual illumination effects between the objects can be neglected. Throughout the paper, symbols [sr], [cd] stand for the steradian and candle power measure unit, respectively; the luminance is expressed in $[cd\cdot m^{-2}]$, the luminous flux in $[lm]\equiv[cd\cdot sr]$ and the illumination in $[lx]\equiv[lm\cdot m^{-2}]$ measure units.

2. Image formation

The line of approach in this section can be stated as follows. First, we give independently the mathematical description of: (i) what we look at (3-dimensional illuminated scene), (ii) what messages we receive (2D luminance fields) and (iii) what is our response to these messages (raw images). Secondly, we give the mathematical explanation how (i) imply (ii); in this way we arrive at what will be called *the luminance equation*. Since the luminance equation describe the fragment of the external world then its form is independent of the viewer. At last we pass to the mathematical description of how the viewer reconstruct the messages (luminances) into the images. This description is given by the interrelation between (ii) and (iii) and is called *the response relation*. The form of the response relation depends on the viewer. Combining the luminance equation and the response relation we obtain the mathematical description of how the images received by the viewer are formed. This description takes also into account the noises and distortions introduced by the signal transformations in the image formation process.

2.1 Primary concepts

What we look at are the finite systems of opaque bodies with piecewise smooth boundaries immersed in a transparent medium. Let the bodies under consideration occupy the part Ω of the region Z in the three-dimensional euclidean space (reference space) as shown in Fig.1. Here and in the sequel we are not interested in the size of the observed bodies but only

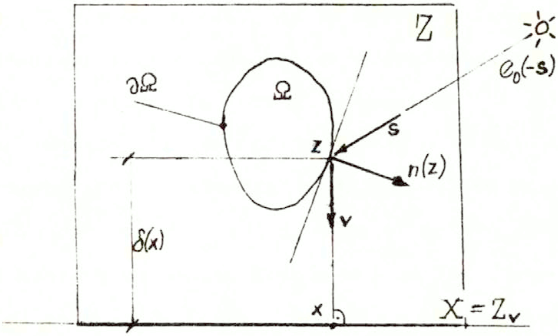

Fig. 1   A surface configuration ∂Ω and a visual plane X

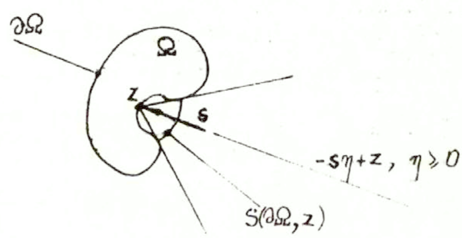

Fig. 2   Illumination rays at z∈∂Ω

in their shapes; hence all length dimensions of Ω and Z in Fig.1 are taken in the scale 1:n which is the image scale.

The important notion of the vision theory is that of the light sources, since we shall observe only illuminated surface configurations. We assume that all light sources are distributed far from the surface configurations as compared with the size of Z; in this situation the position of an arbitrary but fixed single light source observed from an arbitrary point $z\in Z$ can be approximately determined by the unit vector which is independent of z. Hence, in order to describe the position of the light sources we introduce the α-dimensional submanifolds S_α, $\alpha=0,1,2$, of the unit sphere S, $S_\alpha\subset S$ (in the sequel S_0 is the finite set), which determine the positions of the concentrated light sources ($\alpha=0$), light sources distributed along certain lines ($\alpha=1$) and distributed on certain surfaces ($\alpha=2$). Every $s\in -S_\alpha$ (the sets $-S_\alpha$ are defined by $-S_\alpha\equiv\{s\in S\colon -s\in S_\alpha\}$) will be called *the light-source vector*; it is a unit vector directed from the (distant) light source to an arbitrary point of Z. The cases in which one or two from S_0, S_1, S_2 are empty sets can be also taken into account. We shall postulate that in every problem under consideration there exist positive valued functions $-S_\alpha\ni s\to e_\alpha(s)\in R_+$, $\alpha=0,1,2$, where $e_\alpha(s)$ is called *the intensity of illumination* and is measured in $[\mathrm{lx}\cdot \mathrm{m}^{-2}]$ measure unit. Denoting by μ_α the measure of S_α we see that

$$dE_\alpha(s) \equiv e_\alpha(s)d\mu_\alpha(s), \qquad s\in -S_\alpha\ , \quad \alpha = 0,1,2\ , \tag{2.1}$$

is an elementary illumination due to the light source, position of which (related to an arbitrary point $z \in Z$) is determined by the light source vector $s \in -S_\alpha$. For $\alpha=0$ we obtain $d\mu_0(s)=1$ and $dE_0(s)=e_0(s)$, $s \in -S_0$, is the illumination due to the concentrated light source. Since $[lx]=[lm \cdot m^{-2}]$ then it has to be remembered that $dE_\alpha(s)$ is measured in luminous flux measure unit [lm] over the unit area of the plane normal to the light source vector **s**.

Now we shall introduce the concept of *the reflexivity* which describes the reflectance properties of the surface configuration $\partial\Omega$. By the reflectance we mean, roughly speaking, the amounts of incident light reflected in different directions. The reflectance has a local character i.e. is determined for the surface elements $dA(z)$, $z \in \Gamma$, where Γ is a subset (proper or not) of smooth parts of the surface configuration $\partial\Omega$, such that $\bar{\Gamma}=\partial\Omega$. It means that the reflectance properties of the surface configuration are defined on the smooth parts of $\partial\Omega$ except possibly some singular points or lines. Let $\mathbf{n}=\mathbf{n}(z)$ be the unit normal vector outward from $\partial\Omega$ at $z \in \Gamma$. Let us also denote by **v** a unit vector towards the viewer (the view vector) and let **s** be the light source vector as shown in Fig.1. Setting $\cos(i) \equiv \mathbf{n} \cdot \mathbf{s}$, $\cos(e) \equiv \mathbf{n} \cdot \mathbf{v}$, $\cos(p) \equiv \mathbf{v} \cdot \mathbf{s}$ and assuming that $\mathbf{n}=\mathbf{n}(z)$, $\cos(i) \le 0$, $\cos(e) \ge 0$, we shall refer to (i), (e), (p) as to the incident, emittance and phase angle, respectively. Under aforementioned denotations the reflectance properties of $dA(z)$, $z \in \Gamma$, are assumed to be uniquely determined by the non-negative real valued function

$$[-1,1]^3 \ni (\cos(i),\cos(e),\cos(p)) \to \phi_z(\cos(i),\cos(e),\cos(p)) \in \bar{\mathbb{R}}_+ \tag{2.2}$$

satisfying the condition:

$\phi(\cos(i),\cos(e),\cos(p))=0$ if $\cos(i)\in[0,1]$ and/or $\cos(e)\in[-1,0]$, which is called *the reflexivity function*. In order to explain the physical sense of $\phi_{\mathbf{z}}(\cdot)$ we shall introduce the subsets $S(\partial\Omega,\mathbf{z})$, $\mathbf{z}\in\partial\Omega$, of the unit sphere S (cf. Fig.2):

$$S(\mathbf{z},\partial\Omega) \equiv \{\mathbf{s}\in S:\ \{\mathbf{z}-\eta\mathbf{s}\} \cap \bar{\Omega} = \emptyset \text{ for every } \eta>0\}.$$

Let

$$dB^{\mathbf{v}}_{\alpha}(\mathbf{z},\mathbf{n}(\mathbf{z}),\mathbf{s},\partial\Omega) \equiv \tag{2.3}$$

$$\equiv \phi_{\mathbf{z}}(\mathbf{n}(\mathbf{z})\cdot\mathbf{s},\mathbf{n}(\mathbf{z})\cdot\mathbf{v},\mathbf{v}\cdot\mathbf{s})\ \chi_{-S(\partial\Omega,\mathbf{z})}(\mathbf{s})\,dE_{\alpha}(\mathbf{s})$$

where symbol $\chi_{-S(\partial\Omega,\mathbf{z})}(\cdot)$ stands for the characteristic function of the subset $-S(\partial\Omega,\mathbf{z})$ of the unit sphere S, i.e., $\chi_{-S(\partial\Omega,\mathbf{z})}(\mathbf{s})=1$ if $\mathbf{s}\in-S(\partial\Omega,\mathbf{z})$ and $\chi_{-S(\partial\Omega,\mathbf{z})}(\mathbf{s})=0$ if $\mathbf{s}\in S\setminus-S(\partial\Omega,\mathbf{z})$. Then for some $s\in-S_{\alpha}$, $\alpha=0,1,,2$, and $\mathbf{z}\in\Gamma$, we shall interpret $dB^{\mathbf{v}}_{\alpha}$ as an elementary luminance toward the viewer expressed in $[cd\cdot m^{-2}]$ measure unit and related to the unit area of the surface element $dA(\mathbf{z})$ of $\partial\Omega$ at $\mathbf{z}\in\Gamma$ ($dA(\mathbf{z})$ is oriented by the unit normal $\mathbf{n}(\mathbf{z})$). It can be seen that the values of the reflexivity function have to be expressed in $[sr^{-1}]$ (cf. the remarks at the end of Sect.1 and the fact that $dE_{\alpha}(\mathbf{s})$ is measured in $[lx]=[cd\cdot m^{-2}\cdot sr]$) for $\alpha=0,1,2$. Now, setting

$$dI^{\mathbf{v}}_{\alpha}(\mathbf{z},\mathbf{n}(\mathbf{z}),\mathbf{s},\partial\Omega) \equiv \mathbf{v}\cdot\mathbf{n}(\mathbf{z})\cdot dB^{\mathbf{v}}_{\alpha}(\mathbf{z},\mathbf{n}(\mathbf{z}),\mathbf{s},\partial\Omega), \quad \mathbf{z}\in\Gamma, \tag{2.4}$$

we obtain $dI^{\mathbf{v}}_{\alpha}$ as an elementary luminance towards the viewer (expressed in $[cd\cdot m^{-2}]$) but related to the unit area of the plane normal to the view vector $\mathbf{v}$.

Under the aforementioned physical interpretation of $dB^{\mathbf{v}}_{\alpha}(\cdot)$, definition (2.3) also determines the physical meaning of the reflexivity function (2.3) for every $\mathbf{z}\in\Gamma$. For the metallic surfaces the values of $\phi_z(\cdot)$, $z\in\Gamma$, are given by

$$\phi_z(\cos(i),\cos(e),\cos(p)) = \begin{cases} 1 & \text{if} \quad i=e \text{ and } p=i+e , \\ 0 & \text{otherwise} . \end{cases}$$

For the ideal surfaces (diffusers) we assume

(2.5) $$\phi_z = \varkappa(z)\cdot\cos(i) , \qquad z\in\Gamma,$$

where $\varkappa(z)$ is a positive constant, i.e., the values of $\phi(z)$ are independent of cos(e) and cos(p). Now define

(2.6) $$\phi \equiv (\phi_z(\cdot))_{z\in\Gamma} , \qquad e \equiv (e_0(\cdot),e_1(\cdot),e_2(\cdot)).$$

Every fourtuple $(Z,\partial\Omega,\phi,e)$ will be referred to as *the illuminated 3D scene* and is a mathematical model of what we look at. From a formal point of view it is the first primary concept of the approach to the phenomenological vision theory which will be applied in the sequel.

The second primary concept of the presented approach is that of *the luminance field*. In order to describe this field we introduce, for an arbitrary but fixed view vector **v**, $\mathbf{v}\in S$, the orthogonal projection $X\equiv Z_{\mathbf{v}}$ of Z onto the plane normal to the direction of **v** as shown in Fig.1. The luminance field, denoted by $l(\cdot)$, will be a scalar field defined on X (except possibly at some lines and points on X)

(2.7) $$X \supset \mathrm{Dom}(l) \ni x \to l(x) \in \bar{\mathbb{R}}_+, \qquad \overline{\mathrm{Dom}(l)} = \bar{X} ,$$

(for an arbitrary function f by Dom(f) we denote the domain of f and $\bar{\mathbb{R}}_+\equiv\mathbb{R}_+\cup\{0\}$), and its non-negative values $l(x)$ are expressed in $[cd\cdot m^{-2}]$. The luminance field represents, roughly speaking, the 2D messages received from the illuminated 3D scene. The plane region $X=Z_{\mathbf{v}}$ will be referred to as the visual plane (or visual field, [14]). It has to be remembered that the visual plane $X=Z_{\mathbf{v}}$ is oriented in in the reference 3D space by an arbitrary but fixed view vector **v**. Such situation takes place only under assumption that the viewed objects Ω are far from the visual field (cf. remarks at the end of Sect.1).

Remark. More general cases, in which X is not a plane region, can be also considered; for example, in visual perception problems X is a part of a sphere concentric with the eyeball, [14]. In this case as well as in the case of a perspective projection, [1], the view vector related to a visual field is not constant but has to be replaced by a certain vector field $\mathbf{v}(x)$, $x \in X$.

The last primary concept we are going to introduce is that of *the image*. In order to precise this notion we introduce a finite set Y_o, elements of which will be called *sensory units*. From the physical point of view every sensory unit represents a certain small portion of matter, such as a grain on a photographic plate, a retina sensor or a rectangular cell of a TV-screen. We also introduce the concept of the localization of Y_o on the visual plane defined by the injection $\pi: Y_o \to X$ (X is the known region on the plane Ox_1x_2), and we denote $Y \equiv \pi(Y_o)$. For computer vision we can assume $Y \equiv \pi(Y_o) = \{0,\ldots,M-1\} \times \{0,\ldots,N-1\}$, where $v \in \{0,\ldots,M-1\}$ and $h \in \{0,\ldots,N-1\}$ are called vertical and horizontal position variables, respectively, [7]. We also assume that the localization $\pi: Y_o \to X$ is known in every problem under consideration. To define the concept of an image we also introduce the set B which will be called *the brightness chart* (or *the gray level chart*); for the sake of simplicity we assume $B=[0,\beta]$, $\beta>0$, and treat every element of B as a non-dimensional quantity. We introduce two kinds of images: recorded and observed.

By *the recorded image* we shall understand the subset of $Y\times B$ given by

(2.8) $$I = Gr(b) := \{(y,b(y)) \in Y\times B : b \in B^{Y}\} , \qquad Y=\pi(Y_o),$$

where function $b:Y\to B$ will be called *the brightness* (or *the gray level*) *function*. The set of all recorded images which are subsets of $Y\times B$ will be denoted by $\mathcal{I}(Y)$.

The recorded image has a discrete structure and is not what we observe. In order to define *the observed image* as a certain two-dimensional signal we shall interpret a plane region X (the visual plane) as a "background" of this image and we introduce smooth function $p(\cdot)$ defined on X except possibly at some lines or points in X, and with values in B. The function $p(\cdot)$ will be called *the image intensity function*; its values are image intensities (or image intensities observed by a viewer) and X will be now interpreted as the image plane. By the observed image we shall mean the subset of $X\times B$ defined by

(2.9) $$P=Gr(p):=\{(x,p(x)) \in X\times B : x \in Dom(p) \} , \quad \overline{Dom(p)} = \overline{X} .$$

Elements $(x,p)\in Dom(p)\times B$ will be called *observed image elements*. The set of all observed images, with X as the image plane, will be denoted by $\mathcal{I}(X)$.

Summarizing, in the presented approach to the mathematical theory of vision we shall deal with the following primary concepts:

1. The illuminated 3D scene $(Z,\partial\Omega,\phi,e)$ as a mathematical

model of the surface configuration $\partial\Omega$ in the region Z of the reference 3D space, endowed with the reflectance properties described by $\phi\equiv(\phi_{\mathbf{z}})_{\mathbf{z}\in\Gamma}$, $\bar{\Gamma}\subset\partial\Omega$, and illuminated with the intensity $\mathbf{e}\equiv(e_0(\cdot),e_1(\cdot),e_2(\cdot))$.

2. The luminance fields $l(\cdot)$, defined (for every view vector $\mathbf{v}\in S$) on the visual plane $X=Z_{\mathbf{v}}$, which represent 2D messages received by the viewer from the illuminated 3D scene.

3. The recorded image Gr(b) , where $b:Y\to[0,\beta]$ is the image brightness function defined on the finite subset Y of X, and the observed image Gr(p) , where the image intensity $p(\cdot)$ is the function defined almost everywhere on X.

Now we shall pass to the interrelations between the aforementioned primary concepts.

2.2 Governing relations

The interrelation between the illuminated 3D scenes $(Z,\partial\Omega,\phi,\mathbf{e})$ and the luminance fields $l(\cdot)$ defined almost everywhere on an arbitrary but fixed visual plane X, $X=Z_{\mathbf{v}}$, will be referred to as the luminance relation. In order to formulate this relation let us combine (2.1), (2.3) and (2.4). In this way we obtain for every $\mathbf{z}\in\Gamma$:

$$I^{\mathbf{v}}(\mathbf{z},\mathbf{n}(\mathbf{z}),\partial\Omega) = \mathbf{v}\cdot\mathbf{n}(\mathbf{z})\cdot\sum_{\alpha=0}^{2}\int_{-S_\alpha}\phi_{\mathbf{z}}(\mathbf{n}(\mathbf{z})\cdot\mathbf{s},\mathbf{n}(\mathbf{z})\cdot\mathbf{v},\mathbf{v}\cdot\mathbf{s})\cdot\chi_{-S(\partial\Omega,\mathbf{z})}(\mathbf{s})\,e_\alpha(\mathbf{s})\,d\mu_\alpha(\mathbf{s}) \tag{2.10}$$

where $I^{\mathbf{v}}(\mathbf{z},\mathbf{n}(\mathbf{z}),\partial\Omega)$, $\mathbf{z}\in\Gamma$, is the total luminance from a point $\mathbf{z}\in\Gamma$ toward the viewer (expressed in $[cd\cdot m^{-2}]$) related to the unit area of the visual plane $X=Z_{\mathbf{v}}$ normal to the view vector $\mathbf{v}$.

This interpretation of $I^{\mathbf{v}}$ holds true provided that the mutual illumination effects between the objects viewed (represented by the surface configuration $\partial\Omega$) can be neglected and under the condition of distant light sources and viewers from the region Z. If only concentrated light sources are taken into account then (2.8) reduces to the form

$$I^{\mathbf{v}}(\mathbf{z},\mathbf{n}(\mathbf{z}),\partial\Omega) = \mathbf{v}\cdot\mathbf{n}(\mathbf{z})\sum_{-\mathbf{s}\in S_o}\phi_{\mathbf{z}}(\mathbf{n}(\mathbf{z})\cdot\mathbf{s},\mathbf{n}(\mathbf{z})\cdot\mathbf{v},\mathbf{v}\cdot\mathbf{s})\cdot \cdot\chi_{-\mathbf{s}(\partial\Omega,\mathbf{z})}(\mathbf{s})e_o(\mathbf{s}) \tag{2.11}$$

for every $\mathbf{z}\in\Gamma$. In Eqs. (2.10) and (2.11) we tacitly assume that the intensities of illuminations $e_\alpha(\cdot)$, $\alpha=0,1,2$, are known and hence $e_\alpha(\cdot)$ are not arguments of $I^{\mathbf{v}}(\cdot)$.

Now assume that for an arbitrary but fixed view vector $\mathbf{v}$, $\mathbf{v}\in S$, there is known orthogonal projection $X=Z_{\mathbf{v}}$ of Z onto a plane normal to the direction of $\mathbf{v}$, cf. Sec.2.1. Let us also introduce the known concept of the depth function $\delta: X\to\bar{\mathbb{R}}_+$, assuming that:

1. $\delta(x)$ is the minimum distance between the surface configuration $\partial\Omega$ and the point $\mathbf{z}_o+\mathbf{x}$ on the visual plane X, measured along the ray $l_{\mathbf{x}}: -\eta\mathbf{v}+\mathbf{z}_o+\mathbf{x}$, $\eta\geq 0$, normal to this plane, cf. Fig.1, provided that the ray $l_{\mathbf{x}}$ intersects $\partial\Omega$,

2. $\delta(x)\equiv\infty$ if the ray $l_{\mathbf{x}}$ does not intersect $\partial\Omega$. The form of function $\delta(\cdot)$ depends on the choice of point $\mathbf{z}_o\in\mathbb{R}^3$ and on the view vector $\mathbf{v}$.

We assume that a surface configuration $\partial\Omega$ is always situated on the side of the visual plane X oriented by the vector $-\mathbf{v}$. The subset $\Gamma_{\mathbf{v}}$ of Γ, given by

$$\Gamma_{\mathbf{v}} := \{\mathbf{z}\in\Gamma \;:\; \{\mathbf{z}+\eta\mathbf{v}\} \cap \bar{\Omega} = \emptyset \text{ for every } \eta>0\}$$

represents the visible part Γ of the surface configuration $\partial\Omega$, related to the view vector $\mathbf{v}$. Let $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{v})$ be the orthonormal vector basis and for every $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2\in X$, for which $\delta(\cdot)$ is differentiable at x, define $\delta_\beta(x) \equiv \partial\delta(x)/\partial x_\beta$, $\beta=1,2$ and grad $\delta(x)\equiv(\delta_{,1}(x),\delta_{,2}(x))$. Then for every $\mathbf{z}=\mathbf{z}_0+x-\delta(x)\mathbf{v}\in\Gamma_\mathbf{v}$, for which the unit normal $\mathbf{n}(\mathbf{z})$, $\mathbf{z}\in\Gamma_\mathbf{v}$ exists, we obtain

$$(2.12)\qquad \mathbf{n}(\mathbf{z}) = (\delta_{,1}(x),\delta_{,2}(x),-1) \;/\; \sqrt{1+|\mathrm{grad}\ \delta(x)|^2}\ .$$

At the same time, for every $\mathbf{z}\in\Gamma_\mathbf{v}$ we denote

$$(2.13)\qquad I^\mathbf{v}(x,\mathbf{n}(\mathbf{z}),\partial\Omega) \equiv I^\mathbf{v}(\mathbf{z},\mathbf{n}(\mathbf{z}),\partial\Omega)\ ,$$

where x is the orthogonal projection of the point $\mathbf{z}\in\Gamma_\mathbf{v}$ on the visual plane X. On the basis of (2.12) and (2.13) we can introduce the function

$$(2.14)\qquad L(x,\mathrm{grad}\delta(x),\partial\Omega) \equiv I^\mathbf{v}(\mathbf{z},\mathbf{n}(\mathbf{z}),\partial\Omega)\ ,\qquad x\in G\ ,$$

where $\mathbf{z}\in\Gamma_\mathbf{v}$ and G is an orthogonal projection of $\Gamma_\mathbf{v}$ on X. Let the possible noises and disturbances in the signal transmission lead to a certain distortion $l_\varepsilon : X\to\mathbb{R}_+$ of the luminance field, where ε=area(Supp(l_ε))/areaX and ε<<1 (Supp(l_ε) stands for the support of $l_\varepsilon(\cdot)$, i.e., it is a closure of the subset of Dom(l_ε), where $l_\varepsilon(x)\neq 0$). The luminance distortion field $l_\varepsilon(\cdot)$ is not known but has to be taken into account in the image formation process; in most cases Supp(l_ε) consists of a number of small isolated regions. Thus, the 2D luminance field (2.6) will be interrelated with the illuminated 3D scene by means of

$$(2.15)\qquad l(x) = \begin{cases} L(x,\mathrm{grad}\ \delta(x),\partial\Omega) + l_\varepsilon(x) & \text{if } x\in G\ ,\\ l_\varepsilon(x) & \text{if } x\in X\backslash G\ ,\end{cases}$$

where the definitions (2.14), (2.12), and (2.10) have to be taken into account. The dependence of the right-hand sides of Eqs. (2.14), (2.15) on the view vector **v** has to be remembered. Eq.(2.15) holds true if the mutual reflexivity is neglected and for the distant viewers as well as distant light sources (related to the length dimensions of Z). In this case Eq.(2.15) will be called *the luminance equation*. Due to the dependence of $l(x)$ on $\partial\Omega$, it represents a non-local relation.

If there is only one concentrated light source $e_o(s)$ situated near the viewer, $s=(0,0,1)$, then it can be easily shown that

$$L(x,\text{grad}\,\delta(x),\partial\Omega) = L_o(x,|\text{grad}\,\delta(x)|),$$

where $L_o(x,\cdot)$ is the known function, which for the ideal surfaces has the form

$$L_o(x,|\text{grad}\,\delta(x)|) = \frac{\varkappa(x)e_o(0)}{1+|\text{grad}\,\delta(x)|^2}, \quad x\in G\,. \tag{2.16}$$

In this case we deal with the local form of the luminance equation (2.15). Tending with ε to 0 we obtain $l_\varepsilon(x)\to 0$ for every $x\in X$ and the distortion terms $l_\varepsilon(x)$, drop out from Eq.(2.15). In this way we arrive at what will be called *the perfect luminance equation*.

Now we pass to the mathematical explanation of how the viewer reconstructs the luminances $l(x)$, $x\in X$ into the images. This explanation will be given in the form of the so called *response relation*. Roughly speaking, response relations can be interpreted as certain mathematical models of the viewers. We shall deal with two kinds of response relations. The first one describes the interrelation between the luminances and recorded images and will be called *the sensor response relation*. The second kind of response relation interrelates the recorded and observed images and will be referred to as *the visual response relation*.

In order to determine the sensor response relation we introduce for every sensory unit (localized on X, i.e., parameterized by a point $y \in Y$, $Y \subset X$) a neighborhood N(y) of y which represents a small region on X "occupied" by the sensor. Hence, $N(y_1) \cap N(y_2) = \phi$ for every $y_1, y_2 \in Y$ and $y_1 \neq y_2$; for the sake of simplicity we shall also assume that $\bar{X} = \cup \overline{N(y)}$, $y \in Y$. Then, every N(y) will be called the sensory cell on the visual plane. Moreover, let $\nu(y)$, $y \in Y$, be the sensitivity of sensory units, expressed in $[cd \cdot m^{-2}]$. Then, according to the notation

$$[a]^{\beta} \equiv \begin{cases} a & \text{if} \quad a \le 1, \\ \beta & \text{if} \quad a > 1, \end{cases}$$

the sensor response relation b(y) will be postulated in the form

$$(2.17) \qquad b(y) = \left[\frac{1}{\nu(y)} \frac{1}{\text{area } N(y)} \int_{N(y)} l(x)\,dx \right]^{\beta}, \qquad dx \equiv dx_1 dx_2, \; y \in Y,$$

the physical sense of which does not require any comments. Let us define

$$(2.18) \qquad c(x) = \sum_{y \in Y} \chi_{N(y)}(x)\; b(y);$$

hence $c(\cdot)$ is a sectionally constant function defined almost everywhere on X (except possibly at the sensory cell boundaries $\partial N(y) \cap X$, $y \in Y$) and Gr(c) will be referred to as the recorded computer-type image. Passing to the visual response relation, we introduce the concept of the visual acuity, [14]. To this end we introduce, for an arbitrary but fixed viewer, the acuity length parameter ρ which is the least distance on the visual plane X such that all points of a ball $B(x,\rho)$ are, roughly speaking, indistinguishable from a point x, $x\in X$. We also introduce the acuity brightness parameter δ, $\delta\in[0,1)$. Then, the observed image will be determined by the image intensity function $p(\cdot)$ satisfying the relation

$$(2.19) \qquad \Big| \, p(x) - \frac{1}{\text{area } B(x,\rho)} \int_{B(x,\rho)} c(y)\,dy \, \Big| \le \delta \, , \qquad x\in X \, , \qquad \delta\ge 0 \, ,$$

where $c(\cdot)$ is the intensity of the recorded computer-type image, cf. Eq.(2.18). The acuity length parameter ρ is, as a rule, sufficiently small compared with the minimum characteristic length dimension of the visual plane X, but large enough compared with the length dimensions of an arbitrary sensory cell $N(y)$, $y\in Y$, on this plane. Similarly, acuity parameter δ is sufficiently small compared to 1, $\delta\ll 1$. Combining (2.17)-(2.19) we arrive at the resultant response relation which interrelates the luminance field $l(\cdot)$ and the intensity function $p(\cdot)$ of the image which is observed with the visual acuity determined by the parameters (ρ,δ). If $\rho\to 0$ and $\delta\to 0$ then the observed and recorded (computer-type) images coincides.

Summarizing this section we conclude that the image formation process is described mathematically by Eqs.(2.9),(2.12)-(2.14), leading to the luminance equation (2.15), and by Eqs.(2.17)-(2.19) which combine together yield the response relation.

3. Tolerance image matching analysis

In this section an approach to the monochromatic image analysis based on the concept of the tolerance system is proposed, cf. Appendix. The image analysis constitutes the main tool of the computational vision process which begins with a certain existing image. As it is known the aim of the computational vision is to transform "the raw sensed data ... into a meaningful and explicit description of the corresponding scene by a series of inductive steps employing progressively more abstract representations. These steps can be partitioned into three categories, based on the nature of modeling required to carry out the analysis: low-level scene analysis is based on local image properties, intermediate-level scene analysis uses generic and photometric models, and high-level scene analysis is based on the goal-oriented semantic models and relationships", [8]. From the point of view of the possible applications, the approach to the image analysis presented in this section can be used mainly in the low-level and intermediate-level analysis, but it is formulated quite independently of the computational vision process. The main aim of the presented approach is to obtain foundations of the mathematical theory of images. Such theory can be used as a tool in different problems of mathematical modeling where we deal with the formal concept of an image (cf.Sect.2.1) or where it is convenient to introduce this concept into the process of analysis. Hence, this section comprises the formal treatment of images as certain mathematical entities and has nothing in common with their semantic interpretations. The formal image analysis developed in this section can be referred to as *the tolerance image analysis*, as it is based on the notion of the tolerance system, [12]. The examples of application of this formal tool will be given in two subsequent sections of this paper.

3.1 Introductory concepts

The notions of the recorded and observed monochromatic images were explained in Sect.2 on the basis of the image formation process. In this subsection we introduce the basic concepts of the continuum theory of images by means of formal definitions. The leading concepts are these of image, image constraints and image tolerance system.

In order to formulate the definition of an image, we shall denote by X a regular region in $\mathbb{R}^2$ and by $f(\cdot)$ an arbitrary sectionally continuous function defined almost everywhere on X ($f(\mathbf{x})$ may be not defined at certain points and lines in X), such that $\mathrm{Ran}(f)\subset[0,\beta]$ (Ran(f) stands for the range of a function f).

Definition 1. By *the image* (more precisely by *the continuum image*) with *the background* X and *the intensity field* $f(\cdot)$ we shall understand the graph of $f(\cdot)$ given by

$$(3.1)\qquad \mathrm{Gr}(f) \equiv \{(\mathbf{x},f(\mathbf{x}))\in X\times[0,\beta]:\ \mathbf{x}\in\mathrm{Dom}(f)\}\ ,\qquad \overline{\mathrm{Dom}(f)} = \overline{X}\ ,$$

where $f(\cdot)$ satisfies the conditions given above.

This definition coincides with that of the observed image, given by Eq.(2.9). For two images Gr(f), Gr(g) with the background X we shall write Gr(f)=Gr(g) if f(x)=g(x) for almost every $x\in X$. In the sequel we shall also use the denotation $I\equiv Gr(f)$. The set of all images with the background X will be denoted by $\mathcal{I}(X)$ and referred to as the image space.

Definition 2. If $Ran(f) \subset \{0,\beta/N,2\beta/N,\ldots,(N-1)\beta/N,\beta\}$ for some $N\geq 2$ (where $f(\cdot)$ is the image intensity field) then $Gr(f)$ is called *the gray-level image*. For N=2 the gray-level image will be referred to as *the binary image*.

Corollary. If $\beta=1$ then the image intensity field $f(\cdot)$ of every binary image is the characteristic function of the subset $G\equiv\{x\in X:\ f(x)=1\}$ of X :

$$f(x)=\chi_G(x)\ ,\qquad x\in X\ .$$

Let us observe that if $Dom(f)=X$ and $\beta=1$ then the definition of the image coincides with that of the fuzzy set, [8]; hence the binary image can be identified with a pair (G,X) where G is a subset of X.

Let F be a subregion of X, $F\subset X$, and $f|F$ stands for the restriction of the image intensity field $f(\cdot)$ to F. Then $Gr(f|F)$ will be called *the image fragment*. This notion is essential if we are going to specify the decomposition of the background X into regular plane regions F_α, $\alpha\in A$ (A being the known finite index set), such that

(3.2) $\quad \bar{X} = \bigcup_{\alpha\in A} \bar{F}_\alpha$ and $F_\alpha\cap F_\beta=\phi$ for every $\alpha,\beta\in A$ and $\alpha\neq\beta$.

Let us also define $I_\alpha\equiv Gr(f|F_\alpha)$, $\alpha\in A$; every I_α represents a certain fragment of the image $I=Gr(f)$. The set of all image fragments under consideration will be denoted by $\mathcal{F}(X)$; hence $\mathcal{P}(X)\subset\mathcal{F}(X)$.

Definition 3. The family $(I_\alpha)_{\alpha\in A}$ of the image fragments will be referred to as *the image decomposition*. Similarly, the family $D=(F_\alpha)_{\alpha\in A}$ will be called *the decomposition of the image background* X and every F_α, $\alpha\in A$, is said to be *the image cell*.

It will be shown that the image decomposition plays an important role in the tolerance image analysis. Now we shall pass to the concept of the array image.

Definition 4. Let $X=(0,H)\times(0,V)$ be the background of an image $I=Gr(f)$, $A\equiv\{0,1,\dots,V-1\}\times\{0,1,\dots,H-1\}$, (V, H are positive integers) and $F_{vh}\equiv(v,v+1)\times(h,h+1)$, $(v,h)\in A$, be the image cells. If for every $(v,h)\in A$ the condition $(f|F_{vh})(x)=$ $=$const. holds then $I=Gr(f)$ will be called *the array image*.

Hence for an array image there exists the decomposition of X into the $H\cdot V$ rectangular image cells F_{vh}, $(v,h)\in A$, such that the values of the image intensity field on every image cell are constant. Image cells of the array images can be interpreted, from a physical viewpoint, as sensory units. In applications usually $H=V=2^n$, $n=4,5,\dots$, but in the analysis below we shall also deal with more general situations. In the numerical analysis we deal with the array images, which at the same time are gray-level images, cf. Definition 2.

Now we shall pass to a certain generalization of the notion of the array image. To this end let us introduce the decomposition (F_α), $\alpha\in A$, of the image background X, cf. Eq.(3.2), and define

$$f_\alpha \equiv \frac{1}{\mathrm{area}F_\alpha}\int_{F_\alpha} f(x)\,dx\ , \qquad \alpha\in A\ , \qquad dx\equiv dx_1dx_2\ . \tag{3.3}$$

The numbers f_α, $\alpha\in A$, represent the average image intensities in the corresponding image cells F_α.

Definition 5. The image $I_D=Gr(f_D)$, where D is a certain decomposition (3.2) of the image background X and the image intensity field $f_D(\cdot)$ is given by

$$(3.4) \qquad f_D(x) \equiv \sum_{\alpha\in A} f_\alpha \chi_{F_\alpha}(x) \quad \text{for every } x\in \bigcup_{\alpha\in A} F_\alpha$$

where f_α, $\alpha\in A$, are arbitrary reals from the set $[0,\beta]$, will be called *the discretized image*. If $A=\{0,1,...,V-1\}\times\{0,1,...,H-1\}$ and $F_\alpha=F_{(v,h)}=(v,v+1)\times(h,h+1)$ for every $(v,h)\in A$, then the discretized image $Gr(f_D)$ reduces to a certain array image, cf. Definition 4. If f_α, $\alpha\in A$ in (3.4) are defined by means of Eq.(3.3), then the passage from the image $Gr(f)$ to the image $Gr(f_D)$ will be called *the image discretization*.

We end this subsection with the procedure which can be treated as an inverse to the image discretization. To this end we shall introduce a neighborhood $N(\mathbf{O})$ of the point $\mathbf{O}$ on the plane $\mathbf{O}x_1x_2$ and define $N(\mathbf{x})\equiv\mathbf{x}+N(\mathbf{O})$ (in applications the maximum length dimension of $N(O)$ is small compared with the minimum length dimension of X). Then for an arbitrary image $Gr(f)$ we define the new image $Gr(f_N)$ where

$$(3.5) \qquad f_N(x)\equiv \frac{1}{\text{area}(N(x)\cap X)} \int_{N(\mathbf{x})\cap X} f(\mathbf{y})dy \qquad \text{for every } x\in X .$$

The procedure leading from $Gr(f)$ to $Gr(f_N)$ will be called *the image smoothing*. As a neighborhood $N(\mathbf{O})$ of $\mathbf{O}$ we can take the ball on the plane $\mathbf{O}x_1x_2$ with a center $\mathbf{O}$ and a radius ρ, $N(\mathbf{O})=B(\mathbf{O},\rho)$. In this case $N(\mathbf{x})=B(\mathbf{x},\rho)$, where $B(\mathbf{x},\rho)$ is a ball on $\mathbf{O}x_1x_2$ with the center $\mathbf{x}$.

3.2 Image tolerances

The leading role in the approach to the image analysis, which is presented in this paper, is played by the concept of the tolerance system. The formal definition of this concept is given in Appendix where also some examples as well as certain remarks and propositions related to tolerance systems can be found. In this subsection we shall interrelate the notion of the image with that of the tolerance system. We begin with what will be called *the global tolerances of images* which, from a purely formal point of view, inform us to what extent two different images represent the same scene, and hence, can be treated as indistinguishable by the viewer. On the other hand the local tolerances deal with the small image fragments rather than with the images. We restrict ourselves to the description of a few special examples of image tolerance systems which will be applied in the subsequent considerations.

Let $(D(k))$, $k \in K$, be the indexed family of the decompositions of the image background X (cf. Definition 3), such that $(K,<)$ is a certain ordered set and for every $k,l \in K$ and $k<l$, the decomposition $D(l)$ is finer then $D(k)$. The tolerance system $T_k \subset \mathcal{G}(X) \times \mathcal{G}(X)$, $k \in K$, given by

$$(3.6) \qquad (Gr(f),Gr(g)) \in T_k \Leftrightarrow Gr(f_{D(k)}) = Gr(g_{D(k)})$$

will be called *the image decomposition tolerance system*. Thus the images $Gr(f),Gr(g)$ are in the tolerance T_k if and only if the corresponding discretized images $Gr(f_{D(k)})$, $Gr(g_{D(k)})$ coincide.

Independently of the decomposition tolerance system we shall also introduce a general global tolerance system $(T_m)_{m \in M}$, where $M \equiv \{1,2,...,m\}$, $m>1$. To this end we introduce the system

$\{\psi_1(\cdot),\psi_2(\cdot),\dots,\psi_m(\cdot)\}$ of linearly independent functions defined almost everywhere on X. Moreover, for an arbitrary image Gr(g) with the background X we define

$$(3.7)\qquad g_\alpha \equiv \int_X g(x)\psi_\alpha(x)\,dx\ ,\qquad \alpha=1,\dots,m\ .$$

Now we shall specify the global tolerance system $T_n \subset \mathcal{G}(X)\times\mathcal{G}(X)$, $n\in\{1,\dots,m\}$, setting

$$(3.8)\qquad (Gr(f),Gr(g))\in T_n \Leftrightarrow f_\alpha=g_\alpha \qquad \text{for } \alpha=1,\dots,n.$$

System T_n, $n\in\{1,\dots,m\}$ will be called *the general image tolerance system*.

Remark. The image decomposition tolerance system can be treated as a special case of the general image tolerance system by assuming that the decomposition of X into n image cells F_α, $\alpha=1,\dots,n$, is known and by setting

$$\psi_\alpha(x) = \chi_{F_\alpha}(x)/\text{area}F_\alpha \quad \text{for almost every } x\in X\ ,$$

where $\chi_{F_\alpha}(\cdot)$ is the characteristic function of the subset F_α of X.

The different specifications of functions $\psi_1(\cdot),\dots,\psi_m(\cdot)$ lead to many special global image tolerance systems which can be applied to different problems of image analysis.

Now we pass to the concept of *the local image tolerance systems*. To this end for every $\delta\in[0,\delta_o]$ and $\varepsilon\in[0,\varepsilon_o]$ (where δ_o, ε_o are the postulated *a priori* positive constants) we define the tolerances $t_\varepsilon\subset X\times X$, $t^\delta\subset[0,\beta]\times[0,\beta]$, setting

$$(3.9)\qquad \begin{aligned}(x',x'')\in t_\varepsilon &\Leftrightarrow \|x'-x''\|\le\varepsilon\ ,\\ (f',f'')\in t^\delta &\Leftrightarrow |f'-f''|\le\delta\ ,\end{aligned}$$

where $\|x'-x''\|$ is the euclidean distance between points $x',x''\in X$.

In the applications of the theory, ε_o is the length parameter sufficiently small related to the smallest characteristic length dimension of the image background. Hence, every two points x', x'' of X, such that $(x',x'')\in t_\varepsilon$, can be treated as indistinguishable on X by the viewer. Similarly, every two values f', f'' of the image intensity field, satisfying $(f',f'')\in t^\delta$, will be also treated as being too close to be distinguished.

Setting $\Lambda\equiv[0,\varepsilon_o]\times[0,\delta_o]$ and $B\equiv X\times[0,\beta]$ we obtain the tolerance system $t_\lambda\subset B\times B$, $\lambda=(\varepsilon,\delta)\in\Lambda$ defined by $t_{(\varepsilon,\delta)}=t_\varepsilon\times t^\delta$, which will be called *the local image tolerance system*. The tolerance systems t_ε, $\varepsilon\in[0,\varepsilon_o]$, and t^δ, $\delta\in[0,\delta_o]$, will be referred to as *the space local* and *the intensity local tolerance systems*, respectively.

Remark. The above definition of the local tolerance system is motivated by the physical assumption that tolerances t^ε in the visual plane and tolerances t^δ of the image intensity are independent.

The local tolerance system t_λ, $\lambda\in\Lambda$, introduced above has to be treated only as a starting point for the definition of what will be called *the quasi-local image tolerance system*. To formulate this system let us denote by $\mathcal{L}(X)$ and $\mathcal{L}([0,\beta])$ the lattices of subsets of X and $[0,\beta]$, respectively. As it is known (cf. Appendix), every tolerance t_ε on X induces the tolerance $\tilde{t}_\varepsilon$ on $\mathcal{L}(X)$; similarly every local tolerance t^δ on $[0,\beta]$ induces the tolerance $\tilde{t}^\delta$ on $\mathcal{L}([0,\beta])$. The tolerance systems $\tilde{t}_\varepsilon$, $\varepsilon\in[0,\varepsilon_o]$, and $\tilde{t}^\delta$, $\delta\in[0,\delta_o]$, will be called *the space quasi-local* and *the intensity quasi-local tolerance* *systems*. The tolerance system $\tilde{t}_{\lambda}$, $\lambda\equiv(\varepsilon,\delta)\in\Lambda$, given by $\tilde{t}_{(\varepsilon,\delta)}\equiv\tilde{t}_{\varepsilon}\times\tilde{t}^{\delta}$, and restricted to $\mathcal{L}(B)\cap\mathcal{F}(X)$ will be referred to as *the quasi-local image tolerance system*. The main aim of the quasi-local image tolerance system is to characterize the small image fragments which can be interpreted as indistinguishable by the viewer.

3.3 Image constraints

The general idea of constraints as certain postulated *a priori* restrictions imposed on the classes of mathematical entities under consideration has found an important application in the mathematical modeling of physical objects, processes and phenomena. Roughly speaking, the constraints are introduced either in order to formulate the mathematical description of the investigated physical situations in which we deal with certain thresholds which can be attained but cannot be crossed (and hence the constraints are motivated by the physical premises), or to simplify the known mathematical models of physical phenomena on the basis of the extra postulated assumptions (the motivation may be implied by the possible applications of numerical methods). In many problems the constraints can be postulated in the form which makes it possible to pass from the infinite dimensional spaces of mathematical entities to some finite dimensional spaces. In this subsection we apply the concept of constraints to the image analysis mainly in order to simplify the mathematical description of the viewed object and to pass to the numerical treatment of the vision problems.

The sets of the gray scale images and the computer-type images represent the subsets of the image space $\mathcal{G}(X)$ in which on the image intensity fields $f(\cdot)$ (defined almost everywhere on X) the image constraints have been imposed. Among the different possible image constraints the important role will be played by *the n-parameter image constraints*, where $n\geq 1$ is the known positive integer. In order to precise this concept let us denote by $\mathbf{q}\equiv(q^1,\ldots q^n)\in\mathbb{R}^n$ an arbitrary vector in $\mathbb{R}^n$, and let us introduce:

(1) the real-valued function $\xi_n(\mathbf{x},\mathbf{q})$, $\mathbf{q}\in\mathbb{R}^n$, $\mathbf{x}\in X$, piecewise continuous in $\bar{X}$ and differentiable in $\mathbb{R}^n$,

(2) the closed convex subset Q_n in $\mathbb{R}^n$, such that $\xi_n(\mathbf{x},\mathbf{q})\in[0,\beta]$ for almost every $\mathbf{x}\in X$, and every $\mathbf{q}\in Q_n$.

We shall also assume that Q_n has a non-empty interior in $\mathbb{R}^n$. Then the subset of the image set $\mathcal{G}(X)$, given by

$$\mathcal{E}_n(X) \equiv \{Gr(g)\in\mathcal{P}(X): g(\mathbf{x})=\xi_n(\mathbf{x},\mathbf{q}) \text{ for almost every } \mathbf{x}\in X \text{ and some } \mathbf{q}\in Q_n\} \tag{3.10}$$

will be referred to as *the n-parameter image constraints*. An arbitrary component q^a of vector $\mathbf{q}$, $\mathbf{q}=(q^1,\ldots,q^n)\in Q_n$ will be called *the image parameter*. For the sake of simplicity we shall neglect the index "n", setting

$$Q\equiv Q_n,\qquad \xi\equiv\xi_n,\qquad \mathcal{E}(X)\equiv\mathcal{E}_n(X).$$

Remark. In many special problems $Q=\mathbb{R}^n$, i.e., there are no restrictions on the image parameters.

Let us introduce the notation

$$L_a(\mathbf{q})\equiv\int_X \xi(\mathbf{x},\mathbf{q})\frac{\partial\xi(\mathbf{x},\mathbf{q})}{\partial q^a}\,d\mathbf{x},\qquad f_a(\mathbf{q})\equiv\int_X f(\mathbf{x})\frac{\partial\xi(\mathbf{x},\mathbf{q})}{\partial q^a}\,d\mathbf{x},\qquad a=1,\ldots,n. \tag{3.11}$$

Moreover, setting

(3.12) $p(x) = \xi(\mathbf{x},\mathbf{q})$, for almost every $\mathbf{x}\in X$ and some $\mathbf{q}\in Q$,

we define the image Gr(p), such that $Gr(p)\in\mathscr{C}(X)$. The image $Gr(p) \in \mathscr{C}(X)$ will be called *the constrained (n-parameter) model* of the known image $Gr(f) \in\mathscr{I}(X)$ if the differences

(3.13) $$r_\alpha \equiv L_\alpha(\mathbf{q})-f_\alpha(\mathbf{q}) \ , \ \alpha=1,\dots,n \ ,$$

satisfy the subdifferential condition

(3.14) $$\mathbf{r}\in-\partial ind_Q(\mathbf{q}) \ , \quad \mathbf{r}\equiv(r^1,\dots,r^n) \ ,$$

where $\partial ind_Q(\cdot)$ is the indicator function of the set Q.

We tacitly assume here that the reader is familiar with the foundations of the convex analysis (for the particulars cf. I.Ekeland and R.Temam, *Convex analysis and variational problems*, North-Holl. Publ.Comp., 1976). Condition (3.14) is equivalent to the variational inequality (summation convention with respect to $\alpha=1,\dots,n$ holds!).

(3.15) $(v^\alpha-f^\alpha)r_\alpha \geq 0$ for every $\mathbf{v}=(v^1,\dots,v^n)\in Q$, $\mathbf{q}=(q^1,\dots,q^n)\in Q$,

and every $\mathbf{r}=(r_1,\dots,r_n)$ satisfying (3.14) for some $\mathbf{q}\in Q$ will be referred to as *the reaction* to the image constraints C(X) related to the image parameters q^α, $\mathbf{q}\in Q$. Taking into account (3.13) we arrive at the variational inequality for n image parameters q^α, $\alpha=1,\dots,n$:

(3.16) $$(v^\alpha-q^\alpha)(L_\alpha(\mathbf{q})-f_\alpha(\mathbf{q})) \geq 0 \text{ for every } \mathbf{v}=(v^1,\dots,v^n)\in Q, \quad \mathbf{q}=(q^1,\dots,q^n)\in Q.$$

The solution $\mathbf{q}=(q^1,\dots,q^n)$ to the variational inequality (3.14) makes it possible to determine, on the basis of Eq.(3.10), the constrained model Gr(g) of the image Gr(f). If $q\in IntQ$ (IntQ is the interior of Q in $\mathbb{R}^n$, $\overline{IntQ}=Q$) then (3.1) reduces to the system of equations

(3.17) $$L_a(\mathbf{q}) = f_a(\mathbf{q}) \;, \quad a=1,\dots,n$$

which by means of Eqs.(3.11) are equivalent to the conditions

(3.18) $$\frac{\partial}{\partial q^a} \int_X [f(\mathbf{x})-\xi(\mathbf{x},\mathbf{q})]^2 dx = 0 \;, \quad a=1,\dots,n \;,$$

and have the well known analytical meaning. The problem of existence of the solutions to (3.16) depends on the form of the operator $\mathbf{L}(\cdot)\equiv(L_1(\cdot),\dots,L_n(\cdot))$ and is detailed in the recent literature on the convex analysis.

Remark. If $Q=\mathbb{R}^n$ then the variational inequality (3.16) reduces to the system (3.17).

It can be shown that the concept of constraints given by Eqs. (3.10)-(3.14) is closely related to that of the global image tolerance system (3.8). To this end let us assume that $\psi_a(\mathbf{x})=\partial\xi_n(\mathbf{x},\mathbf{q})/\partial q^a$ and $g(\mathbf{x})=\xi_n(\mathbf{x},\mathbf{q})$ for some $\mathbf{q}=(q^1,\dots,q^n)\in Q$. In this case the image Gr(g) in Eq.(3.8) is an element of the n-parameter image constraints $\mathscr{C}(X)=\mathscr{C}_n(X)$ represented by Eq.(3.10) and the constraint parameters q^a, $a=1,\dots,n$, have to satisfy the system of equations (3.13). If this system has a solution $\mathbf{q}=(q^1,\dots,q^n)$ belonging to Q then Gr(f) is in tolerance T_n with Gr(p), where $p(\mathbf{x})=\xi_n(\mathbf{x},\mathbf{q})$ for almost every $\mathbf{x}\in X$.

The image constraints can also be related to the image decomposition tolerance system (3.6); in this case instead of definitions (3.11) we have to introduce the definitions

(3.19) $$L_a(\mathbf{q}) \equiv \int_{F_a} \xi(\mathbf{x},\mathbf{q})\,dx \;, \quad f_a \equiv \int_{F_a} f(\mathbf{x})\,dx \;, \quad a=1,\dots,n \;,$$

where $\bar{X}=\bigcup_{a=1}^{n}\bar{F}_a$, $F_a\cap F_b=\emptyset$ for every $a,b\in\{1,\dots,n\}$ and $a\neq b$ is the postulated *a priori* decomposition of the image background X and hence every F_α is a certain image cell. At the same time we postulate that Eqs.(3.13)-(3.16) have to hold but instead of Eqs.(3.18) we obtain the following system of equations for the image parameters

$$(3.20) \qquad \int_{F_\alpha} \xi(x,q)\,dx = \int_{F_\alpha} f(x)\,dx \ , \quad \alpha=1,\dots,n \ ,$$

which also has the well known interpretation in the numerical analysis. The image $Gr(g) \in \mathscr{E}(X)$ defined by Eq.(3.12) where the image parameters q^α, $\alpha=1,\dots,n$, satisfy the variational inequality (3.16) under denotations (3.19), referred to as the constrained (n-parameter) model of the known image $Gr(f) \in \mathscr{I}(X)$. In order to avoid this ambiguity we have to specify the concept of the reaction to constraints (3.13); under denotations (3.11) the reactions are given by

$$(3.21) \qquad r_\alpha = \int_X [\xi(x,q)-f(x)]\frac{\partial\xi(x,q)}{\partial q^\alpha}dx = \frac{1}{2}\frac{\partial}{\partial q^\alpha}\int_X [\xi(x,q)-f(x)]^2 dx \ , \quad \alpha=1,\dots,n \ ,$$

and will be called *the potential reactions*. If the denotations (3.19) are taken into account then

$$(3.22) \qquad r_\alpha = \int_{F_\alpha} [\xi(x,q)-f(x)]dx \ , \quad \alpha=1,\dots,n \ ,$$

and the reactions will be referred to as *the residual reactions*. Hence the n-parameter constrained models of an arbitrary image introduced above will be called *the potential model* and *the residual model*, respectively.

The potential and residual models can be treated as two special cases of a general (n-parameter) constrained model of an arbitrary image $Gr(f) \in \mathscr{I}(X)$. To define this model we introduce the sequence of functions $\eta_\alpha(x,q)$, $\alpha=1,\dots,m$, defined for every $q\in\mathbb{R}^n$ and almost every $x\in X$. Setting

$$L_\alpha(q) \equiv \int_X \xi(x,q)\eta_\alpha(x,q)\,dx \ , \quad f_\alpha(q) \equiv \int_X f(x)\eta_\alpha(x,q)\,dx \ , \quad \alpha=1,\dots,n \ , \tag{3.23}$$

and assuming that Eqs.(3.13)-(3.16) hold under denotations (3.22), we arrive at *the general constrained (n-parameter) model* of an arbitrary image Gr(f). In this case Eqs.(3.17) have the form

$$\int_X \xi(x,q)\eta_\alpha(x,q)\,dx = \int_X f(x)\eta_\alpha(x,q)\,dx = 0 \ , \quad \alpha=1,\dots,n \ , \tag{3.24}$$

which after the specification of $\eta_\alpha(\cdot)$ leads to (3.18) or (3.20). The general constrained n-parameter model of images can be treated as the basis for the formulation of various special models.

3.4 Image filtering

By the image filtering we shall mean here the transformation of observed images leading to the reduction of noises and distortions introduced in the course of the image formation process. The filtering techniques belong to the low-level image analysis and their general description can be found in [9,10]. In this subsection we propose a special approach to the image filtering in which we apply the concept of the image local tolerance. The proposed procedure can be decomposed into four following steps.

1. We start with a discretization (3.3) of the image Gr(f) leading to the image Gr(f_D); it is assumed that the image cells F_α, $\alpha\in A$, related to Gr(f_D) are sufficiently small compared to

the image background X.

2. We introduce the local image tolerance t_λ, $\lambda=(\varepsilon,\delta)\in\Lambda$, and decompose the image background X into the disjoint regions $G_\mu \equiv G(x_\mu)$, $\mu\in M$, setting

$$x\in G_\mu \Leftrightarrow ((x,f_D(x)),(x_\mu,f_D(x_\mu))) \in \bar{t}_\lambda ,$$

where $\bar{t}_\lambda$ is the transitive closure of t_λ, cf. Appendix. Hence, the image intensities f_α, $\alpha\in A$, in all pairs of adjacent cells are in the tolerance t^δ, $(f_\alpha,f_\beta)\in t^\delta$ if $\bar{F}_\alpha \cap \bar{F}_\beta \neq \emptyset$.

3. Let M_o be a subset of M such that every G_ν, $\nu\in M_o$, is a small (isolated) region on the image background X. To be more exact we assume $M_o \equiv \{\nu\in M \ : \ \text{area}G_\nu/\text{area}X \le \eta\}$, for some postulated *a priori* sufficiently small positive real η, $\eta<<1$. Under the assumption that the image intensity $f(\cdot)$ satisfies the condition : $|f(x)-f_\alpha|\le\delta$ for every $x\in F_\alpha$ and over most of image cells F_α, we shall interpret the regions G_ν, $\nu\in M_o$, as representing unwanted image details due to the effect of noises and distortions.

4. Setting $G_o \equiv \cup G_\nu$, $\nu\in M_o$ and $X_o \equiv X\setminus G_o$, we shall define the reconstructed image $Gr(f_N)$, by applying the smoothing (3.5) in the modified form

$$f_N(x) \equiv \frac{1}{\text{area}(N(x)\cap X_o)} \int_{N(x)\cap X_o} f(y)\,dy \quad \text{for every } x\in X ,$$

where the neighborhood $N(x)$ of x has to satisfy the condition : $\text{area}(N(x)\cap X_o)\neq 0$ for every $x\in G_o$ (and hence for every $x\in X$).

It has to be emphasized that the filtering procedure outlined above yields good results only if the discretization of $Gr(f)$ and smoothing of $Gr(f_D)$ as well as the image local tolerance t_λ are properly chosen.

3.5 Image matching

The fundamental question of image matching is: to what extent two different images represent the same scene? (by the scene we mean here the visible illuminated part of a certain surface configuration). The general answer to this question can be stated as follows : choose the proper global image tolerance system and verify whether the images under consideration are in a tolerance belonging to that system which can be accepted by the viewer. On the other hand the kind of the tolerance depends on the character of both the image and the scene. In this subsection we shall propose a certain formal approach to the image matching problem under the assumption that the following are known:

(i) the image Gr(f) $\in\mathcal{I}(X)$ of the illuminated scene (after the possible filtering),

(ii) the extra information about the scene, represented by parameters q^{α}, $\alpha=1,\ldots,n$, such that $\mathbf{q}\equiv(q^{1},\ldots,q^{n})\in Q$ and Q is a closed and convex subset of $\mathbb{R}^{n}$, having the non-empty interior.

Taking into account the image formation process we can obtain the extra information about the scene (without any noises and distortions) in the form of the n-parameter image constraints $\mathcal{E}(X)$, cf.Eq.(3.10). Thus, the mathematical formulation of the image matching problem is to find the image Gr(p)$\in\mathcal{E}(X)$ which is in a certain (accepted) tolerance with the image Gr(f). If such image exists then the images Gr(f) and Gr(p) represent the same scene and hence the corresponding vector $\mathbf{q}$ (determining Gr(p) by means of $p(\cdot)=\zeta(\cdot,\mathbf{q})$) represents the geometric description of the scene.

The method of the solution to the image matching problem formulated above consists of the following steps:

1. Represent the known *a priori* information about the illuminated scene (determined by an arbitrary vector $q=(q^1,\ldots,q^n)$ belonging to the known subset Q of $\mathbb{R}^n$) in the form of the set $\mathscr{E}_n(X)$ of images, cf. Eq.(3.10). As we have stated above this can be realized by applying the "ideal" image formation process (we neglect the possible noises and distortions). This procedure leads to the function $\xi_n(x,q)$ defined for every $q\in Q$ and almost every $x\in X$.

2. Introduce a certain global image tolerance T_n (for example setting $\psi_\alpha(x)=\partial\xi_n(x,q)/\partial q^\alpha$, $\alpha=1,\ldots,n$ in (3.7)).

3. Calculate $L_\alpha(q)$ and $f_\alpha(q)$, $\alpha=1,\ldots,n$, on the basis of the formulas given by (3.23) (for example setting $\eta_\alpha(x,q)=\partial\phi_n(x,q)/\partial q^\alpha$ in (3.23)).

4. Solve the system of equations (3.17) for the constraint parameters q^α, $\alpha=1,\ldots,n$.

Let $q_o=(q_o^1,\ldots,q_o^n)$ be the obtained solution to the system (3.17). Then we have to consider two following possibilities:

A. If $q_o\in Q$ then we end the procedure because we have found the image $Gr(g)$, $g(\cdot)=\xi_n(\cdot,q_o)$, which is in the tolerance T_n with the known image $Gr(f)$. This also means that the obtained vector q_o describes the visible part of the scene. Such situation always takes place if $Q=\mathbb{R}^n$.

B. If $Q\neq\mathbb{R}^n$ and q_o is not an element of Q then either the image $Gr(f)$ of the scene is distorted by the noises and does not carry any useful information or our information about the scene is wrong or incomplete.

In the second case we also have two possibilities. First, we can verify the information about the illuminated scene. In this case we start the procedure from the very beginning with the modified set Q in $\mathbb{R}^n$ or by introducing the space $\mathbb{R}^s$ where s>n and specifying the set Q in $\mathbb{R}^s$.
Second, we can calculate the constrained model Gr(p) of the image Gr(f) by deriving $\mathbf{q}_o$ from the variational inequality (3.16). We have also to calculate from (3.13) the reaction $r_a^o = L_a(\mathbf{q}_o) - f_a(\mathbf{q}_o)$, to the image constraints. If the norm of $\mathbf{r}^o = (r_1^o, \ldots, r_n^o)$ is, roughly speaking, sufficiently small compared to the norm of $\mathbf{f} = (f_1, \ldots, f_n)$ then the constrained model $Gr(p_o)$, $p_o(\cdot) = \zeta_n(\cdot, \mathbf{q}_o)$, will be referred to as *the tolerance approximation* of the image Gr(f), and the vector $\mathbf{q}_o$ can be interpreted as describing (with a tolerance approximation) the visible part of the illuminated scene under consideration.

After obtaining the image $Gr(p_o)$ which is in the postulated *a priori* tolerance with the known image Gr(f) (or constitutes the constrained model of Gr(f) with the sufficiently small reaction to the image constraints), we can pass to the image matching based on the concept of quasi-global tolerances. In this way we can verify whether the interesting image fragments Gr(f|F), $Gr(p_o|F)$ satisfy the condition

$$(3.25) \qquad (Gr(f|F), Gr(p_o|F)) \in \tilde{t}_\lambda \quad , \lambda = (\varepsilon, \delta)$$

for certain sufficiently small ε and δ. We tacitly assume here that the image Gr(f) on the part F of its background X does not contain any distortions introduced by the image formation process. If the condition (3.25) does not hold then we have to modify the information related to the part of the scene represented by the image fragment $Gr(f|F)$ and we have to start again the image matching procedure from the very beginning.

4. Applications to visual shape perception

The concepts introduced in Sects.2,3 will now be applied to the problem of shape recovering for the visible part Γ_v of a certain surface configuration $\partial\Omega$ ("shape from shading" recovery, cf. [6]). This will be done on the basis of the known (filtered) image $Gr(f)$ of Γ_v and a certain "higher level knowledge" about the presumed shapes of the viewed objects. Thus the underlying assumption is that the visual perception depend not only on the recorded (observed) images but also on what we expect to perceive. This general idea will be formalized in this section into a procedure of image matching based on the concept of a tolerance and introduced in the previous subsection. It has to be emphasized that the visual perception is not related here to the problem of how the brain works, [14], but has to be understood as a certain area of computational vision, [9], restricted to the shape recognition problems.

4.1 Expected images

By *the expected images* of an illuminated surface configuration we shall mean the set of images which can be treated as stored in the memory of a certain natural or artificial system. It is assumed that this system has, roughly speaking, a "finite dimensional memory" and hence every stored image depends on the finite number of real parameters. These are the parameters which describe the visible part of the expected illuminated scene. From the formal point of view this means that the depth function $\delta(\cdot)$ can be introduced in the form

$$\delta(\mathbf{x}) = \sum_{\alpha=1}^{n+1} \sigma_\alpha(\mathbf{x}) d^\alpha , \quad \mathbf{x} \in G , \tag{4.1}$$

where $\sigma_\alpha(\cdot)$, $\alpha=1,\dots,n$, are the known functions and $\mathbf{d} \equiv (d^1,\dots,d^{n+1})$ is an unknown vector in $\mathbb{R}^{n+1}$. A more general form of the interrelation between $\delta(\cdot)$ and $\mathbf{d}$ can be also taken into account. To simplify the considerations let us also assume that the given *a priori* knowledge about the scene comprises the information that the viewed surface configuration is continuous and piecewise smooth. Hence in Eq.(4.1) every $\sigma_\alpha(\cdot)$ is continuous on $\bar{G}$ and piecewise smooth. Without loss of the generality we shall also assume that $\bar{G}=\bar{X}$. Moreover, we posses the following information: (i) the surface configuration is optically ideal and homogeneous with the known coefficient $\varkappa$ in Eq.(2.16); (ii) the scene is illuminated by one concentrated light source, situated near the viewer, which has the known illumination intensity e_o.

The approach presented below and leading to the concept of the perceived image is based on the finite element method adapted to the image analysis problems. The first step of this approach is the decomposition of X into triangle elements E_α,

$\alpha\in A$. Let x^a, $a=1,\ldots,n$ be the nodes of this triangulation and assume that $\sigma_a(\cdot)$ are linear in every E_α, $\alpha\in A$, and satisfy the condition

$$\sigma_a(x^b) = \delta_a^b \quad \text{for every } a,b\in\{1,\ldots,n+1\} . \tag{4.2}$$

It is easy to see that $\sigma_a(\cdot)$ are now the shape functions of the finite element method. It follows that the values of $[\mathrm{grad}\ \delta(x)]^2$ in every finite element E_α, $\alpha\in A$, are independent of $x=(x_1,x_2)$ and equal to

$$(\nabla\delta_\alpha)^2 \equiv (\mathrm{grad}\ \delta(x))^2 = \sum_{a=1}^{n+1}[(\sigma^\alpha_{a,1}(x)d^a)^2+(\sigma^\alpha_{a,2}(x)d^a)^2] \quad \text{for every } x\in E_\alpha,\ \alpha\in A. \tag{4.3}$$

It has to be observed that Eqs.(4.2),(4.3) imply

$$(\nabla\delta_\alpha)^2 = \sum_{b,c=1}^{n} K^\alpha_{bc}(d^b-d^{n+1})(d^c-d^{n+1})/l^2$$

where K^α_{bc} are the known non-dimensional constants and l is a certain length parameter. Hence Eqs.(4.3) involve only n parameters d^a-d^{n+1}, $a=1,\ldots,n$, where d^{n+1} will be treated as an arbitrary constant. Introducing the non-dimensional parameters

$$q^a \equiv (d^a-d^{n+1})/l \ , \quad a=1,\ldots,n \text{ and } \mathbf{q}\equiv(q^1,\ldots,q^n) , \tag{4.4}$$

we obtain

$$(\nabla\delta_\alpha)^2 = K^\alpha_{bc}q^bq^c \ , \quad \alpha\in A ; \tag{4.5}$$

from now on the indices a,b,c run over $1,\ldots,n$ and summation convention with respect to a,b,c is assumed to hold.

It has to be emphasized that in the case of expected images we deal with a certain *ideal situation* in which there are no noises and distortions, where the acuity brightness parameter δ as well as the acuity length parameter ρ tend to zero, and where $N(y)=\{y\}$ in Eq.(2.18). Hence, combining (2.17)-(2.19) (for $\nu(y)=\nu=$const.) and (2.15), (2.16) (for $l_{\varnothing}=0$), under the extra assumption that $\nu\geq\varkappa e_0\beta^{-1}$ and taking into account (4.5), we obtain *the expected image* Gr(p) with the image intensity $p(\cdot)$ given by

$$(4.6)\quad p(x)=\sum_{\alpha\in A}\chi_{E_\alpha}(x)p_\alpha\ ,\quad x\in X\ ,\quad p_\alpha\equiv\frac{\varkappa e_0}{\nu}\,\frac{1}{1+K^{\alpha}_{ab}q^a q^b}\ ,\quad \alpha\in A\ .$$

The set of all expected images Gr(p) will be denoted by $\mathscr{E}(X)$ and represents the constraints in $\mathscr{G}(X)$.

4.2 Perceived images

Now let Gr(f) be the image of the scene under consideration, which is obtained by the optical device as a result of the image formation process described in Sec.1. This means that the possible noises and disturbances cannot be avoided. Image Gr(f) will be referred to as *the recorded image*. We shall assume that this image is known in every problem under consideration.

Now we shall pass to the concept of the perceived image. To this end let us define

$$(4.7)\quad E^a_\alpha\equiv\begin{cases}E_\alpha & \text{if } x^a\in\bar{E}_\alpha\ ,\\ \emptyset & \text{otherwise}\ ,\end{cases}$$

$$F_a\equiv\bigcup_{\alpha\in A}E^a_\alpha\ ,\qquad \mu^a_\alpha\equiv\frac{\mathrm{area}E^a_\alpha}{\mathrm{area}F_a}\ ,\qquad f_a\equiv\frac{1}{\mathrm{area}F_a}\int_{F_a}f(x)\,dx\ ,$$

for every $\alpha\in A$ and $a=1,\ldots,n$. Here Gr(f) is the known recorded image; in the application of the theory we calculate f_a after the filtration of $f(\cdot)$. Using the procedure given in Sec.3.5, we introduce the image decomposition tolerance system (3.6) as the tool for the presented approach. However, it has to be emphasized that now the regions F_a defined by (4.7) are not image cells (in the sense given in Sec.3) because they are not disjointed. Taking into account (3.20), where now $\xi(\mathbf{x},\mathbf{q})$ coincides with the right-hand side of Eq.(4.6), we obtain the following system of equations for the image parameters q^a, $a=1,\dots,n$:

(4.8) $$\sum_{\alpha\in A} \mu^a_\alpha \frac{1}{1+K^\alpha_{bc} q^b q^c} = \frac{\nu f_a}{\varkappa e_o}, \qquad a=1,\dots,n .$$

It is also assumed that the image parameters belong to a certain subset Q of $\mathbb{R}^n$:

(4.9) $$\mathbf{q} = (q^1,\dots,q^n) \in Q ,$$

where Q is the subset of $\mathbb{R}^n$ which is assumed to be known in every special problem. Condition (4.9) together with (4.1) have to represent the extra information (the "higher level knowledge") about the shape of surface configuration. These extra information is necessary because the solution to (4.8), as a rule, is not unique. As an example of Q we can take conditions of the strict convexity of the visible part of the surface configuration, leading to the system of inequalities for $\mathbf{q}=(q^1,\dots,q^n)$.

Let us assume that $\mathbf{q}_o=(q^1_o,\dots,q^n_o)$ is the unique vector which satisfies (4.8) and (4.9). In this case the image $Gr(p_D)$, with the intensity $p_D(\cdot)$ given by Eq.(4.6) for $q^a=q^a_o$, $a=1,\dots,n$, will be referred to as *the discretely perceived image*. Without loss of generality we can assume $d^{n+1}=0$ in Eq.(4.4), and hence, from (4.1) we obtain the function $\delta_o(\cdot)$:

(4.10) $\delta_o(x) = \sigma_a(x) d_o^a, \quad x\in X, \quad d_o^a = 1q_o^a, \quad a=1,\dots,n.$

The function $\delta_o(\cdot)$ represents what will be called *the discretely perceived shape (of the visible part) of the surface configuration.* It has to be remembered that the obtained results of our considerations are restricted to the surface configurations which are continuous on the visual field X. However, the proposed approach can be also applied to more general situations.

Now we shall present a modified form of the system of equations for the image parameters. To this end we introduce the set K of indices which label the straight line segments constituting the boundaries of the particular finite elements E_α. Let us also introduce the new unknowns

(4.11) $\Delta q^k \equiv q^b - q^a, \quad a<b, \quad k\in K, \quad a\in\{1,\dots,n\}, \quad b\in\{1,\dots,n+1\},$

where indices a,b are related to the nodes of the triangulation lattice which determine the straight line segment labelled by the subscript k. Denote by s,t the numbers of elements in A and K, respectively. By means of

$$K^\alpha_{bc} q^b q^c = \sum_{k\in K} G^\alpha_k (\Delta q^k)^2$$

where G^α_k are the known coefficients, we obtain from (4.8) the system of equations for Δq^k, $k\in K$:

(4.12) $$\sum_{\alpha\in A} \mu^a_\alpha \frac{1}{1+\sum_{k\in K} G^\alpha_k (\Delta q^k)^2} = \frac{\nu f_a}{\varkappa e_o}, \quad a=1,\dots,n.$$

Eqs.(4.12) represent the system of n equations for t new unknowns Δq^k, $k=1,\dots,t$. At the same time for every finite element E_α we obtain the obvious interrelation between Δq^k

(4.13) $$\sum_k \delta_k^\alpha \Delta q^k = 0 \ , \quad \alpha=1,\ldots,s$$

where k runs over the three boundary line segments of E_α and δ_k^α are equal to +1 or −1. At last the new image parameters Δq^k, $k=1,\ldots,t$, have to satisfy the extra conditions implied by (4.9) and (4.11). These conditions will be written down in the form

(4.14) $$\Delta \mathbf{q} = (\Delta q^1,\ldots,\Delta q^t) \in \Delta Q \ ,$$

where ΔQ is the known subset of $\mathbb{R}^t$, obtained from (4.9) and (4.11).

Thus we have arrived at the system of n+s equations (4.12),(4.13), for t unknowns Δq^k, $k=1,\ldots,t$, and at the condition (4.14). Let us observe that in triangulation lattices the numbers :

t – of interelement line systems, s – of finite elements and n+1 – of nodes, are interrelated by

$$t = s+(n+1)-1 = s+n \ .$$

It follows that the number of unknowns Δq^k, $k=1,\ldots,t$, is equal to the number n+s of the obtained equations. It can be shown that, under certain condition, the solution to Eqs.(4.12), (4.13) exists but is not unique. Hence, the extra information about the object viewed, which is now expressed in the form (4.14), have to lead to the unique solution $\Delta \mathbf{q}_o \equiv (\Delta q_c^1,\ldots,\Delta q_c^t)$ of (4.12)–(4.14). Then on the basis of (4.11),(4.4) and (4.10) we obtain the discretely perceived shape of the surface configuration.

We end this subsection with the notion of *the perceived image* $Gr(p_p)$. The perceived image will be obtained from the discretely perceived image $Gr(p_D)$ by the smoothing procedure, cf.Sect.3.1, given by

$$p_P(x) = \frac{1}{\text{area}B(x,r)} \int_{B(x,r)\cap X} p_D(y)\,dy \quad \text{for every } y\in X\ , \tag{4.15}$$

where the radius r of the ball $B(x,r)$ on Ox_1x_2 is assumed as the maximum characteristic length dimension of the largest finite element E_α, $\alpha\in A$. The meaning of this notion will be explained in the subsequent subsection.

4.3 Adaptive shape recovering

Now assume that $\mathbf{q}_o=(q_c^1,\dots,q_o^n)$ is the unique solution of Eqs.(4.8),(4.9), which can be also obtained from (4.11) with $\Delta\mathbf{q}=(\Delta q^1,\dots,\Delta q^t)$ as the unique solution of Eqs.(4.12)-(4.14). Vector $\mathbf{q}_o$ determines the discretely perceived image $Gr(p_D)$ and after the smoothing procedure (4.15) we obtain the perceived image $Gr(p_P)$. The general idea of an adaptive shape recovering is based on the tolerance image matching, proposed in Sect.3.5, and takes into account the concepts of the local tolerance system T_λ, $\lambda\in\Lambda$. Let us introduce the known pair $\lambda=(\varepsilon,\delta)$ which determines the local tolerance t_λ that can be accepted by the viewer; it means that any two images $Gr(f),Gr(g)\in\mathcal{G}(X)$ can be treated as indistinguishable if

$$(Gr(f),Gr(g))\in\tilde{t}_\lambda,$$

where $\tilde{t}_\lambda$ is the quasi-global tolerance induced by the local tolerance t_λ, cf.Appendix. The above condition will be used in matching the recorded image $Gr(f)$ to the perceived image $Gr(p_P)$. Hence if

$$(Gr(f),Gr(p_P)) \in \tilde{t}_\lambda\ , \tag{4.16}$$

then the shape recovering obtained *via* the procedure given in Sect.4.2 can be accepted. Now let us assume that the condition (4.16) does not hold. In this case we have to introduce finer decomposition (triangulation) of the image background X onto the set of triangle finite elements then that leading to the images $Gr(p_D)$ and $Gr(p_P)$. This new triangulation, by means of the procedure proposed in Secs.4.1 and 4.2, yields the refined shape of the object viewed and can be examined again by condition (4.16). Hence we obtain *the adaptive shape recovering procedure*, which is controlled by (4.16). This procedure is more efficient if instead of (4.16) we introduce the condition

$$(4.17) \qquad (Gr(f|F), Gr(p_P|F)) \in \tilde{t}_\lambda \;, \quad F \in \mathcal{F} \;,$$

where $\mathcal{F}$ is the postulated *a priori* set of subregions F of X, such that $X \in \mathcal{F}$, cf.Eq.(3.25). If condition (4.17) does not hold only for some F, then we can restrict ourselves to the modification of triangulation only on the part F of X. This procedure has its counterpart in the known adaptive finite element method used in the engineering problems.

4.4 Example

Let $X=(0,n)\times(0,l)$, where n is a positive integer and l is a positive real number. From the computational viewpoint it will be convenient to assume that 1/n is very small compared to 1. In this example we shall investigate an arbitrary cylindrical surface configuration the shape of which depends only on the x_1 coordinate, $x_1 \in (0,n)$. We shall also assume that the conditions mentioned at the beginning of Sect.4.2 concerning optical properties of the viewed objects and the illumination of the scene are fulfilled.

Let us decompose the interval (0,n) into n finite straight line elements $E_a=(a-1,a)$; here and in the sequel index a run over 1,...,n, unless otherwise stated. The points $x_a=a-1$, $a=1,\dots,n$, and $x_{n+1}=n$ will be treated as the nodes of this decomposition. It can be shown, that under denotations

$$f^a \equiv \frac{\nu}{\varkappa e_0}\int_{a-1}^{a} f(x_1)\,dx_1 \ , \qquad a=1,\dots,n \ , \tag{4.18}$$

and setting

$$\Delta q^a \equiv q^{a+1}-q^a \ , \qquad a=1,\dots,n \ , \qquad q^{n+1}\equiv 0 \ , \tag{4.19}$$

the system of equations (4.12) has the form

$$\begin{aligned} &\frac{1}{1+(\Delta q^1)^2} = f^1 \ , \\ &\frac{1}{1+(\Delta q^{a-1})^2} + \frac{1}{1+(\Delta q^a)^2} = f^{a-1}+ f^a \ , \quad a=2,\dots,n \ . \end{aligned} \tag{4.20}$$

The solution of Eqs.(4.20) have the form

$$\Delta q^a = \pm\sqrt{1-f^a} \ , \qquad a=1,\dots,n \ . \tag{4.21}$$

In Sect. 4.1 we have assumed that $\nu/\varkappa e_0 \ge \beta^{-1}$. Because the integral in Eq.(4.18) represents the mean image intensity in the interval (a-1,a), then

$$0 \le \int_{a-1}^{a} f(x_1)\,dx_1 \le \beta \ .$$

From the both aforementioned conditions it follows that

$$f^a \in [0,1] \tag{4.22}$$

and hence all solutions (4.21) are real. Now combining (4.19) and (4.21) we can show that

$$q^{n+1-a}=\pm\sqrt{1-f^{n}}\pm\sqrt{1-f^{n-1}}\pm\cdots\pm\sqrt{1-f^{n+1-a}},\quad a=1,\dots,n, \tag{4.23}$$

are the solutions of the system of equations (4.8). Substituting the right-hand sides of (4.23) (for an arbitrary but fixed system of additions and subtractions of particular terms) we obtain the discretely perceived image intensity function $p_D(\cdot)=p(\cdot)$. However, we are not interested in the form of the image because our aim is to reconstruct the surface configuration. To this end we obtain from (4.4) the values d^a, $a=1,\dots,n+1$, of the depth function at the nodal points $x_a=a-1$, $a=1,\dots,n$; without the loss of the generality we shall assume that $d^{n+1}=0$. Rewriting the right-hand sides of the formula (4.23) in the abbreviated form, we obtain

$$d^{n+1-a}=l\sum_{k=1}^{a}\left(\pm\sqrt{1-f^{n+1-k}}\right),\quad a=1,\dots,n. \tag{4.24}$$

Now taking into account (4.1) and (4.2) we arrive at the reconstructed form of the surface configuration, given by the discretized (piecewise linear) function $\delta(\cdot)$:

$$\delta(x)=\sum_{a=0}^{n}\chi_{[a-1,a]}(x)\left[(a-x)d^{a-1}+(x-a+1)d^{a}\right],\quad x\in[0,n], \tag{4.25}$$

where d^a, $a=1,\dots,n$, are determined by (4.24), $d^{n+1}=0$ and $\chi_{[a-1,a]}(\cdot)$, $a=1,\dots,n$, is, as usual, the characteristic function of the interval $[a-1,a]$. Since the number of solutions (4.2) to the shape recovering problem (equal to $\sum n!/a!(n-a)!$, $a=0,1,\dots,n$) is very large, then the extra information about the shape of the surface configuration, given by condition (4.9), is necessary.

As an example of the aforementioned extra informations (of the "higher level knowledge") let us take the information that the examined surface configuration is convex. In this case we obtain

$$(4.26) \qquad Q = \{q=(q^1,\dots,q^n) \in \mathbb{R}^n \ : \ \Delta q^{\alpha} \geq \Delta q^{\alpha+1} \ \text{for} \ \alpha=1,\dots,n-1\}$$

where Δq^{α} is given by (4.19). Hence using (4.21) we arrive at the conditions

$$(4.27) \qquad \sqrt{1-f^{\alpha}} \geq \sqrt{1-f^{\alpha-1}} \quad \text{for every} \ \alpha=1,\dots,n ,$$

for the mean image intensities (4.18). If (4.27) holds then we obtain the unique reconstruction of the surface configuration shape given by (4.25), where

$$(4.28) \qquad d^{n+1-\alpha} = 1 \sum_{k=1}^{\alpha} \sqrt{1-f^{n+1-k}} \ , \ \alpha=1,\dots,n .$$

If (4.27) does not hold then we deal with the situation which was discussed at the end of Sect.3.4.

5. Applications to geometric modeling of micro-materials

In this section we try to explain how the tolerance image analysis, proposed in Sect.3, can be applied to the mathematical modeling of certain material composite elements. To simplify the considerations we shall confine ourselves to composites which incorporate high-strength fibers situated chaotically in lower strength matrix; moreover, the fibers are assumed to run parallel to a certain line. Hence, every cross-section of a composite by the plane normal to this line can be visualized as a binary image with the intensity function attaining the values 0 and β related to the cross-sections of the matrix and the fibers, respectively. At the same time, the micro-heterogeneous chaotic material structure of the composite imply that the shape of the aforementioned binary images is too complicated to be described by the analytical means or by computer vision. Thus the problem arises how to represent the geometric structure of the composites under consideration in the form which will be useful in the numerical analysis and/or computer vision. This is the main problem of geometric modeling of composites. The possible solution to this problem will be presented below in terms of the tolerance matching analysis of binary images. We shall deal now with the 2-dimensional vision (cross sections of a composite can be treated as plane optically heterogeneous objects) which simplify the image formation process described in Sect.2; however, the possible image distortions cannot be avoided. Thus the recorded images of a composite structure have to be filtered in order to reduce the possible imaging noises. For the sake of simplicity throughout this section all formal results are interpreted in terms of the fibrous composite structures but it has to be emphasized that these results can be also applied to geometric modeling of various micro-materials, e.g., the capillary-porous materials or the materials with chaotically distributed inclusions.

5.1 Micro-heterogeneous images

As it was shown in Sect.3.1, every binary image $I=Gr(f)$ with the background X can be identified with the pair (X,G), where G is a proper subset of X such that areaG>0 and

(5.1) $$f(x) = \chi_G(x) \quad , \quad x \in X \, .$$

At the same time Gr(f) will be referred to as the image of G. For the sake of simplicity here and in the sequel we assume that $\beta=1$, cf.Sect.3.1. Throughout this section the background X will be interpreted as the section across the composite macro-element, normal to the direction of fibers, and G will be the section across the fibers. Hence the subset G of X is composed from a very large number of very small isolated regions. For the recorded images G can be disturbed by possible noises introduced by the image sensing process. In order to describe the highly oscillating and chaotic structure of G we introduce the concept of the heterogeneity parameter of I=Gr(f). To this end define

(5.2) $$\rho(x) \equiv \inf\{r \in R_+ : B(x,r) \cap (X \setminus G) \neq \emptyset \text{ and } B(x,r) \cap G \neq \emptyset\} \, ,$$

for every $x \in X$ and

(5.3) $$\rho = \sup_{x \in X} \rho(x) \, .$$

Moreover, let L stand for the minimum characteristic length dimension of X. The non-dimensional parameter ρ/L will be referred to as the heterogeneity parameter of a binary image I=Gr(f). It can be easily seen that for the composite structures under consideration the heterogeneity parameter is negligibly small compared to 1. Hence, the examined binary images will satisfy the condition

(5.4) $$\frac{\rho}{L} \ll 1 \, ,$$

and will be referred to as *the micro-heterogeneous images.*

At the end of this section we shall also introduce the important concept of macro-elements of the image background.

Let t_λ, $\lambda \in \Lambda$, be the local image tolerance system, $t_\lambda = t_\varepsilon \times t^\delta$, where $\varepsilon \in [0, \varepsilon_o]$, $\delta \in [0, \delta_o]$, cf. Sect. 3.2. Setting $t_\varepsilon x \equiv \{z \in X: (z,x) \in t_\varepsilon\}$ for every $x \in X$, we obtain from the micro-heterogeneous binary image Gr(f) the new image $Gr(f^\varepsilon)$ with the intensity function $f^\varepsilon(\cdot)$:

$$(5.5) \qquad f^\varepsilon(x) \equiv \frac{\operatorname{area}(G \cap t_\varepsilon x)}{\operatorname{area}(t_\varepsilon x)}, \quad x \in X,$$

and we shall refer to $Gr(f^\varepsilon)$ as to the fuzzy image of G in the tolerance t_ε. For every regular subregion F of X and every $\varepsilon \in [0, \varepsilon_o]$ let us define the positive parameter

$$(5.6) \qquad \delta_F(\varepsilon) = \sup_{x', x'' \in F} \{\delta \in R_+ : \delta = |f^\varepsilon(x_1) - f^\varepsilon(x_2)|\} .$$

If

$$(5.7) \qquad \delta_F(\varepsilon) \le \delta$$

then the image fragment Gr(f|F) will be called *macro-homogeneous* in the tolerance t_λ, $\lambda = (\varepsilon, \delta)$. Similarly, if

$$(5.8) \qquad \delta_X(\varepsilon) \le \delta$$

then the image Gr(f) is said to be macro-homogeneous (in the tolerance t_λ). If the condition (5.8) does not hold then Gr(f) will be called *macro-heterogeneous* (in t_λ).

Let L(F) and $\bar{L}(F)$ be the minimum and the maximum characteristic length dimension of F, respectively. The formal definition of the macro-homogeneity (in a certain tolerance t_λ, $\lambda = (\varepsilon, \delta)$) have a physical meaning if $\lambda = (\varepsilon, \delta)$ satisfies the restrictions

(i) $\rho \ll \varepsilon \ll L$ where $L \equiv L(X)$,

(ii) $\delta << 1$,

(iii) $L(F) >> \rho$ for every F such that $\delta_F(\varepsilon)=\delta$.

In the sequel we tacitly assume that every t_λ, $\lambda=(\varepsilon,\delta)$, satisfies the restrictions (i)-(iii).

Let $\mathcal{P}(X)$ be the set of certain specified regular subregions of X (e.g. the set of all quadrangles in X). Then every $F\in\mathcal{P}(X)$ satisfying (5.7) such that $L(F)>>\rho$ and $\bar{L}(F)<<L$, will be called *the macro element* of the image background X.

Remark. The terminology introduced above and related to the micro-heterogeneous images can also be applied to the corresponding micro-heterogeneous material structures.

5.2 Micro-modeling approach

The motivation of geometric modeling of images is based on two facts.

Fact 1. The geometric structure of composites under consideration can be represented by certain micro-heterogeneous (binary) images. Hence, geometric modeling of composite structures can be carried out in terms of the analysis of micro-heterogeneous images.

Fact 2. Micro-heterogeneous images of the investigated (chaotic) composite structures have a very complicated form due to the highly oscillating character of the image intensity field (5.1). Hence, the exact analytical or numerical description of these images as well as their representation in the computational vision is not effective (or even can not be realized) from the point of view of engineering applications.

Thus we arrive at the problem how to represent the geometric structure of micro-heterogeneous images in a certain averaged (approximated) form which can be applicable to the analytical and/or numerical calculations and could be realized by means of computational vision.

The subsequent analysis will be restricted to an arbitrary but fixed macro-element F of the micro-heterogeneous (binary) image G(f). The local image tolerance t_λ, $\lambda=(\varepsilon,\delta)$, which characterizes F, is assumed to fulfill the restrictions (i)-(ii) of the previous subsection. The proposed approach will be based on the tolerance matching analysis introduced in Sect.3.5 and the general procedure will be similar to that of Sect.4 but restricted to the two-dimensional vision and binary images.

Let Gr(f) be the recorded (known *a priori*) image of the plane micro-heterogeneous material structure. Hence Gr(f) is a certain micro-heterogeneous image. Let F be the known macro-element of X and $\mathcal{G}$(F) be the set of all subimages Gr(f|F) of Gr(f) with the background F. Setting

$$(5.9)\quad C(F) \equiv \{Gr(g)\in\mathcal{G}(F) : g(\mathbf{x})=\xi(\mathbf{x},\mathbf{q}) \text{ for almost every } \mathbf{x}\in F \text{ and some } \mathbf{q}\in Q\}$$

with the meaning of $\xi(\cdot)$ similar to that given in Sect.3.3, we introduce the n-parameter image constraints $\mathcal{C}$(F). Here Gr(g) is, for every $\mathbf{q}=(q^1,\dots,q^n)\in Q\subset R^n$, the micro-heterogeneous binary image possessing certain regular geometric structure (for example the periodic structure), and $\xi(\cdot)$ is the postulated *a priori* function. We shall also introduce into considerations the global image tolerance system T_n, $n\in\{1,\dots,m\}$, $T_n\subset\mathcal{G}(F)\times\mathcal{G}(F)$, where

(5.10) $(Gr(f|F), Gr(g|F)) \in T_n \Leftrightarrow f_\alpha = g_\alpha$, $\alpha = 1,\dots,n$,

under the denotations based on those given by Eq.(3.7) :

(5.11) $$f_\alpha \equiv \int_F f(x)\psi_\alpha(x)\,dx \;, \quad \alpha=1,\dots,m \;,$$

and similarly for f_α, where $\{\psi_1(\cdot),\dots,\psi_m(\cdot)\}$ is the postulated system of linear independent functions, defined almost everywhere on F.

The formalized micro-modeling problem can be stated as follows: for the known recorded image Gr(f) of the (chaotic) micro-heterogeneous material structure and for macro-element F of its background find the image Gr(g|F), belonging to the postulated constraints $\mathscr{C}(F)$.Hence, Gr(g|F) is in the global tolerance T_n with the recorded subimage Gr(f|F). The solution to this problem exists if there exists the vector $\mathbf{q}=(q^1,\dots,q^n)$ satisfying the system of equations

(5.12) $$\int_F \xi(\mathbf{x},\mathbf{q})\psi_\alpha(\mathbf{x})\,dx = f_\alpha \;, \quad \alpha=1,\dots,n \;,$$

for f_α previously calculated from (5.11), and fulfilling the condition

(5.13) $$\mathbf{q} = (q^1,\dots,q^n) \in Q \;,$$

where Q is the *a priori* known region in $\mathbb{R}^n$. The form of constrains$\mathscr{C}(F)$, which are determined by the function $\xi(\cdot)$ and the subset Q in $\mathbb{R}^n$, represents a certain "higher level knowledge" about the micro-material composite structure under consideration. In many problems every image $Gr(g) \in \mathscr{C}(F)$ has a periodic structure described by an arbitrary but fixed vector $q=(q^1,\dots,q^n) \in Q$. The solution $q=(q^1,\dots,q^n)$ of Eqs.(5.12), satisfying (5.13), specifies this periodic structure. Thus, the chaotic micro-heterogeneous image $Gr(f|F)$ can be modeled by the periodic micro-heterogeneous image $Gr(g)$, with the intensity field $g(\cdot)\equiv\xi(\cdot,\mathbf{q})$ defined almost everywhere on F. The obtained geometric structure of the image $Gr(g)$ uniquely determines the geometric structure of the mathematical model for the investigated composite material.

5.3 Example

We end this section with a simple illustrative example. Let F be the macro-element of the image background composed of n equi-angular elements F_a; hence $\bar{F}=\cup\bar{F}_a$, $a=1,\dots,n$, $F_a\cap F_b=\emptyset$ for every a,b such that $a\neq b$. Let it be known that the composite structure comprises the fibers with circular cross sections of different radii. In order to specify the tolerance system (5.10) assume that

$$(5.14) \qquad \psi_a(\mathbf{x}) = \frac{1}{\mathrm{area}F_a}\chi_{F_a}(\mathbf{x}) \quad , \ \mathbf{x}\in F \ , \quad a=1,\dots,n \ ,$$

where $\chi_{F_a}(\cdot)$ is the characteristic function of F_a in F. To specify the constraints (5.9) assume that every image $g(\mathbf{x})\in\mathcal{E}(F)$ is the binary image of the set of two-dimensional balls with the centers coinciding with the centers of F_a and with diameters q^a, $a=1,\dots,n$. It means that (l is the length of the F_a side) :

$$(5.15) \qquad Q = \{\mathbf{q}=(q^1,\dots,q^n)\in R^n \ : \ 0\le q^a\le l\sqrt{3}\} \ .$$

Under denotations

$$A \equiv \text{area } F_\alpha \,, \qquad f_\alpha \equiv \frac{1}{A} \int_{F_\alpha} f(x)\,dx \,,$$

the elementary solution of (5.12) yields

$$q^\alpha = \sqrt{\frac{4A}{\pi} f_\alpha} \,, \qquad \alpha = 1,\dots,n \,. \tag{5.16}$$

If the obtained vector $\mathbf{q}=(q^1,\dots,q^n)$ satisfies (5.13) with Q defined by (5.15) then we have formulated a certain geometric micro-model of the chaotic composite structure under consideration in the macro-element F. Moreover, if there exists the real f_o, $f_o \in (0,1)$, such that $(f_o, f_\alpha) \in t_\delta$ for every $\alpha=1,\dots,n$, (such situation takes place in some special cases) then setting

$$d = \sqrt{\frac{4A}{\pi} f_o}$$

we obtain the periodic model of the investigated composite structure in the subregion F of X, in which every fiber cross section has the diameter d. If the condition (5.13) does not hold then we have to pass to more finer decomposition of F into elements or to apply the constraint approach, cf.Sect.3.3. The resulting geometric model can be used as a basis for the formation of engineering models in mechanics of composite structures, *via* different homogenization approaches.

Conclusions

In the Introduction to this contribution we have formulated the leading paradigm which states that *the visual shape perception depends not only on the received (recorded) images but also on our knowledge about the presumed shapes of* *the objects viewed.* The notions of "shape perception", "received (or recorded) images" and "objects viewed" can have different meanings in different physical problems (what was shown in Sects.4 and 5), but the general sense of *the perception paradigm* formulated at beginning of the lecture and quoted above remains unchanged. This hypothesis is a challenge for creation of a mathematical tool which makes it possible to interrelate the experimental facts ("recorded images") with the known *a priori* but uncertain knowledge about the objects under consideration ("higher level knowledge") as the input data in order to obtain the output data describing the interesting features of these objects ("perceived shapes"). This was done in this paper where the formal tool which transformed the perception paradigm into the mathematical form was based on the concepts of *the image constraints* and *the image tolerance systems* introduced and discussed in Sect.3 and in the Appendix. The usefulness of this tool was confirmed by the applications presented in Sects.4 and 5. The proposed approach to the formal realization of the perception hypothesis have some advantages and also some drawbacks. Among the advantages we mention the adaptivity of the formal modeling procedure, which verifies and improves the results obtained at the succeeding stages of this procedure. The main drawbacks of the approach lie in the unprecised choice of the physically accepted tolerances and image constraints which are based rather on the intuition of the researcher than on the clear physical premises. This drawback leads to certain ambiguities in the obtained solutions which should be removed by the verification of the input data.

Looking ahead, the approach proposed in this contribution, based on the concepts of tolerance and constraints, can be treated as the starting point for the formulation of algorithms specifying how the recorded images and the presumed shapes of objects as the input data produce an output representation in the form of reconstructed and/or simplified shapes of these objects.

Appendix. Tolerance systems

The concept of tolerance system was introduced in [12] and constitutes a certain generalization of the notion of the tolerance space, [14]. Here we quote the main ones from the notions discussed in [12].

Let X stand for a nonempty set and t be a tolerance on X, i.e. t is any reflexive and symmetric binary relation on X. We shall write $x_1 t x_2$ for any $x_1, x_2 \in X$ being in a tolerance t.

Definition. A pair $(X,\mathbb{T})$, where $\mathbb{T}$ is a set of tolerances on X satisfying the condition

$$(\forall t_1, t_2 \in \mathbb{T})[t_1 \cap t_2 \in \mathbb{T}] \text{ and } [t_1 \cup t_2 \in \mathbb{T}]$$

will be called *the tolerance system*. The sets X, $\mathbb{T}$ will be referred to as *the underlying set* and *the tolerance lattice* of $(X,\mathbb{T})$, respectively.

Let $\bar{t}$ be the transitive closure of t, i.e. the binary relation in X defined for every $x_0, x \in X$ by $x_0 \bar{t} x$ if $x_0 t x_1$, $x_1 t x_2, \ldots, x_n t x$ for some $x_1, x_2, \ldots, x_n \in X$. If $(\forall \bar{t} \in \mathbb{T})(\forall x_1, x_2 \in X)\cdot$

$\cdot[x_1\bar{t}x_2]$, then the tolerance system $(X,\mathbb{T})$ will be called connected. If $(\mathbb{T},\subset)$ is a chain (for every $t_1,t_2\in\mathbb{T}$ there is either $t_1\subset t_2$ or $t_2\subset t_1$), then $(X,\mathbb{T})$ will be called ordered; every two tolerances of such a system can be compared by inclusion. If $(\forall t_1,t_2\in\mathbb{T})[t_1t_2\in\mathbb{T}]$, then $(X,\mathbb{T})$ is said to be S-closed (closed with respect to the superimposition t_1t_2 of tolerances) where we define $t_1t_2:=\{(x_1,x_2)\,|\,x_1t_1x$ and xt_2x_2 for some $x\in X\}$ as a superimposition of any two tolerances on X; at the same time $t_1t_2\in\mathbb{T}$ if and only if $t_1t_2=t_2t_1$. The tolerance lattice $\mathbb{T}$ of any S-closed tolerance system $(X,\mathbb{T})$ constitutes an Abelian semigroup with respect to the operation of superimposition.

Remark. If $\mathbb{T}=\{t\}$, then the tolerance system $(X,\mathbb{T})$ reduces to the tolerance space of E.C.Zeeman, [14].

Example 1. Let (X,d) be a metric space with a metric distance function $d: X\times X\to\mathbb{R}$, and let $\mathbb{T}=(t_r)_{r\in\mathbb{R}+}$, where $x_1t_rx_2$ if $d(x_1,x_2)<r$, for every $x_1,x_2\in X$. Then $(X,\mathbb{T})$ is a tolerance system and will be called a metric tolerance system. Every such system is ordered and S-closed.

Example 2. Let $(X,\|\cdot\|)$ be a linear normed space and let $\mathbb{T}=(t_\varepsilon)_{\varepsilon\in(0,1)}$, where x_1tx_2 if $\|x_1-x_2\|<\varepsilon(\|x_1\|\wedge\|x_2\|)$ (the symbols $\wedge$, $\vee$ stand for min., max.,respectively), for every $x_1,x_2\in X$, $x_1\neq x_2$. Then $(X,\mathbb{T})$ is a tolerance system and will be called a relative tolerance system.

Example 3. Let $(\mathbb{R}_+,\circ)$ be an Abelian semigroup such that $(\forall\alpha,\beta\in\mathbb{R}_+)[\alpha\circ\beta\geq\alpha\vee\beta]$ and $\delta: X\times X\to\mathbb{R}$ be a function satisfying, for every $x_1,x_2\in X$, the following conditions: $\delta(x_1,x_2)\geq 0$, $\delta(x_1,x_2)=$ $=\delta(x_2,x_1)$, $[\delta(x_1,x_2)=0]\Rightarrow[x_1=x_2]$, $\delta(x_1,x_2)\leq\wedge\delta(x_1,x)\circ\delta(x,x_2)$ (the function δ is said to be a distance function for an operation $\circ$ in $\mathbb{R}_+$). Let $\mathbb{T}=(t_r)_{r\in\mathbb{R}+}$ where x_1tx_2 if and only if $\delta(x_1,x_2)<r$, for every $x_1,x_2\in X$. Then $(X,\mathbb{T})$ is a tolerance system which will be referred to as a distance tolerance system (ultra-metric if $\circ$ stands for $\vee$ or metric if $\circ$ stands for +).

It can be shown that from the known tolerance systems certain new tolerance systems can be easily constructed.

For example if $(X,\mathbb{T}),(Y,\mathbb{S})$ are tolerance systems, then $(X\times Y,\mathbb{T}\times\mathbb{S})$, where for every $(x_1,y_1),(x_2,y_2)\in\mathbb{T}\times\mathbb{S}$ there is $(x_1,y_1)(t\times s)(x_2,y_2)$ if x_1tx_2 and y_1sy_2, is also a tolerance system.

Define $tF=\{x\in X : (x,y)\in t$ for some $y\in F\}$, where $F\subseteq X$. Following [14] we also introduce a tolerance system $(\mathcal{L}(X),\tilde{\mathbb{T}})$ on the lattice $\mathcal{L}(X)$ of subsets of X which is induced by the tolerance system $(X,\mathbb{T})$, setting $(F_1,F_2)\in\tilde{t}$ if $F_1\subset tF_2$ and $F_2\subset tF_1$ for $t\in\mathbb{T}$. For more detailed discussion of the tolerances cf.[12,14].

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Subject Index