Archives of Mechanics • Archiwum Mechaniki Stosowanej • 24, 5-6, pp. 1013-1021, Warszawa 1972

Plastic deformations of thick-walled concrete tubes under dynamic internal pressure

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THE FIRST part of the paper deals with the complete limit analysis of thick-walled concrete tubes reinforced on their exterior surface by steel wires and loaded by internal pressure. In the second part an approximate solution is presented for the calculation of permanent displacements when the tube is subjected to dynamic pressure. In this analysis the elastic deformations and the effects of changes in geometry are neglected the strain-rate sensitivity of steel, however, is taken into consideration.

W pierwszej części pracy zajęto się pełną analizą stanu granicznego żelbetowych rur grubościennych zbrojonych na powierzchni zewnętrznej drutami stalowymi i obciążonych ciśnieniem wewnętrznym. W drugiej części pracy przedstawiono przybliżone rozwiązanie obliczenia przemieszczeń trwałych w przypadku, gdy rura poddana jest działaniu ciśnienia dynamicznego. W obliczeniach tych pomija się wpływ odkształceń sprężystych i zmian geometrii układu, natomiast uwzględnia się wrażliwość stali na prędkość odkształcenia.

В первой части работы занимаются полным анализом предельного состояния железобетонных толстостенных труб, армированных на внешней поверхности стальной проволокой и нагруженных внутренним давлением. Во второй части работы представлено приближенное решение расчета остаточных перемещений в случае, когда труба подвергнута действию динамического давления. В этих расчетах пренебрегается влияниями упругих деформаций и изменений геометрии системы, учитывается же чувствительность стали на скорость деформации.

1. Introduction

RECENTLY, increasing attention has been devoted to the analysis of inelastic deformations in structures subjected to impulsive or pressure loading (see e.g. [1–14]). In these problems the influence of changes in geometry and of strain rate sensitivity may play an important role. Taking into consideration these phenomena, however, exact analysis even of simple structures, results in very complicated calculations. In order to avoid these mathematical difficulties, general theorems and approximate methods have been elaborated which make it possible to arrive at relatively simple solutions (see e.g. [15–27]).

The aim of this paper is also to present a simple approximate solution for thick-walled concrete tubes reinforced on their exterior surface by steel wires and subjected to internal dynamic pressure. In the analysis, concrete is considered as a rigid-plastic, strain-rate insensitive material without tensile strength, while in the wires the viscous effects are also taken into calculation. The geometry change effects are disregarded and the pressure loading is replaced by its impulse. The idea of the approximate method published elsewhere [24–27] is to impose a postulated stationary displacement field on the structure, thus reducing the analysis to solution of the equivalent quasi-static problem and to investigation of a one-degree-of-freedom system.

In the general interpretation, tensor notation and the summation convention will be used. Notations applied are defined as follows:

Notations

A, B interior and exterior radius of the tube,
R, ϑ, z, t cylindrical-coordinates and time,
X, X_0 boundary of two regions of the tube under statical and dynamical conditions,
p internal pressure,
p_0, t_0, I peak value, duration and impulse of internal pressure,
F area of steel wires per unit length of the tube,
S, S_0 force in steel wires per unit length of the tube under statical and dynamical conditions,
σ_{S_0}, σ_S yield stress of steel under statical and dynamical conditions,
D, n viscosity constants of steel,
τ_{c_0} shear-strength of concrete,
ϱ density of concrete per unit volume,
$u_r, u_\vartheta; \dot{u}_r, \dot{u}_\vartheta$ radial and tangential displacements and velocities,
$\varepsilon_r, \varepsilon_\vartheta; \dot{\varepsilon}_r, \dot{\varepsilon}_\vartheta$ radial and tangential strains and strain rates,
$\varepsilon_{\vartheta S}, \dot{\varepsilon}_{\vartheta S}$ strain and strain rate of steel wires,
$\sigma_r, \sigma_\vartheta$ radial and tangential stresses,
W displacement parameter,
V_0 initial velocity,
t_f response time.

dimensionless parameters:

$$b = B/A, \quad r = R/A, \quad x = X/A, \quad x_0 = X_0/A, \quad s = \frac{S}{2A\tau_{c_0}}, \quad s_0 = \frac{S_0}{2A\tau_{c_0}},$$

$$w = \frac{W}{A}, \quad v_0 = \frac{V_0}{A}, \quad \lambda = \frac{2I^2}{\varrho A^2 \tau_{c_0}(b^2-1)^2}, \quad \beta = \frac{v_0}{D}.$$

2. Assumptions and basic relations

Let us consider a thick-walled concrete tube which is reinforced on its exterior boundary by steel wires and subjected to internal pressure (Fig. 1.). It is assumed that the tube is in the state of plane strain and that the effects of changes in geometry and the mass of wires can be disregarded.

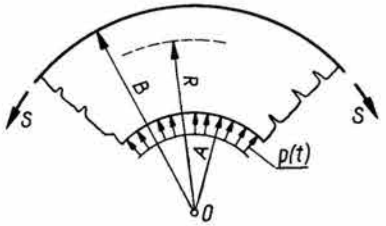

FIG. 1.

Concrete is considered as a rigid-plastic material without tensile strength. Then, in plane strain, the yield condition can be expressed as follows:

$$\left.\begin{array}{ll}\sigma_r \leqslant 0, & \sigma_\vartheta \leqslant 0,\\ \left(\dfrac{\sigma_r-\sigma_\vartheta}{2}\right)^2 - \tau_{c_0}^2 \leqslant 0. & \end{array}\right\}. \tag{2.1}$$

Since the strain-rate sensitivity of concrete is not taken into consideration the shear strength τ_{c_0} is a time-independent constant.

The material of steel wires is assumed to be rigid-viscoplastic. Then, under uniaxial tension the yield condition is of the simple form:

$$\sigma - \sigma_S \leqslant 0. \tag{2.2}$$

Here, as a consequence of strain-rate sensitivity, the yield stress σ_S is the function of strain rate and, following COWPER and SYMONDS [28] can be defined as

$$\sigma_S = \sigma_{S_0}\left(1+\frac{\dot{\varepsilon}_{\vartheta S}}{D}\right)^{1/n}. \tag{2.3}$$

The purpose of our investigations is to determine the maximum permanent displacements of the tube. Since the approximate method presented is based on the solution of the equivalent statical problem, we shall first discuss the limit analysis of the tube. This will be followed by the approximate dynamic analysis.

3. Limit analysis

Let us consider the equivalent problem in which the internal pressure is acting under quasi-static conditions and, for the time being, the strain-rate sensitivity of steel is disregarded. For increased pressure, unrestricted plastic flow can occur only when both the wires and the entire tube are fully plastic. Then, the corresponding stress and velocity field and the collapse load can be determined by using the extremum theorems of limit analysis [29].

3.1. Statical solution

It is known [29] that the equation of equilibrium of thick-walled tubes has the form:

$$\frac{\partial \sigma_r}{\partial R} + \frac{\sigma_r - \sigma_\vartheta}{R} = 0, \tag{3.1}$$

and the boundary conditions are as follows:

$$\sigma_r(A) = -p, \quad \sigma_r(B) = -\frac{S_0}{B}. \tag{3.2}$$

From the point of view of the yield conditions (2.1), the tube can be divided in two regions:

$$\begin{array}{lll}\text{if} & X_0 < R < B: & \sigma_\vartheta = 0, \ \sigma_r < 0.\\ \text{if} & A < R < X_0: & \left(\dfrac{\sigma_r-\sigma_\vartheta}{2}\right)^2 - \tau_{c_0}^2 = 0, \quad \sigma_\vartheta < 0.\end{array} \tag{3.3}$$

While in the wires:

$$S_0 = F\sigma_{S_0}. \tag{3.4}$$

In view of the Eqs. (3.1)–(3.4), and the fact that at $R = X_0$ σ_r must be continuous, in function of dimensionless variables the following results can be obtained:

if $x_0 < r < b$:

$$\sigma_r = -2\tau_{c_0}\frac{s_0}{r}, \quad \sigma_\theta = 0; \tag{3.5}$$

if $1 < r < x_0$:

$$\sigma_r = -2\tau_{c_0}\left(1+\ln\frac{x_0}{r}\right), \qquad \sigma_\theta = -2\tau_{c_0}\ln\frac{s_0}{r}. \tag{3.6}$$

The statically admissible load multiplier

$$p_s = 2\tau_{c_0}\left(\ln x_0 + \frac{x_0}{x_0}\right). \tag{3.7}$$

Here, the parameter x_0 can be determined by using the minimum condition: $\partial p_s/\partial x_0 = 0$, and by considering the restriction that $1 \leqslant x_0 \leqslant b$. Omitting the details, we obtain:

$$\begin{aligned} &\text{if } s_0 \leqslant 1, && x_0 = 1,\\ &\text{if } 1 \leqslant s_0 \leqslant b, && x_0 = s_0,\\ &\text{if } s_0 \geqslant b, && x_0 = b. \end{aligned} \tag{3.8}$$

3.2. Kinematical solution

Let us suppose that the yield mechanism of the tube consists of two parts (Fig. 2.). In the *inner part* ($A < R < X_0$), the velocity field can be assumed as follows:

$$\dot{u}_r = \frac{k}{R}, \quad \dot{u}_\theta = 0. \tag{3.9}$$

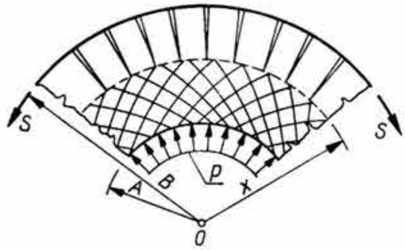

FIG. 2.

Then the strain rates can be obtained:

$$\dot{\varepsilon}_r = \frac{\partial \dot{u}_r}{\partial R} = -\frac{k}{R^2}, \quad \dot{\varepsilon}_\theta = \frac{\dot{u}_r}{R} = \frac{k}{R^2}. \tag{3.10}$$

Here, k denotes a constant. It will be seen that in this part of the tube shear lines develop in 45° relationship to the radius. The maximum shear strain rate along these lines is:

$$|\dot{\gamma}| = |\dot{\varepsilon}_\vartheta - \dot{\varepsilon}_r| = \frac{2k}{R^2}. \tag{3.11}$$

In the *outer part* ($X_0 < A < B$), the velocity field and the strain-rates are assumed as:

$$\begin{aligned} \dot{u}_r &= \frac{k}{X_0}, \quad & \dot{u}_\vartheta &= 0, \\ \dot{\varepsilon}_r &= 0, \quad & \dot{\varepsilon}_\vartheta &= \frac{k}{X_0 R}. \end{aligned} \tag{3.12}$$

This means that in this part of the tube there are developed in radial directions cracks along which the stresses are equal to zero. The elementary sectors bounded by the cracks achieve rigid-body motion.

Finally, from the formulae (3.12) the strain-rate in the wires is $\dot{\varepsilon}_{\vartheta S} = \dfrac{k}{X_0 B}$.

Using the yield mechanism defined above, let us now apply the Principle of Virtual Velocities:

$$\int \tau_{c_0} \dot{\gamma} dV + \int S_0 \dot{\varepsilon}_{\vartheta S} dA = \int p_k \dot{u}_r dA,$$

or substituting the formulae (3.9)–(3.12):

$$\int_0^{2\pi} \int_A^{X_0} \tau_{c_0} \frac{2k}{R^2} R dR d\vartheta + \int_0^{2\pi} S_0 \frac{k}{X_0 B} B d\vartheta = \int_0^{2\pi} p_k \frac{k}{A} A d\vartheta.$$

From this equation, the kinematically admissible load multiplier may easily be obtained:

$$p_k = 2\tau_{c_0}\left(\ln x_0 + \frac{s_0}{x_0}\right). \tag{3.13}$$

Since this is identical with the statically admissible load multiplier expressed by Eq. (3.7), it can be stated that complete solution of problem has been demonstrated and Eqs. (3.7) and (3.8) define the collapse load p_c of the tube.

The results obtained above can be readily applied to the case in which steel is sensitive to strain rate. The only difference is that the yield stress is not a constant but is expressed by the formula (2.3) and, consequently, S_0, s_0, X_0 and x_0 should be replaced in Eqs. (3.4)–(3.13) by S, s, X and x. Thus, the collapse load multiplier is:

$$p_c = 2\tau_{c_0}\left(\ln x + \frac{s}{x}\right), \tag{3.14}$$

where

$$\begin{aligned} &\text{if } s \leqslant 1, & x &= 1, \\ &\text{if } 1 \leqslant s \leqslant b, & x &= s, \\ &\text{if } s > b, & x &= b. \end{aligned} \tag{3.15}$$

4. Approximate dynamic analysis

4.1. Description of the approximate method

Let us consider a rigid-plastic continuum, which is subjected to dynamic surface tractions $T_i = p(t)\, T_i^0(x_i)$, $(i = 1, 2, 3)$ and undergoes plastic displacements $u_i(x_i, t)$. In the case of blast-type loading, the load acts for a very short time or has a rapidly decreasing character, consequently, after a certain time (t_f) the continuum comes to rest. The permanent displacements $u_i^{max} = u_i(x_i, t_f)$ can be determined provided, that at $t = t_f$, $\dot{u}_i = 0$.

Exact analysis of problems of this kind results even for simple structures, in very complicated calculations. The main difficulty arises from the fact that during the response the displacement field is generally not stationary. The idea of the proposed approximate method is to replace the actual displacement field by a stationary one i.e., to express the displacements in the form of a product:

$$u_i(x_i, t) \approx W(t) u_i^k(x_i). \tag{4.1}$$

Here $u_i^k(x_i)$ is any predicted kinematically admissible displacement field and $W(t)$ denotes an unknown displacement parameter function. Using this mode approximation, the dynamic analysis of a continuum or a structure can be reduced to investigation of an equivalent one-degree-of-freedom system. Omitting the proof and details [24–27], the differential equation of motion of this system and the initial conditions are

$$\ddot{W} = K[p(t) - p_k] \tag{4.2}$$

and

$$W(0) = 0, \qquad \dot{W}(0) = 0.$$

Here,

$$K = \frac{\int\limits_A T_i^0 u_i^k \, dA}{\int\limits_V \varrho u_i^k u_i^k \, dV} = \text{const}, \tag{4.3}$$

and p_k is the kinematically admissible load multiplier connected with the displacement field u_i^k. Using rigid-plastic material and taking into consideration only small deflections, p_k is a constant; in view of the influence of strain rate sensitivity, however, p_k is a function of the velocities: $p_k = p_k(\dot{W})$.

A usual approximation replaces the pressure by its impulse

$$I = \int\limits_0^{t_0} p(t)\, dt. \tag{4.4}$$

Then, instead of a pressure loading, an initial velocity field is imposed upon the structure. Consequently, $p(t) \equiv 0$ and the initial conditions should be modified as: $W(0) = 0$, $\dot{W}(0) = V_0$. Here V_0 can be determined from the impulse I under different assumptions.

4.2. Solution of problem

Let us apply the approximate method described above to the dynamic analysis of the tube, which has been solved in the previous Section under statical conditions. Then, p_k is defined by the formulae (3.14) and (3.15) and using the concept of impulsive loading $p(t) \equiv 0$. The constant K can be determined by the formula (4.3) and by the displacement field (3.9) and (3.12). Omitting the details of calculation, the differential equation of motion (4.2) has the form:

$$\ddot{w}+\varrho AK\frac{v_0^2}{\lambda}\left(\ln x+\frac{s}{x}\right) = 0. \tag{4.5}$$

Here,

$$K = \frac{1}{\varrho A[\ln x_0+2(b^2/x_0^2-1)}, \qquad s = s_0\left[1+\left(\frac{\beta}{x_0 b}\right)^{1/n}\left(\frac{\dot{w}}{v_0}\right)^{1/n}\right], \tag{4.6}$$

x and x_0 are defined by Eqs. (3.15) and (3.8), respectively, and v_0 can be calculated from the approximate formula:

$$v_0 = \frac{V_0}{A} = \frac{2I}{\varrho A(b^2-1)}. \tag{4.7}$$

As a consequence of the viscosity of steel, the force arising in the wires, and the resistance displayed by the tube depends on the velocities. Consequently, the second term of Eq. (4.5)

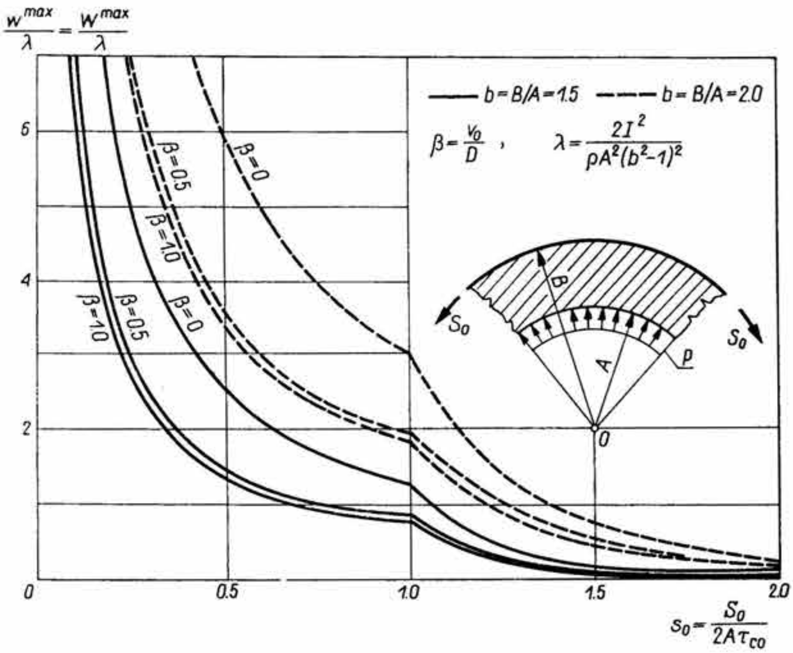

FIG. 3.

is also a function of $\dot{w}$. For this reason, analytical solution of the differential Eq. (4.5) is generally not possible. We have solved the equation numerically, using Runge-Kutta's method. The maximum permanent displacements obtained in function of s_0 and for the parameters $b = 1.5$–2.0, $\beta = 0.0$–0.5–1.0 and $n = 5$ are plotted in Fig. 3.

5. Conclusions

The statical solution presented in Sec. 3 can be applied to the limit analysis and design of post-stressed, thick-walled concrete tubes. The dynamic analysis of Sec. 4, on the other hand, can form a reliable basis for rapid approximate calculations of tubes subjected to internal dynamic pressure or impulse. From Fig. 3, it is seen that the maximum permanent displacements are proportional to λ — i.e., are in quadratic relation with the impulse I. The parameter β significantly influences the response of the tube. This means that the viscosity of steel wires considerably decreases the maximum displacements.

Our further research will be focussed on investigation of the effects of different yield conditions and of elastic and large deformations, respectively.

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Received June 12, 1972