Flow equations of a multiphase mixture with one coherent liquid or gaseous phase

A. SZANIAWSKI (WARSZAWA)

THE FLOW of a multicomponent and multiphase mixture is considered. The mixture is composed of one volumetrically dominant, coherent, gaseous or liquid phase, in which other phases are dispersed in a form of many fractions of particles. After formulating the basic notions and assumptions, the equations for mass, momentum and energy transport were deduced for an element of mixture and of each fraction separately. It was assumed that the perturbation of local equilibrium is sufficiently small to allow linearisation of the equations with respect to the parameters of perturbation.

Rozpatrywany jest przepływ modelu mieszaniny wieloskładnikowej i wielofazowej o jednej ciekłej lub gazowej fazie spójnej, w której rozproszone są w niewielkiej objętościowo ilości pozostałe fazy w postaci wielu frakcji. Po wprowadzeniu podstawowych pojęć i sformułowaniu założeń wyprowadzone zostały równania przenoszenia masy, pędu i energii dla infinitezymalnego elementu mieszaniny oraz dla każdej z frakcji. Założono, że zaburzenie lokalnego stanu równowagi jest niewielkie i dopuszczalna jest linearyzacja równań ze względu na parametry zaburzenia.

Рассматривается течение модельной многокомпонентной смеси многофазного типа, содержащей одну жидкую или газовую сплошную фазу, в которой рассеяны в небольшом объемном количестве остальные фазы в виде многих фракций. Сформулированы основные понятия и предположения теории, на основе которых выведены уравнения переноса масс, количества движения и энергии для бесконечно малого элемента смеси и для каждой из фракций. Предполагается, что отклонения от локального состояния равновесия невелики и допустима линеаризация уравнений по отношению к параметрам возмущения.

1. Introduction

FLOWS of multiphase and multicomponent mixtures are accompanied by various physico-chemical phenomena which present considerable difficulties as regards theoretical description. The phases are separated by surfaces having different geometrical shapes and dimensions. The velocities of elements of adjacent phases are not necessarily the same, and between the phases may occur interchanges of mass, momentum and energy, also not necessarily in equilibrium conditions. These interchanges in the microscale have an important influence on the macroscopic behaviour of a mixture, but on the other hand they may be influenced by other microscopic phenomena, often of a stochastic character, which causes additional deficiency of information necessary to determine the real phenomena.

The complexity and unsatisfactory state of knowledge of the properties of real mixtures necessitates the introduction of very simplified models, based on additional assumptions, taking into account the most important factors and disregarding others of secondary importance.

The very large number of models of multiphase mixtures used for various purposes will not be discussed here (see for instance [1] to [11]). The range of application of such models is always restricted to specified classes of real mixture and real phenomena.

It is not the intention to present in this paper equations applicable to a very large class of mixtures and flows. A limited class of mixtures with one volumetrically dominant phase (such as aerosol, fog, smoke, bubble mixture etc.) will be considered. For a theoretical model of such mixtures, the equations of flow will be deduced. There may also exist certain restrictions concerning the application of these equations to flow problems.

The multicomponent and multiphase mixture considered here is composed of S chemically nonreacting components dissolved in $F+1$ phases. One liquid or gaseous phase is coherent and volumetrically dominant, and the other, F phases are uniformly dispersed in it in the form of a large number of small "particles" of solid, liquid or gaseous phase. These particles are divided into N sorts called here "fractions", each fraction containing identical particles in the same physical state. The number of fractions N may be larger than the number of F dispersed phases, so that the same phase may be dispersed in different fractions (for instance, as drops of different dimensions).

The particles are very small, but they are also sufficiently distant one from another so that the interaction between them may be disregarded, and only the interaction between isolated particles with the surrounding coherent phase is taken into account.

The distance between particles is very large by comparison with the dimensions of particles, but it is very small by comparison with characteristic lengths of the flow. This assumption enables us to regard our model of a multiphase mixture as a continuous medium with an internal "fractional" structure.

In non-equilibrium state, the interaction of adjacent phases causes perturbations of one phase by another, so that the parameters of flow are in general not homogeneous within each phase. In the model of a mixture considered it will be assumed, however, that the internal inhomogeneity of parameters within all particles and within the coherent phase may be disregarded, and that a finite number of parameters, such as in the one-phase case, suffices to determine the local state of each fraction and of the coherent phase. Taking account the interaction between adjacent phases, it will be assumed, however, that the influence of surface tension phenomena may be disregarded. The surface tension causes and additional increase of energy proportional to the size of the contact surface, and on both sides this surface it causes a difference of pressures proportional to the mean curvature. In very highly dispersed phases, the dimensions of particles may be very small, and consequently the surface energy and the pressure differences may be important. But in cases of practical interest the quantitative contribution of these phenomena is of secondary importance, and therefore they will here be disregarded entirely.

The particles are also assumed to be sufficiently large, so that no brownian motion is taken into account.

Two of the more obvious ways of classifying particles into N fractions are: 1) a fraction consists of identical particles with common history, and 2) a fraction consists of particles with different histories but common essential physical and geometrical characteristics, such as for instance drop radii.Since no interchange of particles between fractions will be taken into account, essentially only the first case will be considered.

After formulating the basic assumptions, we shall deduce for the model of a mixture introduced the transport equations of mass, momentum and energy. These macroscopic transport equations will contain certain unknown fluxes of mass momentum and energy from the coherent phase to microscopic particles. To determine these fluxes, other supplementary equations, based on the principles of irreversible thermodynamics, will be deduced. The phenomenological kinetic coefficients involved should be determined from the analysis of interaction phenomena between adjacent phases. This analysis will not be presented here. The sets of equations: macroscopic transport equations and microscopic flux equations constitute a system of flow equations, which, supplemented by constitutive equations (equations of thermodynamical state and formulae for kinetic coefficients), describe the local motion of the model of a mixture considered.

2. Auxiliary notions and basic assumptions

The parameters describing the properties of the multicomponent and multiphase mixture considered will be denoted by indices as follows. The lower index will denote the s components, $s = 1, 2, ..., S$, the upper index—the N fractions of particles $n = 1, 2, ..., N$, or the F dispersed phases $N_f, f = 1, 2, ..., F$. The upper index $n = N_0 = 0$ will be reserved for the volumetrically dominant coherent phase, so that the total number of phases will be equal to $F+1$. The different fractions of particles may belong to the same dispersed phase according to the rule:

if $1 \leqslant n \leqslant N_1$, the n-th fraction belongs to the 1-st phase,

if $N_1+1 \leqslant n \leqslant N_2$, n-th fraction belongs to the 2-nd phase,

. . . .

if $N_{F-1}+1 \leqslant n \leqslant N_F = N$, the n-th fraction belongs to the F-th phase.

A volume V of the mixture is equal to the sum of the volumes of all fractions or phases:

$$(2.1) \qquad V = V^0 + \sum_{n=1}^{N} V^n = V^0 + \sum_{f=1}^{F} V^{N_f}.$$

The mass M of this volume is distributed between S components, $N+1$ fractions or $F+1$ phases:

$$M = \sum_{s=1}^{S} M_s = \sum_{n=0}^{N} \sum_{s=1}^{S} M_s^n = \sum_{n=0}^{N} M^n = \sum_{f=0}^{F} M^{N_f},$$

$$(2.2) \qquad M_s = \sum_{n=0}^{N} M_s^n, \qquad M^n = \sum_{s=1}^{S} M_s^n, \qquad \begin{matrix} n = 0, 1, 2, ..., N, \\ s = 1, 2, ..., S, \end{matrix}$$

$$M^{N_f} = \sum_{n=N_{f-1}+1}^{N_f} M^n, \qquad f = 1, 2, ..., F,$$

where M_s^n is the mass of the s-th component in the n-th fraction, and M_s, M^n, M^{N_f} are the total masses of s-th component, n-th fraction and f-th phase, respectively.

The following mass and volume ratios will be introduced:

$$\xi_s^n = \frac{M_s^n}{M^n}, \quad \xi_s = \frac{M_s}{M}, \quad \xi^n = \frac{M^n}{M}, \quad \xi^{N_f} = \frac{M^{N_f}}{M},$$

(2.3)
$$\varphi^n = \frac{V^n}{V}, \quad \varphi^{N_f} = \frac{V^{N_f}}{V},$$

$$\varphi = 1 - \varphi^0 = \sum_{n=1}^{N} \varphi^n = \sum_{f=1}^{F} \varphi^{N_f}.$$

The density ϱ of the mixture and the densities ϱ^n of each fraction are given by the formulae:

(2.4)
$$\varrho = \frac{M}{V}, \quad \varrho^n = \frac{M^n}{V^n}.$$

If the energy of surface phenomena (surface tension) is disregarded, the internal specific energy of a mixture at rest is equal to the sum of the energies of all fractions:

(2.5)
$$e = \sum_{n=0}^{N} \xi^n e^n, \quad e^n = e^n(p, T^n, \xi_1^n, \xi_2^n, \dots, \xi_S^n),$$

where the specific energy of fraction e^n depends on the parameters of state (p-pressure, T^n—temperature of the n-th fraction), and on the mass ratios ξ_s^n of the n-th fraction only. Without taking into account the surface tension, the pressure p in all phases is the same.

The following evident relations are valid:

(2.6)
$$\sum_{s=1}^{S} \xi_s = 1, \quad \sum_{n=0}^{N} \xi^n = 1, \quad \sum_{f=0}^{F} \xi^{N_f} = 1,$$

$$\sum_{s=1}^{S} \xi_s^n = 1, \quad \sum_{n=0}^{N} \xi^n \xi_s^n = \xi_s,$$

and

(2.7)
$$\varphi^n \varrho^n = \xi^n \varrho,$$

$$\varrho = \sum_{n=0}^{N} \varphi^n \varrho^n, \quad \frac{1}{\varrho} = \sum_{n=0}^{N} \frac{\xi^n}{\varrho^n}.$$

According to assumption, the coherent phase is volumetrically dominant and must be:

(2.8)
$$\varphi^n < \varphi \ll 1, \quad n = 1, 2, \dots, N.$$

The number od parameters determining the state of mixture is very large. We may enumerate here the mass ratios and the parameters of state of each fraction: ξ^n, p, T^n, ξ_1^n, ξ_2^n, ..., ξ_S^n ($n = 0, 1, 2, \dots, N$), which gives us the number of $(N+1)(S+2)+1$ intensive physical quantities. But certain relations may exist between them, and thus not all these parameters are independent.

In the state of equilibrium, the number of independent parameters characterising the state of a mixture is minimal. Three kinds of equilibrium are here to be considered: mechanical, thermal and phasic. In mechanical equilibrium, the mixture may move uniformly

with a constant velocity $\mathbf{u}$ and no relative motion exists between phases. In thermal equilibrium, the temperature of all phases is the same, $T^n = T$. In phase equilibrium, the chemical potentials of each s-th component are in all $F+1$ phases equal one to another. Taking into account the chemical identity of certain fractions of particles, we obtain[1]:

$$\bar{\xi}_s^n = \bar{\xi}_s^{N_f}, \quad n = N_{f-1}+1, N_{f-1}+2, \ldots N_f,$$

(2.9)

$$\mu_s^{N_f}(\bar{p}, T, \bar{\xi}_1^{N_f}, \bar{\xi}_2^{N_f}, \ldots \bar{\xi}_S^{N_f}) = \mu_s^0(\bar{p}, T, \bar{\xi}_1^0, \bar{\xi}_2^0, \ldots \xi_S^0) \equiv \mu_s.$$

Since the parameters of the state of equilibrium of each phase are also not independent, the problem arises of choosing a set of independent parameters of the state of equilibrium of a mixture.

For all $(F+1)$ $S+2$ state parameters of equilibrium of each phase $\bar{p}$, T, $\bar{\xi}_1^{N_f}$, $\bar{\xi}_2^{N_f}$, $\ldots \bar{\xi}_S^{N_f}$ ($f = 0, 1, 2, \ldots F$), and for $N+1$ mass ratios ξ^n ($n = 0, 1, 2, \ldots N$), we obtain from (2.6) and (2.9) the following $(F+1)+SF+1$ relations:

$$\sum_{s=1}^{S} \bar{\xi}_s^{N_f} = 1,$$

(2.10)

$$\mu_s^{N_f}(\bar{p}, T, \bar{\xi}_1^{N_f}, \bar{\xi}_2^{N_f}, \ldots \bar{\xi}_S^{N_f}) = \mu_s^0(\bar{p}, T, \bar{\xi}_1^0, \bar{\xi}_2^0, \ldots \bar{\xi}_E^0),$$

$$\bar{\xi}^0 = 1 - \sum_{n=1}^{N} \bar{\xi}^n = 1 - \sum_{f=1}^{F} \bar{\xi}^{N_f}.$$

The number of independent parameters (degrees of freedom) of a mixture in equilibrium will be equal to the difference between all $FS+N+S+3$ equilibrium parameters and all $FS+F+2$ equilibrium Eqs. (2.10). Thus, we obtain the number $N+S-F+1$ of independent parameters of the state of equilibrium. From these $N+S-F+1$ parameters, N mass ratios $\bar{\xi}^1$, $\bar{\xi}^2$, $\ldots \bar{\xi}^N$ characterise the fractional composition of the mixture, and other $S-F+1$ parameters, resulting from the Gibbs rule, may be chosen, for instance, from $S+1$ quantities $\bar{p}$, T, $\bar{\xi}_1^0$, $\bar{\xi}_2^0$, $\ldots \bar{\xi}_{S-1}^0$. The remaining dependent quantities may now be determined by the equilibrium Eqs. (2.9) and (2.10) or other general formulae (2.3), (2.6), (2.7).

During the flow of a mixture, its equilibrium state is in general perturbed and a perturbed state of an infinitesimal element is determined by a larger number of independent parameters. Now arises an analogous problem—how to choose a set of independent parameters for nonequilibrium states of the model of a mixture considered.

It should be noted that the model employed here assumes small perturbations of the state of equilibrium.

Hence, to each perturbed state we shall subordinate a reference state of equilibrium, with its smaller number of independent parameters, and then we shall introduce the remaining parameters characterising the perturbation state. These parameters of perturbation

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[1] The values of the mass ratios ξ_s^n, $\xi_s^{N_f}$, ξ^n, and of the pressure p for equilibrium conditions will be denoted by bars: $\bar{\xi}_s^n$, $\bar{\xi}_s^{N_f}$, $\bar{\xi}^n$, $\bar{p}$.

should disappear in equilibrium. The equations of flow will be linearised in respect to the parameters of perturbation.

In this way, the $N+1$ velocities of each fraction $\mathbf{u}^0, \mathbf{u}^1, \ldots \mathbf{u}^N$ will be replaced by one reference velocity $\mathbf{u}$ of the centre of mass of the mixture, and N independent velocities $\Delta\mathbf{w}^n$ characterising the relative motion between phases:

$$\mathbf{u} = \sum_{n=0}^{N} \xi^n \mathbf{u}^n,$$

(2.11) $$\mathbf{w}^n = \mathbf{u}^n - \mathbf{u}, \qquad \sum_{n=0}^{N} \xi^n \mathbf{w}^n = 0,$$

$$\Delta\mathbf{w}^n = \mathbf{w}^n - \mathbf{w}^0 = \mathbf{u}^n - \mathbf{u}^0$$

(rejecting higher order terms, we may introduce here $\xi^n \approx \bar{\xi}^n$).

Also, after introducing $N+S-F+1$ thermodynamic parameters of the reference state of equilibrium, we shall choose the parameters of perturbation from the set of $1+N+ +1+(N+1)S+(N+1)$ differences:

(2.12)
$$\Delta p = p - \bar{p}, \qquad \Delta T^n = T^n - T^0, \qquad \Delta T^0 = T^0 - T,$$
$$\Delta\xi_s^n = \xi_s^n - \bar{\xi}_s^n, \qquad \Delta\xi^n = \xi^n - \bar{\xi}^n.$$

Since the state and the mass ratios of each phase determine the state of the mixture as a whole, the $(N+1)(S+2)+1$ parameters $\xi^n, p, T^n, \xi_1^n, \xi_2^n, \ldots \xi_S^n$ of each phase suffice to describe the behaviour of the mixture as a whole. Taking into account the $1+(1+N)$ relations (2.6)

$$\sum_{n=0}^{N} \xi^n = 1, \qquad \sum_{s=1}^{S} \xi_s^n = 1,$$

we obtain the total number of $(N+1)(S+1)$ independent parameters of the perturbed thermodynamic state. The total number $(N+1)(S+1)$ must be equal to the sum of the number $N+S-F+1$ of the equilibrium parameters and the number $NS+F$ of the perturbation parameters. Hence, from the $(N+1)(S+2)+1$ differences $\Delta\bar{\xi}^n, \Delta p, \Delta T^n, \Delta\xi_s^n$, fulfilling the $S+N+1$ linear relations resulting from (2.6)

(2.13) $$\sum_{s=1}^{S} \Delta\xi_s^n = 0, \qquad \sum_{n=0}^{N} (\bar{\xi}^n \Delta\xi_s^n + \bar{\xi}_s^n \Delta\xi^n) = 0,$$

only $NS+F$ may be chosen as parameters of perturbation. The remaining

$$[(N+1)(S+2)+1] - (S+N+1) - (NS+F) = N-F+2$$

differences may be chosen arbitrarily—for instance, as equal to zero.

The manner of choosing the thermodynamic parameters of equilibrium and of perturbed state depends in part on the properties of the mixture considered. It is convenient to choose the mean temperature and the real pressure p as the reference parameters $T, \bar{p}$, but it is not always possible so to do.

For instance, the pressure $\bar{p}$ and the temperature T of a one-component $(S = 1)$ three-phase mixture $(F = 2)$ in equilibrium are given constants, and so for this case the mass ratios $\xi^1, \xi^2, \ldots \xi^N$ $(N \geqslant F, F = 2)$ and the differences Δp, $\Delta T^0, \Delta T^1, \ldots \Delta T^N$ might here be chosen as the N reference equilibrium and $N+2$ perturbation parameters. The remaining N differences $\Delta \xi^1, \Delta \xi^2, \ldots \Delta \xi^N$ may be chosen equal to zero.

In the transport equations, for an infinitesimal element of a mixture some unknown fluxes will appear from the coherent phase to each dispersed fraction. These fluxes: of the mass of each component X_s^n, of momentum $\mathbf{Y}^n$, and of energy Z^n will here be defined as the quantities transferred during a unity of time into a unity of mass of the n-th dispersed fraction.

Since the fluxes of momentum and energy may depend on the velocity of the element considered, the fluxes $\mathbf{Y}^n$ and Z^n introduced here, are defined in the system of reference as moving with the n-th dispersed fraction.

The vector notation will be used, and the product of vectors or tensors will be designated with or without a point, according to the following examples:

$\mathbf{a} \cdot \mathbf{b}$ — scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$,

$\mathbf{ab}$ — diadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$,

$\mathbf{A} \cdot \mathbf{a}$ — contracted product of a tensor $\mathbf{A}$ and a vector $\mathbf{a}$,

$\mathbf{A} : \mathbf{B}$ — double contracted product of two tensors $\mathbf{A}$ and $\mathbf{B}$.

3. Transport equations

Let us consider an element of volume V bounded by a moving surface A. We shall now deduce transport equations for the mixture as a whole contained in the volume V, assuming that the surface A is convected by the velocity field $\mathbf{u}^0$ of the coherent phase. Next, we shall deduce transport equations for the n-th dispersed fraction, assuming that the surface A is convected by the velocity field $\mathbf{u}^n$.

3.1. Transport of mass of the s-th component in the total volume of the mixture

The mass of the s-th component in the volume V may change due to the convection of dispersed fractions through the surface A convected with the velocity field $\mathbf{u}^0$

$$(3.1) \qquad \frac{d}{dt} \int_V \varrho \xi_s \, dV = - \sum_{n=1}^N \oint_A \varrho \xi^n \xi_s^n (\mathbf{u}^n - \mathbf{u}^0) \cdot d\mathbf{A}.$$

The infinitesimal area vector $d\mathbf{A}$ of an element of surface A, moving with velocity $\mathbf{u}^0$, is here directed outward.

Applying the formula

$$(3.2) \qquad \frac{d}{dt} \int_V \tau \, dV = \int_V \left[\frac{\partial \tau}{\partial t} + \nabla \cdot (\mathbf{u}_0 \tau) \right] dV,$$

known from field theory, for an arbitrary scalar, vector or tensor quantity τ, and the Gauss theorem, we may transform (3.1) to the form:

$$\int\limits_V \left\{ \frac{\partial(\varrho\xi_s)}{\partial t} + \nabla\cdot(\mathbf{u}^0\varrho\xi_s) + \sum_{n=1}^{N} \nabla\cdot[(\mathbf{u}^n-\mathbf{u}^0)\varrho\xi^n\xi_s^n] \right\} dV = 0.$$

Since this equation is valid for arbitrary volume V, its subintegral expression should be equal to zero:

$$\frac{\partial(\varrho\xi_s)}{\partial t} + \nabla\cdot\left\{ \varrho\left[\mathbf{u}^0\xi_s + \sum_{n=1}^{N} (\mathbf{w}^n-\mathbf{w}^0)\xi^n\xi_s^n \right] \right\} = 0.$$

Taking into account (2.6), we obtain from it the differential equation:

$$(3.3) \qquad \frac{\partial(\varrho\xi_s)}{\partial t} + \nabla\cdot\left[\varrho\left(\mathbf{u}\xi_s + \sum_{n=0}^{N} \mathbf{w}^n\xi^n\xi_s^n \right) \right] = 0.$$

This set of equations for $s = 1, 2, \ldots S$ may still be transformed as follows. From the sum of all S Eqs. (3.3), taking into account (2.6) and (2.11), we obtain the continuity equation of a mixture:

$$(3.4) \qquad \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\mathbf{u}) = 0.$$

Substracting this equation from (3.3), we obtain the S linearly dependent equations

$$(3.5) \qquad \frac{\partial\xi_s}{\partial t} + \mathbf{u}\cdot\nabla\xi_s = -\frac{1}{\varrho}\sum_{n=0}^{N} \nabla\cdot(\varrho\mathbf{w}^n\xi^n\xi_s^u)$$

which, with the continuity Eq. (3.4), costitute the desired set of transport Eqs. of the mass of each component.

3.2. Transport of momentum in the total volume of the mixture

The increment of momentum in the volume V of the mixture is due to the convection of momentum by all dispersed fractions through the surface A and to the forces acting on the surface A and on the volume V of the element considered:

$$(3.6) \qquad \frac{d}{dt}\int\limits_V \sum_{n=0}^{N} \varrho\xi^n\mathbf{u}^n dV = -\sum_{n=1}^{N} \oint\limits_A \varrho\xi^n\mathbf{u}^n[(\mathbf{u}^n-\mathbf{u}^0)\cdot d\mathbf{A}] + \oint\limits_A \boldsymbol{\tau}^0\cdot d\mathbf{A} + \int\limits_V \varrho\mathbf{f}dV.$$

On the basis of (2.8) $\varphi\ll 1$, the resulting force acting on the element $d\mathbf{A}$ of the surface A is here assumed to be equal to the contracted product of the area vector $d\mathbf{A}$ and the stress tensor $\boldsymbol{\tau}^0$ in the coherent phase only (the perturbing influence of the dispersed phases on the stress tensor is disregarded); $\mathbf{f}$ denotes the mass force.

By proceeding in the same way as in 3.1, we may from (3.6), after some transformations, obtain the differential equation:

$$\frac{\partial(\varrho\mathbf{u})}{\partial t} + \nabla\cdot(\varrho\mathbf{u}\mathbf{u}) = -\nabla\cdot\sum_{n=0}^{N}(\varrho\xi^n\mathbf{w}^n\mathbf{w}^n) + \nabla\cdot\mathbf{\tau}^0 + \varrho\mathbf{f},$$

which, by means of the continuity Eq. (3.4), may be transformed into the classical form of the momentum equation of a fluid:

$$(3.7) \qquad \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{\varrho}\nabla\cdot\mathbf{\tau} + \mathbf{f},$$

where $\mathbf{\tau}$ denotes the stress tensor of a mixture

$$(3.8) \qquad \mathbf{\tau} = \mathbf{\tau}^0 - \sum_{n=0}^{N}\varrho\xi^n\mathbf{w}^n\mathbf{w}^n.$$

It should be emphasised, however, that the stress tensors $\mathbf{\tau}$ and $\mathbf{\tau}^0$ are equal, in linear approximation, in respect to the relative velocities $\mathbf{w}^n$.

3.3. Transport of energy in the total volume of the mixture

The increment of energy in the volume V of the mixture results from the convection of energy by all dispersed fractions through the surface A, the conduction of heat, and the work done by all forces acting on the surface and on the volume of element considered:

$$(3.9) \qquad \frac{d}{dt}\int_V\sum_{n=0}^{N}\varrho\xi^n\left(e^n+\frac{\mathbf{u}^n\cdot\mathbf{u}^n}{2}\right)dV = -\sum_{n=1}^{N}\oint_A\varrho\xi^n\left(e^n+\frac{\mathbf{u}^n\cdot\mathbf{u}^n}{2}\right)(\mathbf{u}^n-\mathbf{u}^0)\cdot d\mathbf{A}$$

$$-\oint_A\mathbf{q}^0\cdot d\mathbf{A} + \oint_A(\mathbf{\tau}^0\cdot d\mathbf{A})\cdot\mathbf{u}^0 + \sum_{n=0}^{A}\int_V\varrho\xi^n\mathbf{f}\cdot\mathbf{u}^n dV.$$

As before, according to the assumption (2.8) ($\varphi\ll 1$), the perturbing influence of the dispersed phases on the heat flux and on the work of stress tensor is disregarded, only the coherent phase being here taken into account.

By proceeding in the same way as before, we may obtain from (3.9) after certain tedious transformations, the differential equation:

$$(3.10) \qquad \frac{\partial}{\partial t}\left[\varrho\left(\tilde{e}+\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right)\right] + \nabla\cdot\left[\mathbf{u}\varrho\left(\tilde{e}+\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right)\right] = -\nabla\cdot\mathbf{q} + \nabla\cdot(\mathbf{\tau}\cdot\mathbf{u}+\mathbf{\tau}^0\cdot\mathbf{w}^0) + \varrho\mathbf{f}\cdot\mathbf{u},$$

where $\tilde{e}$ is the specific internal energy of mixture in motion [cf. (2.5)]:

$$(3.11) \qquad \tilde{e} = \sum_{n=0}^{N}\xi^n\left(e^n+\frac{\mathbf{w}^n\cdot\mathbf{w}^n}{2}\right) = e + \sum_{n=0}^{A}\xi_a^n\frac{\mathbf{w}^n\cdot\mathbf{w}^n}{2},$$

and $\mathbf{q}$ is the total heat flux resulting from the conduction and convection effects:

$$(3.12) \qquad \mathbf{q} = \mathbf{q}^0 + \varrho\sum_{n=0}^{N}\xi^n\left(e^n+\frac{\mathbf{w}^n\cdot\mathbf{w}^n}{2}\right)\mathbf{w}^n.$$

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Substracting from (3.10) Eq. (3.4) multiplied by $(\tilde{e} + \mathbf{u} \cdot \mathbf{u}/2)$, and Eq. (3.7) multiplied by $\varrho\mathbf{u}$, we obtain:

$$(3.13) \qquad \varrho\left(\frac{\partial \tilde{e}}{\partial t} + \mathbf{u} \cdot \nabla\tilde{e}\right) = -\nabla \cdot \mathbf{q} + \boldsymbol{\tau}:\nabla\mathbf{u} + \nabla \cdot (\boldsymbol{\tau}^0 \cdot \mathbf{w}^0).$$

Comparing the transport equations of energy for a mixture and for a one-phase fluid, we see that the difference between them consists not only in another meaning of the specific energy $\tilde{e}$ (3.11), the heat flux $\mathbf{q}$ (3.12) and the stress tensor $\boldsymbol{\tau}$ (3.8), but also in the appearance of an additional term $\nabla \cdot (\boldsymbol{\tau}^0 \cdot \mathbf{w}^0)$. If we consider the linear approximation in respect to the relative velocities, we obtain:

$$\tilde{e} \approx e, \qquad \mathbf{q} \approx \mathbf{q}^0 + \varrho\sum_{n=0}^{N} \xi^n e^n \mathbf{w}^n, \qquad \boldsymbol{\tau} \approx \boldsymbol{\tau}^0;$$

but the additional term $\nabla \cdot (\boldsymbol{\tau}^0 \cdot \mathbf{w}^0)$ should be taken into account, since it does not become negligibly small.

3.4. Transport of mass of the s-th component to the n-th dispersed fraction of the mixture

Let us consider now the volume V of a mixture convected with the velocity field $\mathbf{u}^n$ of the n-th dispersed fraction.

The increment of mass of the s-th component in the n-th dispersed fraction, contained in the volume V, is equal to the total mass flux through the contact surfaces with the coherent phase:

$$(3.14) \qquad \frac{d}{dt}\int\limits_{V} \varrho\xi^n\xi_s^n dV = \int\limits_{V} \varrho\xi^n X_s^n dV.$$

Applying the formula (3.2) with $\mathbf{u}^n$ instead of $\mathbf{u}^0$, and comparing the subintegral expressions, we obtain the equation

$$\frac{\partial}{\partial t}(\varrho\xi^n\xi_s^n) + \nabla \cdot (\varrho\mathbf{u}^n\xi^n\xi_s^n) = \varrho\xi^n X_s^n,$$

which, by substracting the continuity Eq. (3.4) multiplied by $\xi^n\xi_s^n$, may be transformed to the form:

$$(3.15) \qquad \frac{\partial}{\partial t}(\xi^n\xi_s^n) + \mathbf{u}^n \cdot \nabla(\xi^n\xi_s^n) = \xi^n X_s^n - \xi^n\xi_s^n \frac{\nabla \cdot (\varrho\mathbf{w}^n)}{\varrho}.$$

This system of NS equations may be presented in another form by introducing the total flux of mass flowing to the n-th fraction:

$$(3.16) \qquad X^n = \sum_{s=1}^{S} X_s^n, \qquad n = 1, 2, \dots N.$$

Adding all S equations for each n, and taking into account (2.6), we obtain an equation

of the total transport of mass flowing to the n-th fraction:

$$(3.17) \qquad \frac{\partial \xi^n}{\partial t} + \mathbf{u}^n \cdot \nabla \xi_4^n = \xi^n X^n - \xi^n \frac{\nabla \cdot (\varrho \mathbf{w}^n)}{\varrho}, \qquad n = 1, 2, \dots N.$$

Substracting now from (3.15) Eq. (3.17) multiplied by ξ_s^n, we obtain a system of NS linearly dependent equations (each sum of S equations for $n = 1, 2, \dots N$ becomes an identity) of the mass transport of the s-th component to the n-th dispersed fraction:

$$(3.18) \qquad \frac{'\partial \xi_s^n}{\partial t} + \mathbf{u}^n \cdot \nabla \xi_s^n = X_s^n - \xi_s^n X^n.$$

3.5. Transport of momentum to the n-th dispersed fraction of the mixture

Before deducing the transport equation of momentum, we shall consider the fluxes of momentum to the dispersed fraction in different reference systems.

In the reference system connected with the n-th fraction, the fluxes of mass momentum are denoted by X^n and Y^n. In different reference system, in which the n-th fraction moves with the velocity $\mathbf{u}^n$, the flux of mass X^n remains the same, but the flux of momentum may change. This change results from the different values of momentum of an element of mass transported from the coherent phase to the dispersed fraction in different systems of reference. If in the system of reference in which the fraction is at rest the fluxes of mass and momentum are X^n and Y^n, then, in the system of reference in which the fraction moves with velocity $\mathbf{u}^n$ the flux of mass X^n transfers with it also the additional flux of momentum $X^n \mathbf{u}^n$ and the total flux of momentum becomes equal to $Y^n + X^n \mathbf{u}^n$.

Taking into account this resulting flux of momentum and of the volume forces acting on the n-th fraction, we may calculate the increment of the momentum of the n-th fraction:

$$(3.19) \qquad \frac{d}{dt} \int_V \varrho \xi^n \mathbf{u}^n dV = \int_V \varrho \xi^n (\mathbf{Y}^n + X^n \mathbf{u}^n) dV + \int_V \varrho \xi^n \mathbf{f} dV.$$

Applying the formula (3.2), comparing the subintegral expressions, we obtain a differential equation, which, taking into account (3.4) and (3.17), may be transformed to the form:

$$(3.20) \qquad \frac{\partial \mathbf{u}^n}{\partial t} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n = \mathbf{Y}^n + \mathbf{f}.$$

3.6. Transport of energy to the n-th dispersed fraction of the mixture

The flux of energy to the dispersed phase may also depend on the velocity of the system of reference adopted.

All fluxes of mass X^n, momentum $\mathbf{Y}^n$, and energy Z^n were defined in the system of reference connected with the n-th fraction. In different reference system in which the n-th fraction moves with the velocity $\mathbf{u}^n$, the flux of energy Z^n will be increased by two additional terms, which may be found as follows.

Let us consider an element of mass dM^n being a part of $X^n dt$, and passing with its own velocity $\mathbf{c}$ from the coherent phase to the dispersed n-th fraction. The velocity $\mathbf{c}$ may be decomposed into two components $\mathbf{c} = \mathbf{u}^n + \mathbf{C}$, where $\mathbf{C}$ is the velocity of dM^n in the reference system conected with n-th fraction. The momentum and the kinetic energy in the two systems of reference—one moving with the n-th fraction and another in which the n-th fraction moves with the velocity $\mathbf{u}^n$—are related one to another by:

$$(dM_{\mathbf{s}}^n)\mathbf{c} = (dM^n)\mathbf{u}^n + (dM^n)\mathbf{C},$$

$$(dM^n)\frac{\mathbf{c}\cdot\mathbf{c}}{2} = (dM_{\mathbf{s}}^n)\frac{[\mathbf{u}^n\cdot\mathbf{u}^n}{2} + (dM^n)\mathbf{u}^n\cdot\mathbf{C} + (dM^n)\frac{\mathbf{C}\cdot\mathbf{C}}{2}$$

from which, by integrating in respect to dM^n, we obtain the previously presented formula for the total flux of momentum

$$X^n\mathbf{u}_{\mathbf{s}}^n + \mathbf{Y}^n,$$

and the formula for the total flux of energy:

$$X^n\frac{\mathbf{u}^n\cdot\mathbf{u}^n}{2} + \mathbf{Y}^n\cdot\mathbf{u}^n + Z^n.$$

Now, taking into account this resulting flux of energy, we may consider the increment of energy of the n-th fraction, resulting from the flux of energy from the coherent phase and from the work done by volume forces:

$$(3.21) \qquad \frac{d}{dt}\int_V \varrho\xi^n\left(e^n + \frac{\mathbf{u}^n\cdot\mathbf{u}^n}{2}\right)dV = \int_V \varrho\xi^n\left[\left(Z^n + \mathbf{Y}^n\cdot\mathbf{u}^n + X^n\frac{\mathbf{u}^n\cdot\mathbf{u}^n}{2}\right) + \mathbf{f}\cdot\mathbf{u}^n\right]dV.$$

Applying the formula (3.2), and comparing the subintegral expressions, we obtain n differential equations, which, taking into account (3.4), (3.20) and (3.17), may be transformed to the form:

$$(3.22) \qquad \frac{\partial e^n}{\partial t} + \mathbf{u}^n\cdot\nabla e_{\mathbf{s}}^n = Z^n - X^n e^n.$$

4. Entropy production

Into the transport equations obtained, there enter certain new unknown quantities: the fluxes of mass, momentum and energy between the coherent and the dispersed phases X_s^n, $\mathbf{Y}_s^n$, Z^n ($n = 1, 2, \ldots N$, $s = 1, 2, \ldots S$). These fluxes characterise the kinetic of dissipative phenomena, and disappear in equilibrium. To analyse the kinetic of dissipative phenomena, the method of entropy production of the thermodynamics of irreversible processes will be applied here.

Let us consider a volume V of the mixture convected by the velocity field $\mathbf{u}^0$ of the coherent phase. The increase of entropy

$$(4.1) \qquad \frac{d}{dt}\int_V \varrho s\, dV = \int_V\left[\frac{\partial(\varrho s)}{\partial t} + \nabla\cdot(\varrho s\mathbf{u}^0)\right]dV,$$

where

(4.2)
$$s = \sum_{n=0}^{N} \xi^n s^n,$$

(the additivity of entropy of each fraction is assumed) is only partly due to the flux of entropy through the surface A:

(4.3)
$$- \oint_A \frac{\mathbf{q}^0}{T^0} \cdot d\mathbf{A} - \oint_A \sum_{n=1}^{N} \varrho \xi^n s^n (\mathbf{u}^n - \mathbf{u}^0)\, d\mathbf{A} = \int_V \mathbf{\nabla} \cdot \left[-\frac{\mathbf{q}^0}{T^0} - \sum_{n=0}^{N} \varrho \xi^n s^n (\mathbf{w}^n - \mathbf{w}^0) \right] dV.$$

From the difference between the increase of entropy (4.1) and the flux of entropy (4.3), we obtain the nett volumetric entropy production:

(4.4)
$$\theta = \mathbf{\nabla} \cdot \frac{\mathbf{q}^0}{T^0} + \sum_{n=0}^{N} \left[\frac{\partial(\varrho \xi^n s^n)}{\partial t} + \mathbf{\nabla} \cdot (\varrho \xi^n s^n \mathbf{u}^n) \right], \quad \theta \geqslant 0,$$

positive on the basis of the second principle of thermodynamics.

Taking into account principle of thermodynamics and of all the transport equations previously deduced, we may transform this equation to the following form:

(4.5)
$$T^0 \theta = \left[-\mathbf{q}^0 \cdot \frac{\mathbf{\nabla} T^0}{T^0} + (\boldsymbol{\tau}^0 + \mathbf{I}p) : \mathbf{\nabla} \mathbf{u}^0 \right] + \sum_{n=1}^{N} \varrho \xi^n (\mathbf{w}^0 - \mathbf{w}^n) \cdot (\mathbf{Y}^n - \mathbf{w}^0 X^n)$$

$$+ \sum_{n=1}^{N} \sum_{s=1}^{S} \varrho \xi^n T^0 \left(\frac{\mu_s^0}{T^0} - \frac{\mu_s^n}{T^n} \right) X_s^n + \sum_{n=1}^{N} \varrho \xi^n \frac{T^0 - T^n}{T^n} \left(Z^n + \frac{p}{\varrho^n} X^n \right)$$

$$+ \sum_{n=1}^{N} \varrho \xi^n \left(\frac{\mathbf{w}^0 \cdot \mathbf{w}^0}{2} - \frac{\mathbf{w}^n \cdot \mathbf{w}^n}{2} \right) X^n$$

$$+ p \sum_{n=1}^{N} \left\{ \mathbf{\nabla} \cdot [\varphi^n (\mathbf{w}^n - \mathbf{w}^0)] - \varphi^n \left(\frac{\partial \varrho^n}{\varrho^n \partial t} + \mathbf{u}^n \cdot \frac{\mathbf{\nabla} \varrho^n}{\varrho^n} \right) \left(\frac{T^0}{T^n} - 1 \right) \right\},$$

where $\mathbf{I}$ denotes the unity tensor.

As might be expected, this form presents the sum of products of thermodynamic forces by thermodynamic fluxes, but the last term is here more difficult for thermodynamical interpretation. We should emphasize, however, that the model of a mixture introduced is based on certain simplifying assumptions, which may be a source of errors also in the entropy production (4.5). Since we assumed that the volume ratios φ^n are negligibly small (2.8), we have disregarded certain terms in the transport equations and, consequently, we should disregard the last term with $\varphi^n < \varphi \ll 1$ in the entropy production $T^0 \theta$ (4.5).

We shall disregard also the penultimate term in (4.5), because we shall consider only linear relations between thermodynamic forces and thermodynamic fluxes, and the linear part of the thermodynamic force $(\mathbf{w}^0 \cdot \mathbf{w}^0 - \mathbf{w}^n \cdot \mathbf{w}^n)$ is equal to zero.

The two first terms in brackets take into account the heat conduction and viscosity in the coherent phase, which will not be an object of our interest. Thus, we shall consider

here only the part θ_p (also non-negative) of entropy production due to the mass, momentum and energy fluxes between the coherent phase and the dispersed phases:

$$(4.6) \quad T^0\theta_p = \sum_{n=1}^{N}\sum_{s=1}^{S}\varrho\xi^n T^0\left(\frac{\mu_s^0}{T^0} - \frac{\mu_s^n}{T^n}\right)X_s^n + \sum_{n=1}^{N}\varrho\xi^n(\mathbf{w}^0 - \mathbf{w}^n)\cdot(\mathbf{Y}^n - \mathbf{w}^0 X^n)$$

$$+ \sum_{n=1}^{N}\varrho\xi^n\frac{T^0 - T^n}{T^n}\left(Z^n + \frac{p}{\varrho^n}X^n\right), \quad \theta_p \geqslant 0.$$

5. Onsager relations

To the entropy production (4.6) we may apply the principles of thermodynamics of irreversible processes, which enable us to deduce the linear Onsager relations between thermodynamical forces:

$$\varrho\xi^n(\mathbf{w}^0 - \mathbf{w}^n), \quad \varrho\xi^n T^0\left(\frac{\mu_s^0}{T^0} - \frac{\mu_s^n}{T^n}\right), \quad \varrho\xi^n\frac{T^0 - T^n}{T^n}$$

and thermodynamical fluxes

$$(\mathbf{Y}^n - \mathbf{w}^0 X^n), \quad X_s^n, \quad \left(Z^n + \frac{p}{\varrho^n}X^n\right).$$

We should remark, however, that in an isotropic medium no linear coupling between scalar and vector quantities may exist, and that in our model only the interaction between the coherent phase and the dispersed phases is taken into account. Assuming also that the postulate of the symmetry of Onsager coefficients is valid, we may present the linear relations between forces and fluxes in the form:

$$\mathbf{Y}^n - \mathbf{w}^0 X^n = A^n(\mathbf{w}^0 - \mathbf{w}^n),$$

$$(5.1) \quad Z^n + \frac{p}{\varrho^n}X^n = B^n\frac{T^0 - T^n}{T^n} + T^0\sum_{s=1}^{S}C_s^n\left(\frac{\mu_s^0}{T^0} - \frac{\mu_s^n}{T^n}\right),$$

$$X_s^n = C_s^n\frac{T^0 - T^n}{T^n} + T^0\sum_{\sigma=1}^{S}D_{s\sigma}^n\left(\frac{\mu_\sigma^0}{T^0} - \frac{\mu_\sigma^n}{T^n}\right),$$

where the kinetic coefficients A^n, B^n, C_s^n, $D_{s\sigma}^n$, may depend on parameters of equilibrium only.

According to $\theta_p \geqslant 0$, the symmetric matrix

$$(5.2) \qquad\qquad \begin{bmatrix} B^n & C_s^n \\ C_s^n & D_{s\sigma}^n \end{bmatrix}, \quad D_{s\sigma}^n = D_{\sigma s}^n,$$

should be positive and

$$(5.3) \qquad\qquad A^n \geqslant 0.$$

Since only small perturbations of equilibrium are here considered, we should linearise Eqs. (5.1) in respect to the parameters of perturbation, reducting them to the form:

$$\mathbf{Y}^n = -A^n \Delta \mathbf{w}^n,$$

$$(5.4) \qquad X_s^n = -C_s^n \frac{\Delta T^n}{T} - T \sum_{\sigma=1}^{S} D_{s\sigma}^n \Delta \left(\frac{\mu_\sigma^n}{T^n} \right),$$

$$Z^n = -\left(B^n - \sum_{s=1}^{S} \frac{\bar{p}}{\bar{\varrho}^n} C_s^n \right) \frac{\Delta T^n}{T} - T \sum_{s=1}^{S} \left(C_s^n - \sum_{\sigma=1}^{S} \frac{\bar{p}}{\bar{\varrho}^n} D_{\sigma s}^n \right) \Delta \left(\frac{\mu_s^n}{T^n} \right),$$

where in $\Delta(\mu_s^n/T^n)$ only the linear terms should be taken into account:

$$(5.5) \qquad \Delta \left(\frac{\mu_s^n}{T^n} \right) = \frac{\mu_s^n}{T^n} - \frac{\mu_s^0}{T^0} = \frac{\partial(\mu_s/T)}{\partial T} \Delta T^n + \frac{\partial \mu_s}{T \partial p} \Delta p + \frac{1}{T} \sum_{\sigma=1}^{S} \left(\frac{\partial \bar{\mu}_s^n}{\partial \xi_\sigma^n} \Delta \xi_\sigma^n - \frac{\partial \bar{\mu}_s^0}{\partial \xi_\sigma^0} \Delta \xi_\sigma^0 \right).$$

6. Equations of flow

For small perturbations of equilibrium, all previously deduced equations should be linearised in respect to the perturbation parameters (2.11) and (2.12). We shall, however, only partly linearise them, because the full linearisation would somewhat complicate their form of presentation. The consequent linearisation of the equations presented below may be effected during their application to flow problems.

The transport equations may be divided into two sets: for the mixture containing all phases:

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\varrho} \nabla \cdot \tau^0 + \mathbf{f},$$

$$(6.1)$$

$$\varrho \left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e \right) = -\nabla \cdot \mathbf{q} + \tau^0 : \nabla \mathbf{u} + \nabla \cdot (\tau^0 \cdot \mathbf{w}^0),$$

$$\frac{\partial \xi_s}{\partial t} + \mathbf{u} \cdot \nabla \xi_s = -\frac{1}{\varrho} \sum_{n=0}^{N} \nabla \cdot (\varrho \mathbf{w}^n \bar{\xi}^n \bar{\xi}_s^n),$$

and for each dispersed phase:

$$\frac{\partial \xi^n}{\partial t} + \mathbf{u}^n \cdot \nabla \xi^n = \xi^n X^n - \xi^n \frac{\nabla \cdot (\varrho \mathbf{w}^n)}{\varrho},$$

$$\frac{\partial \xi_s^n}{\partial t} + \mathbf{u}^n \nabla \xi_s^n = X_s^n - \bar{\xi}_s^n X^n,$$

$$(6.2)$$

$$\frac{\partial \mathbf{u}^n}{\partial t} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n = \mathbf{Y}^n + \mathbf{f},$$

$$\frac{\partial e^n}{\partial t} + \mathbf{u}^n \cdot \nabla e^n = Z^n - e^n X^n.$$

The interphase fluxes entering here are linear functions of the thermodynamic forces:

$$X_s^n - \bar{\xi}_s^n X^n = -\left(C_s^n - \bar{\xi}_s^n \sum_{\sigma=1}^{S} C_\sigma^n\right)\frac{\Delta T^n}{T} - T\sum_{\sigma=1}^{S}\left(D_{s\sigma}^n - \bar{\xi}_s^n \sum_{\lambda=1}^{S} D_{\lambda s}^n\right)\Delta\left(\frac{\mu_\sigma^n}{T^n}\right),$$

(6.3) $\quad \mathbf{Y}^n = -A^n \Delta \mathbf{w}^n,$

$$Z^n - e^n X^n = -\left[B^n - \left(\frac{\bar{p}}{\varrho^n} + e^n\right)\sum_{s=1}^{S} C_s^n\right]\frac{\Delta T^n}{T}$$

$$-T\sum_{s=1}^{S}\left[C_s^n - \left(\frac{\bar{p}}{\varrho^n} + e^n\right)\sum_{\sigma=1}^{S} D_{\sigma s}^n\right]\Delta\left(\frac{\mu_s^n}{T^n}\right),$$

where the coefficients of dissipation A^n, B^n, C_s^n, $D_{s\sigma}^n$, depend only on equilibrium parameters of coherent and n-th dispersed phases: $\bar{p}$, T, $\bar{\xi}_1^0$, $\bar{\xi}_2^0$, ... $\bar{\xi}_S^0$, $\bar{\xi}_1^{N_f}$, $\bar{\xi}_2^{N_f}$, ... $\bar{\xi}_S^{N_f}$ for $N_{f-1} < n \leqslant N_f$. These coefficients should be given either from experiments or from theoretical considerations of transport phenomena between isolated particles and the surrounding coherent phase.

To describe the physical properties of the mixture, the equations of state for each fraction (phase) should be given:

$$\varrho^n = \varrho^{N_f}(p, T^n, \xi_1^n, ... \xi_S^n), \quad e^n = e^{N_f}(p, T^n, \xi_1^n, ... \xi_S^n),$$

(6.4)
$$\mu_s^n = \mu_s^{N_f}(p, T^n, \xi_1^n, ... \xi_S^n), \quad s = 1, 2, ... S,$$

$$f = 0, \quad n = N_f = 0,$$

$$f = 1, 2, ... F, \quad N_{f-1} < n \leqslant N_f.$$

The density ϱ, the specific energy e and the heat flux $\mathbf{q}$ are now defined by the formulae:

$$\frac{1}{\varrho} = \sum_{n=0}^{N} \frac{\xi^n}{\varrho_1^n}, \quad e = \sum_{n=0}^{N} \xi^n e^n,$$

(6.5)
$$\mathbf{q} = \mathbf{q}^0 + \varrho \sum_{n=0}^{N} \xi^n e^n \mathbf{w}^n.$$

The transport Eqs. (6.1), (6.2), the flux equations (6.3) with given coefficients of dissipation A^n, B^n, C_s^n, $D_{s\sigma}^n$, and the equations characterising the thermodynamical properties of the mixture (6.4), (6.5) constitute a set of flow equations of the model considered of a multiphase mixture with one coherent, volumetrically dominant phase.

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