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An approximate linear theory of thin viscoplastic shells

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AN APPROXIMATE technique is presented which may be used to estimate the dynamic viscoplastic deflections of arbitrarily shaped shells. Emphasis is laid on the proper formulation of the constitutive equations. Explicite formulas are obtained relating the middle surface strain rate tensor and the tensor of rate of change of curvature respectively with stress resultants and stress couples. Unloading criteria are discussed in details and illustrated by way of the example of impulsively loaded cylindrical shell.

Przedstawiona została przybliżona metoda wyznaczania ugięć dynamicznie obciążonych powłok lepkoplastycznych o dowolnym kształcie. Położono nacisk na właściwe sformułowanie równań konstytutywnych. Otrzymano jawne związki łączące składowe uogólnionych tensorów prędkości odkształceń i naprężeń. Przedyskutowano dokładnie warunki odciążania, które zilustrowano na przykładzie cylindrycznej powłoki obciążonej impulsem ciśnienia.

Дан приближённый метод определения прогибов динамически нагруженных вязкопластических оболочек произвольной формы. Основной упор положен на правильную формулировку определяющих уравнений. Выведены явные зависимости между составляющими обобщённых тензоров скоростей деформаций и напряжений. Обсуждены условия разргузки, которые иллюстрируются на примере цилиндрической оболочки, нагруженной импульсом давления.

1. Introduction

A CENTRAL point in the derivation of basic relationship of the theory of thin viscoplastic shells is a transformation of constitutive equations of the material to the space of generalized strain rates and stress resultants and stress couples. In the theory of perfectly plastic bodies the constraint imposed on the stress state in the form of the yield condition renders this transformation difficult. General results are of rather formal character [1, 2] and the existing explicite formulas are restricted to axi-symmetric problems [3].

In the theory of rigid-viscoplastic bodies the constraint on the stress state is relaxed and any stress in excess of the static yield surface is admissible [11]. Taking advantage of this fact, an approximation procedure was suggested in [13] to solve analytically the problem of impulsively loaded circular plate. This method of linearization was subsequently generalized to arbitrary plates [17] and rotationally symmetric shells [15].

It appears that no essential difficulties are encountered in extending further the same method to arbitrarily shaped thin shells. The purpose of the present paper is to outline the approximate theory valid for general thin viscoplastic shells, to show its applications to dynamic problem and to discuss the validity of certain simplifying assumptions. The paper does not make any systematic study of two-dimensional field equations. The foundations of the theory of thin elastic and rigid plastic shells can be found respectively in [6, 7 and 8]. Here, attention is primarily focused on the proper linearization of two sets of equations which involve non-linear terms: the constitutive equations and the unloading condition. The discussion of the unloading criteria not only leads to interesting consequences in the analysis of transient response of shells but may also be of some value to any dissipative media described by a linear flow rule. Starting from the assumption of an approximately plane state of stress parallel to the shell middle surface, the constitutive equations for a rate sensitive plastic material are expressed in Sec. 2 in terms of the generalized stress and strain rates. In the following Section a suitable hypothesis is introduced in order to linearize the problem. An analogy with the appropriate shell equations for an elastic incompressible material is indicated. In Sec. 4, three alternative forms of unloading criteria are formulated and applications of these criteria to the determination of permanent deflections of impulsively loaded cylindrical shell is present in Sec. 5. In the closing Section possible generalization of the present method to the geometrically nonlinear theory of shells is indicated.

2. Constitutive equations

In the derivation of the constitutive equations for rigid-viscoplastic shells we shall follow closely the method developed for rigid-plastic incompressible material [2]. A tensor notation will be used in this Section.

Let x^α $(\alpha = 1, 2)$ denote a system of curvilinear coordinates on the middle surface of the shell and x^3 a coordinate normal to that surface. The metric tensor is denoted by $g_{\alpha\beta}$ the first and second fundamental forms of the surface being respectively $a_{\alpha\beta}$ and $b_{\alpha\beta}$. The common assumption of the thin shell theory the neglect of the normal stresses $\sigma^{33} = 0$ and shear deformations $\dot{\varepsilon}_{\alpha 3} = 0$. As regards the constitutive equations, the stress state in each layer of the shell is described by the tensor $\sigma^{\alpha\beta}$. In curvilinear coordinate system x^α, the second invariant of the stress deviation J_2 is

$$J_2 = \frac{1}{6}[3\sigma^{\alpha\beta}\sigma^{\gamma\delta}g^{\alpha\delta}g^{\gamma\beta} - (\sigma^{\alpha\beta}g_{\alpha\beta})^2]. \tag{2.1}$$

Consider a special case of constitutive equations for rate sensitive plastic material [11]

$$\dot{\varepsilon}_{\alpha\beta} = \gamma\langle F\rangle \frac{\partial F}{\partial\sigma^{\alpha\beta}}, \qquad \langle F\rangle = \begin{cases} F & \text{if} \quad F > 0, \\ 0 & \text{if} \quad F \leqslant 0, \end{cases}$$

where the potential function is defined by

$$F = \frac{\sqrt{J_2}}{k} - 1. \tag{2.3}$$

The incompressibility condition requires that

$$\dot{\varepsilon}_{\gamma\gamma} + \dot{\varepsilon}_{33} = 0. \tag{2.4}$$

In Eqs. (2.2), the elastic properties and strain-hardening of the materials have been disregarded. Substituting (2.3) and (2.1) into (2.2), the constitutive equation in the loading region can be written as

$$\dot{\varepsilon}_{\alpha\beta} = \frac{\gamma}{3k}\left[(3\sigma_{\alpha\beta} - \sigma^\gamma_\gamma g_{\alpha\beta}) - \frac{k}{\sqrt{J_2}}(3\sigma_{\alpha\beta} - \sigma^\gamma_\gamma g_{\alpha\beta})\right]. \tag{2.5}$$

By means of the incompressibility condition (2.4), Eq. (2.5) can be inverted to give

$$\sigma^{\alpha\beta}-\frac{k}{\sqrt{J_2}}\sigma^{\alpha\beta}=\frac{k}{\gamma}[\dot{\varepsilon}_{\varrho\gamma}g^{\varrho\alpha}g^{\gamma\beta}+\dot{\varepsilon}^{\gamma}_{\gamma}g^{\alpha\beta}]. \tag{2.6}$$

The stress resultants $N^{\alpha\beta}$ and stress couples $M^{\alpha\beta}$ are defined as

$$N^{\alpha\beta}=\int_{-h}^{h}\sigma^{\alpha\beta}dx^3, \quad M^{\alpha\beta}=\int_{-h}^{h}\sigma^{\alpha\beta}x^3dx^3, \tag{2.7}$$

while the components of the strain rate tensor $\dot{\varepsilon}_{\alpha\beta}$ are related to the curvature rates $\dot{\varkappa}_{\alpha\beta}$ and extension rates $\dot{\lambda}_{\alpha\beta}$ by

$$\dot{\varepsilon}_{\alpha\beta}=\dot{\lambda}_{\alpha\beta}+x^3\dot{\varkappa}_{\alpha\beta}. \tag{2.8}$$

It is possible to express J_2 entirely in terms of the components of the strain rate tensor, compute then from (2.6) $\sigma^{\alpha\beta}$ and substitute it into (2.7). However the resulting integrals cannot be evaluated by means of elementary functions and in general resulting theory becomes mathematically intractable [9]. It is advantageous to leave (2.6) as it is and integrate it over the shell section, according to (2.7). Using (2.8), we obtain:

$$\begin{aligned} M^{\alpha\beta}-k\int_{-h}^{h}\frac{\sigma^{\alpha\beta}}{\sqrt{J_2}}x^3dx^3 &= \frac{k}{\gamma}\frac{4h^3}{3}[\dot{\varkappa}^{\alpha\beta}+\dot{\varkappa}^{\gamma}_{\gamma}a^{\alpha\beta}],\\ N^{\alpha\beta}-k\int_{-h}^{h}\frac{\sigma^{\alpha\beta}}{\sqrt{J_2}}dx^3 &= \frac{k}{\gamma}h[\dot{\lambda}^{\alpha\beta}+\dot{\lambda}^{\gamma}_{\gamma}a^{\alpha\beta}]. \end{aligned} \tag{2.9}$$

In the loading process, the stress $\sigma^{\alpha\beta}$ follows at a given point (x^{α}, x^3) a certain trajectory which lies entirely outside the static yield surface. At the same time, the point represented by $\sigma^{\alpha\beta}/\sqrt{J_2}$ remains always on the static yield surface. The components $\sigma^{\alpha\beta}/\sqrt{J_2}$, integrated over the thickness of the shell, are denoted by

$$^{*}N^{\alpha\beta}=k\int_{-h}^{h}\frac{\sigma^{\alpha\beta}}{\sqrt{J_2}}dx^3, \quad {}^{*}M^{\alpha\beta}=k\int_{-6}^{h}\frac{\sigma^{\alpha\beta}}{\sqrt{J_2}}x^3dx^3. \tag{2.10}$$

The generalized stresses $^{*}N^{\alpha\beta}$ and $^{*}M^{\alpha\beta}$ are uniquely related to either $(M^{\alpha\beta}, N^{\alpha\beta})$ or $(\dot{\varkappa}^{\alpha\beta}, \dot{\lambda}^{\alpha\beta})$. We shall not derive the corresponding relationship since this would be equivalent to the complicated procedure already discussed. Instead, we note that $^{*}N^{\alpha\beta}$ and $^{*}M^{\alpha\beta}$ defined by (2.10) satisfy the static yield condition

$$F(^{*}N^{\alpha\beta}, {}^{*}M^{\alpha\beta})=0. \tag{2.11}$$

Substituting (2.10), the constitutive equations (2.9) can be written as

$$\begin{aligned} M^{\alpha\beta}-{}^{*}M^{\alpha\beta} &= \frac{k}{\gamma}\frac{4h^3}{3}[\dot{\varkappa}^{\alpha\beta}+\dot{\varkappa}^{\gamma}_{\gamma}a^{\alpha\beta}],\\ N^{\alpha\beta}-{}^{*}N^{\alpha\beta} &= \frac{k}{\gamma}h[\dot{\lambda}^{\alpha\beta}+\dot{\lambda}^{\gamma}_{\gamma}a^{\alpha\beta}]. \end{aligned} \tag{2.12}$$

The above equations are not directly usable because they involve yet unknown functions $^{*}M^{\alpha\beta}$ and $^{*}N^{\alpha\beta}$.

3. Approximation

In a given dynamic boundary value problem, both $(M^{\alpha\beta}, N^{\alpha\beta})$ and $(*M^{\alpha\beta}, *N^{\alpha\beta})$ are variable quantities and are related to each other. However, the variation of $(*M^{\alpha\beta}, *N^{\alpha\beta})$ is limited because of the restriction imposed by the yield condition (2.11) and stress boundary conditions. It is therefore reasonable to assume that $(*M^{\alpha\beta}, *N^{\alpha\beta})$ are fixed in a given boundary value problem. In order to determine the distribution of $*M^{\alpha\beta}$ and $*N^{\alpha\beta}$ in function of x^{α} a hypothesis is introduced as to these unknown quantities being the solution of a static problem for a perfectly plastic shell under the same boundary conditions. This hypothesis applied to circular plates [13] and rotationally symmetric shells [16] was shown to yield good results as compared with the exact solution and experimental data. It should be noted that the Eqs. (2.12), with terms $*M^{\alpha\beta}(x^{\alpha})$, and $*N^{\alpha\beta}(x^{\alpha})$ depending on the shell coordinates x^{α}, were interpreted in [15] as non-associated flow rule. While such a point of view was helpful in discussing the accuracy of the hypothesis introduced, in fact the approximation refers to the whole boundary value problem rather than to the constitutive equations alone.

It will be shown now that instead of the tensor quantities $*M^{\alpha\beta}$, $*N^{\alpha\beta}$, appearing in (2.12), it suffices to know a vector quantity $(*p, *p^{\alpha})$ denoting the components of the static pressure acting on the shell. According to the hypothesis, the generalized stresses $(*M^{\alpha\beta}, *N^{\alpha\beta})$ are in static equilibrium(¹) with $*p$, $*p^{\alpha}$,

$$(3.1)\qquad \left[*n^{\beta\alpha}+\frac{1}{2}b^{\alpha}_{\gamma}{*m^{\beta\gamma}}-\frac{1}{2}b^{\beta}_{\gamma}{*m^{\beta\gamma}}\right]\bigg|_{\beta}+b^{\alpha}_{\gamma}{*m^{\beta\gamma}}\bigg|_{\beta}+{*p^{\alpha}}=0,$$
$$*m^{\alpha\beta}|_{\alpha\beta}-b_{\alpha\beta}{*n^{\alpha\beta}}-{*p}=0.$$

Subtracting (3.1) from the actual equations of motion involving inertia terms, we arrive at:

(3.2)
$$\left[(n^{\beta\alpha}-{*n^{\beta\alpha}})+\frac{1}{2}b^{\gamma}_{\gamma}(m^{\beta\gamma}-{*m^{\beta\gamma}})-\frac{1}{2}b^{\beta}_{\gamma}(m^{\beta\gamma}-{*m^{\beta\gamma}})\right]\bigg|_{\beta}+b^{\alpha}_{\gamma}(m^{\beta\gamma}-{*m^{\beta}_{\gamma}})|_{\beta}+(p^{\alpha}-{*p^{\alpha}})=\mu\ddot{u}^{\alpha},$$
$$(m^{\alpha\beta}-{*m^{\alpha\beta}})|_{\alpha\beta}-b_{\alpha\beta}(n^{\alpha\beta}-{*n_{\alpha\beta}})-(p-{*p})=0.$$

Combining (3.2) with (2.12), the generalized stresses can be eliminated from (3.2) and Eqs. (3.2) can be expressed directly in terms of $(\dot{\varkappa}^{\alpha\beta}, \dot{\lambda}^{\alpha\beta})$ or appropriate gradients of the velocities $(\dot{u}^{\beta}, \dot{u})$. There is still a certain degree of non-uniqueness in the determination of $(*p, *p^{\alpha})$, since the structure can reach the limit state under various distribution of external loads and combination of components p and p^{α}. However, this freedom in choice of the static collapse load turns out to be essential in the formulation of the unloading criteria.

The fact that in the present method of linearization the generalized stresses have been eliminated is of great practical significance. In many particular boundary value problems it is not possible to find the exact solution using the Huber-Mises yields condition. It is,

(¹) In the equations of equilibrium dimensionless stress resultants $n^{\alpha\beta} = N^{\alpha\beta}/2\sigma_0 h$ and stress couples $m^{\alpha\beta} = M^{\alpha\beta}/\sigma_0 h^2$ are used.

however, relatively easy to compute an approximate limit load based on Tresca or some other idealized yield condition.

By letting the yield stress tend to zero, $k \to 0$, the static yield surface shrinks and consequently $^*M^{\alpha\beta} \to 0$, $^*N^{\alpha\beta} \to 0$, $^*p = {}^*p^{\alpha} \to 0$. At that instant the constitutive equations (2.12) reduce to the flow rule for a viscous Newtonian fluid. These equations are formally analogous to the corresponding relations for a shell with an elastic incompressible material:

$$M^{\alpha\beta} = \frac{4Eh^3}{3(1-\nu^2)}[\varkappa^{\alpha\beta}+\varkappa^{\gamma}_{\gamma}a^{\alpha\beta}], \tag{3.3}$$

$$N^{\alpha\beta} = \frac{Eh}{1-\nu^2}[\lambda^{\alpha\beta}+\lambda^{\gamma}_{\gamma}a^{\alpha\beta}],$$

where strain rates are replaced by strains. The elastic analogy indicated above often proves useful in the solutions of particular boundary value problems. A detailed discussion of this problem in the context of the bending theory of thin plates was given in [17]. We shall return to the elastic-viscoplastic analogy in Sec. 5.

4. Unloading criteria

The constitutive equations for the rate sensitive plastic material (2.2) admit the occurrence of rigid regions where $\dot{\varepsilon}_{\alpha\beta} \equiv 0$. Such regions can appear at certain stages of the motion of a dynamically loaded shell and the boundary between rigid and viscoplastic behaviour $\xi(x^{\alpha}, t) = 0$ is in general time variable and should be sought as a part of the solution. This boundary is uniquely determined by a single scalar equation $F = 0$. In the case of the original, nonlinear constitutive equations (2.2), the condition $F = 0$ implies $\dot{\varepsilon}_{\alpha\beta} = 0$ and vice versa. Difficulties arise however in formulating an adequate unloading criterion for shells described by the linearized constitutive equations (2.12). As already pointed out, these equations are characteristic of the given boundary value problem rather than to the material itself. It is therefore reasonable to develop, parallel to the classical approach, some new unloading criteria. *Assume first* that the static yield surface for a shell section is given explicitly by the equation

$$F(M^{\alpha\beta}, N^{\alpha\beta}) = 0 \tag{4.1}$$

and the generalized stresses $^*M^{\alpha\beta}(x^{\alpha})$, $^*N^{\alpha\beta}(x^{\alpha})$ are known and fixed at each point of the shell. Using the constitutive equations (2.12) the unloading condition (4.1), can be expressed entirely in terms of $\dot{\varkappa}^{\alpha\beta}$ and $\dot{\lambda}^{\alpha\beta}$ but now $F = 0$ is no longer equivalent to $\dot{\varkappa}^{\alpha\beta} = \dot{\lambda}^{\alpha\beta} = 0$. This situation is explained on a simple two-dimensional example, Fig. 1. The full line represents the stress trajectory at the prescribed point x^{α} of the cylindrical shell. At the intersection point with the static yield surface $F = 0$, the strain rate vector $(\dot{\varkappa}, \dot{\lambda})$ is finite. Thus, the classical unloading condition applied to the linearized flow rule (2.12) does not lead to continuity of the strain rate vector between loading and unloading regions. The original Eqs. (22) do possess this property.

The continuity can, however, be improved by considering the quantities $^*M^{\alpha\beta}$ and $^*N^{\alpha\beta}$ to be variable in the course of the loading process. If the variation of the external loadings $^*p(x^\alpha, t)$, $^*p^\beta(x^\alpha, t)$ is such that the resulting vectors $(^*M^{\alpha\beta}, {}^*N^{\alpha\beta})$ approach the terminal point of the actual stress trajectory, then the absolute value of the vectors $(\dot{\varkappa}^{\alpha\beta}, \dot{\lambda}^{\alpha\beta})$ would diminish. This situation would correspond to the rotation of the vector (M, N) in Fig. 1 anti-clock-wise. Of course, the increased accuracy of the unloading criterion is achieved at the expense of more complex constitutive equations. Since the

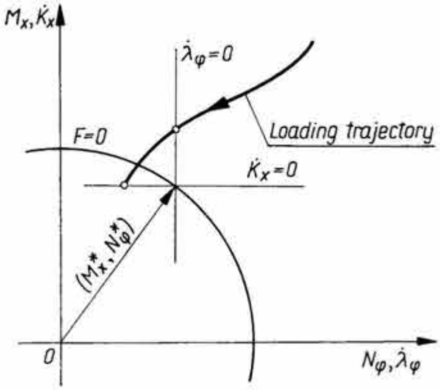

FIG. 1. Illustration of the transition from viscoplastic to rigid behaviour in the case of the linearized constitutive equations.

intersection point of the stress trajectory with the static yield surface is not known a priori, it is not possible, without some additional assumption, that the linearized flow rule (2.12) satisfy strictly the unloading condition (4.1). Such assumptions will be discussed later in this Section.

It should be borne in mind that the explicite forms of the Huber-Mises yield condition are available for very specialized structures and only a few analytic expresions exist for a stress distribution in a statically loaded shells. Thus, the approach discussed above is of a limited applicability and has never been used for the determination of permanent deflections of shells.

The *second possibility* in the formulation of the unloading condition seems to be more natural for problems solved entirely in velocities. It can be assumed that the unloading takes place if the natural norm of the velocity vector with components $(\dot{u}, \dot{u}^\beta)$ vanishes:

$$||\dot{\mathbf{u}}|| = \sqrt{u^2+u^{1^2}+u^{2^2}_{\mathbf{a}}} = 0. \tag{4.2}$$

Again, discontinuities in the generalized strain rates may occur since the equality (4.2) is not equivalent to $\dot{u} = \dot{u}^\beta = 0$, and hence some components of $\dot{\varkappa}^{\alpha\beta}, \dot{\lambda}^{\alpha\beta}$ may be different from zero when the shell section becomes rigid. The advantage of the unloading criterion (4.2) is that in problems with one non-zero component of the velocity vector, Eq. (4.2) does imply that $\dot{\varkappa}^{\alpha\beta} = \dot{\lambda}^{\alpha\beta} = 0$. The converse is not true, since curvature and extension rates are expressed in terms of velocity gradients and hence the condition $\dot{\varkappa}^{\alpha\beta} = \dot{\lambda}^{\beta\alpha} = 0$ admits in general rigid body motion of some parts of the shell. The condition (4.2) was taken as a basic unloading criterion in all existing applications of the linearized flow rule for viscoplastic shells (cf. for example [5, 13, 14]).

The above discussion refers to the unloading criteria at a prescribed point x^α of the shell. In considering dynamic problems, an important question arises as to whether or not all points of the shell stop simultaneously. Such a possibility cannot be eliminated a priori from the analysis of the field equations alone. While in the theory of perfectly plastic shells this problem has never been studied from the general point of view, all experimental results and example solutions for structures deforming in the biaxial state of stress indicate that indeed all points are brought to rest at the some time — called "time to rest" t_f. The propagation of rigid zones are characteristic only for structures which are deformed in the uni-axial state of stress — for example, beams and rods [12], but these are excluded from the present considerations.

Introducing an additional assumption that no rigid zones are formed in the course of the deformation process, great simplifications are obtained because the problem is reduced to the solution of a system of linear partial differential equations in regions with fixed boundaries. One can then take full advantage of the elastic analogy and apply the method of eigenvalue expansion. Having found the solution, it is possible to verify a posteriori the behaviour of the boundary $\xi(x^\alpha, t)$ between rigid and viscoplastic region. Such a simplistic point of view was taken in all existing applications of the present theory to the dynamics of plates and rotationally symmetric shells, [14, 16].

Before showing by an example certain interesting features of the behaviour of viscoplastic shells in the terminal phase of motion, some general properties of the linear field equation describing the motion of a dissipative medium will be discussed. It is known that in linear elastic systems, the principle of superposition holds, while for structures described by the nonlinear constitutive equation (2.2) this principle is no longer valid. It is thus not quite clear whether or not the solution based on the linearized flow rule should satisfy the principle of superposition. The latter principle as applied to dynamics of elastic shells is equivalent to the statement that there is no transmission of energies between subsequent modes of vibrations. As was shown in [17], the eigenfunctions and eigenvalues in the solution of viscoplastic structures are identical with those in elastic problems. The difference is that the time variation of the amplitudes of subsequent modes $A_n(t)$ is of an aperiodic character. Since each amplitude decays at a different rate and reaches zero at various times, the application of the unloading criteria (4.2) implies that the energy is transmitted from lower to higher modes. For example, the velocity may be zero with the amplitude of the first mode positive, and negative amplitude of the remaining modes. There would be no flow of energy frome one mode to the other if negative values of subsequent amplitudes were considered as inadmissible.

Thus, the requirement that no energy is transmitted between modes can be taken as an *alternative definition of the unloading* in the given boundary value problem. According to this definition, the amplitudes of subsequent modes $A_n(t)$ are held at zero for $t > t_f$, where t_f is given by

$$A_n(t_f^n) = 0. \tag{4.3}$$

Since higher modes vanish at early stages of the motion, in the later stage only the first mode persists. The structure is moving then as a one-degree-of-freedom system and at time t^1 all points of the shell stop simultaneously. It should be stressed that the latter property is no longer an additional assumption but is a direct consequence of the assumed unloading criterion. On the other hand, the energy criterion imposes certain restrictions on the choice of the static collapse pressure $^*p$, $^*p^\beta$, a problem which will be discussed in the following Section.

5. Application

It is of interest to investigate to what extent the specific choice of the unloading criterion influences the deflections of a dynamically loaded shell. The comparison will be performed on the example of a simply supported cylindrical shell loaded by an impulse of internal pressure. The distribution of internal forces in the associated static problem is not available, hence the condition (4.1) must be excluded from the present analysis.

In the cylindrical coordinate system (x, r, ϕ) the non-vanishing components of the first and second fundamental forms are:

$$a_{xx} = 1, \quad a^{xx} = 1, \quad a_{\phi\phi} = R^2, \quad a^{\phi\phi} = \frac{1}{R^2}, \qquad (5.1)$$

$$b_{\phi\phi} = R, \quad b^{\phi}_{\phi} = \frac{1}{R}.$$

Since in this case the Christoffel symbols vanish, the covariant differentiation is indistinguishable from the partial differentiation — for example:

$$u_{\alpha}|_{\beta} = u_{\alpha\beta}. \qquad (5.2)$$

The flow rule (2.12) reduces now to:

$$\begin{aligned} M_{xx} - {}^*M_{xx} &= \frac{k}{\gamma}\frac{4h}{3}[2\dot{\varkappa}_{xx} + \dot{\varkappa}_{\phi\phi}], \\ M_{\phi\phi} - {}^*M_{\phi\phi} &= \frac{k}{\gamma}\frac{4h^3}{3}[2\dot{\varkappa}_{\phi\phi} + \dot{\varkappa}_{xx}], \\ N_{xx} - {}^*N_{xx} &= \frac{k}{\gamma}h[2\dot{\lambda}_{xx} + \dot{\lambda}_{\phi\phi}], \\ N_{\phi\phi} - {}^*N_{\phi\phi} &= \frac{k}{\gamma}h[2\dot{\lambda}_{\phi\phi} + \dot{\lambda}_{xx}]. \end{aligned} \qquad (5.3)$$

The above particular form of constitutive equations, valid for rotationally symmetric shells, was first derived in [15].

The physical components of the strain rate tensor are

$$\begin{aligned} &\dot{\lambda}_{xx} = \dot{u}_{x,x}, \quad \dot{\lambda}_{\phi\phi} = \frac{\dot{w}}{R}, \quad \dot{\lambda}_{\phi x} = \dot{\lambda}_{x\phi} = 0, \\ &\dot{\varkappa}_{xx} = -\dot{w}_{,xx}, \quad \dot{\varkappa}_{\phi\phi} = 0, \quad \dot{\varkappa}_{\phi x} = \varkappa_{\phi\varphi} = 0. \end{aligned} \qquad (5.4)$$

If the loading is axisymmetric and the shell can freely slide on the supports in the axial direction, $M_{\phi\phi} = 0$, $N_{xx} = 0$ and the only meaningful equation of motion is

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{N_{\phi\phi}}{R} - p(x, t) + \mu\ddot{w} = 0, \qquad (5.5)$$

where w denotes radial displacement of the shell middle-surface. The corresponding equations of static equilibrium reads

$$\frac{\partial^2 M^*_{xx}}{\partial x^2} + \frac{N^*_{\phi\phi}}{R} - {}^*p(x, t) = 0. \qquad (5.6)$$

Equations (5.3)–(5.6) furnish a complete set of relations describing the axi-symmetric dynamic deformation of cylindrical shells. In the case of a shell simply supported at both ends $(0, 2L)$ the boundary conditions are

$$(5.7)\qquad \begin{aligned} &M_{xx}(0, t) = w(0, t) = 0,\\ &Q_{xx}(L, t) = \frac{\partial \dot{w}(x, t)}{\partial x}\bigg|_{x=L} = 0. \end{aligned}$$

For a uniformly distributed impulse I, the initial conditions are

$$(5.8)\qquad \dot{w}(x, 0) = \frac{I}{\mu}; \quad w(x, 0) = 0, \quad p(x, t) = 0.$$

It is convenient to introduce the following dimensionless quantities

$$(5.9)\qquad \dot{w} = \frac{W}{R}, \quad x = \frac{x}{L}, \quad p' = \frac{{}^{*}pR}{N_0}, \quad N_0 = 2\sigma_0 h, \quad \alpha = \frac{\mu R^2}{N_0}.$$

Now, eliminating all unknowns between Eqs. (5.3)–(5.6) except velocity, the equation for w becomes [10]

$$(5.10)\qquad \frac{16}{9c^4}\frac{\partial^4 \dot{w}}{\partial x^4} + \dot{w} + \overline{\gamma}p(x, t) - \overline{\gamma}\alpha\ddot{w} = 0,$$

where $c^2 = 2L^2/Rh$ ($2L$ — length, R — radius of the shell) and $\overline{\gamma} = \dfrac{2}{\sqrt{3}}\gamma$.

First approach. It is assumed that rigid zones are not propagating and thus the solution is sought in the region with a fixed boundary. (4.2) is taken as the unloading criterion formula. Finally, it is assumed that the static distribution of moments ${}^{*}M_{xx}$ and axial forces N^{*}_{xx} is equilibrated by the uniformly distributed pressure ${}^{*}p(x)$. The approximate formula for the collapse pressure, determined by Hodge [4], is

$$(5.11)\qquad {}^{*}p = 1 + \frac{2}{c^2}.$$

The solution to the initial-boundary value problem (5.7)–(5.10), obtained in [10], can be put in the simple form

$$(5.12)\qquad \dot{w}(x, \tau) = \frac{2I'}{\pi\alpha p}\sum_{n=1,3}^{\infty}\frac{2}{n}[(1+\beta_n)e^{-\tau/\beta_n} - \beta_n]\sin\frac{n\pi}{2}x,$$

where

$$\beta_n = \frac{\alpha\overline{\gamma}p}{I'}\frac{a}{\left(\dfrac{n\pi}{2}\right)^4 + a} \qquad a = \frac{9c^4}{16}, \qquad \tau = \frac{tI}{p}.$$

The rate of convergence of the above series depends significantly on the parameter a and hence c^2. For small c (short shells), the series (5.12) is rapidly convergent while for large c (long shells), the convergence is much slower. The unloading condition (4.2) reduces now to:

$$(5.13)\qquad \dot{w}(x, \tau_f) = 0.$$

Equations (5.12), (5.13) have been solved numerically for various values of the parametre c and β. The solution represents on the plane (x, τ_f) a certain curve which is a boundary between zones of positive and negative velocities, Fig. 2. Thus, the assumption that boundaries of the deformation process are fixed leads to the occurrence of negative velocities

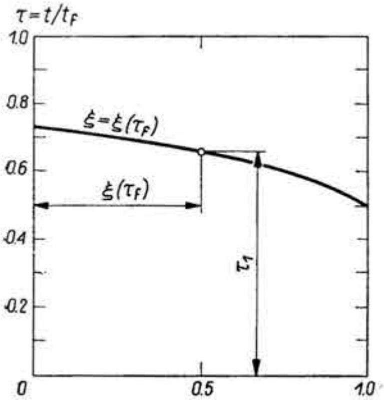

FIG. 2. Propagation of a boundary between regions of positive and negative velocities in the shell.

in the terminal phase of the shell motion. While such regions are not physically admissible, if was found that within a few percent all shell points do stop simultaneously so that the violation of the obvious physical fact is not great. A simple one-dimensional explanation of the unloading criterion considered, as applied to the linearized constitutive equation, is shown in Fig. 3.

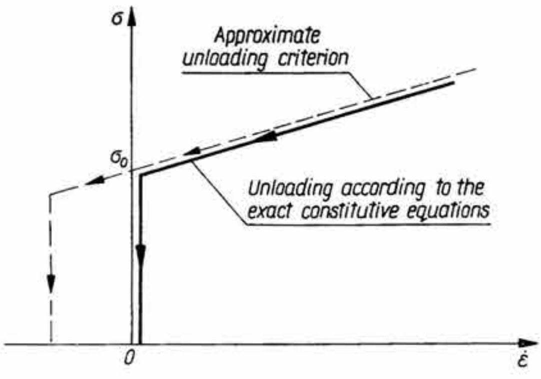

FIG. 3. Exact and approximate paths for rigid-viscoplastic material.

Integrating (5.12) in time, and using the second initial condition (5.8), an expression for the deflection $w(x, \tau)$ is obtained:

$$w(x, \tau) = \frac{2I'^2}{\pi p \alpha} \sum_{n=1,3}^{n} \frac{2}{n} \beta_n [(1+\beta_n) e^{-\tau/\beta_n} - \tau] \sin \frac{n\pi}{2} x. \tag{5.14}$$

The permanent deflection $w^f(x) = w(x, \tau_f)$ is computed from (5.2) after substituting $t = \tau(x)$, according to (5.13). The resulting curves of normalized deflection shapes $w^f(x)/w(0)$ are plotted in Fig. 4 (full line) for the following values of the shell geometry: $c = 1.5; 9; 250; 3000$ and the fixed parameter $\beta = 0.2$.

Second approach. The assumption that no energy is transmitted between subsequent modes automatically implies that all points of the shell are simultaneusly brought to rest.

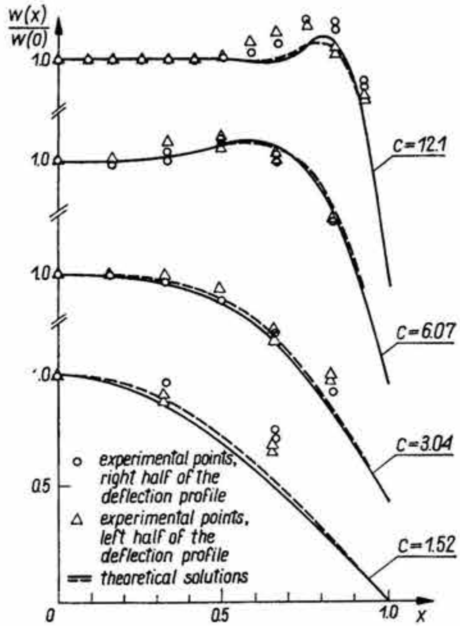

FIG. 4. Comparison of final deflection profiles in a cylindrical shell for two alternative forms of the unloading criteria.

A fixed boundary problem is again formulated for the Eq. (5.10) with the unloading criterion (4.3). The solution for velocities (5.12) still applies, with the exception that the n-the amplitude $A_n(t) = \dfrac{2}{n}[(1+\beta_n)e^{-t/\beta_n}-\beta_n]$ should be equated to zero for $t > t^f_n$, where according to (4.3) the time t^n_f is

$$t^n_f = \beta_n \ln\left(1+\frac{1}{\beta}\right). \tag{5.15}$$

Substituting (5.15) back into (5.14) a simple formula is obtained for the permanent deflection:

$$w^f(x) = \frac{2I'^2}{\pi p \alpha}\sum_{n+1,3}^{\infty}\frac{2}{n}\left[\beta_n-\beta_n^2\ln\left(1+\frac{1}{\beta_n}\right)\right]\sin\frac{n\pi}{2}x. \tag{5.16}$$

The above solution is valid if the static pressure distribution is considered as time variable. More specifically, in order to satisfy the equation of motion the pressure term should take the form:

$$p(x,\tau) = p^*\sum_{n=1,3}^{\infty}\frac{2}{n}[H(\tau-\tau^n_f)-H(\tau^n_f)]\sin\frac{n\pi}{2}x. \tag{5.17}$$

where $H(\tau)$ denotes the Heaviside function.

The above expression implies time discontinuous change in the inhomogeneous term entering Eq. (5.10). Consequently the acceleration field is discontinuous but with the unloading criterion (4.3) the resulting velocity field is continuous.

The series (5.16) has been summed up numerically for the same values of parameters c and β. Corresponding normalized deflection profiles are plotted in Fig. 4 as dashed lines. It is seen that both unloading criteria lead virtually to identical results. In addition, all curves follow with reasonable accuracy the trend of experimental points (circles and triangles) taken from the author's previous paper [16].

In conclusion, it can be stated that both approximate unloading conditions give satisfactory results at least in the case of the particular boundary value problem considered here. However, the energy criterion leads to a much simpler analytic answer and thus is recommended in further applications.

6. Generalization

The application of the linearized theory of viscoplastic shells is severely restricted by the assumption concerning linear geometrical relations valid for small displacements. In reality, maximum permanent deflections attained in the course of a dynamic process may reach several thicknesses of the structure.

The present method can be extended to shallow shells undergoing moderately large deflections (small finite deflections in the Koiter nomenclature [7]). The appropriate simplified strain measures are:

$$\lambda_{\alpha\beta} = \frac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha) - b_{\alpha\beta} w + \frac{1}{2} w_{,\alpha} w_{,\beta}, \tag{6.1}$$
$$\varkappa_{\alpha\beta} = w|_{\alpha\beta},$$

and corresponding equations of motion are given by

$$n^{\alpha\beta}|_\beta = 0, \tag{6.2}$$
$$m^{\alpha\beta}|_{\alpha\beta} - (b_{\alpha\beta} + w|_{\alpha\beta}) n^{\alpha\beta} - p = \mu \ddot{w},$$

which now replace Eq. (3.1). The appropriate equations for starred quantities have the form:

$$^*n^{\alpha\beta}|_\beta = 0, \tag{6.3}$$
$$^*m^{\alpha\beta}|_{\alpha\beta} - (b_{\alpha\beta} + w^*|_{\alpha\beta})^* n^{\alpha\beta} - {}^*p(w) = 0,$$

where the static load-carrying capacity is an increasing function of the central deflection W^*. It is seen by using the constitutive equations (2.12) that moments and forces between (6.2) and (6.3) can be eliminated only if derivatives of deflection profiles in the dynamic and corresponding static problem are assumed to be identical, $w|_{\alpha\beta} = w^*|_{\alpha\beta}$. The validity of this additional simplifying assumption was discussed in [14] in connection with the problem of impulsively loaded clamped circular plate.

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