Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 5-6, pp. 931-940, Warszawa 1972

Creep in slightly redundant structures

J. HULT (GÖTEBORG)

ENGINEERING structures like an I-beam subject to plane bending and a thin-walled sphere loaded by pressure are examples of slightly redundant structures. The stress fields in such nearly isostatic structures depend only slightly on the constitutive properties of the material. A perturbation method may then be used to analyze the stress field during stationary creep without having to specify the creep in detail. It is shown that for some standard structures the deviation from the isostatic case is an effect to the second order in a geometrically linear measure of redundancy.

Takie konstrukcje inżynierskie jak belki dwuteowe zginane w płaszczyźnie lub cienkościenne powłoki kuliste, poddane ciśnieniu wewnętrznemu, stanowią przykłady ustrojów słabo statycznie niewyznaczalnych. Rozkłady naprężenia w takich prawie statycznie wyznaczalnych konstrukcjach zależą jedynie w nieznacznym stopniu od własności konstytutywnych materiału. Można zatem zastosować do analizy procesu ustalonego pełzania takich konstrukcji metodę perturbacji bez wchodzenia w szczegóły samego pełzania. Wykazano, że w pewnych standardowych konstrukcjach tego rodzaju odchylenie od przypadku statycznie wyznaczalnego staje się efektem drugiego rzędu względem liniowej miary statycznej niewyznaczalności.

Примерами слабо статически неопределимых сооружений являются такие инженерные конструкции, как двутавровые балки, изгибаемые в плоскости, или тонкостенные сферические оболочки под внутренним давлением. Распределения напряжений в таких, почти статически определимых сооружениях, зависят лишь в незначительной степени от физических свойств материала. Поэтому для анализа процессов установившейся ползучести в таких сооружениях можно употребить метод возмущений, не входя в подробности самого процесса ползучести. Показано, что в некоторых стандартных конструкциях такого рода отклонения от статической определимости являются эффектами второго порядка по отношению к линейной мере статической неопределимости.

1. Introduction

THE THEORY of creep is becoming an important tool in the design of high temperature machinery. Its basic foundations are now rather well established, both on the microscopic and the macroscopic level.

The microscopic theory describes various creep phenomena in terms of previously known physical entities and laws of interaction. The macroscopic theory is a branch of continuum mechanics. Both these aspects are essential to applications in design work.

Creep rupture is known to start in small isolated regions, and so the conditions for creep rupture depend strongly on phenomena on a microscopic level. Steady creep deformation on the other hand is governed by certain average material properties. Hence design rules against creep rupture call for a deeper physical understanding of the creep process than do design rules against excessive creep deformation.

The two aspects of creep are in no way opposed to one another. On the contrary, they depend strongly on each other, and their common ground seems to increase as our understanding of the creep phenomena deepens. Physical theory cannot disregard certain general relations derived on the macroscopic level and the mechanical theory must not contain any violations of basic physical principles.

Design against creep deformation is complicated for two reasons as compared with traditional practice: time enters as a new variable, and the constitutive laws are usually nonlinear. As a result creep calculations normally require an electronic computer. This is strongly reflected in a recent creep design monograph by PENNY and MARRIOTT (1971), which simply takes the availability of a computer for granted.

A primary goal for the creep designer is to determine the stress distribution in the structure. Once the stress field is known the deformation may be calculated, and the time to creep rupture may be estimated.

Most engineering structures are redundant (hyperstatic) and so the constitutive equations enter in the stress calculation. The complexity of the stress calculation is then largely governed by the complexity of the constitutive equations. In certain cases, however, the situation is simplified. For all isostatic structures, subject to prescribed surface tractions, the stress field is determined by equilibrium requirements alone.

Examples of such isostatic structures are the idealized I-beam with infinitely thin flanges and negligible web area subject to bending in the plane of the web, and the infinitely thin-walled sphere subject to internal or external pressure. Both these structures represent limiting cases, which may not be realized in practice. If, however, the flange and wall thicknesses are small but finite, the stress distributions may be expected to deviate only slightly from those in the isostatic cases. The magnitude of this deviation will depend on the geometry of the structure and on the constitutive law of the material. One may anticipate that the influence of the material properties will be smaller the closer the structure approximates the limiting isostatic case. It is the object of this paper to study such nearly isostatic cases in some detail.

2. Simple truss

As an introductory example a simple plane, three-bar truss will be examined, cf. Fig. 1. It is easily analyzed, and it displays some features common to all hyperstatic structures. The state of stress in this truss will be determined under the following assumptions:

1) the loading force P is applied at zero time and then kept constant;

2) the deformations are so small that equilibrium conditions may be stated for the undeformed state;

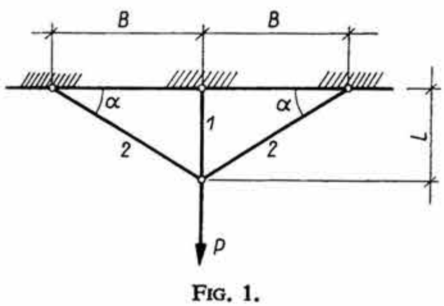

FIG. 1.

3) the deformations are so small that all second-order terms may be neglected in the compatibility conditions.

Denoting the cross sectional area by A, and the stresses by σ, we then obtain the equilibrium condition

$$\sigma_1 A + 2\sigma_2 A \sin\alpha = P, \tag{2.1}$$

where index 1 and 2 refer to the central bar and the outer bars respectively.

With ε denoting the strain follows the compatibility condition

$$\varepsilon_1 \sin^2\alpha - \varepsilon_2 = 0. \tag{2.2}$$

The constitutive equation of the bar material is assumed to belong to the class

$$\varepsilon = \varepsilon^{(e)} + \varepsilon^{(c)} \tag{2.3}$$

with

$$\varepsilon^{(e)} = G(\sigma) \tag{2.4}$$

and

$$\dot{\varepsilon}^{(c)} = F(\sigma). \tag{2.5}$$

Here $\varepsilon^{(e)}$ denotes time-independent elastic strain and $\varepsilon^{(c)}$ denotes creep strain. The dot in Eq. (2.5) denotes the time derivative, and hence Eq. (2.5) predicts a constant rate of creep strain for all cases of constant stress. With P = constant then follows (HULT, 1962) that a constant state of stress will be reached in the truss after a certain stress redistribution has taken place. With σ = constant the constitutive equations (2.3)–(2.5) may be replaced by the simpler one

$$\dot{\varepsilon} = F(\sigma). \tag{2.6}$$

As shown by HOFF (1954) the same stress field would be obtained if the constitutive equation were of the type

$$\varepsilon = F(\sigma). \tag{2.7}$$

This may be interpreted as the constitutive equation of a nonlinearly elastic material. Since $\dot{\varepsilon}^{(c)} = 0$ when $\sigma = 0$, it follows from Eq. (2.5) that

$$F(0) = 0. \tag{2.8}$$

The following analysis, which aims at finding the stationary state of stress, will therefore be based on this simpler constitutive equation, i.e.

$$\varepsilon_1 = F(\sigma_1), \quad \varepsilon_2 = F(\sigma_2). \tag{2.9}$$

The stresses σ_1 and σ_2 may now be determined from the four Eqs. (2.1), (2.2), $(2.9)_1$ and $(2.9)_2$. Two limiting cases are of interest:

I) $\alpha \to 0$. From Eq. (2.1) then follows $\sigma_1 = P/A$, and from Eqs. (2.2), (2.8), $(2.9)_2$ follows $\sigma_2 = 0$ for any material behavior of type (2.7) and (2.8). The truss here degenerates to acting as only one load carrying member, and hence this system may be termed isostatic.

II) $\alpha \to \pi/2$. From Eq. (2.2) then follows $\varepsilon_1 = \varepsilon_2$ and hence from Eqs. $(2.9)_1$ and $(2.9)_2$ follows $\sigma_1 = \sigma_2$ irrespective of the shape of the function F. From Eq. (2.1) then follow the stresses $\sigma_1 = \sigma_2 = P/3A$. In this limiting case the truss again degenerates to only one load carrying member, viz. a bar with cross sectional area $3A$.

Hence when $\alpha = 0$ and $\alpha = \pi/2$ the stresses in this truss are independent of the material properties, provided they belong to the class (2.7) and (2.8).

When α is near 0 or $\pi/2$ the stresses are then likely to depend only weakly on the material properties. On the other hand, the dependence on the material properties will be a maximum at some intermediate slope α. We shall consider the latter case first.

Two forms of the function F will be assumed:

1. $F(\sigma) = B\sigma^n$. This corresponds to the creep law usually associated with NORTON (1929). A closed form solution is obtained, and, in particular, the largest stress is found to be

$$\sigma_1 = \frac{P/A}{1+2\sin^{1+2/n}\alpha}. \tag{2.10}$$

As shown elsewhere (HULT, 1962) this proportionality between load and stresses is obtained only when F is a power function.

The stress σ_1 is never larger than the corresponding stress in the linearly elastic truss

$$\sigma_1^* = \frac{P/A}{1+2\sin^3\alpha}. \tag{2.11}$$

Plotting the ratio

$$R = \sigma_1/\sigma_1^* \tag{2.12}$$

for various n-values, with α varying between zero and $\pi/2$, we obtain the diagram in Fig. 2.

For any given n-value the ratio R is a minimum for a certain slope α. It is seen that this optimum α-value is fairly independent of n. When α falls in this range the truss displays

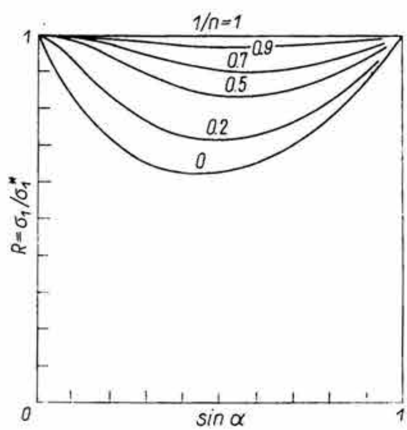

FIG. 2.

a maximum dependence on the material property n. We may denote this α-interval the *region of maximum redundancy*.

2. $F(\sigma) = K\text{sh}(\sigma/\sigma_0)$. This corresponds to the creep law often associated with PRANDTL (1928). No closed-form solution is obtained, and the stresses are no longer proportional

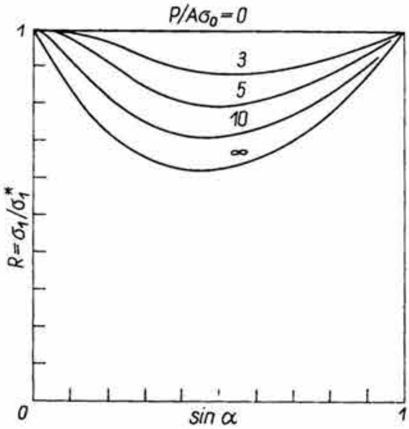

FIG. 3.

to the load. The stress ratio (2.12) will therefore depend on the load P as shown in Fig. 3. Again the truss shows maximum redundancy in a rather limited α-range. It should be noted that the limiting curve corresponding to infinite load is identical with the limiting curve in Fig. 2 corresponding to the rigid plastic material.

The case of nearly parallel bars will now be considered. We shall term this a case of *slight redundancy*, even if this terminology may not pass entirely without objections. When the bars are nearly parallel

(2.13) $$\sin\alpha \approx 1-B^2/2L^2$$

and we shall use the small parameter

(2.14) $$\theta^2 = B^2/2L^2 \ll 1$$

in the calculations. The basic equations (2.1) and (2.2) then take the forms, to order θ^4

(2.15) $$\begin{gathered}\sigma_1 = 2(1-\theta^2)\sigma_2 = P/A,\\ (1-2\theta^2)\varepsilon_1-\varepsilon_2 = 0.\end{gathered}$$

Instead of the constitutive equations $(2.9)_1$ and $(2.9)_2$, we shall use their inverses

(2.16) $$\begin{gathered}\sigma_1 = F^{-1}(\varepsilon_1) \equiv f(\varepsilon_1),\\ \sigma_2 = F^{-1}(\varepsilon_2) \equiv f(\varepsilon_2).\end{gathered}$$

When $\theta \to 0$ it then follows that

(2.17) $$\begin{gathered}\varepsilon_1 = \varepsilon_2 = F(P/3A) \equiv \varepsilon_0,\\ \sigma_1 = \sigma_2 = P/3A = f(\varepsilon_0).\end{gathered}$$

With $0 < \theta \ll 1$ we now put

(2.18) $$\varepsilon_1 = \varepsilon_0+\delta.$$

From the compatibility relation $(2.15)_2$ then follows to order θ^4

$$\varepsilon_2 = \varepsilon_0+\delta-2\theta^2\varepsilon_0-2\theta^2\delta. \tag{2.19}$$

The Taylor series expansion

$$\sigma = f(\varepsilon) = f(\varepsilon_0)+(\varepsilon-\varepsilon_0)f'(\varepsilon_0)+\frac{1}{2}(\varepsilon-\varepsilon_0)^2 f''(\varepsilon_0)+ \ldots \tag{2.20}$$

then yields with Eqs. $(2.17)_2$, (2.18) and (2.19) denoting $f'(\varepsilon_0)$ by f' and $f''(\varepsilon_0)$ by f''

$$\begin{aligned} \sigma_1 &= P/3A+\delta f'+\frac{1}{2}\delta^2 f''+\ldots \\ \sigma_2 &= P/3A+(\delta-2\theta^2\varepsilon_0-2\theta^2\delta)f'+\frac{1}{2}(\delta-2\theta^2\varepsilon_0-2\theta^2\delta)^2 f''+\ldots . \end{aligned} \tag{2.21}$$

If, finally, these stresses are required to fulfill the equilibrium Eq. $(2.15)_1$, it follows that

$$\delta = 2\theta^2 P/9Af'+4\theta^2\varepsilon_0/3 \tag{2.22}$$

and hence

$$\begin{aligned} \sigma_1 &= (P/3A)(1+2\theta^2/3)+4\theta^2\varepsilon_0 f'/3+O(\theta^4 f''), \\ \sigma_2 &= (P/3A)(1+2\theta^2/3)-2\theta^2\varepsilon_0 f'/3+O(\theta^4 f''). \end{aligned} \tag{2.23}$$

The form of the creep rate function F in Eq. (2.5) has been studied for a large number of structural materials. It is found that, invariably $F'(\sigma) > 0$, $F''(\sigma) > 0$ and hence, cf. Fig. 4,

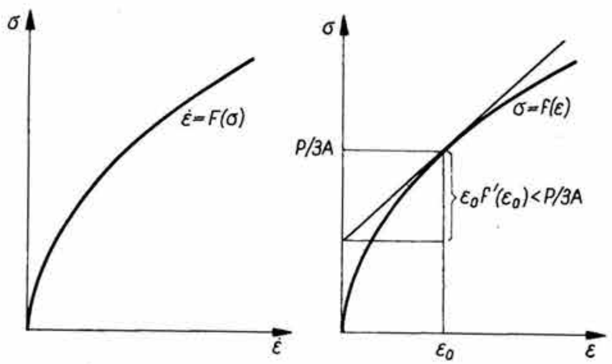

FIG. 4.

$$f'(\varepsilon) > 0, \quad f''(\varepsilon) < 0, \tag{2.24}$$

i.e., $f(\sigma)$ is convex upwards.

From Eqs. $(2.21)_1$ and $(2.23)_1$ and Fig. 4 then follows for the maximum stress

$$P/3A < \sigma_1 < (P/3A)(1+2\theta^2) \tag{2.25}$$

or, considering Eq. (2.14)

$$P/3A < \sigma_1 < (P/3A)(1+B^2/L^2). \tag{2.26}$$

Hence, independently of the creep law, the maximum stress deviates from the limiting value $P/3A$ by a factor less than $1+B^2/L^2$. This implies that detailed creep stress calculations are unnecessary when $B/L < 1/5$, say.

Our result illustrates the "guardsmen effect", so named by LECKIE (1971). Forcing the bars in a truss to fulfil both equilibrium and compatibility requirements implies that the relative influence of their material properties is suppressed, just as the individual personalities among guardsmen are suppressed, when they are forced to march in line and keep in step.

3. I-section beam

The maximum stress during creep in an I-section beam according to Fig. 5 will now be determined. The loading consists of a constant bending moment M, and the constitu-

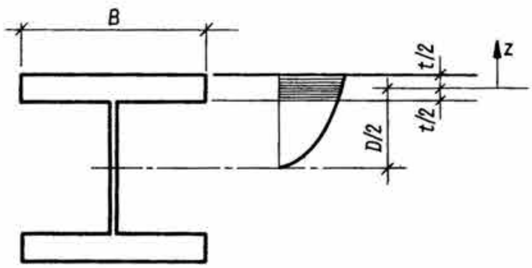

FIG. 5.

tive equation is given by Eqs. (2.3), (2.4) and (2.5). A limiting, constant state of stress will then arise, corresponding to the constitutive equation (2.7) with the inverse

$$\sigma = f(\varepsilon). \tag{3.1}$$

With standard Euler-Bernoulli assumptions and the notation of Fig. 5 follows the longitudinal strain in an arbitrary fibre

$$\varepsilon = \varkappa(D/2+z) = \varepsilon_0(1+\theta\zeta), \tag{3.2}$$

where

$$\begin{aligned} \varkappa &= \text{curvature of neutral plane,} \\ \varepsilon_0 &= \varkappa D/2 = \text{strain in flange midplane,} \\ \theta &= t/D = \text{flange thickness ratio} \ll 1, \\ \zeta &= 2z/t = \text{dimensionless coordinate.} \end{aligned} \tag{3.3}$$

The Taylor series expansion of Eq. (3.1), given by Eq. (2.20), then yields the stress field

$$\sigma = f(\varepsilon_0)+\varepsilon_0\theta\zeta f'(\varepsilon_0)+\frac{1}{2}\varepsilon_0^2\theta^2\zeta^2 f''(\varepsilon_0)+ \ldots. \tag{3.4}$$

Neglecting the web area we obtain the equilibrium equation

$$M = 2\int \sigma(z)(D/2+z)B\,dz, \tag{3.5}$$

where the integral extends from $-t/2$ to $t/2$. Change of variables and insertion of the stress field (3.4) then yields (with the same abbreviations as before)

$$M = \theta BD^2[f+\varepsilon_0\theta^2 f'/3+\varepsilon_0^2\theta^2 f''/6]+O(\theta^4). \tag{3.6}$$

It follows that, to order θ^2

$$f = M/\theta BD^2 \tag{3.7}$$

and hence the maximum stress, according to Eq. (3.4) for $\zeta = 1$, is found to be

$$\sigma_{\max} = M/\theta BD^2+\varepsilon_0\theta f'+\ldots = (M/\theta BD^2)\,(1+\theta^2 BD^2\varepsilon_0 f'/M)+\ \ldots. \tag{3.8}$$

Again, the deviation from the maximum stress in the limiting idealized case is of order θ^2, where θ is the relative flange thickness. It also follows, since $f'' < 0$, that Eq. (3.8) represents an upper boundary for $\sigma_{\max}$.

4. Spherical pressure vessel

Pressure vessels under creep conditions have been extensively studied in recent literature. The spherical and cylindrical vessels are particularly accessible to analysis, permitting in some cases even closed-form solutions, cf. ODQVIST & HULT (1962). We shall here determine the stress field in a moderately thinwalled spherical vessel loaded by a constant internal pressure p. The constitutive equations of the shell material are the multiaxial counterparts of Eqs. (2.3), (2.4) and (2.5), assuming isotropy, incompressibility and second invariant (Huber-Mises) theory. A limiting, constant state of stress will then arise, and may be found by using the corresponding relation between effective stress σ_e and effective strain ε_e

$$\sigma_e = f(\varepsilon_e). \tag{4.1}$$

Here

$$\sigma_e^2 = (3/2)s_{ij}s_{ij} \tag{4.2}$$

and

$$\varepsilon_e^2 = (2/3)e_{ij}e_{ij}, \tag{4.3}$$

s_{ij} and e_{ij} being the stress and strain deviation tensors, respectively. Because of the spherical symmetry the scalar relation (4.1) is the only constitutive equation needed.

From the assumption of incompressibility follows, using the notation of Fig. 6,

$$\varepsilon_e = Cr^{-3} = C(R+x)^{-3} = \varepsilon_e^0(1+\theta\xi)^{-3} = \varepsilon_e^0(1-3\theta\xi+6\theta^2\xi^2-\ \ldots), \tag{4.4}$$

where

$$\begin{aligned} \varepsilon_e^0 &= \text{strain in midsurface},\\ \theta &= h/2R = \text{wall thickness ratio} \ll 1,\\ \xi &= 2x/h = \text{dimensionless coordinate}. \end{aligned} \tag{4.5}$$

The Taylor series expansion of Eq. (4.1)

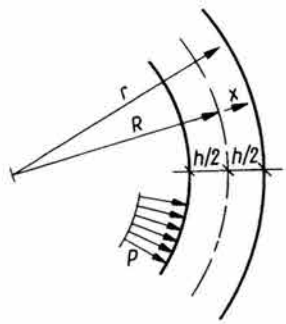

FIG. 6.

$$\sigma_e = f(\varepsilon_e^0)+(\varepsilon_e-\varepsilon_e^0)f'(\varepsilon_e^0)+\frac{1}{2}(\varepsilon_e-\varepsilon_e^0)^2 f''(\varepsilon_e^0)+\ldots \tag{4.6}$$

then gives the effective stress field

$$\sigma_e = f+(-3\theta\varepsilon_e^0\xi+6\theta^2\varepsilon_e^0\xi^2)f'+\frac{1}{2}(-3\theta\varepsilon_e^0\xi+6\theta^2\varepsilon_e^0\xi^2)f''+\ldots \tag{4.7}$$

where f, f', f'' are short notations for $f(\varepsilon_e^0), f'(\varepsilon_e^0), f''(\varepsilon_e^0)$.

Next, the circumferential stress $\sigma_\phi(x)$ is expanded in a power series

$$\sigma_\phi = C_0+C_1x+C_2x^2+C_3x^3+\ldots . \tag{4.8}$$

The overall equilibrium relation for the half sphere

$$\int \sigma_\phi\cdot 2\pi(R+x)dx = p\pi(R-h/2)^2, \tag{4.9}$$

where the integral extends from $-h/2$ to $h/2$, requires

$$C_0+(1/3)(C_1R+C_2R^2)\theta^2+(1/5)C_3R^3\theta^4+\ldots-(p/4\theta)(1-\theta)^2 = 0. \tag{4.10}$$

The differential equilibrium relation

$$(R+x)d\sigma_r/dx+2\sigma_r = 2\sigma_\phi \tag{4.11}$$

yields, with σ_ϕ given by Eq. (4.8),

$$\begin{aligned}\sigma_r = {} & C_0-C_1R/3+C_2R^2/6-3C_3R^3/20\\ & +(2C_1/3-C_2R/3+3C_3R^2/10)x+(C_2/2-9C_3R/20)x^2+\ldots .\end{aligned} \tag{4.12}$$

The effective stress $\sigma_e = \sigma_\phi-\sigma_r$ then follows from Eqs. (4.8) and (4.12), and after replacing x by $h\xi/2$,

$$\begin{aligned}\sigma_e = {} & C_1R/3-C_2R^2/6+3C_3R^3/20+\\ & +(C_1/3+C_2R/3-3C_3R^2/10)h\xi/2+(C_2/2+9C_3R/20)h^2\xi^2/4+\ldots .\end{aligned} \tag{4.13}$$

The two expressions for σ_e, Eqs. (4.7) and (4.13), are finally identified term by term for equal powers of ξ to yield

$$\begin{aligned} C_1R &= 2f-3\varepsilon_e^0 f',\\ C_2R^2 &= -f+3\varepsilon_e^0 f'+(9/2)\varepsilon_e^{02} f'',\\ C_3R^3 &= (10/9)f+10\varepsilon_e^0 f'+5\varepsilon_e^{02} f''.\end{aligned} \tag{4.14}$$

The stresses may now be calculated at any chosen point. The circumferential stress will be a maximum at the inner or outer surface. From Eqs. (4.8), (4.10) and (4.14) follows

$$\sigma_\phi\binom{\text{outer}}{\text{inner}} = (p/4\theta)(1-\theta)^2 \pm \theta(2f - 3\varepsilon_e^0 f') + O(\theta^2) \tag{4.15}$$

and hence σ_ϕ will be a maximum at the outer surface, if

$$\varepsilon_e^0 f'(\varepsilon_e^0) < (2/3) f(\varepsilon_e^0) \tag{4.16}$$

and at the inner surface otherwise.

For a Norton material this corresponds to $n > 3/2$, which is a well-known result.

As expected, the expressions for maximum stress in the shell, Eq. (4.15), and in the I-beam, Eq. (3.8), show marked similarities. The deviation from the maximum stress in the limiting idealized case is again of order θ^2. Hence even for moderately thinwalled shells there is no urgent need to do very precise stress field analyses. Irrespective of the complexity of the creep law the maximum stress will deviate very little from that in the corresponding elastic shell provided the wall thickness ratio is small.

5. Conclusions

The maximum stress in three slightly redundant structures subject to stationary creep has been calculated by a standard perturbation technique. This stress value has been compared to its counterpart in a corresponding fully isostatic structure. In all cases these stresses differ by an amount which is of the second order in a characteristic geometrically linear measure of redundancy. This result holds for all materials subject to stationary creep, irrespective of the form of the creep law.

References

1. N. J. HOFF, Quart. App. Math., **12**, 49–55, 1954.
2. J. HULT, IUTAM Symposium Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics (Proc. Ed. M. REINER and D. ABIR), Pergamon, p. 352, Oxford 1964.
3. F. LECKIE, *Thermal stresses and thermal fatigue* (Proc. Ed. D. J. LITTLE), Butterworths, p. 420, London 1971.
4. F. H. NORTON, *Creep of steel at high temperatures*, McGraw-Hill, New York 1929.
5. F. K. G. ODQVIST and J. HULT, *Kriechfestigkeit metallischer Werkstoffe*, Springer, Berlin 1962.
6. R. K. PENNY and D. L. MARRIOTT, *Design for creep*, McGraw-Hill, London 1971.
7. L. PRANDTL, Zeitschr. angew. Math. u. Mech., **8**, 85–106, 1928.

CHALMERS UNIVERSITY OF TECHNOLOGY, GÖTEBORG

Received May 31, 1972