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BRIEF NOTES

A note on finite elastic-plastic bending

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Finite bending of a hyper elastic-plastic strip is considered. A continuous transition from hyperelastic solution to perfectly plastic one is obtained. The results are compared with the available solutions of the strip bending problem.

1. Introduction

ELASTIC-PLASTIC deformations are usually studied within the framework of infinitesimal theories. Displacements are then assumed to be small so that the initial configuration and the configuration at yielding do not need to be discerned. Situations when yielding starts and continues at large elastic strains call, however, for an appropriate analysis.

There are two aspects in the analysis of large elastic-plastic deformations, regarding respectively the constitutive law and the yield function. The main problem of a finite elastic-plastic strain theory consists in developing a suitable set of constitutive relations. The second, uncomparably simpler problem is that of transition of a nonlinearly elastic material into the plastic state. This question was not studied so far within the concepts and notions of nonlinear theories. Available studies account for large deformations through the logarithmic strain measure, but otherwise preserve the structure of a linear theory.

The present note concerns finite bending of a hyperelastic-plastic strip. The material is assumed to yield when the true stress satisfies an appropriate yield condition. Under continued bending plastic zones develop and spread out. A continuous transition from the hyperelastic solution, GREEN and ZERNA [5], to that of bending of a perfectly plastic circular segment, HILL [6], is obtained. The results are compared with the available solutions of the strip bending problem by DE BOER [1], BRUHNS [2], BRUHNS and THERMANN [3] and CELEP and HARTUNG [4], where the Hencky stress-strain relation was used in the elastic range (i.e., when logarithmic strains are related to physical stress by Hooke's law). As regards the elastic response, the analysis is kept within the hyperelastic model for the stored energy function of the Murnaghan type [7, 8].

2. Basic relations

We consider a plate strip of incompressible hyperelastic-plastic material. The strip of initial geometry as shown in Fig. 1a is bent into a cylindrical sector in conditions of plane deformation, Fig. 1b. The problem is considered in convected coordinates. The metric tensor in a current configuration is denoted by G_{ij}, whereas the initial state has

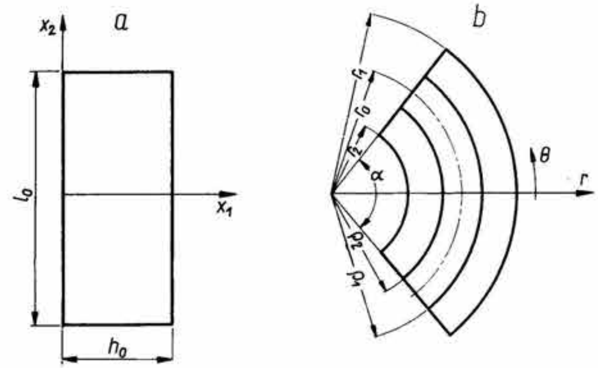

FIG. 1. Strip geometry in bending: a) initial configuration, b) current configuration.

the metric tensor g_{ij}. The notations used are essentially those of GREEN and ZERNA [5] and of Fig. 1.

For the considered deformation of an incompressible material, the metric tensors are

$$(2.1) \qquad G_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_{ij} = \begin{bmatrix} A^2r^2 & 0 & 0 \\ 0 & A^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$(2.2) \qquad A = \frac{2h_0}{r_1^2 - r_2^2}.$$

The strain invariants entering the stored energy function $W = W(I_1, I_2, I_3)$ have the form

$$(2.3) \qquad \begin{aligned} I_1 &= g^{ij}G_{ij} = A^2r^2 + \frac{1}{A^2r^2} + 1 = I, \\ I_2 &= G^{ij}g_{ij}I_3 = A^2r^2 + \frac{1}{A^2r^2} + 1 = I, \quad I_3 = \det G_{ij}/\det g_{ij} = 1. \end{aligned}$$

The stresses in an incompressible hyperelastic material express in terms of the stored energy function $W = W(I_1, I_2)$ to within a hydrostatic pressure p as follows

$$(2.4) \qquad \tau^{ij} = 2\frac{\partial W}{\partial G_{ij}} = \phi g^{ij} + \psi B^{ij} + pG^{ij},$$

where

$$(2.5) \qquad \phi = 2\frac{\partial W}{\partial I_1}, \quad \psi = 2\frac{\partial W}{\partial I_2}, \quad B^{ij} = I_1 g^{ij} + g^{ir}g^{js}G_{rs}.$$

In the considered deformation $\phi = \psi$ in view of (2.3).

In the motion studied the physical stress components take the same numerical values as the mixed stress components σ^i_j

$$(2.6) \qquad \sigma^i_j = \tau^{ir}G_{rj}.$$

These guaranties will be used when studying the outset of yielding. The yield condition will be referred to current configurations and expressed therefore in terms of the true stresses in form of a scalar function

$$f(\sigma_i^i, \sigma_j^i\sigma_i^j, \sigma_j^i\sigma_r^j\sigma_i^r) = 0. \tag{2.7}$$

The stress field is subjected to the equilibrium requirements. In the considered case the only relevant relation is

$$\frac{d\sigma_1^1}{dr} + \frac{\sigma_1^1 - \sigma_2^2}{r} = 0, \tag{2.8}$$

if expressed in terms of the mixed stress components (2.6) (representing the physical components as well). Moreover, the following stress boundary conditions are imposed

$$\sigma_1^1(r_1) = \sigma_1^1(r_2) = 0. \tag{2.9}$$

For further reference, we write the following geometrical relations, applicable in the case of considered incompressible deformation and straightforward to obtain (Fig. 1)

$$\begin{aligned} r_1 &= \frac{l_0}{\alpha}\sqrt{\alpha s + \sqrt{1+\alpha^2 s^2}}, \quad r_2 = \frac{l_0^2}{r_1\alpha^2}, \\ \frac{2h_0}{r_1^2 - r_2^2} &= \frac{\alpha}{l_0}, \quad s = \frac{h_0}{l_0}, \quad A^2 = \frac{1}{r_1 r_2}. \end{aligned} \tag{2.10}$$

3. Outset of yielding

Within the hyperelastic range the stresses (2.4), subjected to the requirements (2.8) and (2.9), take eventually the form

$$\begin{aligned} \sigma_1^1 &= W + C = W - W_0, \\ \sigma_2^2 &= 2\phi\left(A^2r^2 - \frac{1}{A^2r^2}\right) + W - W_0, \quad \sigma_3^3 = \phi\left(A^2r^2 - \frac{1}{A^2r^2}\right) + W - W_0, \end{aligned} \tag{3.1}$$

where $C = -W_0 = -W(r_1) = -W(r_2)$. It appears that on surface $\theta = \pm\alpha/2$ the stress resultant vanishes while the stress couple (per unit length) acting on each of the surfaces initially at $x_2 = \pm l_0/2$ is

$$M = \int_{r_0}^{r_1} \sigma_2^2 (r - r_0)\, dr + \int_{r_0}^{r_2} \sigma_2^2 (r - r_0) dr. \tag{3.2}$$

The physical components of the stress deviator $\mathbf{s}^{ij}$ are found to be

$$s_1^1 = -s_2^2 = \phi\left(\frac{1}{A^2r^2} - A^2r^2\right), \quad s_3^3 = 0. \tag{3.3}$$

The solution (3.1) applies until the stresses remain within the yield surface (2.7). To be specific we assume that the considered hyperelastic material complies with the Huber-Mises criterion

$$s^i_j s^j_i = \frac{2}{3}\sigma_0^2, \tag{3.4}$$

where σ_0 stands for the yield stress in the deformed configuration i.e. the "true" yield stress in tension.

Making use of (3.3) and of the last relation (2.10), we obtain

$$3\phi^2\left(\frac{r_1 r_2}{r^2}-\frac{r^2}{r_1 r_2}\right)^2 = \sigma_0^2. \tag{3.5}$$

An analogous relation is obtained when the criterion of maximum shear stress is used, except that the right-hand side constant takes a different numerical value.

To obtain explicitly the stresses and to evaluate the shape of the strip at the outset of yielding, the stored energy function W has to be specified. For simplicity, we assume a truncated form of the Murnaghan potential, namely

$$W = \frac{\lambda+2\mu}{8}(I_1-3)^2+\mu(I_1-3)-\frac{\mu}{2}(I_2-3), \tag{3.6}$$

where λ and μ denote the Lamé constants of an infinitesimal theory. The relation (3.6) means that the strain energy function is of the form known from the linear elasticity (i.e., it is quadratic in terms of the strain tensor components), except that the strains are no longer considered infinitesimal. The potential (3.6) defines the considered hyperelastic material.

In the case of strip bending, Eq. (3.6) takes the form

$$W = \frac{\lambda+2\mu}{8}\left(A^2r^2+\frac{1}{A^2r^2}-2\right)^2+\frac{\mu}{2}\left((A^2r^2+\frac{1}{A^2r^2}-2\right) \tag{3.7}$$

and it follows further from (2.5) that

$$\phi = \frac{\lambda+2\mu}{2}\left(A^2r^2+\frac{1}{A^2r^2}-2\right)+\mu. \tag{3.8}$$

The yield condition (3.5) becomes

$$\left(\frac{\lambda}{2}+\mu\right)\left(A^4r^4-\frac{1}{A^4r^4}\right)+(\lambda+\mu)\left(A^2r^2-\frac{1}{A^2r^2}\right) = \pm k, \tag{3.9}$$

where $k = \sigma_0/\sqrt{3}$. It can be established that (3.9) takes extremal values simultaneously at r_1 and r_2. The outer fibers thus yield at the same instant.

The strip geometry at the outset of yielding is specified by the relation

$$\left(\frac{\lambda}{2}+\mu\right)\left(\frac{r_1^2}{r_2^2}-\frac{r_2^2}{r_1^2}\right)+(\lambda+\mu)\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right) = k. \tag{3.10}$$

In view of the relations (2.10) and denoting

$$s\alpha_0+\sqrt{1+s^2\alpha_0^2} = x, \quad P = \frac{2(\lambda+\mu)}{\lambda+2\mu}, \quad Q = \frac{2k}{\lambda+2\mu}, \tag{3.11}$$

the following equation is eventually obtained, involving the initial slenderness s and the sectional angle α_0 at yield

$$x^4 - Px^3 - Qx^2 + Px - 1 = 0. \tag{3.12}$$

The results computed for two sets of material constants λ, μ and k are plotted in Fig. 2.

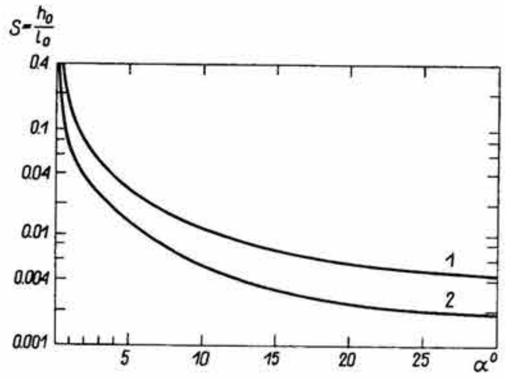

FIG. 2. Outset of yielding: 1) stell, 2) aluminum.

The bending moment within the elastic range is found from (3.3), while making use of (3.1), (3.6) and (3.8)

$$\frac{M}{h_0^2} = \frac{\lambda+2\mu}{8s^2\alpha^2}(B^3+B^{-3}) - \frac{4\lambda+5\mu}{8s^2\alpha^2}(B^2+B^{-2}) + \frac{7\lambda+10\mu}{8s^2\alpha^2}(B+B^{-1}) - \frac{\lambda+2\mu}{10s^2\alpha^2}\times \tag{3.13}$$

$$\times(B^{5/2}+B^{-5/2}) + \frac{\lambda}{6s^2\alpha^2}(B^{3/2}+B^{-3/2}) + \frac{\lambda+\mu}{s^2\alpha^2}(B^{1/2}+B^{-1/2}) - \frac{1}{s^2\alpha^2}\left(\frac{47}{15}\lambda + \frac{67}{20}\mu\right)$$

where $B = s\alpha + \sqrt{1+s^2\alpha^2}$.

4. Elastic-plastic bending

In bending to an angle $\alpha > \alpha_0$ plastic zones develop within the range $\varrho_1 \leqslant r \leqslant r_1$ and $r_2 \leqslant r \leqslant \varrho_2$, Fig. 1. The yield condition (3.4) allows to integrate the equilibrium equation (2.8) in the plastic zones since $\sigma_1^1 - \sigma_2^2 = \text{const}$. Under the stress boundary condition (2.9) the results are, HILL [6],

$$\begin{aligned} &\sigma_1^1 = -2k\ln\frac{r_1}{r}, \quad \sigma_2^2 = 2k\left(1-\ln\frac{r_1}{r}\right), \quad \varrho_1 \leqslant r \leqslant r_1,\\ &\sigma_1^1 = -2k\ln\frac{r}{r_2}, \quad \sigma_2^2 = -2k\left(1+\ln\frac{r}{r_2}\right), \quad r_2 \leqslant r \leqslant \varrho_2, \end{aligned} \tag{4.1}$$

and $2\sigma_3^3 = \sigma_1^1 + \sigma_2^2$.

Within the elastic zone the stresses are to be determined from the strain energy function. The formulas (3.1) apply for $\varrho_2 \leqslant r \leqslant \varrho_1$, except that the constant has to be appropriately evaluated from a condition on the elastic-plastic interface. At $r = \varrho_1$ and $r = \varrho_2$ the radial stress continuity requirement, $[\sigma_1^1] = 0$, and the yield condition (3.5) furnish the necessary relations to determine the radii of elastic-plastic boundaries and the respective constant in (3.1),

$$\varrho_1 = \frac{l_0}{\alpha}\xi, \quad \varrho_2 = \frac{l_0}{\alpha}\frac{1}{\xi}, \quad C = -2k\ln\frac{r_1}{\varrho_1} - W(\varrho_1), \tag{4.2}$$

where $\xi^2 = x$ and x denotes the solution of (3.12).

Computations show that there is no tensile force on the faces $\theta = \pm\alpha/2$ in the elastic-plastic range as well. The resulting stress couple of pure bending is given by the following expression

$$\begin{aligned} \frac{M}{h_0^2} = \frac{k}{2s^2\alpha^2}\left[2\sqrt{s^2\alpha^2+1} - (\xi^2+\xi^{-2}) + (\xi^2+3\xi^{-2}-8\xi^{-1}+4)\ln\frac{\xi}{\sqrt{B}}\right] \\ + \frac{\lambda+2\mu}{8s^2\alpha^2}(\xi^6+\xi^{-6}) - \frac{\lambda+2\mu}{10s^2\alpha^2}(\xi^5+\xi^{-5}) - \frac{4\lambda+5\mu}{8s^2\alpha^2}(\xi^4+\xi^{-4}) + \frac{\lambda}{6s^2\alpha^2}(\xi^3+\xi^{-3}) \\ + \frac{7\lambda+10\mu}{8s^2\alpha^2}(\xi^2+\xi^{-2}) + \frac{\lambda+\mu}{s^2\alpha^2}(\xi+\xi^{-1}) - \frac{1}{s^2\alpha^2}\left(\frac{47}{15}\lambda + \frac{67}{20}\mu\right), \end{aligned} \tag{4.3}$$

where B as in (3.13).

The above result passes into (3.13) when $\varrho_1 \to r_1$ and $\varrho_2 \to r_2$. If $\varrho_1 \to \varrho_2$ the bending moment tends asymptotically to the value

$$M = \frac{kh^2}{2}, \tag{4.4}$$

corresponding to a perfectly plastic material, HILL [6].

The strip changes its thickness according to the law

$$h = \frac{l_0}{\alpha}\left(\sqrt{s\alpha + \sqrt{1+s^2\alpha^2}} - \sqrt{-s\alpha + \sqrt{1+s^2\alpha^2}}\right) \tag{4.5}$$

as originally established by BRUHNS [1]. The variation of thickness is independent of the strain energy function.

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