

## A gradient theory of materials with memory and internal changes

O. W. DILLON Jr. (LEXINGTON) and P. PERZYNA (WARSZAWA)

The objective of this paper is a construction of the thermodynamic theory of a material with memory and internal changes when higher gradients of the deformation and temperature are taken into account. The fading memory phenomena is used to describe the rheological properties and internal state variables are introduced to describe the internal structural changes during plastic deformation of a material. Higher gradients of the deformation and temperature are included to describe fine structure or the local behaviour observed in the electron microscope. General and linearized theories are presented and physical motivations are given.

Celem obecnej pracy jest konstrukcja termodynamicznej teorii materiałów z pamięcią i z wewnętrznymi zmianami strukturalnymi przy uwzględnieniu wyższych gradientów deformacji i temperatury. Zjawisko zanikania pamięci wykorzystano do opisu własności reologicznych materiału, natomiast parametry wewnętrzne wprowadzono w celu opisu wewnętrznych zmian strukturalnych, wywołanych deformacją plastyczną materiału. Wyższe gradienty deformacji i temperatury zostały włączone, aby opisać lokalne zachowanie się zaobserwowane pod mikroskopem elektronowym. Przedstawiono zarówno teorię ogólną jak i zlinearyzowaną oraz podano motywacje fizyczne.

Целью данной работы является создание термодинамической теории наследственных материалов с внутренними структурными изменениями путём учёта высших градиентов деформации и температуры. Явление затухания наследственности использовано для описания реологических свойств материалов, тогда как внутренние параметры введены для описания внутренних структурных изменений, вызванных пластической деформацией материала. Высшие градиенты деформаций и температуры включены в описание для того, чтобы учесть локальное поведение, наблюдаемое под электронным микроскопом. Наряду с общей теорией представлен её линейный вариант, а также приведены некоторые физические мотивировки.

### 1. Introduction

IN RECENT years, the thermodynamic theory of continua has been extensively used to describe the properties of many kinds of materials. There is now a trend to base this description on good physical and experimental foundations. This is especially true when the plastic properties of the material are involved. A modern theory of plastic flow must consider the microscopic investigations, because plastic deformations change not only the external shape of a body but also its internal structure<sup>(1)</sup>. One of the most basic of experimental facts concerning plastic deformations is the non-homogeneity of the response when viewed at the microscopic level<sup>(2)</sup>. That is, the microscopic level response is non-homogeneous in situations when the macroscopic response is uniform. In the past it has been tacitly assumed that one can ignore the microscopic structure (and especially the non-homogeneity

<sup>(1)</sup> There are several papers published in which plasticity is based on the theory of dislocations (cf. EISENBERG [13], FOX [14], DILLON and KRATOCHVIL [12], LARDNER [26], MURA [29], and SEDOV and BERDICHEVSKI [37]).

<sup>(2)</sup> Cf. here the recent investigations published by ASHBY [1].

in the microstructure) in developing a theory to be used in practical applications. Undoubtedly, in some problems this is still true. However, some investigators now believe that it is (possibly) very much more realistic to consider at least the effect of small scale response on the large scale behavior. This permits the response functions or functionals to be used in a much wider class of problems. Such reasoning is the very basis for much modern solid state physics research and metallurgical studies.

The objective of this paper is an attempt to develop a fundamental framework within which there is *consistency* of a wide variety of experimental data and mathematical assumptions. It is likely that practical problems (as distinct from basic phenomena) will require a tremendously simplified version of this paper. In this case, our objective is to bring the necessary assumptions into the open rather than to have them be implicit.

A secondary purpose is to show how to introduce some aspects of the small scale response into a continuum theory of plasticity.

We shall construct the thermodynamic theory of a material with memory and internal changes when higher gradients of the deformation and temperature are taken into account<sup>(3)</sup>. The fading memory phenomena will be used to describe the rheological properties and internal state variables will be introduced to describe the internal structural changes during plastic deformation of a material. These internal structural changes are the continuum analogue of atomistic rearrangements.

For many purposes and almost all practical design procedures, internal variables as described below will suffice for introducing the effect of the small scale response on the large scale deformations. However, by their very nature, internal variables are inadequate for reproducing the fine structure in the deformation field during a standard tensile or compressive test. When one finds it necessary to actually include this small scale structure itself (as contrasted to just its effects), we suggest that higher gradients of the deformation and temperature be used in the constitutive relations<sup>(4)</sup>. The reason that higher deformation gradients in the constitutive relations lead to a fine structure in the deformation fields associated with the standard experiments requires experience with this type theory.

There are other heuristic arguments which suggest that higher gradients cannot always be neglected in plasticity. The most important of these is the observation that plasticity means changes in the defect structure and defects in turn mean high strain gradients. A second major argument is indirectly contained in theories of work hardening especially in

---

<sup>(3)</sup> The thermodynamic theory of materials with memory has been developed by COLEMAN [7, 8], cf. also COLEMAN and MIZEL [10] and GURTIN [19]. Thermodynamics with internal state variables has been proposed by COLEMAN and GURTIN [9] and VALANIS [40]. The formulation of the thermodynamic theory of a rate sensitive plastic material within the framework of thermodynamics of material with internal state variables was given by PERZYNA and WOJNO [35], cf. also generalizations presented by PERZYNA [33]. A similar theory of rate dependent plasticity was formulated independently by KRATOCHVIL and DILLON [23, 24]. They assumed that quantities which are related to the dislocation motion and the dislocation arrangement in a material play the role of the internal state variables. The construction of the thermodynamic theory of a material with both memory and internal changes and its application to the description of an elastic-viscoplastic material has been presented by PERZYNA [34].

<sup>(4)</sup> Similar conception has been used for an elastic material by GREEN and RIVLIN [16, 17], for a material with fading memory by ZAHORSKI [42] and GURTIN [19] and for an elastic-plastic material by GREEN, McINNIS and NAGHDI [18] and DILLON and KRATOCHVIL [12].

those based on Seeger's model. In these theories work hardening is assumed to be due to interacting dislocations. These interactions occur over a large number of atomistic or dislocation spacings. The associated force field in these models is therefore inherently "non-local". Even in the elastic range when non-local force fields are an essential feature of the phenomena, higher gradients of the deformation function should be included in the continuum theory. Furthermore, the whole theory known as the continuous distribution of dislocations<sup>(5)</sup> has for some years been based on residual stresses being related to gradients of part (or all) of the rotation field. In fact, one of the major weaknesses in the continuous distribution of dislocations is its inability to describe the case of a bar in tension where average rotations do not change. Gradient theories can represent the changes in the residual stresses in this case. However, the most convincing argument is to examine the results obtained in special cases.

The difficulty with using higher gradients is the absence of a criterion for deciding what order derivatives are likely to be most important in a specific material.

The response of real materials is very complex when one wishes to simultaneously consider elastic, viscous and plastic effects and when one is further interested (of necessity) in the microscopic mechanisms which occur during the deformation process. The consideration of this complex behavior may not be necessary in a specific problem but it is our judgement that consistency between mathematical models and features of real materials requires a framework that is large enough to contain all of these effects and thereby is complex.

## 2. General theory

Let us consider a body  $\mathcal{B}$  with particle  $X$  and assume that this body can deform inelastically and conduct heat.

The deformation of a particle  $X$  in body  $\mathcal{B}$  is described by the function of motion  $\chi(X, t)$ . This function determines the spatial position  $x$  occupied by the material point  $X$  at time  $t$ , which in the reference configuration  $\mathcal{R}$  occupied the position  $X$ , i.e.,

$$(2.1) \quad x = \chi(X, t).$$

The components of the function  $\chi$  are assumed to be jointly  $n+1$  — times continuously differentiable with respect to the material coordinates  $X$  and twice continuously differentiable with respect to time  $t$ .

Let  $\mathcal{N}(X)$  be a small neighborhood of  $X$  in  $\mathcal{B}$  and let  $Y$  denote a typical particle in  $\mathcal{N}(X)$ . The motion at  $Y$  can be described by the function of motion at  $X$  and by a sequence of deformation gradients<sup>(6)</sup>

$$(2.2) \quad F = \nabla\chi(X, t), \nabla F = \nabla^2\chi(X, t), \dots, \nabla^j F = \nabla^{j+1}\chi(X, t), \dots, \nabla^n F = \nabla^{n+1}\chi(X, t),$$

<sup>(5)</sup> Cf. here for instance BILBY [2], KRÖNER [25], MINDLIN [28], MINDLIN and TIERSTEN [27] and TOUPIN [39].

<sup>(6)</sup> One can use the fading property in space to describe the same phenomena (non-local effects). Essentially, we base our description on the principle of local action but in nonlinear sense. Classical models of a material, of course, have local properties in linear sense only. This occurs when we put  $n = 0$ .

where the operator  $\nabla$  denotes the differentiation with respect to the material coordinates  $X$ . Thus we can write the motion of a neighboring particle  $Y$  in terms of the generic one

$$(2.3) \quad \chi(Y, t) = \chi(X, t) + \sum_{j=0}^n \frac{1}{(j+1)!} \nabla^j F(X, t) (Y-X)^{j+1} + o(|Y-X|^{n+1}),$$

where  $o(|Y-X|^{n+1})$  denotes the error which has the property that<sup>(7)</sup>

$$(2.4) \quad \lim_{|Y-X|^{n+1} \rightarrow 0} \frac{o(|Y-X|^{n+1})}{|Y-X|^{n+1}} = 0.$$

Similarly, to describe the distribution of the temperature in the body  $\mathcal{B}$  with the error  $o(|Y-X|^n)$  we need to know higher gradients of temperature, i.e.,

$$(2.5) \quad \vartheta(Y, t) = \vartheta(X, t) + \sum_{j=0}^n \frac{1}{j!} \nabla^j \vartheta(X, t) (Y-X)^j.$$

For the sake of convenience, we choose to use

$$(2.6) \quad C = F^T F$$

as the deformation tensor and to introduce the deformation-temperature pair as follows

$$(2.7) \quad A = (C, \vartheta).$$

Thus, to describe the deformation and the distribution of temperature in a body  $\mathcal{B}$  with the error  $o(|Y-X|^n)$ , it suffices to know  $n$  gradients of  $A$  in the particle  $X$ , i.e.,

$$(2.8) \quad \nabla^j A = (\nabla^j C, \nabla^j \vartheta), \quad j = 0, 1, 2, \dots, n.$$

A thermodynamic process within the body  $\mathcal{B}$  is described by a set of functions which contain parameters that are of three different types. The first are the generalized deformation-temperature pairs  $\nabla^j A, j = 0, 1, 2, \dots, n$ , given in (2.8), the second are those dependent variables  $\Pi$  which occur in the balance laws. The third type are known as *internal state variables*  $\omega$ . We introduce the internal state variables to describe those effects which are due to internal changes occurring in a material but which are not explicitly considered in the conservation of energy expression. In other words, the internal state variables serve to describe the part of internal dissipation implied by structural changes of a material during inelastic deformations.

Thus a thermodynamic process in  $\mathcal{B}$  will be described by the set of functions

$$(2.9) \quad \{\nabla^j A(X, t), \Pi(X, t), \omega(X, t)\}, \quad j = 0, 1, 2, \dots, n,$$

defined for every particle  $X$  in  $\mathcal{B}$  and for every time  $t$ . For the material considered herein, it suffices to let  $\Pi$  represent the following variables

$$(2.10) \quad \Pi(X, t) = \{\psi(X, t), \eta(X, t), T_{R(j)}(X, t), q_R(X, t)\}, \quad j = 0, 1, 2, \dots, n,$$

(7) We know of no criterion which determines which values of  $n > 0$  are responsible for which physical effects to develop a mathematical criterion for the error that can be tolerated by a truncation of the series in (2.3).

where  $\psi(X, t)$  denotes the specific free energy per unit mass, and  $\eta(X, t)$  the specific entropy. The  $T_{R(j)}(X, t)$ ,  $j = 0, 1, 2, \dots, n$ , are the generalized stresses<sup>(8)</sup> and  $q_R(X, t)$  is the heat flux vector per unit surface in the reference configuration  $\mathcal{R}$ .

In order to specify the response of a material in a body  $\mathcal{B}$ , we shall use the notion of a generalized particle  $p$  (cf. WANG [41]). In a gradient theory of a material with memory and internal changes a generalized particle  $p$  is characterized as follows:

1) it is described by the set of functions

$$(2.11) \quad g(s) = \{\nabla^j A^t(t-s); \omega(t)\}, \quad s \in [0, \infty), \quad j = 0, 1, 2, \dots, n,$$

where  $\nabla^j A^t(t-s)$ ,  $s \in [0, \infty)$ , denotes the history of the  $j$ -th order gradient of the deformation-temperature pair  $A = (C, \vartheta)$ ;

2) it is equipped with response functionals which define the properties of the materials by the constitutive equations as follows<sup>(9)</sup>

$$(2.12) \quad \Pi(t) = L(g(s)), \quad \dot{\omega}(t) = \Omega(g(s)), \quad s \in [0, \infty).$$

The response functional  $L$  represents the free energy functional  $\psi$ , the entropy functional  $N$ , the generalized stress functionals  $T_{(j)}$  ( $j = 0, 1, 2, \dots, n$ ) and the heat flux functional  $Q$ , i.e.,

$$(2.13) \quad L = \{\psi, N, T_{(j)}, Q\}, \quad j = 0, 1, 2, \dots, n.$$

Let us denote by  $\mathcal{D}$  a common domain of functionals  $L$  and  $\Omega$ .

A thermodynamic process described by (2.9) is said to be admissible in  $\mathcal{B}$  if it is compatible with the constitutive assumption (2.12) at each generalized particle  $p$  of  $\mathcal{B}$  and at all time  $t \in (-\infty, \infty)$ .

We now explain why in the function  $g(s)$  which describes the generalized particle  $p$  [see Eq. (2.11)] we need both the history and internal variable parameters. Our basic con-

<sup>(8)</sup> The generalized stresses  $T_{R(j)}(X, t)$ ,  $j = 0, 1, 2, \dots, n$ , we understand in the following way. For a body  $\mathcal{B}$  we consider that we know the resultant generalized forces  $f_{(j)}(\mathcal{B})$ ,  $j = 0, 1, 2, \dots, n$ . For  $j = 0$ , the force  $f_{(0)}(\mathcal{B})$  will be a vector, for  $j = 1$ ,  $f_{(1)}(\mathcal{B})$  will be a second order tensor, etc. We assume that all set functions  $f_{(j)}(\mathcal{B})$ ,  $j = 0, 1, 2, \dots, n$  are  $\sigma$ -finite measures on  $\mathcal{B}$ . Then the Lebesgue Decomposition Theorem (see HALMOS [22], p. 134) yields

$$f_{(j)}(\mathcal{B}) = \int_{\mathcal{B}} b_{(j)}(x, t) dV(x) + \int_{\partial\mathcal{B}} t_{(j)}(x, t) dA(x), \quad j = 0, 1, 2, \dots, n.$$

The integrable functions  $b_{(j)}(x, t)$ ,  $j = 0, 1, 2, \dots, n$ , represent the generalized body forces and the integrable functions  $t_{(j)}(x, t)$ ,  $j = 0, 1, 2, \dots, n$ , represent the generalized contact forces. According to a Theorem of Cauchy (see GURTIN, MIZEL and WILLIAMS [21]), we can write the contact forces as,

$$t_{(j)}(x, t) = T_{(j)}(x, t) \mathbf{n}(x, t), \quad j = 0, 1, 2, \dots, n,$$

where  $\mathbf{n}(x, t)$  is a unit vector. We shall introduce the new generalized stress tensors in the reference configuration

$$T_{R(j)} = JF^{-(j+1)} T_{(j)} (F^{-1})^T, \quad j = 0, 1, 2, \dots, n,$$

where  $J = \det F > 0$ . All generalized stress tensors  $T_{R(j)}$  ( $j = 0, 1, 2, \dots, n$ ) are invariant upon the change of frame, i.e.,  $T_{R(j)} \rightarrow T_{R(j)}$ ,  $j = 0, 1, 2, \dots, n$ .

We understand that during the thermodynamic process the generalized body forces  $b_{(j)}(x, t)$ ,  $j = 0, 1, 2, \dots, n$ , can be determined by balance laws. In other words, there are additional equations of motion (cf. here for instance TOUPIN [29] and DILLON and KRATOCHVIL [12]).

<sup>(9)</sup> We have assumed that all constitutive equations describing the physical properties of a material satisfy the principle of material frame-indifference as formulated by NOLL [30].

cept is that the history may describe anelastic response of a material while the internal state variables can be used to consider inelastic effects with plastic flow included. As shown below, this splits the internal dissipation of our material into two parts. We assume that the first reason for the internal dissipation is internal friction and the second is the internal-changes generated by plastic flow<sup>(10)</sup>.

This interpretation suggests that a convenient set of the internal state variables for the material considered herein is as follows<sup>(11)</sup>

$$(2.14) \quad \omega = \{\kappa, P, \Gamma^{(i)}\}, \quad i = 1, 2, \dots, m,$$

where the scalar  $\kappa$  represents the work-hardening parameter, the second-order tensor  $P$  denotes the inelastic deformation tensor, and the tensors  $\Gamma^{(i)}$  ( $i = 1, 2, \dots, m$ ) are the dislocation arrangement tensors. For definiteness we assume here that the internal state variables remain invariant upon the change of the reference frame, i.e.,

$$(2.15) \quad \kappa \rightarrow \kappa, \quad P \rightarrow P \quad \text{and} \quad \Gamma^{(i)} \rightarrow \Gamma^{(i)}, \quad i = 1, 2, \dots, m.$$

Thus the response functional  $\Omega$  in (2.12) represents the functional  $K$  describing the rate of the work-hardening parameter  $\dot{\kappa}$ , the functional  $G$  determining the rate of the inelastic deformation tensor  $\dot{P}$  and the functionals  $Z^{(i)}$  defining the rate of change of the arrangement tensors  $\dot{\Gamma}^{(i)}$ , i.e.,

$$(2.16) \quad \Omega = \{K, G, Z^{(i)}\}, \quad i = 1, 2, \dots, m.$$

The energy balance equation (the first law of thermodynamics) can be written as follows

$$(2.17) \quad \frac{1}{2\varrho_R} \sum_{j=0}^n \{T_{R(j)} \cdot \dot{\nabla}^j C\} - \text{div} q_R - \varrho_R(\dot{\psi} + \vartheta \dot{\eta} + \dot{\vartheta} \eta) + \varrho_R r = 0,$$

where the dot denotes the material differentiation with respect to time  $t$ ,  $\varrho_R$  is the mass density in the reference configuration  $\mathcal{R}$ , the operator  $\text{div}$  is computed with respect to the material coordinates,  $r$  denotes the heat supply per unit mass and unit time, and the expression<sup>(12)</sup>

$$(2.18) \quad \frac{1}{2\varrho_R} \sum_{j=0}^n \{T_{R(j)} \cdot \dot{\nabla}^j C\} = w$$

represents the net working per unit mass of the generalized stresses on the rates of gradients of deformation tensor.

The function  $r(X, t)$  is then uniquely determined by Eq. (2.17). We shall require that

<sup>(10)</sup> This concept has been first introduced in Refs. [33, 34].

<sup>(11)</sup> There is no assumed connection between the total deformation tensor  $C$  and the inelastic deformation tensor  $P$ . The deformation tensor  $C$  is described by the function of motion while the inelastic deformation tensor  $P$  occurs as an internal state variable and is therefore determined by the solution of the initial-value problem for an ordinary first-order differential equation (cf. PERZYNA and WOJNO [35] and PERZYNA [33, 34]).

<sup>(12)</sup> The result of the operator denoted by dot is a scalar.

for a real thermodynamic process in  $\mathcal{B}$  for any time  $t$  an additional thermodynamic postulate be satisfied. This is equivalent to the following inequality<sup>(13)</sup>

$$(2.19) \quad -\dot{\psi} + \frac{1}{2\rho_R} \sum_{j=0}^n \{T_{R(j)} \cdot \nabla^j \dot{C}\} - \eta \dot{\vartheta} - \frac{1}{\rho_R \vartheta} q_R \cdot \nabla \vartheta \geq 0$$

which must be satisfied for every generalized particle  $p$  in  $\mathcal{B}$ .

Let us write the constitutive assumption in the form

$$(2.20) \quad \begin{aligned} \Pi(t) &= L(\nabla^j A_r^t(s), \nabla^j A(t); \omega(t)), \quad s \in (0, \infty), \\ \dot{\omega}(t) &= \Omega(g(s)), \quad s \in [0, \infty), \end{aligned}$$

where  $\nabla^j A_r^t(s)$ ,  $s \in (0, \infty)$  is the reduced history of  $\nabla^j A$ .

To investigate the restrictions imposed on the constitutive equations by the thermodynamic postulate, we shall need the principle of fading memory for a material with both memory and internal changes.

Let  $h(s)$  denote a fixed influence function, i.e., a continuous monotone decreasing function with  $s^2 h(s)$  integrable in  $s$  on  $[0, \infty)$ ; let us define the norms

$$(2.21) \quad \|\nabla^j A(s)\| = \int_0^\infty |\nabla^j A(s)|^2 h(s) ds, \quad j = 0, 1, 2, \dots, n,$$

where

$$(2.22) \quad |\nabla^j A(s)| = \{\nabla^j A \cdot \nabla^j A\}^{\frac{1}{2}} = \{\nabla^j C \cdot (\nabla^j C)^T + \nabla^j \vartheta \cdot (\nabla^j \vartheta)^T\}^{\frac{1}{2}};$$

let  $\mathcal{H}_h$  denote the set of all measurable real-valued functions  $\{\nabla^j A(s)\}$  on  $[0, \infty)$  for  $j = 0, 1, 2, \dots, n$  with  $\sum_{j=0}^n \|\nabla^j A(s)\| < \infty$ .

The fading memory assumption can be stated as follows<sup>(14)</sup>:

1. For each fixed  $\{\nabla^j A, \omega\}$ ,  $j = 0, 1, 2, \dots, n$ , the response functionals (represented by  $L$ ) regarded as functions of  $\nabla^j A_r^t(s)$ ,  $s \in (0, \infty)$ ,  $j = 0, 1, 2, \dots, n$ , have for their common domain  $\mathcal{D}$  a neighborhood in  $\mathcal{H}_h$  of the rest history.

2. The functionals are Fréchet-differentiable throughout  $\mathcal{D}$  with respect to the  $h$ -norm.

3. For each fixed  $\nabla^j A_r^t(s)$ ,  $s \in (0, \infty)$ ,  $j = 0, 1, 2, \dots, n$ , in  $\mathcal{D}$  the functionals regarded as functions of  $\nabla^j A$ ,  $j = 0, 1, 2, \dots, n$ , and  $\omega$  are piecewise continuously differentiable with respect to their natural norms.

4. All derivatives are piecewise continuous functions of  $g(s)$  in  $\mathcal{D}$ .

It follows from the fading memory postulate that the free energy functional  $\psi$  has the chain-rule property, so the time derivative of  $\psi$  exists at  $t$  and obeys the formula

$$(2.23) \quad \dot{\psi}(t) = \sum_{j=0}^n (\partial_{\nabla^j C} \psi \cdot \nabla^j \dot{C}) + \sum_{j=0}^n (\partial_{\nabla^j \vartheta} \psi \cdot \nabla^j \dot{\vartheta}) + \sum_{j=0}^n \delta_j \psi(g(s) | \nabla^j A_r^t(s)) + \partial_\omega \psi(g(s)) \cdot \dot{\omega}(t),$$

<sup>(13)</sup> This inequality is implied by the Clausius-Duhem inequality (cf. COLEMAN and NOLL [6]).

<sup>(14)</sup> Cf. COLEMAN [7], and COLEMAN and MIZEL [10].

where  $\partial_{\nabla^j C} \psi$  is the derivative of  $\psi$  with respect to the present value  $\nabla^j C$  holding all  $\nabla^j \vartheta$ , all reduced histories  $\nabla^j A_r^i$  and  $\omega$  fixed,  $\partial_{\nabla^j \vartheta} \psi$  has an analogous meaning,  $\delta_j \psi$  is the Fréchet-derivative of  $\psi$  with respect to the reduced history  $\nabla^j A_r^i$  holding the present values  $\nabla^j A$  ( $j = 0, 1, 2, \dots, n$ ) and  $\omega$  fixed, and  $\partial_\omega \psi$  is the derivative of  $\psi$  with respect to the present value of the internal state variable  $\omega$  holding the present values  $\nabla^j A$  ( $j = 0, 1, 2, \dots, n$ ) and the reduced histories  $\nabla^j A_r^i$  ( $j = 0, 1, 2, \dots, n$ ) fixed.

The thermodynamic postulate (2.19) becomes

$$(2.24) \quad \frac{1}{2\rho_R} \sum_{j=0}^n \{ [T_{R(j)} - 2\rho_R \partial_{\nabla^j C} \psi] \cdot \overline{\nabla^j C} \} - (\partial_\omega \psi + \eta) \dot{\vartheta} - \sum_{j=0}^n (\partial_{\nabla^k \vartheta} \psi \cdot \overline{\nabla^k \vartheta}) - \sum_{j=0}^n \delta_j \psi(g(s) | \overline{\nabla^j A_r^i(s)}) - \partial_\omega \psi(g(s)) \cdot \dot{\omega}(t) - \frac{1}{\rho_R \vartheta} q_R \cdot \nabla \vartheta \geq 0.$$

Choosing arbitrary values  $\overline{\nabla^j C}$  and  $\overline{\nabla^j \vartheta}$  for  $j = 0, 1, 2, \dots, n$ , it is possible to determine an admissible thermodynamic process in a body  $\mathcal{B}$ <sup>(15)</sup>. Hence, the fulfillment of the inequality (2.24) yields the following fundamental results

$$(2.25) \quad \partial_{\nabla^k \vartheta} \psi = 0, \quad k = 1, 2, \dots, n,$$

$$(2.26) \quad T_{R(j)} = 2\rho_R \partial_{\nabla^j C} \psi(g^*(s)), \quad j = 0, 1, 2, \dots, n,$$

$$(2.27) \quad \eta = -\partial_\omega \psi(g^*(s)),$$

$$(2.28) \quad \sum_{j=0}^n \delta_j \psi(g^*(s) | \overline{\nabla^j A_r^i(s)}) + \partial_\omega \psi(g^*(s)) \cdot \dot{\omega}(t) + \frac{1}{\rho_R \vartheta} q_R \cdot \nabla \vartheta \leq 0,$$

where

$$(2.29) \quad g^*(s) = \{ \nabla^j A_r^i(s), \nabla^j C(t), \vartheta(t); \omega(t) \}, \quad j = 0, 1, 2, \dots, n, \quad s \in (0, \infty), \\ t \in (-\infty, \infty).$$

The inequality (2.28) is called the general dissipation inequality.

It will be useful to introduce the internal dissipation functional of the material as follows

$$(2.30) \quad \sigma(g(s)) = -\frac{1}{\vartheta} \left\{ \sum_{j=0}^n \delta_j \psi(g^*(s) | \overline{\nabla^j A_r^i(s)}) + \partial_\omega \psi(g^*(s)) \cdot \Omega(g(s)) \right\}.$$

This illustrates our assumption concerning the splitting of the internal dissipation into two parts. From the expression (2.30) it is clear that the first part of the internal dissipation is generated by memory effects and the second by internal changes.

Let us now additionally assume that the internal changes occur only during plastic deformations<sup>(16)</sup>. To satisfy this condition we postulate that the functionals  $K$  and  $Z^{(i)}$ ,  $i = 1, 2, \dots, m$  are linear with respect to the rate of inelastic deformation tensor  $\dot{P}$ , i.e.,

$$(2.31) \quad \dot{\kappa} = K(g(s)) = \{ H(g(s)) \cdot \dot{P} \},$$

$$(2.32) \quad \dot{I}^{(i)} = Z^{(i)}(g(s)) = S^{(i)}(g(s)) [\dot{P}].$$

<sup>(15)</sup> Cf. COLEMAN and NOLL [6], COLEMAN [7], COLEMAN and GURTIN [9], COLEMAN and MIZEL [10], GURTIN [19] and VALANIS [40].

<sup>(16)</sup> A different assumption has been made by KRATOCHVIL and DILLON [24].



Due to this assumption, the internal dissipation functional as defined by (2.30) can be written as follows

$$(2.33) \quad \sigma(g(s)) = -\frac{1}{\vartheta} \sum_{j=0}^n \delta_j \psi(g^*(s) | \nabla^j \dot{A}_r^t(s)) + \left\{ \left[ \partial_{\kappa} \psi(g^*(s)) H(g(s)) + \partial_P \psi(g^*(s)) + \sum_{i=0}^m \partial_{\Gamma^{(i)}} \psi(g^*(s)) S^{(i)}(g(s)) \right] \cdot G(g(s)) \right\}.$$

This expression shows that the functional  $G(g(s))$  describing the rate of the inelastic deformation tensor  $\dot{P}$  plays an important part in the thermodynamic theory of a material with memory and internal changes which are implied by plastic deformations.

We will now specify the functional  $G(g(s))$  for the case of an elastic-viscoplastic material. By an elastic-viscoplastic material we shall mean a material which before yielding has the properties of fading memory and after yielding has additionally the rate sensitive plastic properties<sup>(17)</sup>. To do this let us introduce the quasi-static criterion of yielding for a material with memory. This criterion is defined by the quasi-static yield functional

$$(2.34) \quad \bar{\mathfrak{F}}(g(s)) = \frac{f(g^{\circ}(s))}{\varkappa} - 1,$$

where

$$(2.35) \quad g^{\circ}(s) = \{ \nabla^j A_r^t(s), \nabla^j A(t); P(t), \Gamma^{(i)}(t) \}, \quad j = 0, 1, 2, \dots, n; \quad i = 1, 2, \dots, m; \\ s \in (0, \infty), \quad t \in (-\infty, \infty).$$

We postulate that the rate of the inelastic deformation tensor  $\dot{P}$  is a function of the excess of the stress over the quasi-static yield condition<sup>(18)</sup>, i.e.,

$$(2.36) \quad \dot{P} = \gamma(\vartheta) \langle \Phi(\bar{\mathfrak{F}}(g(s))) \rangle M(g(s)),$$

where  $\gamma(\vartheta)$  denotes a temperature dependent viscosity coefficient of a material. The symbol  $\langle \Phi(\bar{\mathfrak{F}}) \rangle$  is defined as follows

$$(2.37) \quad \langle \Phi(\bar{\mathfrak{F}}) \rangle = \begin{cases} 0 & \text{for } \bar{\mathfrak{F}} \leq 0, \\ \Phi(\bar{\mathfrak{F}}) & \text{for } \bar{\mathfrak{F}} > 0, \end{cases}$$

and  $M(g(s))$  is a symmetric, second-order tensor functional.

Due to assumption (2.36), the response functionals regarded as functions of  $\omega$  may have jump discontinuities at the quasi-static yield surface defined by

$$(2.38) \quad \bar{\mathfrak{F}}(g(s)) = 0.$$

The constitutive assumption (2.36), in the case when  $\bar{\mathfrak{F}} > 0$ , can be rearranged to give the following dynamical yield criterion

$$(2.39) \quad \bar{\mathfrak{F}}(g^{\circ}(s)) = \varkappa \left\{ 1 + \Phi^{-1} \left[ \frac{(\text{tr} \dot{P}^2)^{\frac{1}{2}}}{\gamma(\vartheta)} (\text{tr} M^2)^{-\frac{1}{2}} \right] \right\}.$$

<sup>(17)</sup> For this definition see Ref. [34].

<sup>(18)</sup> For a thorough discussion of general foundations of viscoplasticity see Refs. [31–35].

This relation may be interpreted as a description of the change of the yield surface during the thermodynamical process. This change is caused by work-hardening effects, by the influence of the rate of inelastic deformations and temperature and by fading memory effects when higher gradients of the deformation and temperature are taken into account.

The differential equations for  $\kappa$  and  $I^{(i)}$  can now be written in the form

$$(2.40) \quad \dot{\kappa}(t) = \gamma(\vartheta) \langle \Phi(\tilde{\mathfrak{F}}(g(s))) \rangle \text{tr} \{ H(g(s)) M(g(s)) \},$$

$$(2.41) \quad \dot{I}^{(i)}(t) = \gamma(\vartheta) \langle \Phi(\tilde{\mathfrak{F}}(g(s))) \rangle S^{(i)}(g(s)) [M(g(s))].$$

The internal dissipation functional now takes the form

$$(2.42) \quad \sigma(g(s)) = -\frac{1}{\vartheta} \left\{ \sum_{j=0}^n \delta_j \psi(g^*(s) | \overline{\nabla^j A_i^j}(s)) + \gamma(\vartheta) \langle \Phi(\tilde{\mathfrak{F}}(g(s))) \rangle \text{tr} \left[ \left( \partial_P \psi(g^*(s)) + \partial_{\kappa} \psi(g(s)) H(g(s)) \sum_{i=1}^m \partial_{I^{(i)}} \psi(g^*(s)) S^{(i)}(g(s)) \right) M(g(s)) \right] \right\}.$$

### 3. Physical motivation

To illustrate the physical phenomena being studied, consider the example of a specimen subjected to controlled average shearing histories. Let the specimen be deformed in one direction and then in the reversed sense. If the load is released at the correct point, the stress-strain curve might look like that shown in Fig. 1. At this point, there is: (1) no

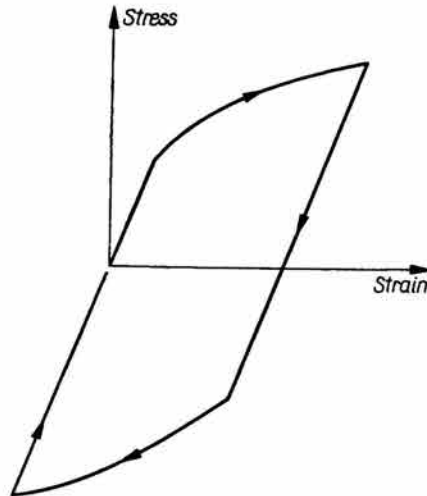


FIG. 1. A typical stress-strain curve for simple shear.

overall strain, and: (2) a changed microstructure as observed especially in the electron microscope. An essential feature of the changes in the microstructure is the non-homogeneity of the local deformations even when the large scale response is uniform. When the ex-

ternal load is reapplied, the stress-strain behavior will be altered from that shown in Fig. 1. The change in the stress-strain behavior can be correlated with microstructural parameters observed by metallurgists and metal physicists. Usually, this is done by developing "models" of the material which involve a mixture of continuum and atomistic concepts. Frequently, the logic is not always tight, there exist inconsistencies and the limitations imposed by the assumptions of the model are not clear. An example of these models is one known as thermally activated dislocations (sometimes just one) which are assumed to control the inelastic material response for specific ranges of temperature and strain-rate<sup>(19)</sup>. Comprehensive reviews of this model are contained in the papers of EVANS and RAWLINGS [15] and CONRAD [11]. Thermally activated dislocations are presumed to be a major factor in the over-stress between the static stress-strain curve and the dynamic one. There are many other models in the literature. Typical of the wide variety of responses of real materials is the case of mild steel. As described by ROSENFELD and HAHN [36] and more recently by CAMPBELL and FERGUSON [4], there are (at least) four different regions in the temperature strain-rate spectrum of mild steel that reflect different mechanisms of yielding. Thus an analytical representation which encompasses the whole spectrum of responses for this and other materials will be complex indeed.

If one needs to solve a boundary value problem in which every material particle undergoes about the same type deformation history as every other particle, classical plasticity is adequate or can be made to be so by modest alterations. However, when different particles are subjected to greatly different types of deformation histories, it is unlikely that classical plasticity is an adequate framework. It is argued (with varying degrees of confidence) that the models mentioned above or other defect parameter data can better correlate the wide variety of responses of real materials than classical plasticity.

Thus the authors have long detected a need for a general theoretical framework which simultaneously considers a wide variety of material response such as creep, relaxation, fatigue, impact and which, therefore, also covers the dimensions seen in the electron microscope and those of the standard tensile specimen. In particular, we believe that a framework which is logical and which will correlate the stress-strain curves in Fig. 2 with the one shown in Fig. 1 is needed. It is our judgement that any improvements to classical plasticity theory will come from combining the models of the metallurgist into a continuum mechanics framework. Furthermore, we believe that the field known as the continuous distribution of dislocations is mathematically correct but is not physically adequate.

In considering how to make an improved framework, it is important to distinguish between incorporating the effect of the non-homogeneous local deformation seen in the electron microscope and reproducing this part of the deformation field in the analytical model. If one needs to know how changes in the number of dislocations or their arrangement affect the stress-strain curve or the relaxation function, one can use average dislocation density data for these parameters as internal state variables. The metallurgical models then serve as guides on the constitutive equations for these parameters. By properly formulating these equations, one can then develop a specific boundary value problem for solution whenever it is needed. Examples of this approach are contained in the KRATOCHVIL and

---

<sup>(19)</sup> Cf. SEEGER [38].

DILLON [24] paper, where the mobile dislocation density and the dislocation loop density are used as internal variables. Such parameters are measurable on the electron microscope. A critical discussion of the experimental technique and the interpretation of the data needed for the internal variables is given in the paper by BRANDON and KOMEM [3].

The Seeger model of work hardening is related to dislocations interacting with one another. This in turn means that non-local force fields develop at the atomic level. At

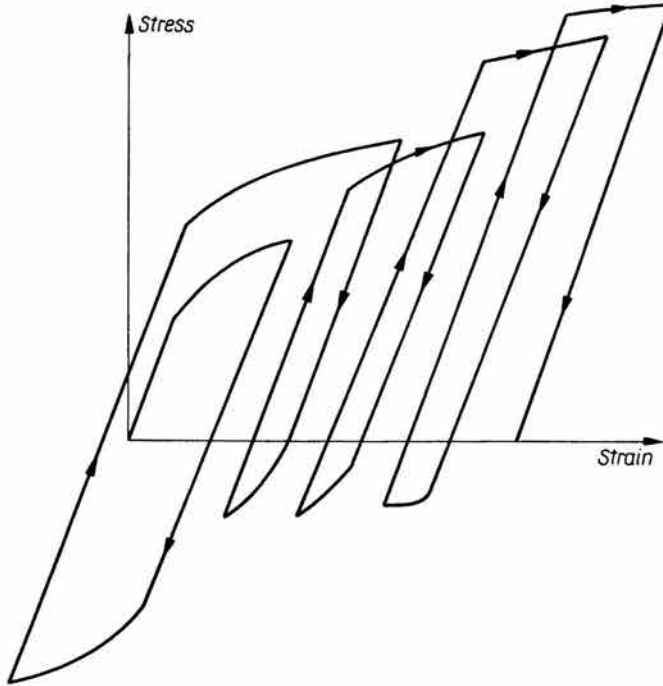


FIG. 2. A schematic stress-strain curve obtained from a complicated process.

this level, these fields are frequently split into “long” and “short” range ones. These non-local force fields are related to the nonhomogeneous local deformations mentioned earlier. The major role of the small scale deformations is the storage of energy in the form of residual stresses and thereby are responsible for the work hardening of the material. Thus one is faced with finding a way of describing nature as a continuum but at the same time having non-local force fields to represent the dislocation interactions. In the elastic range when non-local force fields are an essential feature of the phenomena, higher gradients of the deformation function are included in the continuum theory. Therefore, we suggest a similar procedure for the inelastic range. It requires some experience with higher gradient theories to see how they reproduce the fine structure of the deformation field. Succinctly, they raise the order of the displacement equations of motion, so that in the linear range oscillatory solutions are included in addition to the classical deformations. These oscillatory deformations store energy in the form of residual generalized stresses.

In summary, to incorporate metallurgical models such as the viscous effects attributed to thermally activated dislocations, one can use an internal variable theory. In order to

include damping effects and some aspects of dislocation scattering, one may prefer to use a history as description of the material. However, if one wants more details which represent the local behavior observed in the electron microscope, one also includes higher deformation gradients. The present framework considers all of these at the same time and thereby is more complete than any specific case.

#### 4. Linearized theory

Let us define a thermodynamic equilibrium state for our material with memory and internal changes. The function

$$(4.1) \quad g^\#(s) = \{\nabla^j A_r^\#(s), \nabla^j A^\#(t); \omega^\#(t)\}, \quad j = 0, 1, 2, \dots, n,$$

is said to describe the generalized particle  $p$  in the state of thermodynamic equilibrium, if and only if,

$$(4.2) \quad \mathfrak{F}(g^\#(s)) \leq 0,$$

and

$$(4.3) \quad \begin{aligned} \nabla^j A_r^\#(s) &= \{1(s), \vartheta^\circ(s)\} \text{ for } j = 0 \\ &= \{0, 0\} \text{ for } j = 1, 2, \dots, n, \\ \nabla^j A^\#(t) &= \{1(t), \vartheta^\circ(t)\} \text{ for } j = 0 \\ &= \{0, 0\} \text{ for } j = 1, 2, \dots, n. \end{aligned}$$

In other words, the state of equilibrium is characterized by the fact that all rates of the internal variables vanish and the history of the generalized particle  $p$  does not change. The history of the generalized particle  $p$  is held constant at all times in its reference configuration.

We define here the smallness in the sense that<sup>(20)</sup>

$$(4.4) \quad \sup_{-\infty < \tau < t} \left\{ \sum_{j=0}^n |\nabla^j A(\tau) - \nabla^j A^\#(\tau)| + |\omega(\tau) - \omega^\#(\tau)| \right\} < \delta$$

with  $\delta$  small.

The relation for the generalized stresses [cf. Eqs. (2.26) and (2.29)] can be written in the form

$$(4.5) \quad T_{R(j)} = T_{(j)}(\nabla^j A_r^\#(s), C(t), \vartheta(t), \nabla^k C(t); \omega(t)), \\ j = 0, 1, 2, \dots, n; \quad k = 1, 2, \dots, n.$$

For a case when the condition (4.4) is satisfied by the fading memory assumptions, we have

$$(4.6) \quad T_{R(j)} = T_{(j)}(g^\#(s)) + \partial_C T_{(j)}(g^\#(s))[C(t) - 1] + \partial_\vartheta T_{(j)}(g^\#(s))[\vartheta(t) - \vartheta^\circ] \\ + \sum_{k=1}^n \partial_{\nabla^k C} T_{(j)}(g^\#(s))[\nabla^k C] + \partial_\omega T_{(j)}(g^\#(s))[\omega(t) - \omega^\#] \\ + \sum_{l=0}^n \delta_l T_{(j)}(g^\#(s))[\nabla^l A_r^\#(s) - \nabla^l A_r^\#(s)].$$

<sup>(20)</sup> Cf. COLEMAN [8] and GURTIN and PIPKIN [20].

Let us denote by

$$(4.7) \quad T_{(j)}^{\text{(thel)}}\{C, \vartheta\} = T_{(j)}(g^\#(s)) + \partial_C T_{(j)}(g^\#(s))[C(t) - 1] + \partial_\vartheta T_{(j)}(g^\#(s))[\vartheta(t) - \vartheta^\circ],$$

$$(4.8) \quad T_{(j)}^{\text{(grad)}}\{\nabla^k C\} = \sum_{k=1}^n \partial_{\nabla^k C} T_{(j)}(g^\#(s))[\nabla^k C].$$

The functional  $\delta_l T_{(j)}(g^\#(s)|\nabla^l A_r^l(s) - \nabla^l A_r^{\#l}(s))$  for each particular  $j$  and  $l$  is both a linear and continuous functional over Hilbert space  $\mathcal{H}_h$ . Due to the Riesz's representation theorem, we can write

$$(4.9) \quad \sum_{l=0}^n \delta_l T_{(j)}(g^\#(s)|\nabla^l A_r^l(s) - \nabla^l A_r^{\#l}(s)) = \sum_{l=0}^n \int_0^\infty \mathfrak{R}_{(lj)}(s)[\nabla^l A_r^l(s) - \nabla^l A_r^{\#l}(s)] ds.$$

For each particular  $j$  and  $l$ ,  $\mathfrak{R}_{(lj)}(s)$  is a square-integrable linear transformation and is called the generalized stress-relaxation modulus.

We can also write

$$(4.10) \quad \partial_\omega T_{(j)}(g^\#(s))[\omega(t) - \omega^\#] = \mathfrak{G}_{(j)}(t)[\omega^* + \int_{t_0}^t \Omega(g(s, \xi)) d\xi],$$

where we denote by

$$(4.11) \quad \mathfrak{G}_{(j)}(t) = \partial_\omega T_{(j)}(g^\#(s)),$$

$$(4.12) \quad \omega^* = \omega_0 - \omega^\#,$$

and  $\omega_0$  is given initial value.

Due to (4.7)–(4.12), the result (4.6) can be written in the form as follows:

$$(4.13) \quad T_{R(j)} = T_{(j)}^{\text{(thel)}}\{C, \vartheta\} + T_{(j)}^{\text{(grad)}}\{\nabla^k C\} + \sum_{l=0}^n \int_0^\infty \mathfrak{R}_{(lj)}(s)[\nabla^l A_r^l(s) - \nabla^l A_r^{\#l}(s)] ds \\ + \mathfrak{G}_{(j)}(t) \left[ \omega^* + \int_{t_0}^t \Omega(g(s, \xi)) d\xi \right].$$

It is worth noting that some terms in the representation (4.13) will vanish for initially isotropic material due to symmetry requirements. For the heat flux vector, we have a following relation:

$$(4.14) \quad q_R(t) = Q(\nabla^j A_r^l(s), \nabla^j A(t); \omega(t)).$$

Using the same procedure as for generalized stresses, we can write

$$(4.15) \quad q_R(t) = Q(g^\#(s)) + \sum_{j=0}^n \partial_{\nabla^j A} Q(g^\#(s))[\nabla^j A - \nabla^j A^\#] + \sum_{j=0}^n \delta_j Q(g^\#(s)|\nabla^j A_r^l(s) \\ - \nabla^j A_r^{\#l}(s)) + \partial_\omega Q(g^\#(s))[\omega(t) - \omega^\#].$$

It can be proved that<sup>(21)</sup>

$$(4.16) \quad Q(g^\#(s)) = 0.$$

<sup>(21)</sup> Cf. for instance GURTIN and PIPKIN [20].

We denote by

$$(4.17) \quad q_1\{\nabla^j A\} = \sum_{j=0}^n \partial_{\nabla^j A} Q(g^\#(s))[\nabla^j A - \nabla^j A^\#],$$

$$(4.18) \quad m(t) = \partial_\omega Q(g^\#(s)),$$

and we can write

$$(4.19) \quad \partial_\omega Q(g^\#(s))[\omega - \omega^\#] = m(t)\left[\omega^* + \int_{t_0}^t \Omega(g(s, \xi)) d\xi\right].$$

Using the Riesz's representation theorem for Hilbert space  $\mathcal{H}_h$ , we have

$$(4.20) \quad \sum_{j=0}^n \delta_j Q(g^\#(s))[\nabla^j A_r^t(s) - \nabla^j A_r^{\#t}(s)] = \sum_{j=0}^n \int_0^\infty e_{(j)}(s)[\nabla^j A_r^t(s) - \nabla^j A_r^{\#t}(s)] ds.$$

Due to (4.16)–(4.20), the relation for heat flux vector can be written in the form

$$(4.21) \quad q_R(t) = q_1\{\nabla^j A\} + \sum_{j=0}^n \int_0^\infty e_{(j)}(s)[\nabla^j A_r^t(s) - \nabla^j A_r^{\#t}(s)] ds + m(t)\left[\omega^* + \int_{t_0}^t \Omega(g(s, \xi)) d\xi\right].$$

Of course, some terms in this representation will vanish for initially isotropic material.

The result for heat flux vector (4.21) shows that thermal disturbances in our material will propagate with finite wave speeds<sup>(22)</sup>.

## 5. Particular cases

The theory formulated above is broad enough to obtain many well known theories of materials as special cases. Our procedure for showing this is very simple and obvious. We place some limitations on the function  $g(s)$  which describes the generalized particle  $p$  and then study the restrictions imposed on the response functionals. With these reasonable limitations, we can obtain a number of previous theories of materials which are of general interest.

We illustrate this procedure in two examples, namely for a material of grade 2 with fading memory and for an elastic-plastic material.

### 5.1. Material of Grade 2

A generalized particle  $p$  for a material of grade 2 with fading memory is described by

$$(5.1) \quad \begin{aligned} g(s) &= \{A^t(s), \nabla A^t(s); \omega(t)\}, \\ s &\in [0, \infty), \quad t \in (-\infty, \infty). \end{aligned}$$

The constitutive equations (2.12) for this case take the form

$$(5.2) \quad \Pi(t) = L(g(s)), \quad \dot{\omega}(t) = \Omega(g(s)), \quad s \in [0, \infty),$$

<sup>(22)</sup> Cf. GURTIN and PIPKIN [20] and CHEN and GURTIN [5].

where the response functional  $\Pi$  represents now the following functionals

$$(5.3) \quad \Pi(t) = \{\psi(t), \eta(t), T_0(t), T_1(t), Q(t)\},$$

and the functional  $\Omega$  has the same form as in our general case (2.16).

Due to (2.25)–(2.29), we have the following results:

$$(5.4) \quad \partial_{\nabla\vartheta}\psi = 0,$$

$$(5.5) \quad T_{R(0)}(t) = 2\rho_R \partial_C \psi(g^*(s)),$$

$$(5.6) \quad T_{R(1)} = 2\rho_R \partial_{\nabla C} \psi(g^*(s)),$$

$$(5.7) \quad \eta(t) = -\partial_\vartheta \psi(g^*(s)),$$

$$(5.8) \quad \delta_0 \psi(g^*(s)|\dot{A}_r^t(s)) + \delta_1 \psi(g^*(s)|\dot{\nabla A}_r^t(s)) + \partial_\omega \psi(g^*(s)) \cdot \Omega(g(s)) + \frac{1}{\rho_R \vartheta} Q(g(s)) \cdot \nabla \vartheta \leq 0,$$

where

$$(5.9) \quad g^*(s) = \{A_r^t(s), \nabla A_r^t(s), C, \nabla C, \vartheta, \omega(t)\}, \quad s \in (0, \infty), \quad t \in (-\infty, \infty).$$

The internal dissipation functional (2.30) for a material of grade 2 becomes

$$(5.10) \quad \sigma(g(s)) = -\frac{1}{\vartheta} \{\delta_0 \psi(g^*(s)|\dot{A}_r^t(s)) + \delta_1 \psi(g^*(s)|\dot{\nabla A}_r^t(s)) + \partial_\omega \psi(g^*(s)) \cdot \Omega(g(s))\}.$$

For an elastic-viscoplastic material of grade 2, the internal dissipation functional (5.10) can be written in the form (cf. 2.42)

$$(5.11) \quad \sigma(g(s)) = -\frac{1}{\vartheta} \left\{ \delta_0 \psi(g^*(s)|\dot{A}_r^t(s)) + \delta_1 \psi(g^*(s)|\dot{\nabla A}_r^t(s)) \right. \\ \left. + \gamma(\vartheta) \langle \Phi(\tilde{\delta}(g(s))) \rangle \operatorname{tr} \left[ \left( \partial_p \psi(g^*(s)) + \partial_* \psi(g^*(s)) H(g(s)) \right) \right. \right. \\ \left. \left. + \sum_{i=1}^m \partial_{R^{(i)}} \psi(g^*(s)) S^{(i)}(g(s)) \right) M(g(s)) \right] \right\}.$$

If we assume that there is no internal changes in the material considered, i.e.,

$$(5.12) \quad \dot{\omega}(t) = 0 \quad \text{for } t \in (-\infty, \infty),$$

then our results may be compared with those published by GURTIN [19]. It appears that he implicitly assumed the conventional net working expression in the first law of thermodynamics while the constitutive equations contained the possibility of dependence on  $\nabla A$ . From the thermodynamic postulate he reached the conclusion that the free energy functional for a material of grade 2 does not depend on the second gradient of the motion.

## 5.2. Elastic-plastic material

Let us assume that the generalized particle  $p$  is described by the set of functions

$$(5.13) \quad g(t) = \{\nabla^j A(t); \omega(t)\}, \quad t \in (-\infty, \infty), \quad j = 0, 1, 2, \dots, n,$$

and the temperature, dependent viscosity coefficient  $\gamma(\vartheta)$  tends to infinity, i.e.,

$$(5.14) \quad \gamma(\vartheta) \rightarrow \infty.$$



Due to these assumptions, the rate of the plastic deformation tensor  $P$  is defined by the relation

$$(5.15) \quad \dot{P}(t) = \xi M(g(t)),$$

where

$$(5.16) \quad \xi = \gamma(\vartheta) \langle \Phi(\tilde{\delta}(g(t))) \rangle$$

denotes a new coefficient.

In this case,

$$(5.17) \quad f(g^\circ(t)) = \varkappa,$$

where

$$(5.18) \quad g^\circ(t) = \{\nabla^j \Lambda(t); P(t), \Gamma^{(i)}(t)\}, \quad j = 0, 1, 2, \dots, n; \quad i = 1, 2, \dots, m.$$

The differential equations for  $\varkappa$  and  $\Gamma^{(i)}$  ( $i = 1, 2, \dots, m$ ) have the form (cf. (2.40) and (2.41))

$$(5.19) \quad \dot{\varkappa}(t) = \xi \operatorname{tr}\{H(g(t)) M(g(t))\},$$

$$(5.20) \quad \dot{\Gamma}^{(i)}(t) = \xi S^{(i)}(g(t)) [M(g(t))].$$

Let us consider an equivalent yield condition in the form

$$(5.21) \quad f(T_{R(j)}, \nabla^j \vartheta; P(t), \Gamma^{(i)}(t)) = \varkappa(t), \quad i = 1, 2, \dots, m; \quad j = 0, 1, 2, \dots, n.$$

It can be shown that the following criterion of loading is valid

$$(5.22) \quad f = \varkappa \text{ and } \sum_{j=0}^n [(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla^j \vartheta} f \cdot \dot{\nabla^j \vartheta})] > 0.$$

Similarly, the criteria

$$(5.23) \quad f = \varkappa \text{ and } \sum_{j=0}^n [(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla^j \vartheta} f \cdot \dot{\nabla^j \vartheta})] \leq 0$$

define the unloading and neutral state, respectively.

The parameter  $\xi$  can be determined from the condition that the point representing the actual state of loading and temperature lies on the yield surface defined by (5.21). It is sufficient to fulfil the equation  $\dot{f} = \dot{\varkappa}$ , i.e.,

$$(5.24) \quad \sum_{j=0}^n \{(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla^j \vartheta} f \cdot \dot{\nabla^j \vartheta})\} + \operatorname{tr}(\partial_P f \dot{P}) + \sum_{i=1}^m \{\partial_{\Gamma^{(i)}} f \cdot S^{(i)}[\dot{P}]\} = \operatorname{tr}(H \dot{P}).$$

This yields the result

$$(5.25) \quad \xi = \lambda \sum_{j=0}^n [(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla^j \vartheta} f \cdot \dot{\nabla^j \vartheta})],$$

where we denoted

$$(5.26) \quad \lambda = \left\{ \left[ \left( H - \partial_P f - \sum_{i=1}^m \partial_{\Gamma^{(i)}} f S^{(i)} \right) \cdot M \right] \right\}^{-1} > 0.$$

Equations (5.15) and (5.25) give

$$(5.27) \quad \dot{P}(t) = \lambda \left\langle \left\{ \sum_{j=0}^n [(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla j \vartheta} f \cdot \nabla^j \dot{\vartheta})] \right\} \right\rangle M(g(t)),$$

where the symbol  $\langle \{ \} \rangle$  is defined by

$$(5.28) \quad \langle \{ \} \rangle = \begin{cases} \{ \} & \text{if } f = \kappa \text{ and } \{ \} > 0, \\ 0 & \text{if } f = \kappa \text{ and } \{ \} \leq 0 \\ \text{or if } f < \kappa. \end{cases}$$

The full system of constitutive equations describing an elastic-plastic material has the form

$$(5.29) \quad \psi(t) = \psi(\nabla^j C, \vartheta; P, \Gamma^{(i)}),$$

$$(5.30) \quad T_{R(j)}(t) = 2Q_R \partial_{\nabla j C} \psi,$$

$$(5.31) \quad \eta(t) = -\partial_{\vartheta} \psi,$$

$$(5.32) \quad q_R(t) = Q(\nabla^j A(t); \kappa P, \Gamma^{(i)}),$$

$$(5.33) \quad \dot{P}(t) = \lambda \left\langle \left\{ \sum_{j=0}^n [(\partial_{T_{R(j)}} f \cdot \dot{T}_{R(j)}) + (\partial_{\nabla j \vartheta} f \cdot \nabla^j \dot{\vartheta})] \right\} \right\rangle M(g(t)),$$

$$(5.34) \quad \dot{\kappa}(t) = \lambda \langle \{ \} \rangle \text{tr}[H(g(t)) M(g(t))],$$

$$(5.35) \quad \dot{\Gamma}^{(i)}(t) = \lambda \langle \{ \} \rangle S^{(i)}(g(t)) [M(g(t))].$$

The general dissipation inequality is as follows

$$(5.36) \quad \sigma - \frac{1}{Q_R \vartheta^2} Q \cdot \nabla \vartheta \geq 0,$$

where  $\sigma$  denotes now the internal dissipation function for an elastic-plastic material, i.e.,

$$(5.37) \quad \sigma(t) = -\frac{\lambda}{\vartheta} \langle \{ \} \rangle \left\{ \left[ (\partial_P \psi + \partial_{\kappa} \psi H(g(t)) + \sum_{i=1}^n \partial_{\Gamma^{(i)}} \psi S^{(i)}(g(t))) \cdot M(g(t)) \right] \right\}.$$

If we additionally assume that our elastic-plastic material is a material of grade 2, put  $m = 1$  and denote the third-order tensor  $\Gamma^{(i)}$  by  $\Gamma$ , then our results are comparable with those published by GREEN, MCINNIS and NAGHDI [18].

## 6. Discussion

The basic postulates of this paper are contained in Eqs. (2.12) and the main results are given as (2.25)–(2.28). The simultaneous consideration of memory effects (viscosity), average internal changes in the microstructure (plasticity) and gradients (fine deformation structure itself) evidently require such a general framework.

Many open questions remain within the framework presented above. The special cases considered and the linearized approximation are intended to illustrate how one specializes the general results. They also aid in interpretation and thereby provide partial answers

to some of the open questions. Since the special results themselves represent very general material behavior, additional limits must be imposed in order to reduce to classical plasticity or in interpreting experiments. This appears to be consistent with what the authors understand to be the physical situation and therefore is desirable. It should be mentioned that the linearization considered was in the deviation of the process from thermodynamic equilibrium and not in either the geometric aspect of the deformation parameters or material constitutive relations per se. The splitting effect obtained in (4.13) has been helpful in understanding some aspects of the general framework.

The assumptions represented by Eqs. (2.31)–(2.32) and (2.36) are special but these or similar statements are required in order to represent the physical phenomena. The sharp limits between reversible processes and irreversible structural changes represented by the brackets in (2.36) idealize well known experimental observations. A function which permits a smooth but rapid change in the dominant part of the process could be considered. The results obtained will not be changed except in the transition region itself.

The most difficult aspect of the postulates to evaluate concerns the role of the gradients ( $\nabla^j A$ ). To a large extent this is due to the general lack of experience and absence of results with such theories. As mentioned above, it is compounded by the absence of any criteria which can uniquely associate specific order gradients with specific physical phenomena. Clearly, if one believes that gradients might be needed in the constitutive relations, they must also be included in the first law as well. Furthermore, they alter slightly the linear and angular momentum equations, which have not been presented here.

After considerable study and thought, we rejected the existing forms of the theory of continuously distributed dislocations as being inadequate for describing the actual process of plastic deformations. We have adopted instead the gradients as the means for including the small scale non-homogeneous deformations that develop in simple tension or shear experiments. In this manner, we have attempted to include what dislocations really are when viewed in the electron microscope, as well as their overall effect on such phenomena as work hardening. For most problems in continuum mechanics, it appears to us that the internal variables are sufficient and in fact that only a few of these will be needed. However, the explicit awareness of what one does when one ignores higher gradients seems to us to be vital in discussions with metal physicists and others who actually perform the small scale experiments.

### Acknowledgement

One of the authors (P. PERZYNA) wishes to express his gratitude to the U.S. National Science Foundation for the Senior Foreign Scientist Fellowship which permitted him to collaborate in this work. Both of the authors are indebted to Professors J. KRATOCHVIL and S. M. VOGEL for valuable discussions of some parts of this work.

### References

1. M. F. ASHBY, *The deformation of plastically non-homogeneous materials*, Phil. Mag., **21**, 399–424, 1970.

2. B. A. BILBY, *Continuous distributions of dislocations*, Prog. Solid. Mech., **1**, 331–398, 1960.
3. D. G. BRANDON and Y. KOMEM, *Quantitative analysis of dislocation distributions*, Metallography, **3**, 111–126, 1970.
4. J. D. CAMPBELL and W. G. FERGUSON, *The temperature and strain-rate dependence of the shear strength of mild steel*, Phil. Mag., **21**, 63–82, 1970.
5. P. J. CHEN and M. E. GURTIN, *On second sound in materials with memory*, ZAMP, **21**, 232–241, 1970.
6. B. D. COLEMAN and W. NOLL, *The thermodynamics of elastic materials with heat conduction and viscosity*, Arch. Rat. Mech. Anal., **13**, 167–178, 1963.
7. B. D. COLEMAN, *Thermodynamics of material with memory*, Arch. Rat. Mech. Anal., **17**, 1–46, 1964.
8. B. D. COLEMAN, *On thermodynamics, strain impulses, and visco-elasticity*, Arch. Rat. Mech. Anal., **17**, 230–254, 1964.
9. B. D. COLEMAN and M. E. GURTIN, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.
10. B. D. COLEMAN and V. J. MIZEL, *A general theory of dissipation in materials with memory*, Arch. Rat. Mech. Anal., **27**, 255–274, 1968.
11. H. CONRAD, *Thermally activated deformation of metals*, J. Metals, **16**, 582–588, 1964.
12. O. W. DILLON and J. KRATOCHVIL, *A strain gradient theory of plasticity*, Int. J. Solid and Structures (1970) in print.
13. M. A. EISENBERG, *On the relation between continuum plasticity and dislocation theories*, Int. J. Engng. Sci., **8**, 261–271, 1970.
14. N. FOX, *On the continuum theories of dislocations and plasticity*, Quart. J. Mech. and Appl. Math., **21**, 67–75, 1968.
15. A. G. EVANS and R. D. RAWLINGS, *The thermally activated deformation of crystalline materials*, Phys. Stat. Sol., **34**, 9–31, 1969.
16. A. E. GREEN and RIVLIN, *Simple force and stress multiplies*, Arch. Rat. Mech. Anal., **16**, 325–353, 1964.
17. A. E. GREEN and R. S. RIVLIN, *Multipolar continuum mechanics*, Arch. Rat. Mech. Anal., **17**, 113–147, 1964.
18. A. E. GREEN, B. C. MCINNIS and P. M. NAGHDI, *Elastic-plastic continua with simple force dipole*, Int. J. Engng. Sci., **6**, 373–394, 1968.
19. M. E. GURTIN, *On the thermodynamics of materials with memory*, Arch. Rat. Mech. Anal., **28**, 40–50, 1968.
20. M. E. GURTIN and A. C. PIPKIN, *A general theory of heat conduction with finite wave speeds*, Arch. Rat. Mech. Anal., **31**, 113–126, 1968.
21. M. E. GURTIN, V. J. MIZEL and W. O. WILLIAMS, *A note on Cauchy's stress theorem*, J. Math. Anal. Appl., **22**, 398–401, 1968.
22. P. R. HALMOS, *Measure theory*, Van Nostrand, New York 1950.
23. J. KRATOCHVIL and O. W. DILLON, *Thermodynamics of elastic-plastic materials as a theory with internal state variables*, J. Appl. Phys., **40**, 3207–3218, 1969.
24. J. KRATOCHVIL and O. W. DILLON, *Thermodynamics of crystalline elastic-visco-plastic materials*, J. Appl. Phys., **41**, 1470–1479, 1970.
25. E. KRÖNER, *Dislocation field theory*, Theory of Crystal Defects, Proc. Summer School, September 1964, Academic Press, New York 1966, pp. 231–256.
26. R. W. LARDNER, *Dislocation dynamics and the theory of the plasticity of single crystals*, ZAMP, **20**, 514–529, 1969.
27. R. D. MINDLIN and TIERSTEN, *Effects of couple-stresses in linear elasticity*, Arch. Rat. Mech. Anal., **11**, 415–438, 1962.
28. R. D. MINDLIN, *Micro-structure in linear elasticity*, Arch. Rat. Mech. Anal., **17**, 51, 1964.
29. T. MURA, *Continuum theory of dislocations and plasticity*, Proc. IUTAM—Symposium on the Generalized Cosserat Continuum and the Continuum Theory of Dislocations with Applications, 1967; Springer 1968, **11**, 269–278.
30. W. NOLL, *A mathematical theory of the mechanical behavior of continuous media*, Arch. Rat. Mech. Anal., **2**, 197–226, 1958.

31. P. PERZYNA, *The constitutive equations for rate sensitive plastic materials*, Quart. Appl. Math., **20**, 321–332, 1963.
32. P. PERZYNA, *Fundamental problems in viscoplasticity*, Advances in Applied Mechanics, **9**, 243–377, 1966.
33. P. PERZYNA, *Thermodynamics of rheological materials with internal changes*, J. Mecanique, **10**, 391–408, 1971.
34. P. PERZYNA, *Memory effects and internal changes of a material*, Int. J. Non-linear Mechanics, **6**, 707–716, 1971.
35. P. PERZYNA and W. WOJNO, *Thermodynamics of a rate sensitive plastic material*, Arch. Mech. Stos., **20**, 499–511, 1968.
36. A. R. ROSENFELD and G. T. HAHN, *Numerical descriptions of the ambient low-temperature, and high-strain rate flow and fracture behavior of plain carbon steel*, Tran. ASM, **59**, 962–980, 1966.
37. L. I. SEDOV and V. L. BERDITCHEVSKI, *A dynamic theory of continual dislocations*, Proc. IUTAM — Symposium on the Generalized Cosserat Continuum and the Continuum Theory of Dislocations with Applications, 1967; Springer 1968, pp. 214–238.
38. A. SEEGER, *The generation of lattice defects by moving dislocations, and its application to the temperature dependence of the flow-stress of f.c.c. crystals*, Phil. Mag., **46**, 1194–1217, 1955.
39. R. A. TOUPIN, *Elastic materials with couple-stresses*, Arch. Rat. Mech. Anal., **11**, 385–414, 1962.
40. K. C. VALANIS, *Unified theory of thermomechanical behavior of viscoelastic materials*, Symp. on Mechanical Behavior of Materials under Dynamic Loads, 1967 San Antonio; Springer 1968, pp. 343–364.
41. C. C. WANG, *Generalized simple bodies*, Arch. Rat. Mech. Anal., **32**, 1–28, 1969.
42. S. ZAHORSKI, *On motion and thermodynamics of non-simple continua with microstructure*, Arch. Mech. Stos., **19**, 25–44, 1967.

DEPARTMENT OF ENGINEERING MECHANICS, UNIVERSITY OF KENTUCKY  
LEXINGTON, KENTUCKY, USA  
and  
INSTITUTE OF FUNDAMENTAL TECHNICAL RESEARCH, POLISH ACADEMY  
OF SCIENCES, WARSAW, POLAND

Received February 12, 1972