Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 1, pp. 69-86, Warszawa 1973

Extremum principles in the dynamics of rigid-plastic bodies and mathematical programming

M. I. ERKHOV (MOSCOW)

Two variational principles for dynamically loaded rigid-plastic bodies are established as an extension of corresponding theorems of static limit analysis. Certain alternative forms of these principles are suggested, allowing for a rapid integrated estimation of an avarage permanent displacement and response time of the structure. The problem of dynamic loading is also formulated as a problem of mathematical programming. As an example, the problem a simply supported circular plate acted on by a uniformely distributed rectangular pressure pulse is solved. Good agreement with the available exact solution was obtained.

Ustanowione zostały dwie zasady wariacyjne dla dynamicznie obciążonych sztywno-plastycznych ciał. Zasady te są odpowiednim uogólnieniem znanych twierdzeń z teorii nośności granicznej konstrukcji. Podane są również alternatywne sformułowania powyższych zasad pozwalające na szybką ocenę uśrednionych trwałych ugięć i czasu pracy konstrukcji. Problem dynamicznego obciążenia jest sformułowany również jako zadanie matematycznego programowania. Jako przykład rozwiązano numerycznie swobodnie podpartą płytę kołową obciążoną równomiernie rozłożonym prostokątnym impulsem ciśnienia. Otrzymano dobrą zgodność ze znanym dokładnym rozwiązaniem tego problemu.

Установлены два вариационных принципа для динамически нагруженных жестко-пластических тел. Эти принципы соответствующим образом обобщают известные теоремы теории несущей способности сооружений. Даются варианты формулировок этих принципов, позволяющие получать простым путем интегральные оценки остаточных прогибов и времени работы сооружения. Задача о динамическом нагружении формулируется также, как задача математического программирования. В качестве примера дано численное решение задачи об импульсном нагружении равномерно распределенным давлением свободно опертой круглой пластинки. Получено хорошее совпадение с известным точным решением этой задачи.

Introduction

The classical theorems of limit analysis has proved to be the basic approximate method of solution for structures loaded statically into the plastic range. These theorems provide upper and lower bounds for the collapse load by means of a proper technique of minimalization (maximalization) of certain functionals. The foundations of the static theory of limit analysis, together with proofs of the extremal properties of the limit load, weie laid by G. Kazinczy [1], A. A. Gvozdiev [2], S. M. Feinberg [3], D. C. Drucker, H. J. Greenberg, W. Prager [4], and R. Hill [5].

This theory is related to extremum principles for rigid-plastic bodies under static loading established by A. A. Markov [6], S. M. Feinberg [3], W. Prager and P. G. Hodge [7], R. Hill [5] and W. Koiter [8].

However, in the case of dynamic problems, when inertia forces cannot be disregarded, the existing extremal principles and the static theory of limit analysis is no longer applicable. It is also not possible to use a purely "static" or purely "kinematic" method of solution. On the other hand, the proper reformulation of the extremum principle to cover dynamic problems for rigid-plastic bodies is highly desirable from both the practical and the theoretical points of view.

It should be noted that accelerations in rigid-plastic bodies can be determined approximately, using the methods of analytic mechanics. In particular, W. P. TAMUZH [9] applied the Gauss principle of least action, while A. R. RZHANITZIN [10] suggested the Hamilton principle in the case of a velocity profile stationary in time.

The second line of approach was developed by J. B. MARTIN [11] who established the upper bound on the permanent deformation and the lower bound for the response time of impulsively loaded structures (structures with prescribed initial velocity). In Martin's theorems were introduced two time independent statically and kinematically admissible fields. These give approximation to the actual stresses and velocities. The corresponding inequalities involve uknnown (not prescribed) quantities (permanent displacement and time for rest) and hence the theorems are of bounding rather than variational character. The bound would coincide with the exact solution only if the latter is given by a single mode velocity pattern. Since the Martin theorems refer above all to the case of impulsively loaded bodies, it should be noted that in all available exact solutions the velocity and deflection profiles appeared to be time variable. An improved method of solution for impulsively loaded structures, utilizing a one-degree-of-freedom velocity field, was developed by J. B. MARTIN and S. S. SYMONDS [12]. The method was illustrated on the examples of beams. Analyzing the stability condition, the authors were able to prove uniqueness of the velocity field.

In the present paper, extremum principles of the dynamics of rigid-plastic bodies under arbitrary loading are formulated. These principles are equivalent to the statement that certain functionals depending on the approximate fields of stresses, velocities and acceleration attain maximum or minimum.

The suggested principles provide criteria for the existence of the solution and a choice of an approximate solution converging to the exact solution. They also enable the application of various variational methods and even the formulation of the given dynamic problem as an appropriate problem of mathematical programming. It is essential that minimum (or maximum) values of the functionals introduced be always equal to zero.

In addition, some extremal properties for loadings and accelerations are considered, leading to determination of the bounds on these quantities by means of approximate values of loads and accelerations. This approach is also of a variational character. The general dynamic problem for rigid-plastic bodies has been formulated in this paper as a problem of linear and also non-linear programming. Thus, the response of any dynamically loaded structure can be determined with great accuracy on a high-speed electronic computer. As an illustration of the applicability of the method of linear programming, a simply supported circular plate is solved and the results obtained are compared with the exact solution presented by H. G. HOPKINS and W. PRAGER [13]. The method of mathematical programming is used to solve the dynamic problem of square plate.

The general theorems proved in the present paper are illustrated on the exemplary solution of a shallow rotationally symmetric shell loaded dynamically into the plastic range.

No exact solution of this problem is available in the literature and it would be difficult to derive such solution by means of classical methods used, for instance, in [13].

In what follows it is assumed that the yield condition does not depend on the deformation process and the shearing stresses do not enter into the constitutive equations. The rotary inertia is disregarded and deformations are considered small.

1. Extremum principles

A solution of the dynamic problem should satisfy the following conditions:

a) equation of motion and boundary conditions on the body surface S

$$\sigma_{ij,j}+X_i-\gamma\ddot{u}_i = 0, \qquad \sigma_{ij}n_j = p_i;$$

b) yield condition $f(\sigma_{ij}) \leqslant 0$;

c) components of the acceleration vector $\ddot{u}_i = \partial\dot{u}_i/\partial t$;

d) components of the velocity vector should satisfy the incompressibility condition and kinematic boundary conditions, $\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i})$;

e) fields of displacements u_i, velocities $\dot{u}_i(\dot{\varepsilon}_{ij})$ and accelerations $\ddot{u}_i$ should conform to the initial conditions;

f) the yield function $f(\sigma_{ij})$ is taken as a plastic potential for the strain rate tensor and the associated flow rule is assumed.

In the above formulae, σ_{ij} denotes components of a stress tensor, X_i and p_i denote respectively vectors of body force and surface tractions; γ is mass density, and n_j is a unit normal vector to the surface S, t denotes time and $i, j = 1, 2, 3$.

Let us introduce the following notation:

$$\int_V \sigma_{ij}\dot{\varepsilon}_{ij}dV - \int_V X_i\dot{u}_i dV = F(\sigma_{ij}, \dot{u}_i),$$

$$\int_V \gamma\ddot{u}_i\dot{u}_i dV = J(\ddot{u}_i, \dot{u}_i), \qquad \int_V p_i\dot{u}_i dS = I(p_i, \dot{u}_i),$$

where V denotes the volume of the body. The body forces X_i are regarded as known quantities. Using the symbols defined above, the principle of virtual velocities takes the form:

$$I(p_i, \dot{u}) = F(\sigma_{ij}, \dot{u}_i)+J(\ddot{u}_i, \dot{u}_i). \tag{1.1}$$

It should be emphasized that in (1.1) the quantities σ_{ij}, X_i, p_i and $\ddot{u}_i$, satisfying the equation of equilibrium, are in general quite independent of the kinematic quantities $\dot{u}_i$ and $\dot{\varepsilon}_{ij}$. This means that the conditions c) and f) should not necessarily be satisfied by quantities entering Eq. (1.1). However, the fields $\dot{u}_i$ and $\dot{\varepsilon}_{ij}$ must meet the requirement d); any velocity field satisfying the latter condition is called admissible and is indicated by "stars" (for example $\dot{\varepsilon}^*_{ij}, \dot{u}^*_i$). At the same time, the quantities σ_{ij}, $\ddot{u}_i$, X_i and p_i should necessarily satisfy conditions a) and b). The stress field σ_{ij} falling in the class defined above is called admissible and is denoted by σ^0_{ij}. The determination of p_i and u_i will be described later.

We shall make use of the following Drucker postulate [8]:

$$(\sigma_{ij}-\sigma^0_{ij})\dot{\varepsilon}_{ij} \geqslant 0 \quad \text{or} \quad (\sigma^*_{ij}-\sigma^0_{ij})\dot{\varepsilon}^*_{ij} \geqslant 0, \tag{1.2}$$

where σ_{ij}, $\dot{\varepsilon}_{ij}$, σ^*_{ij}, $\dot{\varepsilon}^*_{ij}$, σ^0_{ij} are functions of time and σ_{ij} and σ^*_{ij} are related respectively to $\dot{\varepsilon}_{ij}$ and $\dot{\varepsilon}^*_{ij}$ by an associated flow rule.

Suppose to a certain admissible velocity field $\dot{u}_i^{**}$ there corresponds an acceleration field $\ddot{u}_i^{**}$. Now, with these accelerations $\ddot{u}_i^{**}$ a stress field σ_{ij}^0 can be found so as to satisfy the condition a). According to (1.1),

$$I(p_i, \dot{u}_i^*) = F(\sigma_{ij}^0, \dot{u}_i^*) + J(\ddot{u}_i^{**}, \dot{u}_i^*), \tag{1.3}$$

where $\dot{u}_i^* \neq \dot{u}_i^{**}$. A question arises as to the existence of the equilibrium set $(\sigma_{ij}^0, \ddot{u}_i^{**}, p_i)$ different from the actual solution $(\sigma_{ij}, \ddot{u}_i, p_i)$. The existence of such a system $(\sigma_{ij}^0, \ddot{u}_i^{**}, p_i)$ can be shown, if these quantities are taken to be an exact solution for a certain yield surface inscribed in the given yield surface. Indeed, let $\tilde{\sigma}_{ij}$, $\tilde{\ddot{u}}_i$, $\tilde{\dot{u}}_i$, p_i denote the exact solution obtained for the "inscribed" yield function satisfying all the requirements a)–f). The stress field $\tilde{\sigma}_{ij}$ appears to be admissible for the exact yield surface — i.e., it can be denoted $\tilde{\sigma}_{ij} = \sigma_{ij}^0$. This field together with the acceleration $\tilde{\ddot{u}}_i$ and loading p_i satisfies the condition a). Since $\tilde{\ddot{u}}_i$ and $\tilde{\dot{u}}_i$ satisfy respectively conditions c) and d), they can be denoted by $\tilde{\ddot{u}}_i = \ddot{u}_i^{**}$, $\tilde{\dot{u}}_i = \dot{u}_i^{**}$. This completes the proof of the existence of the functions $(\sigma_{ij}^0, \ddot{u}_i^{**}, p_i)$. An example of the determination of such functions is given in Sec. 3. It is understood, as is usually assumed in the theory of plasticity, that the solution of a given dynamic problem exists.

The kinematically admissible velocity field $\dot{u}_i^*$ is, in general, distinct from that resulting from the acceleration field $\dot{u}_i^* \neq \dot{u}_i^{**}$. It is clear, however, that $\dot{u}_i^{**}$, as kinematically admissible, can be chosen to be the desired field. Denoting now $\ddot{u}_i^{**}$ and $\dot{u}_i^{**}$ respectively, by means of one star subscripts $\ddot{u}_i^*$, $\dot{u}_i^*$, the identity (1.3) can be written as

$$I(p_i, \dot{u}_i^*) = F(\sigma_{ij}^0, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*). \tag{1.4}$$

Equation (1.4) together with (1.2) yields

$$F(\sigma_{ij}^*, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*) - I(p_i, \dot{u}_i^*) \geqslant 0, \tag{1.5}$$

which is the desired minimum principle. It states that for an actual solution the functional (1.5) attains minimum and is equal to zero.

Equation (1.4) can be rewritten with p_i substituted by p_i^{0*}, where $p_i^{0*} \neq p_i$,

$$I(p_i^{0*}, \dot{u}_i^*) = F(\sigma_{ij}^0, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*). \tag{1.6}$$

The existence of the system of functions $(\sigma_{ij}^0, \ddot{u}_i^*, p_i^{0*})$ can be proved similarly as was done for the system $(\sigma_{ij}^0, \ddot{u}_i^*, p_i)$, entering (1.3). For given $F(\sigma_{ij}^*, \dot{u}_i^*)$ and $J(\ddot{u}_i^*, \dot{u}_i^*)$ one can always find such $I(p_i^*, u_i^*)$ as to satisfy the equality

$$I(p_i^*, \dot{u}_i^*) = F(\sigma_{ij}^*, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*). \tag{1.7}$$

The quantities $(\ddot{u}_i^*, \sigma_{ij}^*, p_i^*)$ may violate the condition a), the only requirement being that p_i^* satisfy (1.7), (an example of such function will be given again in Sec. 3). Hence Eq. (1.7), by contrast with Eqs. (1.3), (1.4) and (1.6), is not a principle of virtual velocities. In particular, the choice of $(\sigma_{ij}^0, \ddot{u}_i^*, p_i^{0*})$ can be made so that $p_i^* = p_i$, where p_i is the actual loading. Equation (1.7) then takes the form:

$$I(p_i, \dot{u}_i^*) = F(\sigma_{ij}^*, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*). \tag{1.8}$$

In view of the condition (1.2), Eq. (1.8) yields:

$$F(\sigma_{ij}^0, \dot{u}_i^*) + J(\ddot{u}_i^*, \dot{u}_i^*) - I(p_i, \dot{u}_i^*) \leqslant 0, \tag{1.9}$$

where σ_{ij}^0, $\ddot{u}_i^*$ and p_i^{0*}, according to (1.6), satisfy the condition a). The above inequality expresses the following maximum principle: the functional (1.9) attains maximum and is equal to zero for the actual solution.

Assuming the system of functions according to (1.5) and (1.9), the exact solution (or in view of mathematical difficulties an approximation to this solution) is obtained by a suitable minimalization or maximalization of the functionals (1.5) or (1.9), using well established mathematical methods.

In the derivation of both principles according to (1.3), $\ddot{u}_i^*$ can be replaced by $\ddot{u}_i^{**}$, for it is known that the solution of dynamic problems is unique. Similarly to (1.5) and (1.9), various other forms of the extremum principles can easily be established.

For accelerations equal to zero, the principles obtained reduce to the known extremal static principles of rigid-plastic bodies.

2. Alternative forms of dynamic principles

An addition to the results established in the preceding section, a new form of extremal principles can be suggested which does not involve acceleration and velocity fields. Such forms of the theorems are of considerable interest from the practical point of view.

The time integration of the inequality (1.5) from 0 to t gives

$$\int_0^t F(\sigma_{ij}^*, \dot{u}_i^*)dt + \frac{1}{2} J(\dot{u}_i^*, \dot{u}_i^*) - \int_0^t I(p_i, \dot{u}_i^*)dt \geqslant 0, \tag{2.1}$$

where $u_i^* = \dot{u}_i^* = 0$ for $t = 0$.

An alternative form of the minimum principle is obtained from (2.1) if $t \geqslant t_k$, where t_k is so called "time to rest":

$$F(\sigma_{ij}^*, u_i^*) - \int_0^{t_k} F(\dot{\sigma}_{ij}^*, u_i^*)dt - I(p_i, u_i^*) + \int_0^{t_k} I(\dot{p}_i, u_i^*)dt \geqslant 0. \tag{2.2}$$

The expression (2.2) msy be linearized. Integrating in a similar way the expression (1.9) from 0 to t, a corresponding maximum principle is obtained not involving accelerations or velocities. These forms of extremum principles describe in particular the cases of a removed loading $p_i = 0$, or an impulsive loading, where $p_i = 0$, but an initial velocity field is prescribed. The extremal principles of dynamics derived above constitute the fundamental relations of the dynamic theory of limit analysis (such a theory should be more properly called "the theory of limit strength"), since they provide criteria for the rational choice of the solution. The method of the approximate solution of dynamic problems is indicated by the very structure of the principles which determines the choice of the stress accelerations and velocity fields [14].

3. Example

The application of the principles derived in Sec. 1 requires minimalization or maximalization of certain functionals depending on approximate fields of velocities, accelerations and stresses. The value of the functionals computed for these approximate functions would, of course, differ from zero, so that the difference can be taken as a certain measure of the accuracy of the solution. On the other hand, the degree of approximation can be estimated by means of certain integral relations for accelerations and appropriate relations for the loading. Such relations express the extremal properties of accelerations and loadings.

Substituting $p_i = p_i^*$ in (1.7) and using (1.3), we obtain:

$$J(\ddot{u}_i^{**}, \dot{u}_i^*) \geqslant J(\ddot{u}_i^*, \dot{u}_i^*). \tag{3.1}$$

It can further be shown that the $\ddot{u}_i^{**}$ (integrated over the volume) is an upper bound for $\ddot{u}_i$ while $\ddot{u}_i^*$ is a corresponding lower bound.

Comparison of (1.6) and (1.7) yields:

$$I(p_i^*, \dot{u}_i^*) \geqslant I(p_i^{0*}, \dot{u}_i^*), \tag{3.2}$$

where either $p_i^* = p_i$ or $p_i^{0*} = p_i$.

The procedure described above will be explained on the example of a shell loaded by a rectangular pressure pulse. Consider a rigid-plastic shallow spherical cap loaded by a uniformly distributed pressure p. It is assumed that p is suddenly applied at $t = 0$, then held constant, and removed at $t = t_1$ i.e., $p = 0$ for $t \geqslant t_1$. The shell is simply supported, initial velocities and deflections are zero.

The equations of motions are (Fig. 1)

$$\begin{gathered} \frac{\partial}{\partial \varrho}(\varrho N_1) - N_2 = 0, \qquad -\frac{\partial}{\partial \varrho}(\varrho M_1) - \varrho Q_1 + M_2 = 0, \\ \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho Q_1) - \frac{N_1}{R} - \frac{N_2}{R} - p + \gamma \ddot{w} = 0, \end{gathered} \tag{3.3}$$

where M, N and Q denote respectively internal bending moments and axial and shear forces; w is radial deflection, γ denotes the mass density per unit area of the shell middle surface, and R and ϱ are defined in Fig. 1. The indices 1 and 2 refer respectively to the radial and circumferential directions.

We shall use equations of an approximate yield surface inscribed and circumscribed on the exact one [15]

$$|n_1| = |n_2| = n \leqslant k_1, \quad m_2 = m = \sqrt{k_1^2 - n^2}, \quad m_1 \leqslant m, \quad k_1^2 \leqslant 0.75, \tag{3.4/1}$$

$$|n_1| = |n_2| = n = k_2, \quad m_2 = m = k_3, \quad m_1 \leqslant k_3, \quad |k_2| \geqslant 1, \quad |k_3| \leqslant 1, \tag{3.4/2}$$

where $n = N/2\sigma_s h$, $m = M/\sigma_s h^2$, σ_s being the yield stress and $2h$ is the wall thickness of the shell.

Consider the case of the "intermediate" pressures, defined by

$$\frac{p}{p_s} < 2+2n\varrho_0^2/(3m_2Rh-2n\varrho_0^2), \tag{3.5}$$

where p_s denotes the static collapse load. According to (3.4), bounds on the static loads are:

$$\frac{6h^2}{\varrho_0^2}\sqrt{k_1^2-n^2}-\frac{4hn}{R} = \frac{p_s^0}{\sigma_s} \leqslant \frac{p_s}{\sigma_s} \leqslant \frac{p_s^*}{\sigma_s} = \frac{6h^2}{\varrho_0^2}k_3-\frac{4h}{R}k_2, \tag{3.5/1}$$

where $n = -2k_1\varrho_0^2/\sqrt{9R^2h^2+4\varrho_0^4}$.

Pressures exceeding the limiting value defined by (3.5) are called "high".

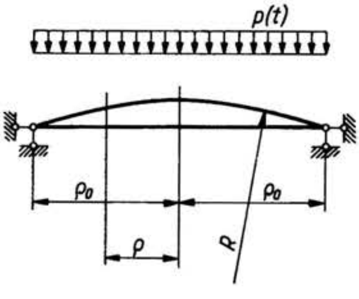


FIG. 1.

The kinematically admissible field of deflection in the phase $0 \leqslant t \leqslant t_1$ is assumed in the form:

$$w^* = w_0(t)\left(1-\frac{\varrho}{\varrho_0}\right), \quad u^* = 0, \tag{3.6}$$

where u^* denotes the horizontal component of the displacement vector.

The components of the generalized strain rate vector are:

$$\dot{\varepsilon}_1 = \frac{\partial\dot{u}}{\partial\varrho}-\frac{\dot{w}}{R} = -\frac{\dot{w}_0(t)}{R}\left(1-\frac{\varrho}{\varrho_0}\right), \quad \dot{\varepsilon}_2 = \frac{\dot{u}}{\varrho}-\frac{\dot{w}}{R} = \dot{\varepsilon}_1, \quad \dot{\varkappa}_2 = -\frac{\partial\dot{w}}{\varrho\partial\varrho} = \frac{\dot{w}_0(t)}{\varrho\varrho_0},$$

$$\dot{\varkappa}_1 = -\frac{\partial^2\dot{w}}{\partial\varrho^2} = 0.$$

Internal forces can now be computed from (3.4) and (3.3):

$$m_2 = m, \quad n_1 = n_2 = n,$$
$$m_1 = m-\frac{p\varrho^2}{6\sigma_s h^2}+\frac{\gamma\ddot{w}_0\varrho^2(2-\varrho/\varrho_0)}{12\sigma_s h^2-2n\varrho_0^2/3Rh}. \tag{3.7}$$

The condition (3.5) is obtained directly from $\partial m/\partial\varrho|_{\varrho=0} \leqslant 0$. Since $m_1 = 0$ at the outer edge $\varrho = \varrho_0$, the amplitude w_0 can easily be computed from (3.7):

$$w_0 = \frac{(p-p_s)t^2}{\gamma}. \tag{3.8}$$

Taking in the second phase $t_1 \leqslant t \leqslant t_k$ the same form (3.6) for the function w^*, and using the continuity conditions of $w(t_1)$ and $\dot{w}(t_1)$, the permanent deformation and response time are given by

$$w^*(\varrho, t_k) = \frac{pt_1^2(p-p_s)}{\gamma p_s}\left(1-\frac{\varrho}{\varrho_0}\right), \quad t_k = t_1\frac{p}{p_s}. \tag{3.9}$$

It is easy to check that the boundary condition at $\varrho = \varrho_0$ imposes the following condition:

$$\frac{p_s}{\sigma_s} \geqslant -\frac{8nh}{R} \quad \text{or} \quad m \geqslant -\frac{2\varrho_0^2 n}{3Rh}. \tag{3.10}$$

The above condition can be satisfied by taking for m and n the respective values: $0 \leqslant \leqslant |n| = k_1 \leqslant 0.866$ for the inscribed yield surface $(3.4)_1$ and $n \leqslant -1$, $m \geqslant 1$ for the circumscribed yield condition.

The case of "high" pressures can easily be treated in a similar way.

In the approximate solution discussed above, approximate stress, velocity and acceleration functions were introduced, satisfying the necessary conditions defined in Sec. 1. The evaluation of the functionals appearing in the principles proved would give some indication as to the accuracy of the approximate solution. [The admissible stresses correspond to the surface $(3.4)_1$, the pressure p and acceleration $\ddot{w}^*$ being in equilibrium. The strain rates $\dot{\varepsilon}_2^*$, $\dot{\varepsilon}_1^*$, $\dot{\varkappa}_2^*$, resulting from $\dot{w}^*$, are computed from the flow rule associated with the yield surface $(3.4)_2$].

Substituting in (3.9) $p_s = p_s^0$ or $p_s = p_s^*$, according to $(3.5)_1$ and (3.1), we obtain respectively the upper and lower bound on the function $\ddot{w}$ (and hence on w), integrated over the surface of the shell. It can be shown, using (3.2), that accuracy of the computed bounds is determined by the values p_s^0 and p_s^*.

4. Formulation of the mathematical programming problem

In recent years, considerable attention has been paid in the theory of limit analysis to certain practical methods of solution by means of linear programming, which is characterized by the necessary accuracy and high degree of automatization in computations. The corresponding mathematical methods are well developed and routine computer programms are now available for carrying out difficult and time-consuming calculations. The method of convex programming has also proved to be of practical interest. By comparison with static problems, only a limited member of dynamic problems for rigid-plastic solids have been solved. The application of the methods of mathematical programming to such problems is therefore of considerable importance. A typical initial-boundary value problem for rigid-plastic structure will now be formulated as a problem of linear and convex programming.

The functional to be minimalized is given by (1.5). Taking into account that initial ($t = 0$) and final ($t = t_k$) velocities are zero, the functional (1.5) can be replaced by (2.2)

$$Z = \int_0^{t_k}\int_V a\,dV\,dt - \int_0^{t_k}\int_S p_i\dot{u}_i^*\,dS\,dt, \quad a = \sigma_{ij}^*\dot{\varepsilon}_{ij}^*. \tag{4.1}$$

We shall be concerned with plates and shells, hence, generalized stresses E_j and strain rates $\dot{e}_j$ will be used rather than stresses σ_{ij} and strain rates $\dot{\varepsilon}_{ij}$. The dissipation $a = \sigma_{ij}^* \varepsilon_{ij}^*$, appearing in (4.1), should then be replaced by $a = E_j^* \dot{e}_j^*$, where $j = 1, 2, \ldots$

The yield condition is in general represented in the space E_j by a certain non-concave hypersurface. The nonlinear yield surface is approximated by a set of hyperplanes defining a piece-wise linear yield surface. It can be shown [16], that the equality $a = E_j^* \dot{e}_j^*$ is now replaced by a system ofi nequalities with the properties of dissipation functions a retained:

$$a \geqslant E_j^{(m)} \dot{\varepsilon}_j^*, \tag{4.2}$$

where $m = 0, 1, 2, \ldots$ denotes the number of corner points in the piece-wise linear yield hypersurface, $E_j^{(m)}$ is a generalized stress at the m-th corner point (the associate flow rule holds). Now, the dynamic problem for rigid-plastic plates or shells can be reduced to the corresponding problem of linear programming and can be solved by means of the simplex method. To this end, the time and space coordinates are discretized by means of finite differences [in the case of plates and shells the volume integration in (4.1) is replaced by integration over the middle surface S]. Consequently, the continuous fields of stresses, displacements and rate of energy dissipation are described by a finite number of corresponding parameters, defined in the nodal points of the space-time net.

The problem of linear programming can finally be stated as follows: Find a minimum of the objective function Z (4.1) under the restrictions a) for E_j^0 and $\ddot{u}_i^*$, b) for E_j^0 and (4.2) for a. The function Z is linear with respect to u_i^* in nodal points, E_j and u_i^* being free variables while a is not a free variable ($a \geqslant 0$).

The problem of linear programming formulated above can be reduced to the corresponding problem of convex programming. The method consists now in solving the problem by consecutive steps in time t. For each step (for example for the step $t_n - t_{n-1}$) a minimum is sought for the functional (2.1)

$$Z_n = \int_{t_n}^{t_{n+1}} \int_S a dS dt + \frac{1}{2} \int_S \gamma \{[\dot{u}_i^*(t_{n+1})]^2 - [\dot{u}_i^*(t_n)]^2\} ds - \int_{t_n}^{t_{n+1}} \int_S p_i \dot{u}_i^* dS dt,$$

under the same restriction as in the preceeding case, formulated for each time step (the restriction a) should now be modified; instead, the corresponding equation is solved for stresses).

5. Bounding theorems

An approximate value of permanent displacements occurring in dynamically loaded structures can be determined by the methods described above. It is possible to find certain direct bounds on maximum deflections and response time.

For simplicity, consider a structure with initial velocity and displacement equal to zero. Let the structure be loaded by a rectangular-pressure pulse of intensity p_i and duration time t_1. Integrating over time the equation of motion and boundary conditions a) and yield condition b) in the interval 0 to t_i, we obtain:

$$\int_0^{t_k} (\sigma_{ij,j}) dt + \int_0^{t_k} X_i dt = 0, \qquad \int_0^{t_k} \sigma_{ij} n_j = p_i t_1 \text{ on } S, \qquad t_k f(\sigma_{ij}) \leqslant 0, \tag{5.1}$$

where t_k is the so called time for rest.

Equations (5.1) are fulfilled by a certain fictitious static solution. Following the first theorem of limit analysis concerning the lower bound on the collapse load, we conclude from (5.1) that

$$p_i t_1 \leqslant p_i^s t_k, \tag{5.2}$$

where p^s is the static load-carrying capacity of the structure acting in the same direction as p_i. Hence, the bound on the response time t_k, computed from (5.2) is:

$$t_k \geqslant t_1 \frac{p_i}{p_i^s}. \tag{5.3}$$

In order to obtain bounds on permanent displacements, we shall rewrite the functional (4.1) and the system (4.2), taking actual velocities $\dot{u}_i, \dot{e}_j$:

$$\int_0^{t_k}\int_S a dS dt - \int_S p_i u_i(t_1) dS = 0, \quad a \geqslant E_j^m \dot{e}_j, \quad m = 0, 1, 2, \ldots \tag{5.4}$$

Integrating the dissipation inequality (5.4) over the shell surface and time, we have

$$\int_0^{t_k}\int_S a dS dt = \int_S E_j^m e_j(t_k) dS, \quad m = 0, 1, 2, \ldots. \tag{5.5}$$

The left-hand side of (5.5) is greater than or equal to the energy dissipated in the course of an actual motion leading to the permanent displacement $u_i(t_k)$. Hence, the upper bound theorem for the limit load implies

$$\int_0^{t_k}\int_S a dS dt \geqslant \int_S p_i^s u_i(t_k) dS, \tag{5.6}$$

where p_i^s denotes the static load-carrying capacity of the structure, where the pressure p_i^s acts in the direction of p_i.

Comparison of (5.4) and (5.6) yields the final results:

$$\int_S p_i u_i(t_1) dS \geqslant \int_S p_i^s u_i(t_k) dS. \tag{5.7}$$

Inequalities (5.3) and (5.7) can be extended to other loading conditions.

6. Example

The method of linear programming has not yet been applied to dynamic problems for structures. As an example of application of this method, consider a simply supported circular plate. The exact solution of the same problem is according to H. G. HOPKINS and W. PRAGER. Comparison of the two solutions referred to above would give indications as to the efficiency of the proposed method and accuracy of the results obtained. Thus, we shall present an examplary solution merely as an illustration of the application of a linear programming method to the solution of dynamic problems for rigid-plastic bodies. Emphasis is placed on discussion of the usefulness of the new method by comparison with the existing ones, since the solution itself, as known in the literature, is not of any special interest.

It is assumed that uniformly distributed pressure p is applied at $t = 0$, held constant afterwards and removed at $t = t_1$. The velocities resulting from the rectangular pressure pulse are continuous; the acceleration may however suffer discontinuities at $t = 0$, $t = t_1$ and $t = t_k$. The jump of acceleration at $t = t_1$ can be described by finite difference approximation, the acceleration at $t = t_k$ appears to be averaged. This introduces a negligible error in computations, since plate velocity is zero at $t = t_k$. The equation of motion of the plate is

$$\frac{\partial}{\partial r}(rM)-N = -\int\limits_0^r (P-\gamma\ddot{w})rdr, \tag{6.1}$$

where M, N denote respectively radial and circumferential bending moment, w denotes vertical deflection and r is a radial coordinate, Fig. 2.

According to the Tresca-Saint Venant yield condition

$$|M| \leqslant M_s, \quad |N| \leqslant M_s, \quad |M-N| \leqslant M_s, \quad \sigma_s h^2 = M_s, \tag{6.2}$$

where σ_s is yield stress in simple tension and $2h$ denotes thickness of the plate.

The load intensity is taken to be $P = 3P^s$, where P^s denotes static collapse load. Let us divide the radius of the plate R into five equal sections, each of length $R/5$, Fig. 3. According to (5.3), the response time is estimated as $t_k = 5\frac{2}{3}t_1$. Now the interval $t_1 \leqslant t \leqslant t_k$ is divided into sevene quals teps of length $2/3\ t_1$, Fig. 3. All integrals over the surface and time are replaced by corresponding sums. The partial derivatives with respect to r and t are replaced by central differences (at the plate centre, curvature rates in radial and circumferential direction coincide, $\dot{K}_r = \dot{K}_\theta$).

Displacements are defined at the points 0, 1, 2, 3, 4 on the radial coordinate and at the points $t_1, t_{1-2}, t_{2,3}$ ... on the time coordinate. Thus,

$$\ddot{w}^{0-1} = \frac{2w^1}{t_1^2}, \quad \dot{w}^1 = \frac{2w^1}{t_1}, \quad w^{1-2} = \frac{5}{3}w^1, \quad w^{0'-1} = \frac{1}{3}w^1.$$

The point O' is introduced to describe a jump in acceleration at $t = t_1$.

Upper and lower indices refer to the number of mesh steps respectively in the time and space coordinate.

The velocity at the points t_2 and t_3 is defined by

$$\dot{w}^2 = \frac{3}{2t_1}\left(w^{2-3}-\frac{5}{3}w^1\right), \quad \dot{w}^{\,3} = \frac{3}{2t_1}(^{3-4}-w^{2-3}),$$

the corresponding expressions at point t_4, t_5, t_6 are analogous to that defined for t_3.

Equation (6.1) is written for intermediate points in the radial direction (points 0–1, 1–2, etc.) and intermediate points in time (points $t_{1-2}, t_{2,3}$, etc.), and separately at the point t_1. Moments M and N are associated with points 0, 1, 2, 3, 4 on the radial axis and with points $t_1, t_{1-2}, t_{2-3}, t_{3-4}, t_{4-5}/t_{5-6}$ and t_{6-7} on the time axis. The boundary conditions yield $M_0 = N_0$, $M_5 = 0$.

The accelerations at points t_{1-2}, t_{2-3} and t_{3-4} are given respectively by

$$\ddot{w}^{2-3} = \frac{9}{4t_1^2}\left(w^{3-4} - 2w^{2-3} + \frac{5}{3}w^1\right), \qquad \ddot{w}^{1-2} = \frac{9}{4t_1^2}(w^{2-3} - 3w^1),$$

$$\ddot{w}^{3-4} = \frac{9}{4t_1^2}(w^{4-5} - 2w^{3-4} + w^{2-3}).$$

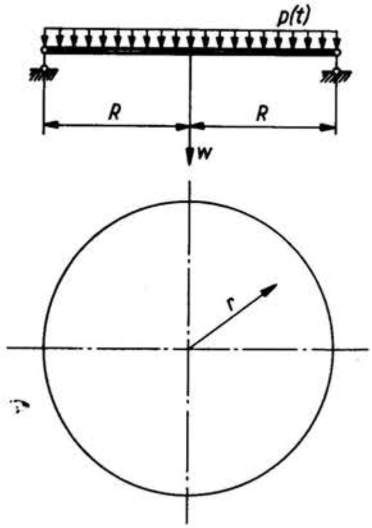
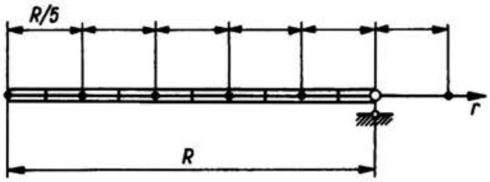


FIG. 2.



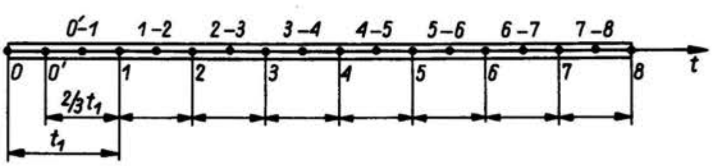


FIG. 3.

Similar expressions, corresponding to points t_{4-5}, t_{5-6} and t_{6-7}, are constructed by analogy to $\ddot{w}^{3-4}$. The boundary condition of a simple support requires that $w_5 = 0$, while the requirement of zero velocity at the end of the motion t_7 yields $w^{6-7} = w^{7-8}$.

Now, the objective function together with the imposed restriction can be constructed.

The results of computations, performed by means of the method of linear programming, are gathered in Table 1. A comparison of these results for w with the earlier exact solution [13] indicates the high accuracy and hence the efficiency of the method applied (the computed response time $t_k = 3t_1$ coincides with the exact value). Consequently, the method of linear programming can be successfully applied to the solution of other boundary value problems.

Table 1. Numerical values $\frac{wR^2\gamma}{\sigma_s h^2 t_1^2}$

t \ r/R	0	0.2	0.4	0.6	0.8	1.0
t_1	9.57	9.57	7.18	4.79	2.39	0
$2t_1$	26.21	24.16	18.12	12.08	6.04	0
$2\frac{2}{3}t_1$	31.16	28.12	21.09	14.08	7.03	0
$3t_1$	31.16	28.12	21.09	14.08	7.03	0

Table 2. True values $\frac{wR^2\gamma}{\sigma_s h^2 t_1^2}$

t \ r/R	0	0.2	0.4	0.6	0.8	1.0
t_1	9.0	9.0	7.75	4.95	2.50	0
$2t_1$	25.50	23.30	18.60	12.10	6.20	0
$2\frac{2}{3}t_1$	30.80	27.60	21.80	14.20	7.30	0
$3t_1$	31.50	28.10	22.20	14.55	7.40	0

7

Consider the dynamics of a rigid-plastic square plate, simply supported on its boundary (Fig. 4). A uniformly distributed pressure of intensity P kg/cm^2 acts within the time interval $0 \leqslant t \leqslant t_1$, while for $t \geqslant t_1$ we have $P = 0$. It is required to determine the motion and the permanent deflections of the plate. This problem has not so far been solved. We shall apply to the solution of this problem the LP.

The qualitative characteristics of the problem are the same as in the problem of motion of the plastic circular plate considered in the preceding Section (the continuity of the velocities, jumps of accelerations and uniformly accelerated motion for $0 \leqslant t \leqslant t_1$). As in the preceding Section the time interval $t_1 \leqslant t \leqslant t_k$ is split into 7 segments, each of length $\frac{2}{3}t_1$ and we denote $t_k = 5\frac{2}{3}t_1$.

The differential equation of motion of the plate has the form

$$\frac{\partial^2 M}{\partial x^2} - 2\frac{\partial^2 S}{\partial x\,\partial y} + \frac{\partial^2 N}{\partial y^2} + P - \gamma \ddot{w} = 0, \tag{7.1}$$

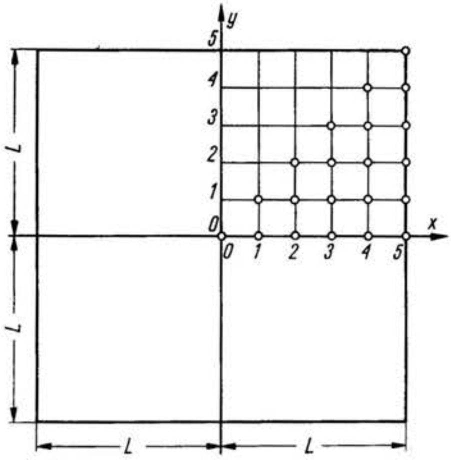


FIG. 4

where M and N are the bending moments in the directions of the x and y-axes, respectively (Fig. 4), S is the torsional moment; the remaining notations are the same as in the preceding Section.

The material of the plate is assumed to obey the Tresca-St. Venant yield condition so that

$$(7.2)\qquad \begin{aligned} -\sigma_s h^2 \leqslant \frac{M+N}{2} \pm \sqrt{\left(\frac{M-N}{2}\right)^2 + S^2} \leqslant \sigma_s h^2, \\ -\sigma_s h^2 \leqslant 2\sqrt{\left(\frac{M-N}{2}\right)^2 + S^2} \leqslant \sigma_s h^2, \end{aligned}$$

we assume moreover that the linearized equation of the yield surface has the form:

$$(7.3)\qquad \begin{aligned} -\sigma_s h^2 \leqslant \pm M \mp N + 2S \leqslant \sigma_s h^2, \quad -\sigma_s h^2 \leqslant M \pm S \leqslant \sigma_s h^2, \\ -\sigma_s h^2 \leqslant N \pm S \leqslant \sigma_s h^2. \end{aligned}$$

The surface (7.3) is inscribed into the surface (7.2) and is shown in Fig. 5.

In view of the symmetry of the plate it is sufficient to investigatei ts one eigths (Fig. 4). We introduce the lattice with step $\partial x = 0.2\,l$, i.e. we divide each side of the plate into 10 parts (l is one half of the side of the plate).

Integrals over the area and time are replaced by the appropriate sums.

The system of inequalities (4.2) is written in the form

$$(7.4)\qquad \begin{aligned} A \geqslant \pm\sigma_s h^2 \dot{\varkappa}_x, \quad A \geqslant \sigma_s h^2 \dot{\varkappa}_y, \quad A \geqslant \pm\sigma_s h^2 (\dot{\varkappa}_x + \dot{\varkappa}_y), \\ A \geqslant \pm\frac{1}{2}\sigma_s h^2 (\dot{\varkappa}_x + \dot{\varkappa}_y + 2\dot{\varkappa}_{xy}), \quad A \geqslant \pm\frac{1}{2}\sigma_s h^2 (\dot{\varkappa}_x + \dot{\varkappa}_y - 2\dot{\varkappa}_{xy}), \end{aligned}$$

where $\dot{\varkappa}_x$, $\dot{\varkappa}_y$ are the velocities of change of the curvatures in the directions of the axes x and y and $\dot{\varkappa}_{xy}$ is the velocity of the relative torsion of the surface of the plate with respect to the axes x and y; these quantities are

$$\dot{\varkappa}_x = -\frac{\partial^2 \dot{w}}{\partial x^2}, \quad \dot{\varkappa}_y = \frac{\partial^2 \dot{w}}{\partial y^2}, \quad \dot{\varkappa}_{xy} = \frac{\partial^2 \dot{w}}{\partial x \partial y}.$$

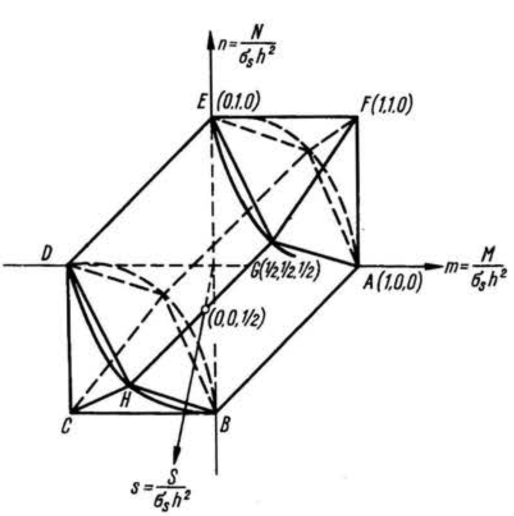


FIG. 5

The derivatives of M, N, S, w with respect to x, y and t are expressed in terms of the finite differences, as shown in the preceding Section. It is taken into account that on the diagonal 0–55 of the plate $M = N$ whereas on the axes x and y we have $S = 0$, $M_{5j} = 0$, $N_{i5} = 0$ ($i = 0, 1, 2, 3, 4, 5$; $j = 0, 1, 2, 3, 4, 5$; the lower subscript denotes hereafter the corresponding nod of the difference lattice, while the upper subscript denotes the time instant; in the double notation of the points the first number denotes the point along the x-axis, while the second along the y-axis, Fig. 4).

The system (7.4) is constructed for the time instants t_1, t_2, t_3, t_4, t_5 (the velocity of the deflection at instant t_6 is assumed to vanish). The deflection is prescribed at nods of the difference lattice at instants $t_1, t_{1-2}, t_{2-3}, t_{3-4}, t_{4-5}, t_{5-6}$. In accordance with the boundary condition $w_{5j} = 0$, $w_{i5} = 0$ ($i = 0, 1, 2, 3, 4, 5$; $j = 0, 1, 2, 3, 4, 5$); the distribution of the deflections is symmetric with respect to the axes x and y and the diagonal 0–55.

For definiteness we assume that the plate is subject to the pressure $P = 3P^s$, where P^s is the limiting static pressure. To determine the quantity P^s by the LP method we solved the problem of load-carrying capacity of the square plate; we made use of the basic relations of the preceding Section for accelerations $\ddot{w} = 0$. By means of the determination of the maximum of the value of P with the conditions (7.1) and the inequalities (7.3), we obtained the quantity $P^s = 5.716\sigma_s h^2/l^2$.

Computer calculations of the considered problem yield the following results. The values of the reduced deflections $w\gamma l^2/\sigma_s h^2 t_1^2$ of the plate at the nods of the difference lattice for the time instant $t_1, t_{1-2}, t_{2-3}, t_{3-4}, t_{4-5}, t_{5-6}$ are given in Table 2. Since $w^{4-5} = w^{5-6}$, with the assumed discretization the derived time of the motion must be assumed to be $t_5 = 3\frac{2}{3}t_1$. The permanent deflections of the plate are equal to the deflections at the instant $t_{4-5} = 3\frac{1}{3}t_1$ (Table 3). The deformed plate is shown in Fig. 6.

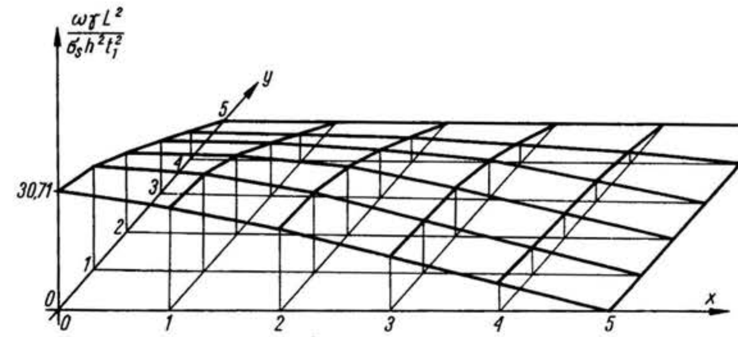


FIG. 6

Table 3

Time instant / Point numbers	t_1	$t_{1-2} = 1\frac{1}{3}t_1$	$t_{2-3} = 2t_1$	$t_{3-4} = 2\frac{2}{3}t_1$	$t_{4-5} = 3\frac{1}{3}t_1$	$t_{5-6} = 4t_1$
00	8.913	14,855	25.111	30.396	30.710	30.710
10	8,913	14.855	23.054	27.278	27.529	27,529
20	6.915	11.525	17.665	20.830	21.018	21.018
30	4.645	7.742	11.825	13.929	14.054	14.054
40	2.323	3.872	5.901	6.947	7.009	7.009
11	8.151	13.585	21.783	26.007	26.258	26.258
21	6.915	11.525	17.665	20.830	21.018	21.018
31	4.645	7.742	11.822	13.924	14.049	14.049
41	2.323	3.872	5.890	6.930	6.991	6.991
22	6.809	11.348	16.640	19.367	19.529	19.529
32	4.645	7.742	11.472	13.394	13.508	13.508
42	2.323	3.872	5.744	6.709	6.767	6.767
33	3.778	6.297	9.107	10.555	10.641	10.641
43	3.106	3.510	4.848	5.537	5.578	5.578
44	1,053	1.775	2.424	2.769	2.790	2.790

In accordance with the above, the values of the dissipation function A vanished at the following nods at the following time instants:

$$A^1_{00} = A^1_{40} = A^1_{41} = A^1_{51} = A^2_{10} = A^2_{20} = A^3_{10} = A^3_{20} = A^3_{10} = A^4_{20} = A^5 = 0.$$

At these points the velocities of the curvatures are equal to zero (these points may be assumed to be "rigid".

We do not present here all values of M, N, S at the nods of the lattice at various instants of time, we note however that except at the rigid points the state of stress corresponds to the edges FAG, EFG, the ribs AF, FG, EF, EG, GH and the corner points F, G (Fig. 5).

It is noteworthy that in the first stage of the motion there exists a zone of equal deflections which decreases and vanishes in the following stages. A similar character of the motion is observed in the case of a circular plate [13] after it has been subject to "high" pressures.

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Received January 21, 1972