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Remarks concerning the "plateau" in dynamic plasticity

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IN THE PAPER a discussion is given of the possible existence of a "plateau" in one-dimensional dynamic plasticity, using rate-type constitutive equations. A necessary condition to be satisfied by the coefficient functions entering rate-type constitutive equations is given. It is shown that with the usual forms of a semilinear model, only an "asymptotic plateau" can occur. This means that all three functions (stress, strain and particle velocity) tend asymptotically towards plateaus, i.e. it is not possible for even one of the functions to be exactly constant, while the others approach a plateau. With the quasi-linear model, in addition to the possibility that all three functions can approach plateaus asymptotically, it is possible that one, two or even all three functions can have absolute plateaus.

W pracy przedstawiono dyskusję problemu ewentualnego istnienia "plateau" w jednowymiarowej dynamicznej teorii plastyczności opierając się na równaniach konstytutywnych, uwzględniających wrażliwość na prędkość odkształceń. Podano warunek konieczny, jaki spełniać powinny współczynniki funkcyjne, pojawiające się w równaniach konstytutywnych tego rodzaju materiałów. Wykazano, że w przyjmowanych zazwyczaj prawie liniowych modelach ciała pojawiać się może jedynie "asymptotyczne plateau". Znaczy to, że wszystkie trzy funkcje (naprężenie, odkształcenie i prędkość cząstek ciała) zmierzają asymptotycznie do swych plateau, a więc żadna z tych funkcji nie może być dokładnie stała, podczas gdy pozostałe zmierzają do plateau. Natomiast w modelu quasi-liniowym, poza możliwością asymptotycznego zmierzania wszystkich trzech funkcji do plateau, istnieje także możliwość, że jedna, dwie lub nawet wszystkie trzy funkcje osiągają swe absolutne plateau.

В работе представлено обсуждение проблемы возможного существования "плато" в одномерной динамической теории пластичности опираясь на определяющие уравнения, учитывающие чувствительность на скорость деформаций. Дается необходимое условие какому должны удовлетворять функциональные коэффициенты, появляющиеся в определяющих уравнениях этого рода материалов. Доказано, что в принимаемых обыкновенно полулинейных моделях тела может появиться только "асимптотическое плато". Это означает, что все три функции (напряжение, деформация и скорость частиц тела) асимптотически стремятся к своим плато, значит ни одна из этих функций не может быть точно постоянной, когда остальные стремятся к плато. В моделях же квази-линейных, кроме возможности асимптотического стремления всех трех функций к плато, существует также возможность, что одна, две или даже все три функции достигают свое абсолютное плато.

1. Introduction

IT WAS OBSERVED from the early papers on propagation of plastic waves [KÁRMÁN-DUWEZ (1950), TAYLOR (1946), RAKHMATULIN (1945)] that near the impacted end of a semi-infinite bar, impacted with constant velocity, the maximum strain is constant. This result was obtained theoretically for finite stress-strain laws and instantaneous impacts. The early experimental results have also shown a certain strain plateau [see LEE (1956)].

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It was, however, only after the publication of two papers by MALVERN (1951a, 1951b) that an extensive discussion concerning the possible existence of a plateau began. MALVERN, after introducing a semi-linear rate-type constitutive equation to describe the propagation of plastic waves, observed that his numerical examples given did not show any plateau. For many years the presence or absence of a plateau was considered to be one of the main arguments in favor of one or the other of the two theories of plastic wave propagation then in existence: rate-type and rate-independent.

Generally the constant-velocity experimental results have always shown some kind of plateau, though in some of the cases, very close to the impacted end a slightly higher strain than that of the plateau has been reported [for additional bibliography see CRISTESCU (1967, 1968)].

The early computations done by MALVERN (1951a, 1951b) showed no plateau for a semi-linear rate-type theory. RUBIN (1954) showed that in an asymptotic manner, when $t \to \infty$, one can obtain a plateau. The computations by CRISTESCU (1965), this time for a finite bar, have again shown no plateau. WOOD and PHILLIPS (1967), EFRON and MALVERN (1969) and KUKUDJANOV (1967) have obtained, however, certain kinds of "asymptotic plateaus", with semi-linear rate-type theories.

It has been already established that finite-form stress-strain laws will predict plateaus for stress, strain and velocity. In the present paper, the possibility of describing some kinds of plateaus for one or more of the three mentioned functions, will be analyzed for quasi-linear and semi-linear equations of the rate type.

2. Basic equations

The system of equations governing the motion is composed of the equation of motion

$$\varrho \frac{\partial v}{\partial t} = -\frac{\partial \sigma}{\partial x}, \tag{2.1}$$

the compatibility condition

$$\frac{\partial v}{\partial x} = -\frac{\partial \varepsilon}{\partial t}, \tag{2.2}$$

and the constitutive equation

$$\frac{\partial \sigma}{\partial t} = \varphi(\sigma, \varepsilon)\frac{\partial \varepsilon}{\partial t} + \psi(\sigma, \varepsilon). \tag{2.3}$$

Here ϱ is the mass density, v is the particle velocity, σ is the stress (force per unit initial area), $\varepsilon = -\partial u/\partial x$ is the strain, u is displacement, and σ and ε are positive in compression. In (2.3) the coefficient function φ governs the instantaneous response [see CRISTESCU (1967)]. In particular, when $\varphi = E = \text{const}$, (2.3) is reduced to the semi-linear Malvern Model. A particular form of this model which will be used in what follows is

$$\dot{\sigma} = E\dot{\varepsilon} - k[\sigma - f(\varepsilon)], \tag{2.4}$$

where k is a constant and $\sigma - f(\varepsilon)$ is the "overstress" above a typical curve for each material, which will be called the relaxation boundary [CRISTESCU (1972)].

A procedure will be given to integrate the constitutive equation with respect to time [SULICIU (1972)]. First, one eliminates the stress from the system (2.1)–(2.3) in order to have in the system of equations governing the motion, kinematic variables only. For simplicity this procedure will be presented for the semilinear form (2.4). A similar procedure, somewhat involved, is, however, possible for the quasi-linear version of the (2.3) constitutive equation as well.

By leaving aside in a first step the last term from the right side in (2.4), and integrating from an initial time t_0 (x is fixed but arbitrary), we have

$$\sigma(x, t) = E[\varepsilon(x, t)-\varepsilon(x, t_0)]+\sigma(x, t_0),$$

where $\varepsilon(x, t_0) = \varepsilon_0$ is the initial strain and $\sigma(x, t_0) = \sigma_0$ is the initial stress. In order to find a solution for the entire equation (2.4) the term $\sigma(x, t_0)$ in the last equation will be replaced by a function $\tau(x, t)$, which is to be determined using a Lagrangian method of variation of parameters. Thus the last formula will be written as

$$\sigma = \mathscr{F}(\varepsilon, \varepsilon_0, \tau) = E[\varepsilon-\varepsilon_0]+\tau. \tag{2.5}$$

Differentiating (2.5) with respect to time and combining with (2.4) gives the following equation to determine τ

$$\dot{\tau} = -k[\tau+E(\varepsilon-\varepsilon_0)-f(\varepsilon)] \tag{2.6}$$

with $\tau(t_0) = \sigma_0$. By straight integration of (2.6) τ can be determined as a functional of ε on any interval (t_0, t).

Thus the integrated form of (2.4) can be written as

$$\sigma(t) = E\varepsilon(t)+e^{-k[t-t_0]}\Big\{\sigma_0-E\varepsilon_0-k\int_{t_0}^{t}[E\varepsilon(s)-f(\varepsilon(s))]e^{k[s-t_0]}ds\Big\}. \tag{2.7}$$

It is obvious from (2.5) and (2.7) that τ is determined by the explicit form of both φ and ψ and depends on the deformation history. For this reason τ will be called the "history parameter". The structure of the function $\mathscr{F}$ in (2.5) is determined by the function φ alone, and for ε_0 and τ fixed this function describes the instantaneous response.

A similar procedure can be applied to the quasi-linear constitutive equation (2.3). The analogue of formula (2.5) is this time

$$\sigma = \mathscr{F}(\varepsilon, \varepsilon_0, \tau), \tag{2.8}$$

where $\mathscr{F}$ is determined as a solution of the differential equation $d\sigma/d\varepsilon = \varphi(\varepsilon, \sigma)$. The analogue of the formula (2.6) is

$$\frac{\partial\mathscr{F}}{\partial\tau}\dot{\tau} = \psi\big(\varepsilon, \mathscr{F}(\varepsilon, \varepsilon_0, \tau)\big) \tag{2.9}$$

with $\tau(t_0) = \sigma_0$.

Combining (2.7) with (2.1) one obtains, for the semi-linear Eq. (2.4),

$$\varrho\frac{\partial v}{\partial t}+E\frac{\partial\varepsilon}{\partial x} = ke^{-k[t-t_0]}\Big\{\frac{\partial\sigma_0}{\partial x}-E\frac{\partial\varepsilon_0}{\partial x}+\int_{t_0}^{t}[E-f'(\varepsilon(s))]\frac{\partial\varepsilon(s)}{\partial x}e^{k[s-t_0]}ds\Big\}, \tag{2.10}$$

which together with (2.2) forms the system of equations describing the motion, using the kinematic variables only.

A somewhat more involved procedure also leads from Eq. (2.9) for the quasi-linear constitutive Eq. (2.3) to the analogue of (2.10):

$$(2.11)\qquad \varrho\frac{\partial v}{\partial t}+\varphi(\varepsilon,\mathscr{F}(\varepsilon,\tau))\frac{\partial\varepsilon}{\partial x}=$$
$$-\frac{\partial\mathscr{F}}{\partial\tau}\exp\left[\int_0^t\frac{\partial\mu}{\partial\tau}(\varepsilon(s),\tau(s))\,ds\right]\int_0^t\left[\exp-\int_0^s\frac{\partial\mu}{\partial\tau}(\varepsilon(s_1),\tau(s_1))\,ds_1\right]\frac{\partial\mu}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial x}\,ds,$$

where

$$\mu(\varepsilon,\tau)=\frac{\psi(\varepsilon,\mathscr{F}(\varepsilon,\tau))}{\dfrac{\partial\mathscr{F}}{\partial\tau}(\varepsilon,\tau)}$$

and therefore

$$(2.12)\qquad \frac{\partial\mu}{\partial\varepsilon}=\frac{\dfrac{\partial\psi}{\partial\varepsilon}+\dfrac{\partial\psi}{\partial\varrho}\varphi-\psi\dfrac{\partial\varphi}{\partial\sigma}}{\dfrac{\partial\mathscr{F}}{\partial\tau}},\qquad \frac{\partial\mu}{\partial\tau}=\frac{\dfrac{\partial\psi}{\partial\sigma}\left(\dfrac{d\mathscr{F}}{d\tau}\right)^2-\psi\dfrac{\partial^2\mathscr{F}}{\partial\tau^2}}{\left(\dfrac{\partial\mathscr{F}}{\partial\tau}\right)^2}.$$

In (2.11) in order to simplify the writing it has been assumed that $t_0=0$, $\sigma_0=0$, $\varepsilon_0=0$. If not mentioned otherwise, this assumption will be maintained from now on.

3. The necessary condition for the existence of a plateau

In the formulas (2.10) and (2.11) appears the term $\partial\varepsilon/\partial x$, which is related to quantities that can be measured experimentally. The parameter that can be measured is the "surface angle" α, i.e. the angle between the position of the normal to the lateral surface during the bar deformation and the initial position of the same normal. The variation of this angle during the whole process of dynamic deformation can be determined experimentally [see BELL (1969)].

Using the mass conservation theorem, one can obtain

$$(3.1)\qquad \tan\alpha=R_0\sqrt{\varrho}\,\frac{1}{2\bar{\varrho}^{3/2}(1-\varepsilon)^{5/2}}\left[\frac{\partial\bar{\varrho}}{\partial x}-\varepsilon\frac{\partial\bar{\varrho}}{\partial x}-\bar{\varrho}\frac{\partial\varepsilon}{\partial x}\right],$$

where R_0 is the initial radius of the bar, ϱ the initial density and $\bar{\varrho}$ the actual density.

Simplified versions of this formula have been given by various authors. Thus the formula

$$(3.2)\qquad \alpha=-\frac{R_0}{2}\frac{\partial\varepsilon}{\partial x}$$

has been given by BELL, the formula

$$(3.3)\qquad \alpha=-\frac{R_0}{2}\frac{\partial\varepsilon}{\partial x}\left(1+\frac{5}{2}\varepsilon\right)$$

was given by FILBY, and

$$\alpha = -\frac{R_0}{2}\frac{1}{(1-\varepsilon)^{5/2}}\frac{\partial \varepsilon}{\partial x} \tag{3.4}$$

by CRISTESCU [for literature see BELL (1969)]. All these formulas can be obtained from (3.1) with some additional simplifying assumptions (incompressibility and by neglecting higher powers of ε).

Knowing the way in which α varies in a single loading experiment [see BELL (1968)], using (3.4), one concludes that $\partial\varepsilon/\partial x$ in (2.10) or (2.11) preserves the same sign.

We will consider a plateau to exist when there is a domain in the characteristic plane xOt, where one of the derivatives $\partial\omega/\partial t$ or $\partial\omega/\partial x$ is zero, or both are simultaneously zero. Here ω stands for any one of the functions v, ε, σ. One might find a plateau either for a single one of these functions or for more than one (possibly all three) of them. We will consider it a "plateau in time" if one of the derivatives $\partial\omega/\partial t = 0$ and "plateau in space" if one of the derivatives $\partial\omega/\partial x = 0$, and we will examine the cases when both derivatives are simultaneously zero.

By analyzing the formula (2.11) one can see that $\partial\mathscr{F}(\varepsilon, \tau)/\partial\tau > 0$. This results from the way in which $\mathscr{F}$ is determined in (2.8). From all the other terms which enter the right-hand side of Eq. (2.11), the only one which may be assumed to change sign is $\partial\mu/\partial\varepsilon$, given by $(2.12)_1$. In order for it to change sign, there must be a point t^* in the integration interval, where

$$\frac{\partial\psi}{\partial\varepsilon}+\frac{\partial\psi}{\partial\sigma}\varphi-\psi\frac{\partial\varphi}{\partial\sigma} = 0, \tag{3.5}$$

where all quantities are computed in the point $\Big(\varepsilon(x, t^*), \mathscr{F}\big(\varepsilon(x, t^*), \tau(x, t^*)\big)\Big)$. Therefore the condition (3.5) is a necessary condition for the existence of a plateau. Examples and further discussion are given in the following two Sections.

4. Numerical examples

The existence of the plateau has generally been discussed in connection with the impact with constant velocity of a semi-infinite bar. However, such plateaus can be found even for more realistic problems as for instance the symmetric impact of two identical finite bars. For such a problem assuming that the specimen is impacted with a velocity of $V = 1600$ in/sec and that the other end is free, several solutions have been obtained with constitutive equations of the form (2.3) in [CRISTESCU (1972)].

For one of these cases, the one called XII in CRISTESCU (1972), the three curves of Fig. 1 show the variation in time of σ, ε and v at various cross-sections of the bar. From these figures one can see when any one of these three variables is constant either in time and space or in time only. The plotting has been done only for those cross-sections of the bar where there are one or several plateaus. Of the three functions involved, it is the strain which exhibits plateaus in time in the greatest part of the bar.

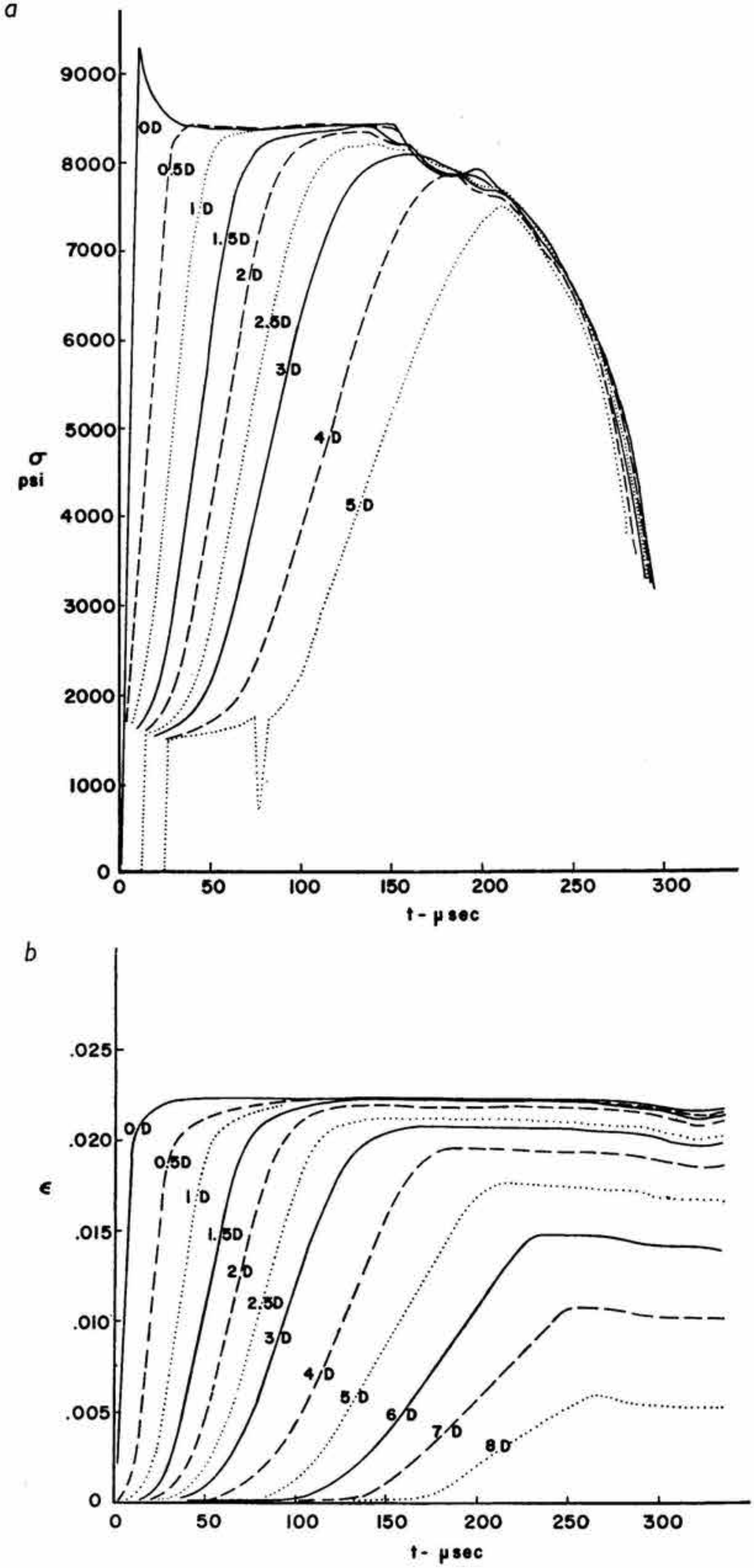
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FIG. 1.

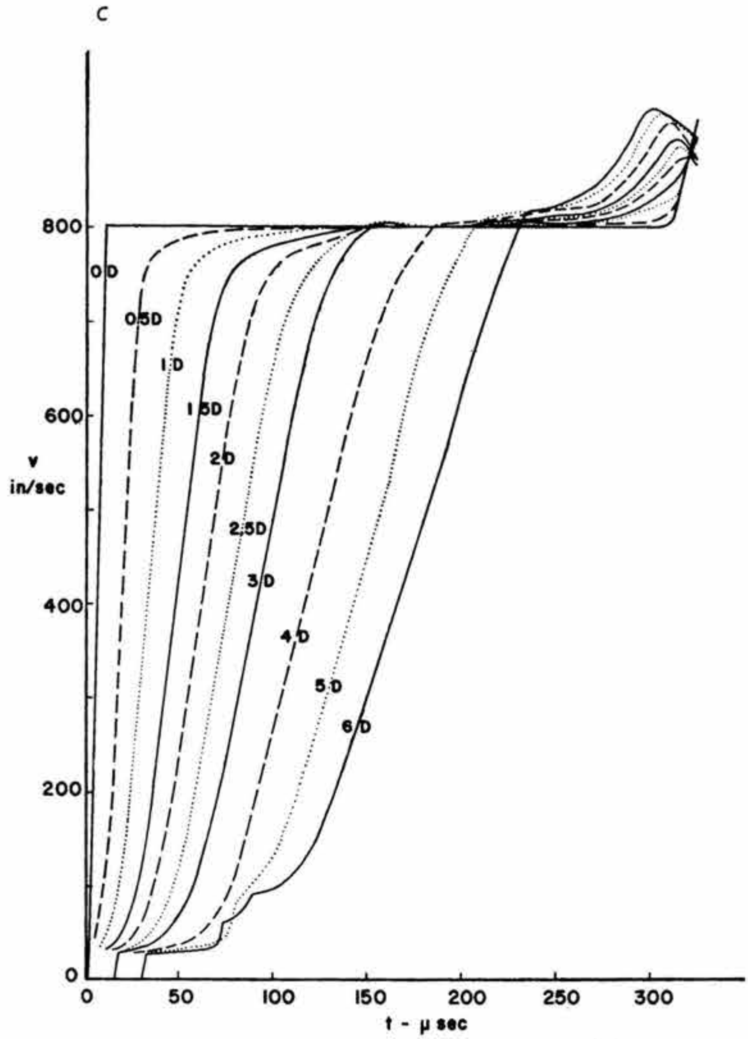


FIG. 1. Quasi-linear constitutive equation predictions:
(a) stress-time, (b) strain-time, (c) particle velocity-time curves at various cross-sections.
(2*D* means two inches from impact end, etc.).

Let us write the necessary condition for the existence of the plateau, Eq. (3.5), for the particular case under consideration. In [Cristescu (1972)] the following explicit expression for the constitutive Eq. (2.3)

$$\dot{\sigma} = \frac{E}{1+E\Phi}\dot{\varepsilon} - \frac{E\psi}{1+E\Phi} \tag{4.1}$$

was used, where

$$\psi(\sigma, \varepsilon) = \begin{cases} \dfrac{k(\varepsilon)}{E}[\sigma - f(\varepsilon)] & \text{if} \quad \sigma > f(\varepsilon) \quad \text{and} \quad \varepsilon \geqslant \varepsilon_Y \\ 0 & \text{if} \quad \sigma \leqslant f(\varepsilon) \quad \text{or} \quad \varepsilon < \varepsilon_Y \end{cases} \tag{4.2}$$

$\sigma = f(\varepsilon)$ is the "relaxation boundary", and

$$k(\varepsilon) = k_0\left[1 - \exp\left(-\frac{\varepsilon}{\hat{\varepsilon}}\right)\right] \tag{4.3}$$

with k_0 and $\hat{\varepsilon}$ constants. The function $\Phi(\varepsilon)$ was defined for increasing stress processes as

$$\Phi(\varepsilon) = \frac{3\left[\varepsilon - \varepsilon_Y - \varepsilon^* + \left(\dfrac{a}{3E}\right)^{3/2}\right]^{2/3}}{a} - \frac{1}{E}, \tag{4.4}$$

where

$$a = m + n\sqrt{\varepsilon} \tag{4.5}$$

and $m, n, \varepsilon_Y, \varepsilon^*$ were constants.

With the previously given notations, the condition (3.5) can be written as

$$\frac{1}{\left(\dfrac{1}{E}+\Phi\right)^2}\left[\frac{\partial\psi}{\partial\varepsilon}\left(\frac{1}{E}+\Phi\right) - \psi\frac{\partial\Phi}{\partial\varepsilon} + \frac{\partial\psi}{\partial\sigma}\right] \tag{4.6}$$

$$\approx \frac{k_0\left[1-\exp\left(-\dfrac{\varepsilon}{\hat{\varepsilon}}\right)\right]}{[1+E\Phi]^2}\left[1 - \left(\frac{1}{E}+\Phi\right)f'(\varepsilon) - \frac{2\{\sigma - f(\varepsilon)\}}{a\left[\varepsilon - \varepsilon_Y - \varepsilon^* + \left(\dfrac{a}{3E}\right)^{3/2}\right]^{1/3}}\right]$$

$$+ \frac{k_0}{1+E\Phi}\left[\left\{\frac{1}{E}+\Phi\right\}\frac{1}{\hat{\varepsilon}}\{\sigma - f(\varepsilon)\}\exp\left(\frac{\varepsilon}{\hat{\varepsilon}}\right)\right.$$

$$\left. - \left\{1 - \exp\left(-\frac{\varepsilon}{\hat{\varepsilon}}\right)\right\}\{\sigma - f(\varepsilon)\}\frac{3n}{2\sqrt{3Ea\varepsilon}}\frac{1}{\left[\varepsilon - \varepsilon_Y - \varepsilon^* + \left(\dfrac{a}{3E}\right)^{3/2}\right]^{1/3}}\right].$$

In writing the right-hand term in (4.6) some terms of a much lower order of mgnitude were neglected.

If in the right-hand term from (4.6) one makes $\varepsilon = \varepsilon_Y$ (beginning of the plasticdeformation), this term is positive, while at the states close to the maximum value of Φ the same term is negative, i.e., meantime it has changed sign. This result is obvious since for

$\varepsilon = \hat{\varepsilon}$ (strain at the inflection point of the $\varepsilon - t$ curve at $x = 0$) the term $\left(\frac{1}{E}+\Phi\right)f'(\varepsilon) \approx$ ≈ 1.37. The condition is verified at $x = 1D$ as well. All the other terms in the right-hand side of (4.6) are at $\varepsilon = \bar{\varepsilon}$ either also negative or negligible.

Thus the main conclusion is that for quasi-linear constitutive equations of the form (4.1), the right-hand term in (2.11) can be zero for a certain $t^* > 0$ and therefore an absolute plateau in time and space is possible. In other words, with the quasi-linear constitutive equation there can be a domain in the characteristic plane, where $v(x, t) = \text{const}$, $\varepsilon(x, t) = \text{const}$, and $\sigma(x, t) = \text{const}$. The examples plotted in Fig. 1 show that in fact a plateau does occur.

5. Absolute and asymptotic plateaus

Let us show first that semi-linear models in the form (2.4) most often used, do not exhibit an absolute plateau.

In order to show this let us assume that above a certain curve Γ in the characteristic plane we have

$$v = \text{const}, \qquad \varepsilon = \text{const}. \tag{5.1}$$

Then by combining (5.1) with (2.10) one obtains for any point above Γ (with $\sigma_0 = 0$, $\varepsilon_0 = 0$)

$$e^{-k[t-t_0]}\int_{t_0}^{t^*} [E-f'(\varepsilon(s))]\frac{\partial \varepsilon(s)}{\partial x} e^{k[s-t_0]}ds = 0. \tag{5.2}$$

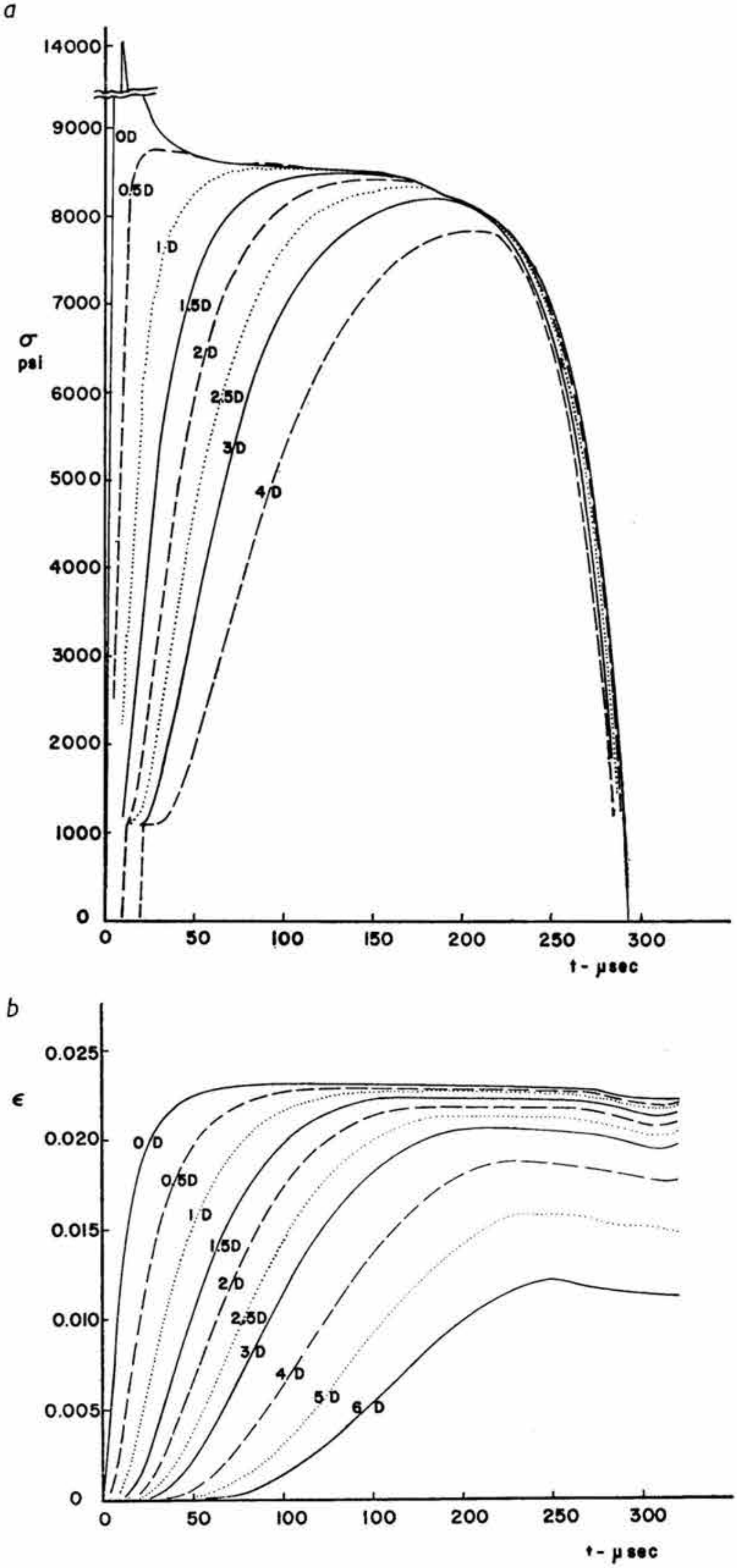
Taking into account the formulas of the type (3.4) and knowing from experiment that α does not change sign, one comes to the conclusion that (5.2) is impossible. Thus an absolute plateau is impossible with the usual linear overstress form of the function ψ assumed in (2.4).

An absolute plateau is also impossible with certain other forms that have been proposed for the function ψ, such as $\psi = -k\left[\exp\left(\frac{\sigma-f(\varepsilon)}{a}\right)-1\right]$ or $\psi = -k\sqrt{\sigma-f(\varepsilon)}$, as is easy to see from direct checking the formula (3.5) with $\varphi = E$. However, it might be possible to verify the condition (3.5) even for a semi-linear model, by choosing in some other appropriate way the function ψ.

Because of the negative exponential in front of the integral in (5.2), it might appear that (5.2) could approach zero asymptotically for $t \gg t^*$. But we will now show that, with the semi-linear model (2.4), the equations of motion cannot be satisfied everywhere in the x, t-plane by assuming that one of the functions ε, v, σ is constant while the other two asymptotically approach a plateau.

Let us assume that above a certain curve Γ in the characteristic plane, one of the required functions, say v, is a constant (in time and space). For this case let us examine in what conditions, in an asymptotic manner, some other function, say ε, could reach a plateau. If

$$v = \text{const} \quad \text{above } \Gamma, \tag{5.3}$$


a
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9000
8000
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5000
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1000
0
σ psi
0D
0.5D
1.D
1.5D
2/D
2.5D
3/D
4/D
0 50 100 150 200 250 300
t - μsec
b
0.025
0.020
0.015
0.010
0.005
0
ε
0/D
0.5D
1.D
1.5D
2/D
2.5D
3/D
4/D
5 D
6 D
0 50 100 100 200 250 300
t - μsec

Fig. 2.

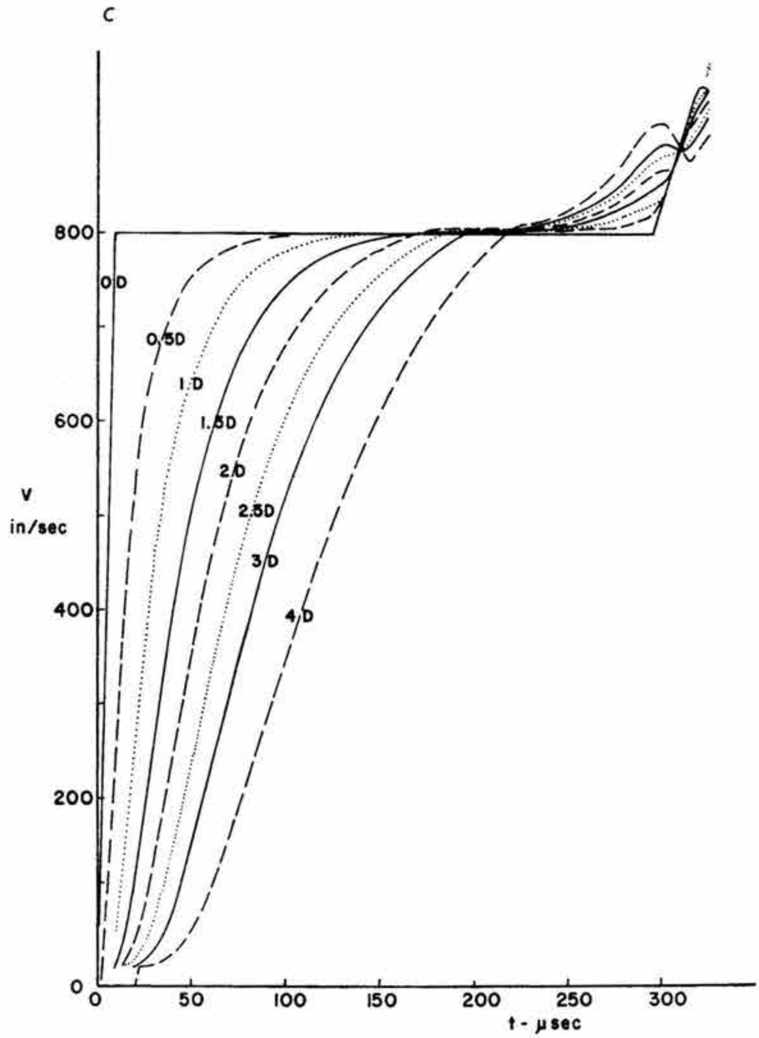

FIG. 2. Semi-linear constitutive equation predictions:
(a) stress-time, (b) strain-time, (c) particle velocity-time curves at various cross-section.
(2*D* means two inches from impact end, etc.).

then from (2.2) results

$$\varepsilon = h(x). \tag{5.4}$$

Denoting by (x^*, t^*) the coordinates of the points on Γ, and integrating (2.10) up to $t > t^*$ we get

$$\begin{aligned}0 = -f'\big(\varepsilon(t^*, x^*)\big)\frac{\partial\varepsilon(t^*, x^*)}{\partial x} + k\exp\big(-k(t-t_0)\big)\times \\ \times\Bigg\{\int\limits_{t_0}^{t^*}[E-f'\big(\varepsilon(s, x^*)\big)]\frac{\partial\varepsilon(s, x^*)}{\partial x}\exp\big(k(s-t_0)\big)ds \\ -\frac{1}{k}[E-f'\big(\varepsilon(t^*, x^*)\big)]\frac{\partial\varepsilon(t^*, x^*)}{\partial x}\exp\big(k(t^*-t_0)\big)\Bigg\}.\end{aligned} \tag{5.5}$$

This identity is of the form $a+b\exp\big(-k(t-t_0)\big) = 0$ with $a = \text{const}$ and $b = \text{const}$. We will see that this identity cannot in general be satisfied for $t > t^*$ either exactly or in an asymptotic manner.

In order to satisfy exactly (5.5), one must have

$$\begin{aligned}f'\frac{\partial\varepsilon}{\partial x} = 0 \quad \text{and} \quad \int\limits_{t_0}^{t^*}[E-f'\big(\varepsilon(s, x^*)\big)]\frac{\partial\varepsilon(s, x^*)}{\partial x}\exp\big(k(s-t_0)\big)ds \\ -\frac{1}{k}[E-f'\big(\varepsilon(t^*, x^*)\big)]\frac{\partial\varepsilon(t^*, x^*)}{\partial x}\exp\big(k(t^*-t_0)\big) = 0.\end{aligned} \tag{5.6}$$

Equation (5.6) is satisfied in very particular cases only. First, $(5.6)_1$ is satisfied if $f'\big(\varepsilon(t^*, x^*)\big) = 0$ and then from $(5.6)_2$ results

$$k\int\limits_{t_0}^{t^*}\frac{\partial\varepsilon(s, x^*)}{\partial x}\exp\big(k(s-t_0)\big)ds = \frac{\partial\varepsilon(t^*, x^*)}{\partial x}\exp\big(k(t^*-t_0)\big).$$

That is, (5.6) is satisfied if no work-hardening is present and if a very special exponential law for the variation of strain is imposed.

Secondly, $(5.6)_1$ is satisfied also if $\partial\varepsilon(t^*, x^*)/\partial x = 0$, i.e.

$$\int\limits_{t_0}^{t^*}[E-f'\big(\varepsilon(s, x^*)\big)]\frac{\partial\varepsilon(s, x^*)}{\partial x}\exp\big(k(s-t_0)\big)ds = 0$$

which is impossible (see (5.2)).

In order to satisfy (5.5) asymptotically, $(5.6)_1$ must be satisfied (in the particular conditions already discussed) and simultaneously one has to find a particular big value of **t** for which (5.2) is to be negligible. But then it is observed that in the strip $(t^*, \mathbf{t})$, the equation of motion is not satisfied.

Therefore the assumption concerning the absolute plateau for one of the functions v, ε, or σ [assumption (5.3)] is incompatible with a semilinear model which would not satisfy at least the necessary condition (3.5). (A similar discussion follows from assuming that it is ε or σ that is constant.)

Figures 2a–2c show that for a semi-linear model, though nearly horizontal plateaus are visible, these are in fact slowly but steadily increasing or decreasing. The figures 1a–1c for the quasi-linear model, however, show a much better approximation of an absolute plateau in the first two diameters near the impacted end.

Conclusions

It was shown that in order to possess a plateau, the coefficient functions entering a semilinear or quasi-linear constitutive equation must satisfy a certain necessary condition, Eq. (3.5).

With the semi-linear model, "asymptotic plateau" means that all three functions tend asymptotically toward plateaus, since we have seen that it is not possible for even one of the functions to be exactly constant while the others approach a plateau. With the quasi-linear model, in addition to the possibility that all three functions approach plateaus asymptotically, it is possible that one, two or even all three functions can have absolute plateaus.

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