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Yield conditions in plasticity

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NO SATISFACTORY form seems to exist for the yield condition in aelotropic bodies. It is proposed that at yield the mapping of the macro-element becomes singular giving rise to a yield condition in terms of invariants of the strain tensor, J_1, J_2, J_3 in the form $8J_3-4J_2+2J_1-1 \to 0$. This is transformed into the invariants of the stress tensor and it is shown that in a large number of cases the yield condition reduces to a generalized form of Tresca yield condition given by $\tau_{11}-k_0\tau_{33} = k_1$.

Wydaje się, że brak jest zadowalających warunków plastyczności dla ciał anizotropowych. Przyjmuje się, że przy uplastycznieniu odwzorowanie makro-elementu staje się osobliwe, co prowadzi do warunku plastyczności, wyrażonego przez niezmienniki tensora odkształcenia J_1, J_2, J_3 w postaci $8J_3-4J_2+2J_1-1 \to 0$. Po przekształceniu tego warunku do postaci zawierającej niezmienniki tensora naprężenia okazuje się, że w wielu przypadkach warunek plastyczności sprowadza się do uogólnionego warunku Treski $\tau_{11}-k_0\tau_{33} = k_1$.

Кажется, что отсутствуют удовлетворительные условия пластичности для анизотропных тел. Предполагается, что при пластичности отображение макро-элемента становится особенным, что ведет к условию пластичности выраженному через инварианты тензора деформации J_1, J_2, J_3 в виде: $8J_3-4J_2+2J_1-1 \to 0$. После преобразования этого условия к виду, который содержит инварианты тензора напряжения оказывается, что в многих случаях условие пластичности сводится к обобщенному условию Треска: $\tau_{11}-k_0\tau_{33} = k_1$.

1. Introduction

IT IS customary to treat the plastic state independent of the elastic one from which it emanates. An immediate consequence is that some yield condition such, as that of von Mises or Tresca, has to be introduced. Such a procedure has given a number of useful results. But this has involved a semi-empirical hypothesis which may or may not exist [1]. Also, it does not show that the plastic state is only an ideal state, which should be obtained as a limiting case from an elastic-plastic solution. If the elastic space is denoted by A, the plastic by B, the elastic-plastic by T, then T is an asymptotic sub-space of A and B. The fully plastic state is the limit of T when the response coefficient, called the Poisson's ratio, approaches the value 1/2.

In an isotropic medium, the von Mises or Tresca yield conditions have been widely used. For aelotropic bodies the corresponding hypothesis is not available. R. HILL suggests for orthotropic materials the yield condition [2]:

$$(1.1) \quad C_{11}(\tau_{22}-\tau_{33})^2+C_{22}(\tau_{33}-T_{11})^2+C_{33}(\tau_{11}-\tau_{22})^2 +2C_{44}\tau_{23}^2+2C_{55}\tau_{34}^2+C_{66}\tau_{12}^2 = 1,$$

where C's are parameters characteristic of the current state of aelotropy, and the principal axes of aelotropy are the axes of reference. He disregards the Bauchinger effect, and assumes that (1.1) should reduce to von Mises form when the aelotropy is negligible small.

This is not a function of the invariants either of the stress-tensor or of the response-coefficients tensor.

For the general aelotropic condition of 21 response coefficients, no yiedl condition seems to have been suggested by any author.

Thus arises the need of a new scientific approach to the yield conditions. In any new formulation it has to be borne in mind, as R. HILL has pointed out, that constitutive inequalities, such as the yield condition, depend upon the strain measure used [3]. Moreover, its invariance (i) under arbitrary orthogonal coordinate transformations and (ii) under group symmetry operations of the crystal must be preserved.

If a start is made with the stress tensor field, we soon run into a number of difficulties without getting any satisfactory results. The foundations of plasticity, therefore, should be based on the geometry of the field. As spin or rotation effects become prominent in all interaction fields, the second degree terms in the strain tensor, which represent the rotation effects, and which are disregarded in the classic linear theory, should not now be neglected. Moreover, for plastic flow problems the current state is important and hence a deformed framework should be employed.

2. The yield condition

Since yield is an asymptotic phenomenon, it imposes a constraint on the invariants of the field tensor. The change from elastic to plastic deformation can be interpreted as a mapping of one space into another. If the elastic strain field is e_{ij}, whose invariants are J_1, J_2, J_3, then the asymptotic behaviour may be represented by the existence of a functional relation of the type

$$f(J_1, J_2, J_3) = 0,$$

in which J's are independent of one another in the normal part of the field. This does not fix the nature of the function f for which we can invoke the additional geometric condition that yielding can result from infinite contraction or expansion of a macro-element. This shows that the transformation matrix should become singular. We use this concept to arrive at the yield condition in the strain tensor field.

If u^r is the deformation vector and X^r the deformed coordinates, the modulus of transformation is given by

$$\frac{\partial(x-u, y-v, z-\omega)}{\partial(x, y, z)}. \tag{2.1}$$

Expanding and squaring it, we find that its vanishing gives the condition:

$$8J_3-4J_2+2J_1 \to 1, \tag{2.2}$$

where J's are the invariants of the Almansi strain tensor, which in the Cartesian system is given by:

$$2e_{ij} = u_{i,j}+u_{j,i}-u_{\alpha,i}u_{\alpha,j}. \tag{2.3}$$

The values of J's are

$$J_1 = \frac{1}{1!}\delta^i_j e^j_i, \quad J_2 = \frac{1}{2!}\delta^{ik}_{jl} e^l_k e^j_i, \tag{2.4}$$
$$J_3 = \frac{1}{3!}\delta^{ikm}_{jln} e^n_m e^l_k e^j_i.$$

Referred to principal axes, the relation (2.2) reduces to

$$(1-2e_{11})(1-2e_{22})(1-e_{33}) \to 0. \tag{2.5}$$

For generalized measure, this takes the form [1]:

$$(1-ne_{11})(1-ne_{22})(1-ne_{33}) \to 0. \tag{2.6}$$

This shows that at yielding

$$e_{11}, e_{22}, e_{33} \to \left(\frac{1}{n}\right). \tag{2.7}$$

These conditions hold good for any medium — homogeneous, heterogeneous, isotropic or aelotropic. Thus they can be made the starting point to obtain the yield condition in terms of the stress invariants, I's. This is not always easy. Firstly, some constitutive equation has to be assumed and then except, in the isotropic case, I's cannot be expressed in terms of J's. Thus, though (2.2) is linear in J's it is found that, even for the isotropic case, the condition in terms of I's is cubic. For the aelotropic case, it is found to be of the sixth degree in I's. All these conditions include the classical yield conditions as particular cases.

2.1. Isotropic body

Assuming a linear stress-strain tensor relation, which is found to be adequate for our purpose, we have

$$e_{ij} = E^{-1}[(1+\sigma)\tau_{ij} - \sigma\delta_{ij}\tau_{\alpha\alpha}], \tag{2.8}$$

where E is Young's modulus and σ the Poisson's ratio. In the transition region they change and become the response coefficients. In fact $\sigma \to 1/2$ as the fully plastic state is reached. From (2.8) we readily obtain the following relations:

$$EJ_1 = (1-2\sigma)I_1,$$
$$E^2 J_2 = [(1+\sigma)^2 I_2 - \sigma(2-\sigma)I_1^2], \tag{2.9}$$
$$E^3 J_3 = [(1+\sigma)^3 I_3 - \sigma(1+\sigma)^2 I_1 I_2 + \sigma^2 I_1^3].$$

Substituting these values in (2.2) and making use of the relation

$$I_2^1 = 2I_1^2 - 6I_2, \tag{2.10}$$

the transition condition for elastic-plastic deformation is found to be

$$(1-2\sigma)E^{-1}I_1 + \frac{2}{3}E^{-2}\left[\frac{1}{2}(1+\sigma)^2 I_2^1 - (1-2\sigma)I_1^2\right] \tag{2.11}$$
$$+4E^{-3}\left[(1+\sigma)^3 I_3 + \frac{1}{6}\sigma(1+\sigma)^2 I_1 I_2^1 - \frac{1}{3}\sigma(1\cdot\ \sigma+\sigma^3 I_1^3)\right] = \frac{1}{2}.$$

In (2.11), the elastic effect is present through E and σ. For the fully plastic state, $\sigma \to 1/2$ and we obtain:

$$3I_2^1+2\left[27I_3+\frac{3}{2}I_2^1 I_1-I_1^3\right]=2, \tag{2.12}$$

where the invariants I's have been made non-dimensional with respect to E. E can now be expressed in terms of the yield stress in tension. Putting τ_{22}, $\tau_{33}=0$ in (2.12), we obtain:

$$3(\tau_{11}/E)^2+2(\tau_{11}/E)^3 = 1, \tag{2.13}$$

which gives

$$\tau_{11}=\frac{1}{2}E \quad \text{or} \quad -E. \tag{2.14}$$

If Y is the yield stress in tension, then

$$Y=\frac{1}{2}E.$$

From (2.14) we note the Bauschinger effect — the yield stress in compression is different from that in tension and is twice the value of the latter.

In terms of the principal stresses, (2.12) can be put in the following convenient forms:

$$3I_2^1+2(3\tau_{11}-I_1)(3\tau_{22}-I_1)(3\tau_{33}-I_1) = 2, \tag{2.15}$$

$$\sum(\tau_{11}^1)^2+6\tau_{11}^1\tau_{22}^1\tau_{33}^1=\frac{2}{9}, \tag{2.16}$$

where τ_{11}^1, ... etc. is the deviatoric stress in a non-dimensional form.

In a large number of classical problems of torsion, flexure, plane stress and plane strain, I_3 vanishes identically and from (2.12) we obtain the yield condition in the form [4]:

$$3I_2^1 = 2(1-I_1+I_1^2), \quad I_1 \neq -1. \tag{2.17}$$

This should be used for such cases as that of combined loads in the form of tension, torsion, flexure. From the classical point of view, it will be interpreted as showing work-hardening due to the presence of the terms I_1 and I_1^2 on the right-hand side. STASSI [5] also uses a similar condition.

For the particular cases of the Haar-Kármán [6] hypothesis and the principal line theory [7], where $\tau_{33}=\tau_{22}$ and $\tau_{33}=\frac{1}{2}(\tau_{11}+\tau_{22})$, respectively, we see that (2.15) and (2.16) reduce to the Tresca form. For plane stress, we readily obtain [8]:

$$\tau_{11}-\frac{1}{2}\tau_{22}=Y, \quad \tau_{11}-2\tau_{22} = -2Y, \tag{2.18}$$

for tension and compression respectively. Thus we see that the condition (2.12) contains all the classical conditions. It also shows that elastic failure depends on the type of deformation and can occur in any of the following cases when:

(i) the elastic energy of deformation reaches a critical value,

(ii) the principal strain becomes a maximum,
(iii) the principal stress becomes a maximum,
(iv) the principal stress difference becomes a maximum.

3. Aelotropic body

In this case, the linear stress-strain tensor relation is

(3.1) $$\Delta e_{ij} = C_{ij}^{hk}\tau_{hk}, \quad \Delta = |C_{ij}|,$$

C_{ij} being elastic response coefficients.

If the strain energy function exists, the 36 response coefficients reduce to 21 as

(3.2) $$C_{ij}^{hk} = C_{ji}^{hk}.$$

Now the principal axes of strain and stress are not parallel except in the orthotropic case. We take the axes of principal strains as the coordinate axes, of which at least one set exists. As indicated in (2.7), at the elastic-plastic transition we can take $e_{11} \to \frac{1}{2}$. Thus we obtain the four linear equations in τ_{hk} as

(3.2') $$\frac{1}{2}\Delta = C_{11}^{hk}\tau_{hk}, \quad 0 = C_{12}^{hk}\tau_{hk}$$
$$0 = C_{23}^{hk}\tau_{hk}, \quad 0 = C_{13}^{hk}\tau_{hk}.$$

We also have the first stress-invariant given by

(3.3) $$I_1 = \tau_{11}+\tau_{22}+\tau_{33}.$$

Equations (3.2) and (3.3) are linear in the six components of τ_{ij}. We can determine five of them in terms of the sixth, (say τ_{11}) and I_1 and C's. We substitute their values in the remaining two invariants:

(3.4) $$I_2 = \tau_{11}I_1-\tau_{11}^2+\tau_{22}\tau_{33}-\tau_{23}^2-\tau_{31}^2-\tau_{12}^2,$$

(3.5) $$I_3 = \tau_{11}I_2-\tau_{11}^2I_1+\tau_{11}^3+2\tau_{12}\tau_{23}\tau_{31}+\tau_{12}^2(\tau_{11}-\tau_{33})+\tau_{31}^2(\tau_{11}-\tau_{22}).$$

Thus we arrive at the following two equations in τ_{11}:

(3.6) $$I_2 = \alpha_1\tau_{11}^2+\alpha_2\tau_{11}+\alpha_3,$$

(3.7) $$I_3 = \beta_1\tau_{11}^3+\beta_2\tau_{11}^2+\beta_3\tau_{11}+\beta_4,$$

where α's and β's are functions of I_1, I_2 and C's. τ_{11} can easily be eliminated between (3.6) and (3.7).

Thus we obtain the transition invariant relation:

(3.8) $$[(\alpha_3-I_2)(\alpha_1\beta_3-\alpha_3\beta_1+\beta_1I_2)-\alpha_1\alpha_2(\beta_4-I_3)]\times$$
$$\times[\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)-\alpha_1(\alpha_1\beta_3-\alpha_3\beta_1+\beta_1I_2)] = [\alpha_1^2(\beta_1-I_3)-(\alpha_3-I_2)(\alpha_1\beta_2-\alpha_2\beta_1)].$$

This is of the sixth degree in I's. By imposing on the response coefficients C's a suitable condition, we can get the plasticity condition. This may be illustrated by taking the simpler case of the orthotropic body in which the principal axes of stress and strain are parallel.

4. Orthotropic body

Referred to the principal axes, the relation (3.1) becomes

(4.1) $$\Delta e_{ij} = C_{ij}^{hk}\tau_{hk}, \quad C_{ij}^{hk} = 0, \quad h \neq k.$$

There are now three Young's moduli and six Poisson's ratios [9]. For example, for tension across the plane 1, the corresponding values are [9]:

(4.2) $$E_1 = \Delta/C_{11}^{11}, \quad \sigma_{12} = -C_{11}^{22}/C_{11}^{11}, \quad \sigma_{13} = -C_{11}^{33}/C_{11}^{11}.$$

It may be noted that

$$\sigma_{21} = -C_{11}^{22}/C_{22}^{22},$$

which is not equal to σ_{12}.

The general value of the modulus of compression, k, is

(4.3) $$\frac{1}{k} = \frac{1}{E_1}(1-\sigma_{12}-\sigma_{13})+\frac{1}{E_2}(1-\sigma_{21}-\sigma_{23})+\frac{1}{E_3}(1-\sigma_{31}-\sigma_{32}).$$

For the isotropic case

(4.4) $$\frac{1}{k} = \frac{3(1-2\sigma)}{E}.$$

Since E's are independent of each other, $1/k$ will vanish only if simultaneously we have:

(4.5) $$\sigma_{ij}+\sigma_{ik} = 1, \quad i \neq j, \quad i \neq k.$$

These are the plasticity conditions corresponding to $\sigma \to \frac{1}{2}$ of the isotropic case. For the transition condition (3.8), we notice that now

$$I_3 = \tau_{11}^3-\tau_{11}^2 I_1+\tau_{11} I_2,$$

so that

(4.6) $$\beta_1 = 1, \quad \beta_2 = -I_1, \quad \beta_3 = I_2, \quad \beta_4 = 0.$$

Substituting these values, we see that (3.8) reduces to

(4.7) $$[\alpha_1\alpha_2 I_3+(\alpha_3-I_2)(\alpha_1 I_2+I_2-\alpha_3)][\alpha_1(\alpha_1 I_2+I_2-\alpha_3) + \alpha_2(\alpha_1 I_1+\alpha_2)]+[(\alpha_3-I_2)(\alpha_1 I_1+\alpha_2)-\alpha_1^2 I_3]^2 = 0,$$

where α's are given by

[4.8] $$\begin{aligned} \alpha_1+1 &= D^2(1+\sigma_{12})(1+\sigma_{13}),\\ \alpha_2 &= D^2\left[\frac{1}{2}E_1(2+\sigma_{12}+\sigma_{13})+I_1(\sigma_{12}+\sigma_{13}+\sigma_{12}^2+\sigma_{12}^2)\right],\\ \alpha_1 I_1+\alpha_2 &= D^2\left[\frac{1}{2}E_1(2+\sigma_{12}+\sigma_{13})+I_1(1-\sigma_{12}\sigma_{13})\right],\\ \alpha_3 &= -\frac{1}{4}D^2(E_1+\sigma_{12}I_1)(E_1+\sigma_{13}I_1),\\ 1/D^2 &= (\sigma_{12}-\sigma_{13})^2 = (\sigma_{12}+\sigma_{13})^2-4\sigma_{12}\sigma_{13}. \end{aligned}$$

These contain only I_1, the Young's modulus and the Poisson's ratios.

In the particular case when $I_3 = 0$, the relation (4.7) reduces to

$$I_2 = \alpha_3, \tag{4.9}$$

$$[(\alpha_1+1)I_2-\alpha_3][\alpha_1 I_2(\alpha_1+1)-\alpha_1\alpha_3+\alpha_2(\alpha_1 I_1+\alpha_2)]+(\alpha_3-I_2)(\alpha_1 I_1+\alpha_2)^2 = 0. \tag{4.10}$$

The relation (4.9) is of the von Mises type. Also (4.10) is now of the fourth order in I's.

The response coefficients may be expressed in terms of the yield stress in tension and compression. Putting $\tau_{22}, \tau_{33} = 0$ in (4.7), we obtain:

$$\begin{aligned} \tau_{11} &= \frac{1}{2}E_1 = Y, \\ \tau_{11} &= -\frac{1}{2}\frac{E_1}{\sigma_{12}} = -\frac{1}{2}\frac{E_2}{\sigma_{21}} = Y_1, \\ \tau_{11} &= -\frac{1}{2}\frac{E_1}{\sigma_{13}} = -\frac{1}{2}\frac{E_2}{\sigma_{31}} = Y_2. \end{aligned} \tag{4.11}$$

Thus all α's in (4.8) can be expressed in term of Y, Y_1 and Y_2. For the fully plastic case when $\sigma_{12}+\sigma_{13} = 1$, we obtain:

$$\frac{1}{Y}+\frac{1}{Y_1}+\frac{1}{Y_2} = 0. \tag{4.12}$$

The following particular cases may be noted:

a) $\tau_{22} = \tau_{33}$. Now we have for tension

$$\tau_{11}-\tau_{33} = \frac{\frac{3}{2}E_1-I_1(1-\sigma_{12}-\sigma_{13})}{2+\sigma_{12}+\sigma_{13}},$$

which reduces to the Tresca's form

$$\tau_{11}-\tau_{33} = Y,$$

when $\sigma_{12}+\sigma_{13} = 1$.

For compression, we have:

$$\tau_{11}-\tau_{33} = Y_1 \quad \text{or} \quad Y_2. \tag{4.13}$$

b) Plane stress. Now we can put $\tau_{3i} = 0$, $i = 1, 2, 3$. The corresponding yield conditions for tension and compression are found to be

$$\frac{\tau_{11}}{Y}+\frac{\tau_{22}}{Y_2} = 1$$

and

$$\frac{\tau_{11}}{Y_1}+\frac{\tau_{22}}{Y_2} = 1. \tag{4.14}$$

c) Plane strain. Now $e_{33} = 0$, we have:

$$I_2 = \sum(\tau_{11}-\tau_{22})^2 = \frac{9}{4}E_1^2(1+\sigma_{12}^2+\sigma_{13}^2), \qquad \sigma_{12}+\sigma_{13} = 1. \tag{4.15}$$

But

$$\sigma_{31}(\tau_{33}-\tau_{11}) = \sigma_{32}(\tau_{22}-\tau_{33}).$$

Thus we have the reduced form:

$$\tau_{11}-\tau_{22} = \frac{Y}{1-\sigma_{13}\sigma_{31}}. \tag{4.16}$$

d) When $\tau_{22} = \frac{1}{2}(\tau_{11}+\tau_{33})$, we again obtain a result similar to that in (a).

In all these cases, we see that the orthotropic yield conditions are of the Tresca type.

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