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Small vibrations of elastic medium deforming in time

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Isotropic elastic material is subject to finite strains in such a manner that the elongations in three mutually perpendicular directions are proportional to time. The equations for a small additional motion are constructed and several types of possible vibrations are analyzed. On the basis of the condition of propagation it is demonstrated that three principal directions of propagation exist connected with longitudinal and transversal waves.

Izotropowy materiał sprężysty poddany jest skończonym odkształceniom w ten sposób, że wydłużenia w trzech wzajemnie prostopadłych kierunkach są proporcjonalne do czasu. Buduje się równania dla małego dodatkowego ruchu, a następnie analizuje kilka możliwych drgań. W oparciu o warunek propagacji pokazuje się, że istnieją trzy główne kierunki propagacji, którym odpowiadają fale poprzeczne lub podłużne.

Изотропный упругий материал подвержен конечным деформациям таким образом, что удлинения в трех взаимно перпендикулярных направлениях пропорционально изменяются во времени. Выводятся уравнения, описывающие малое дополнительное движение, которые затем подвергаются анализу с точки зрения различных видов возможных колебаний. На основе условия распространения волн показано, что существуют три основных направления распространения волн, которым соответствуют продольные или поперечные колебания.

Small vibrations of elastic media have been extensively treated in the literature. In particular it is known that a complete coincidence can be established between the theory of vibrations and the condition of propagation of an acoustical wave provided the vibrations are infinitesimal and the finite initial deformation is homogeneous and stationary [1]. Thus, the coincidence also holds true in the linear theory of elasticity where—according to the definition—the initial deformation does not exist. In the present paper, we consider a situation more general than those hitherto dealt with. The finite, initial deformation varies in time; small additional vibrations are now superposed on that deformation and certain particular forms of vibrations are investigated.

1. Fundamental motion and additional motion

Let us introduce fixed Cartesian coordinate system. The coordinates of a typical point of the body under consideration in the natural state B_R are denoted by X^α, $\alpha = 1, 2, 3$. Let us consider the motion $\boldsymbol{\chi}(t)$ given by the relations

(1.1) $$x^1 = \lambda_1 X^1, \quad x^2 = \lambda_2 X^2, \quad x^3 = \lambda_3 X^3,$$

where λ_K are certain functions of time t only,

(1.2) $$\lambda_K = \lambda_K(t),$$

At the instant $t = 0$, the body is in a natural state B_R, and hence $\lambda_K(0) = 1$. Superposition of a rigid translation (but no rotation) upon the motion (1.1) does not influence the subsequent relations of this paper.

Let us pass to determination of the strain gradients $x^i{}_{,\alpha}$, the strain tensor B^{ik}, its invariants I_K, $K = 1, 2, 3$, and the density ϱ. All calculations of this section are based on the relations and notation given in [2]. According to (1.1), we have

$$x^i{}_{,\alpha} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ & \lambda_2 & 0 \\ & & \lambda_3 \end{bmatrix}, \tag{1.3}$$

$$B^{ik} = x^i{}_{,\alpha} x^{k,\alpha} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ & \lambda_2^2 & 0 \\ & & \lambda_3^2 \end{bmatrix}, \tag{1.4}$$

$$\begin{aligned} I_1 &= B^r_r = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{1}{2}(I_1^2 - B^r_s B^s_r) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \frac{1}{3} B^r_s B^s_p B^p_r - \frac{1}{2} B^r_r B^s_p B^p_s + \frac{1}{6}(B^r_r)^3 = \lambda_1^2\lambda_2^2\lambda_3^2; \end{aligned} \tag{1.5}$$

$$\varrho = \frac{1}{\lambda_1\lambda_2\lambda_3}\varrho_R, \tag{1.6}$$

ϱ_R being the density in the state B_R. Raising and lowering of indices is performed by means of the metric tensors of the Cartesian coordinate system previously introduced:

$$g_{ij} = g^{ij} = \delta_{ij}, \qquad g_{\alpha\beta} = g^{\alpha\beta} = \delta_{\alpha\beta}. \tag{1.7}$$

Further considerations will be confined to homogeneous and isotropic elastic materials. For such materials, there exists the elastic potential σ (referred to unit mass) which is a function of the strain invariants I_K only, $\sigma = \sigma(I_K)$, and the Piola-Kirchhoff stress tensor $T_{Ri}{}^{\alpha}$ is defined by

$$T_{Ri}{}^{\alpha} = \varrho_R \frac{\partial \sigma}{\partial x^i{}_{,\alpha}}, \tag{1.8}$$

where σ is a function of the gradients $x^i{}_{,\alpha}$ through the invariants I_K. Using the relations following from Eqs. (1.4), (1.5),

$$\begin{aligned} \frac{\partial B^{rs}}{\partial x^i{}_{,\alpha}} &= \delta^r_i x^{s}{}_{,}{}^{\alpha} + \delta^s_i x^{r}{}_{,}{}^{\alpha}, \\ \frac{\partial I_1}{\partial B^{rs}} &= g^{rs}, \qquad \frac{\partial I_2}{\partial B^{rs}} = I_1 g_{rs} - B_{rs}, \\ \frac{\partial I_3}{\partial B^{rs}} &= B_{rp} B^p_s - I_1 B_{rs} + I_2 g^{rs}, \end{aligned} \tag{1.9}$$

we obtain

$$(1.10)\qquad T_{Rk}{}^{\alpha} = 2W_1 x_{k,}{}^{\alpha} + 2W_2 (I_1 x_{k,}{}^{\alpha} - B_{ks} x^{s}{}_{,}{}^{\alpha}) + 2W_3 (B_{kp} B^{pq} x_{q,}{}^{\alpha} - I_1 B_{kp} x^{p}{}_{,\alpha} + I_2 x_{k,}{}^{\alpha}),$$

$$W_K = \frac{\partial W}{\partial I_K}, \quad W = \varrho_R \sigma, \quad K = 1, 2, 3.$$

Substituting now Eqs. (1.3), (1.4), (1.5) into Eq. (1.8), we obtain:

$$(1.11)\qquad \begin{aligned} T_{R1}{}^{1} &= 2[\lambda_1 W_1 + \lambda_1 (\lambda_2^2 + \lambda_3^2) W_2 + \lambda_1 \lambda_2^2 \lambda_3^2 W_3], \\ T_{R2}{}^{2} &= 2[\lambda_2 W_1 + \lambda_2 (\lambda_3^2 + \lambda_1^2) W_2 + \lambda_2 \lambda_3^2 \lambda_1^2 W_3], \\ T_{R3}{}^{3} &= 2[\lambda_3 W_1 + \lambda_3 (\lambda_1^2 + \lambda_2^2) W_2 + \lambda_3 \lambda_1^2 \lambda_2^2 W_3], \\ T_{R1}{}^{2} &= T_{R2}{}^{1} = T_{R1}{}^{3} = T_{R3}{}^{1} = T_{R2}{}^{3} = T_{R3}{}^{2} = 0. \end{aligned}$$

Since $T_{Ri}{}^{\alpha}$ are independent of the coordinates X^{α}, the left-hand side of the equations of motion

$$(1.12)\qquad T_{Ri}{}^{\alpha}{}_{,\alpha} = \varrho_R \ddot{x}_i,$$

is equal to zero, which yields the conclusion that also the acceleration $\ddot{x}_i$ is equal to zero. The motion (1.1) is then possible provided that

$$(1.13)\qquad \lambda_1 = 1 + c_1 t, \quad \lambda_2 = 1 + c_2 t, \quad \lambda_3 = 1 + c_3 t,$$

where c_1, c_2, c_3 are fixed parameters. In the subsequent considerations it is assumed that Eq. (1.13) has been subsituted in Eq. (1.1).

Let us now pass to the consideration of a perturbed motion $\boldsymbol{\chi}^*(t)$, differing only slightly from the motion $\boldsymbol{\chi}(t)$ [cf. Eqs. (1.1) and (1.13)]—that is, the motion

$$(1.14)\qquad \begin{aligned} x^{*1} &= \lambda_1 X^1 + u^1 (X^{\alpha}, t), \\ x^{*2} &= \lambda_2 X^2 + u^2 (X^{\alpha}, t), \\ x^{*3} &= \lambda_3 X^3 + u^3 (X^{\alpha}, t). \end{aligned}$$

The quantity $u^i(X^{\alpha}, t)$ is the displacement of the perturbation. TOUPIN and BERNSTEIN [3] derived the following equation for u^i

$$(1.15)\qquad (A_i{}^{\alpha}{}_k{}^{\beta} u^k{}_{;\beta})_{;\alpha} = \varrho_R \ddot{u}_i,$$

where the functions

$$(1.16)\qquad A_i{}^{\alpha}{}_k{}^{\beta} = \varrho_R \frac{\partial^2 \sigma}{\partial x^i{}_{,\alpha} \partial x^k{}_{,\beta}},$$

are calculated for the fundamental motion $\boldsymbol{\chi}(t)$. In the Cartesian coordinate system introduced here and under a consistent application of the independent variables X^{α} (and not x^i), all differentiations (1.15) may be replaced by partial differentiations with respect to X^{α} and t.

In order to obtain Eq. (1.16) in an explicit form in the case of isotropic materials, the necessary differentiations should be performed; it should be born in mind that σ depends on $x^i{}_{,\alpha}$ through the invariants I_K. Applying Eqs. (1.9) once again, we obtain

$$(1.17)\qquad A_k{}^{\alpha}{}_m{}^{\beta} = 2W_1 g_{km} g^{\alpha\beta} + 2W_2 [2x_{k,}{}^{\alpha} x_{m,}{}^{\beta} - g_{km} x_{r,}{}^{\alpha} x^{r}{}_{,}{}^{\beta} - x_{k,}{}^{\beta} x_{m,}{}^{\alpha} + (I_1 g_{km} - B_{km}) g^{\alpha\beta}] +$$

$$+2W_3[(g_{kr}B_{sq}x^q{}_{,}{}^{\alpha}+B_{kr}x_{s,}{}^{\alpha}-g_{rs}B_{kq}x^q{}_{,}{}^{\alpha}-I_1g_{kr}x_{s,}{}^{\alpha}+I_1g_{rs}x_{k,}{}^{\alpha}-B_{rs}x_{k,}{}^{\alpha})\times$$
$$\times(g^r{}_m x^s{}_{,}{}^{\beta}+g^s{}_m x^r{}_{,}{}^{\beta})+(B_{kp}B^p{}_m-I_1B_{km}+I_2g_{km})g^{\alpha\beta}]$$
$$+4\{W_{11}x_{k,}{}^{\alpha}x_{m,}{}^{\beta}+W_{22}(I_1x_{k,}{}^{\alpha}-B_k{}^px_{p,}{}^{\alpha})(I_1x_{m,}{}^{\beta}-B_m{}^rx_{r,}{}^{\beta})$$
$$+W_{33}(B_{kp}B^{pq}x_{q,}{}^{\alpha}-I_1B_k{}^qx_{q,}{}^{\alpha}+I_2x_{k,}{}^{\alpha})(B_{mr}B^{rs}x_{s,}{}^{\beta}-I_1B_m{}^rx_{r,}{}^{\beta}+I_2x_{m,}{}^{\beta})$$
$$+W_{23}[(I_1x_{k,}{}^{\alpha}-B_k{}^px_{p,}{}^{\alpha})(B_{mr}B^{rs}x_{s,}{}^{\beta}-I_1B_m{}^sx_{s,}{}^{\beta}+I_2x_{m,}{}^{\alpha})$$
$$+(B_{kp}B^{pq}x_{q,}{}^{\alpha}-I_1B_k{}^qx_{q,}{}^{\alpha}+I_2x_{k,}{}^{\alpha})(I_1x_{m,}{}^{\beta}-B_{mr}x^r{}_{,}{}^{\beta})]$$
$$+W_{31}[(B_{kp}B^{pq}x_{q,}{}^{\alpha}-I_1B_k{}^qx_{q,}{}^{\alpha}+I_2x_{k,}{}^{\alpha})x_{m,}{}^{\beta}+x_{k,}{}^{\alpha}(B_{mr}B^{rs}x_{s,}{}^{\beta}-I_1B_m{}^rx_{r,}{}^{\beta}+I_2x_{m,}{}^{\beta})$$
$$+W_{12}[(I_1x_{k,}{}^{\alpha}-B_{kp}x^p{}_{,}{}^{\alpha})x_{m,}{}^{\beta}+x_{k,}{}^{\alpha}(I_1x_{m,}{}^{\beta}-B_{mr}x^r{}_{,}{}^{\beta})],\qquad W_{kL}=\frac{\partial^2W}{\partial I_{k,}\partial I_L}.$$

The functions A_{ik} are symmetric neither in Latin nor Greek indices. Due to Eq. (1.16), however, the symmetry $A_{ik}=A_{ki}$ occurs.

Substituting Eqs. (1.3), (1.4), (1.5) into (1.17), we finally obtain:

$$
\begin{aligned}
A_1{}^1{}_1{}^1 &= 4\lambda_1^2W_{11}+4\lambda_1^2(\lambda_2^2+\lambda_3^2)^2W_{22}+8\lambda_1^2\lambda_2^4\lambda_3^4W_{33}+8\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_2^2+\lambda_3^2)W_{23}\\
&\quad+8\lambda_1^2\lambda_2^2\lambda_3^2W_{31}+4\lambda_1^2(\lambda_2^2+\lambda_3^2)W_{12}+2W_1+2(\lambda_2^2+\lambda_3^2)W_2+2\lambda_2^2\lambda_3^2W_3,\\
A_1{}^2{}_1{}^2 &= 2(W_1+\lambda_3^2W_2),\\
A_1{}^3{}_1{}^3 &= 2(W_1+\lambda_2^2W_2),\\
A_1{}^1{}_2{}^2 &= 4\lambda_1\lambda_2W_{11}+4\lambda_1\lambda_2(\lambda_1^2+\lambda_3^2)(\lambda_2^2+\lambda_3^2)W_{22}+4\lambda_1^3\lambda_2^3\lambda_3^4W_{33}\\
&\quad+4\lambda_1\lambda_2\lambda_3^2(\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2+2\lambda_1^2\lambda_2^2)W_{23}+4\lambda_1\lambda_2\lambda_3^2(\lambda_1^2+\lambda_2^2)W_{31}\\
&\quad+4\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2+2\lambda_3^2)W_{12}+4\lambda_1\lambda_2W_2+4\lambda_1\lambda_2\lambda_3^2W_3,\\
A_1{}^1{}_3{}^3 &= 4\lambda_1\lambda_3W_{11}+4\lambda_1\lambda_3(\lambda_1^2+\lambda_2^2)(\lambda_2^2+\lambda_3^2)W_{22}+4\lambda_1^3\lambda_2^4\lambda_3^3W_{33}\\
&\quad+4\lambda_1\lambda_2^2\lambda_3(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+2\lambda_3^2\lambda_1^2)W_{23}+4\lambda_1\lambda_2^2\lambda_3(\lambda_1^2+\lambda_3^2)W_{31},\\
&\quad+4\lambda_1\lambda_3(\lambda_1^2+\lambda_3^2+2\lambda_2^2)W_{12}+4\lambda_1\lambda_3W_2+4\lambda_1\lambda_2^2\lambda_3W_3,\\
A_1{}^2{}_2{}^1 &= -2\lambda_1\lambda_2W_2-2\lambda_1\lambda_2\lambda_3^2W_3,\\
A_1{}^3{}_3{}^1 &= -2\lambda_1\lambda_3W_2-2\lambda_1\lambda_3\lambda_2^2W_3.
\end{aligned}
\tag{1.18}
$$

All the remaining functions $A_1{}^{\alpha}{}_k{}^{\beta}$ vanish. The functions $A_2{}^{\alpha}{}_k{}^{\beta}$ and $A_3{}^{\alpha}{}_k{}^{\beta}$ may be obtained from those given by cyclic change of indices at the elongations λ_K.

It is seen that the tensor $A_i{}^{\alpha}{}_k{}^{\beta}$ is independent of (x^k, X^{α}) and is the function of the only variable t. Owing to that property, the tensor may be taken out of the parantheses in Eq. (1.15). The differentiation indicated in that formula may be reduced—in the Cartesian coordinate system—to partial differentiation (with fixed X^{α}); hence, we obtain:

$$A_i{}^{\alpha}{}_k{}^{\beta}\frac{\partial^2u^k}{\partial X^{\alpha}\partial X^{\beta}}=\varrho_R\frac{\partial^2u_i}{\partial t^2}. \tag{1.19}$$

This equation is the equation sought for, describing the small motion superposed on the fundamental motion (1.1).

2. Derivation in convective coordinates

The derivation of Eqs. (1.19) presented above in the case of small additional motion is complete. Since, however, that in solving certain particular problems (e.g., stability), convective coordinates are extensively used in the literature, let us employ these coordinates in the present paper. Such an approach has the advantage that the corresponding formulae are — particularly in the case of isotropic materials—well known from the relevant literature (cf., e.g., [4]).

In addition to the Cartesian reference frames x^i and X^α, let us now introduce the convective coordinate system θ^i which coincides in $\mathring{B}$ with the system X^α

$$\theta^1 = X^1, \quad \theta^2 = X^2, \quad \theta^3 = X^3. \tag{2.1}$$

The time-dependent metric tensor corresponding to θ^i is denoted by G^{ij}, G_{ij}, by contrast with fixed g_{ij}, g^{ij}, $g_{\alpha\beta}$, $g^{\alpha\beta}$.

Calculations of the invariants I_K lead to relations already given (1.5). Passing now to the stress tensor, we obtain

$$\tau^{11} = \Psi_1 + (\lambda_2^2+\lambda_3^2)\Psi_2 + \frac{1}{\lambda_1^2}\Psi_3, \quad \tau^{12} = 0, \tag{2.2}$$

with the notations

$$\Psi_1 = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad \Psi_2 = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad \Psi_3 = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}, \tag{2.3}$$

in [3] the corresponding notations were Φ, Ψ, p), W denoting the elastic energy referred to unit volume in B_R. This function is identical with the function $W = \varrho_R\sigma$ introduced in the preceding section. The remaining components of the stress tensor are obtained from those defined by Eq. (1.11) by means of cyclic interchange of indices.

On the basis of relations given in [3], let us consider the motion $\mathbf{R}^*(t) = \mathbf{R}(t)+\mathbf{w}(t)$, $\mathbf{w}$ being small in comparison with $\mathbf{R}$. The quantities appearing in Eqs. (2.2)–(2.3) are now subject to certain increments. Their linear components are denoted by the same kernel letter as those connected with the motion $\mathbf{R}(t)$ and marked by a prime. With the notations

$$\mathbf{W} = u\mathbf{G}^1 + v\mathbf{G}^2 + w\mathbf{G}^3 \tag{2.4}$$

the physical components of $\mathbf{w}$ are, according to (2.2), the quantities u/λ_1, v/λ_2, w/λ_3. We now have

$$\tau'^{11} = \frac{2}{\lambda_1\lambda_2\lambda_3}\Big[2W_{11} + 2W_{22}(\lambda_2^2+\lambda_3^2)^2 + 2W_{33}\lambda_2^4\lambda_3^4 + 4W_{23}\lambda_2^2\lambda_3^2(\lambda_2^2+\lambda_3^2)$$
$$+4W_{31}\lambda_2^2\lambda_3^2 + 4W_{12}(\lambda_2^2+\lambda_3^2) - W_1\frac{1}{\lambda_1^2} - W_2\frac{\lambda_2^2+\lambda_3^2}{\lambda_1^2} - W_3\frac{\lambda_2^2\lambda_3^2}{\lambda_1^2}\Big]u_X$$
$$+\frac{2}{\lambda_1\lambda_2\lambda_3}\Big[2W_{11} + 2W_{22}(\lambda_1^2+\lambda_3^2)(\lambda_2^2+\lambda_3^2) + 2W_{33}\lambda_1^2\lambda_2^2\lambda_3^4$$
$$+2W_{23}\lambda_3^2(\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2+2\lambda_1^2\lambda_2^2) + 2W_{31}\lambda_3^2(\lambda_1^2+\lambda_2^2) + 2W_{12}(\lambda_1^2+\lambda_2^2+2\lambda_3^2)$$

$$(2.5)\qquad -W_1\frac{1}{\lambda_2^2}+W_2\frac{\lambda_2^2-\lambda_3^2}{\lambda_2^2}+W_3\lambda_3^2\Bigg]v_Y+\frac{2}{\lambda_1\lambda_2\lambda_3}\Bigg[2W_{11}+2W_{22}(\lambda_1^2+\lambda_2^2)(\lambda_2^2+\lambda_3^2)$$
$$+2W_{33}\lambda_1^2\lambda_2^4\lambda_3^2+2W_{23}\lambda_2^2(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+2\lambda_1^2\lambda_3^2)+2W_{31}\lambda_2^2(\lambda_1^2+\lambda_3^2)$$
$$+2W_{12}(\lambda_1^2+\lambda_3^2+2\lambda_2^2)-W_1\frac{1}{\lambda_3^2}+W_2\frac{\lambda_3^2-\lambda_2^2}{\lambda_3^2}+W_3\lambda_2^2\Bigg]W_Z,$$

$$\tau'^{12}=-\frac{2}{\lambda_1\lambda_2\lambda_3}(W_2+\lambda_3^2W_3)(u_Y+v_X),$$

$$\tau'^{13}=-\frac{2}{\lambda_1\lambda_2\lambda_3}(W_2+\lambda_2^2W_3)(u_z+w_X),$$

where

$$(2.6)\qquad W_{KL}=\frac{\partial^2W}{\partial I_K\partial I_L}.$$

The remaining increments of the stress tensor are obtained from those given above by cyclic permutation of indices of all quantities except W_{KL}.

To the equations of motion constructed by means of convective coordinates, there enter, in addition to the increments already mentioned, the increments of the Christoffel symbols Γ'^i_{jk} and of the acceleration $\mathbf{a}'$. In order to determine $\mathbf{a}'$, let us differentiate $\mathbf{R}^*(t)$ twice with respect to time, which finally yields

$$(2.7)\qquad a'^1=\frac{1}{\lambda_1}\frac{D^2}{Dt^2}\left(\frac{u}{\lambda_1}\right),\quad a'^2=\frac{1}{\lambda_2}\frac{D^2}{Dt^2}\left(\frac{v}{\lambda^2}\right),\quad a'^3=\frac{1}{\lambda_3}\frac{D^2}{Dt^2}\left(\frac{w}{\lambda_3}\right).$$

Here D/Dt denotes the differentiation with respect to t at fixed values of θ^i.

Using the formulae derived in [2], the increments of Γ'^i_{jk} are determined (Christoffel symbols Γ^i_{jk} vanish);

$$(2.8)\qquad \begin{aligned}\Gamma'^1_{11}&=\frac{1}{\lambda_1^2}u_{XX},\quad \Gamma'^1_{22}=\frac{1}{\lambda_1^2}u_{YY},\quad \Gamma'^1_{33}=\frac{1}{\lambda_1^2}u_{ZZ},\\ \Gamma'^1_{23}&=\frac{1}{\lambda_1^2}u_{YZ},\quad \Gamma'^1_{31}=\frac{1}{\lambda_1^2}u_{ZX},\quad \Gamma'^1_{12}=\frac{1}{\lambda_1^2}u_{XY}.\end{aligned}$$

The remaining increments Γ'^i_{jk} are obtained from (2.14) by cyclic permutation of indices and (u, v, w).

Inserting now Eqs. (2.7) and (2.8) in the equations of motion we finally obtain:

$$(2.9)\qquad \Bigg[2W_{11}+2W_{22}(\lambda_2^2+\lambda_3^2)^2+2W_{33}\lambda_2^4\lambda_3^4+4W_{23}\lambda_2^2\lambda_3^2(\lambda_2^2+\lambda_3^2)+4W_{31}\lambda_2^2\lambda_3^2$$
$$+4W_{12}(\lambda_2^2+\lambda_3^2)+\frac{1}{\lambda_1^2}W_1+\frac{\lambda_2^2+\lambda_3^2}{\lambda_1^2}W_2+\frac{\lambda_2^2+\lambda_3^2}{\lambda_1^2}W_3\Bigg]u_{XX}+\left(\frac{1}{\lambda_1^2}W_1+\frac{\lambda_3^2}{\lambda_1^2}W_2\right)u_{YY}$$
$$+\left(\frac{1}{\lambda_1^2}W_1+\frac{\lambda_2^2}{\lambda_1^2}W_2\right)u_{ZZ}+[2W_{11}+2(\lambda_1^2+\lambda_3^2)(\lambda_2^2+\lambda_3^2)W_{22}+2\lambda_1^2\lambda_2^2\lambda_3^4W_{33}$$
$$+2\lambda_3^2(\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2+2\lambda_1^2\lambda_2^2)W_{23}+2\lambda_3^2(\lambda_1^2+\lambda_2^2)W_{31}+2(\lambda_1^2+\lambda_2^2+2\lambda_3^3)W_{12}$$

$$+W_2+\lambda_3^2W_3]v_{XY}+[2W_{11}+2(\lambda_1^2+\lambda_2^2)(\lambda_2^2+\lambda_3^2)W_{22}+2\lambda_1^2\lambda_2^4\lambda_3^2W_{33}$$
$$+2\lambda_2^2(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+2\lambda_1^2\lambda_3^2)W_{23}+2\lambda_2^2(\lambda_1^2+\lambda_3^2)W_{31}+2(\lambda_1^2+\lambda_3^2+2\lambda_2^2)W_{12}$$
$$+W_2+\lambda_2^2W_3]w_{XZ}=\frac{1}{2\lambda_1}\varrho_R\frac{D^2}{Dt^2}\left(\frac{u}{\lambda_1}\right).$$

The remaining two equations may be obtained from the above by cyclic permutation of the indices and functions u, v, w. By introducing the quantities $u_1 = u/\lambda_1$, $u_2 = v/\lambda_2$, $u_3 = w/\lambda_3$, we obtain the system (1.19), derived in a different manner. In the subsequent sections of the paper, the analysis of solutions of that system will be presented.

3. Small vibrations of the medium

The coefficients of the system (1.19) are functions of time, which makes its general analysis somewhat complicated. It is possible, however, to find several particular solutions which are presented below.

Let us first consider the vibrations corresponding to a plane wave. To this end, the additional displacements u^i are assumed to have the form

$$u^i = l^i\varphi(P,t), \tag{3.1}$$

where

$$P = X^\alpha N_\alpha, \qquad N_\alpha N^\alpha = 1, \qquad l^i = \text{const}, \qquad l^r l_r = 1. \tag{3.2}$$

The function $\varphi(P, t)$ represents the length of the displacement vector u^i. From Eq. $(3.2)_3$, it follows that this vector has a fixed direction in space. On the material surfaces having normals N_α in the state B_R, the absolute value of that vector depends exclusively on time.

Substituting (3.1) into (1.19), we obtain:

$$B_{km}l^m\frac{\partial^2\varphi}{\partial P^2} = \varrho_R l_k\frac{\partial^2\varphi}{\partial t^2}, \tag{3.3}$$

where

$$B_{km} = A_k{}^\alpha{}_m{}^\beta N_\alpha N_\beta. \tag{3.4}$$

The symmetry $A_k{}^\alpha{}_m{}^\beta = A_m{}^\beta{}_k{}^\alpha$ makes B_{km} a symmetric tensor, $B_{km} = B_{mk}$. Assuming that $\partial^2\varphi/\partial p^2 \neq 0$, let us divide Eq. (3.3) by $\partial^2\varphi/\partial P^2$,

$$B_{km}l^m = \varrho_R l_k\frac{\partial^2\varphi}{\partial t^2}\bigg/\frac{\partial^2\varphi}{\partial P^2}. \tag{3.5}$$

From Eq. (3.5) it follows that the direction of vibrations is the eigenvector of the B_{km} tensor. B_{km} is, however, a function of time, l^m being time-independent, according to our assumption. This means that the vibrations (3.1) are possible for a prescribed N_α only in the case in which $B_{km}(N_\alpha)$ has at least one time-independent eigenvector. The eigenvalue corresponding to this vector may, on the other hand, be time-dependent.

In order to find the form of B_{km} necessary to make possible the vibrations (3.1), let us assume $b_i^{(K)}(t)$, $K = 1, 2, 3$, to be normed eigenvectors. Owing to the symmetry $B_{km} = B_{mk}$, it can be assumed that these vectors are mutually orthogonal. Denoting their corresponding eigenvalues by $\overset{K}{\varkappa}(t)$, we obtain

$$(3.6) \qquad B_{km} = \sum_K \overset{K}{\varkappa}(t)\overset{K}{b}_k(t)\overset{K}{b}_m(t).$$

According to (3.4), the motion (3.1) is possible provided at least one vector $\overset{K}{b}_k(t)$ is time-independent, $\overset{K}{b}_k(t) = l_k = \text{const}$. In such a case

$$(3.7) \qquad B_{km} = \varkappa(t)l_kl_m+\overset{2}{\varkappa}(t)\overset{2}{b}_k(t)\overset{2}{b}_m(t)+\overset{3}{\varkappa}(t)\overset{3}{b}_k(t)\overset{3}{b}_m(t)$$

and the only possible vibrations of the form (3.1) have the direction l_k. If two vectors $\overset{K}{b}_k(t)$ and $\overset{L}{b}_k(t)$ are time-independent, then, due to the orthogonality of the triple $\overset{K}{b}_k$, $K = 1, 2, 3$, also the third vector must be time-independent and assume the form

$$(3.5) \qquad B_{km} = \sum_{K=1,1,3} \overset{K}{\varkappa}(t)\overset{K}{l}_k\overset{K}{l}_m.$$

Three mutually orthogonal possible directions of vibrations are found to exist. If these vibrations were, for each time t, orthogonal or parallel to the material plane $X^\alpha N_\alpha = \text{const}$, they might be called longitudinal or transversal vibrations. Since the planes rotate (except the planes $X^\alpha = \text{const}$), they are, in general, neither longitudinal nor transversal.

The situation in which B_{km} has the form (3.7) or (3.8) is special. In general, no vibrations of the form (3.1) can exist for prescribed values of N_α. The important particular case occurs when $N_\alpha = \delta_{\alpha\varrho}$ for a fixed $\varrho = 1, 2$ or 3. For the sake of simplicity let us assume $\varrho = 1$; then, in accordance with Eq. (3.4), we may write

$$(3.9) \qquad B_{km} = \begin{bmatrix} A_1{}^1{}_1{}^1 & 0 & 0 \\ & A_2{}^1{}_2{}^1 & 0 \\ & & A_3{}^1{}_3{}^1 \end{bmatrix},$$

and the tensor B_{km} for each material and each $\lambda_1, \lambda_2, \lambda_3$, has the form (3.8), the constant eigenvectors being $\overset{1}{l}_i = (1, 0, 0)$, $\overset{2}{l}_i = (0, 1, 0)$, $\overset{3}{l}_i = (0, 0, 1)$. The planes $X_1 = \text{const}$ do not rotate in time and hence the first direction corresponds to longitudinal vibrations, the remaining two—to transversal vibrations.

In the case of spherically symmetric deformation $\lambda_1 = \lambda_2 = \lambda_3$ the tensor B_{km} for each N_α has the form (3.8). Then $A_{11}^{11} = A_{22}^{22} = A_{33}^{33} = \overset{1}{\psi}(t)$, $A_{11}^{22} = A_{11}^{33} = \ldots = A_{33}^{22} = \overset{2}{\psi}(t)$, $A_{12}^{(12)} = A_{13}^{(13)} = \ldots = A_{32}^{(32)} = [\overset{1}{\psi}(t)-\overset{2}{\psi}(t)]/2$, whence

$$(3.10) \qquad B_{km} = \overset{2}{\psi}(t)\,\delta_{km}+\big(\overset{1}{\psi}(t)-\overset{2}{\psi}(t)\big)\delta_k^\alpha\delta_m^\beta N_\alpha N_\beta.$$

Each time-independent vector $l_k \perp \delta_k^\alpha N_\alpha$ and $l_k = \delta_k^\alpha N_\alpha$ is now an eigenvector of the tensor B_{km}.

Assuming now the necessary condition (3.7) to hold true, Eq. (3.3) is written in the form:

$$\varkappa(t)\frac{\partial^2\varphi}{\partial P^2} = \varrho_R\frac{\partial^2\varphi}{\partial t^2}. \tag{3.11}$$

Separation of variables yields

$$\varphi(P, t) = \alpha(P)\chi(t), \tag{3.12}$$

$$\frac{\ddot{\chi}}{\varkappa(t)\chi} = \frac{\alpha''}{\varrho_R\alpha}. \tag{3.13}$$

Since the left-hand side of Eq. (3.12) depends on t only and the right-hand side on p, the following equations are true:

$$\alpha''+k^2\varrho_R\alpha = 0, \qquad \ddot{\chi}+k^2\varkappa(t)\chi = 0, \tag{3.14}$$

k^2 being the coupling coefficient. The solutions of the first equation are

$$\alpha_1 = e^{ik\sqrt{\varrho_R}P}, \qquad \alpha_2 = e^{-ik\sqrt{\varrho_R}P}. \tag{3.15}$$

The solutions of Eq. $(3.14)_2$ are to be found for a known value of $\varkappa(t)$ — i.e., the elastic potential W. Let us denote two linearly independent real solutions of $(3.14)_2$ by χ_1 and χ_2. Owing to the linearity and homogeneity of the equation, also the following expressions represent the solutions:

$$\begin{aligned} \overline{\chi}_1 &= \chi_1+i\chi_2 = |\chi_1+i\chi_2|e^{i\arg(\chi_1+i\chi_2)}, \\ \overline{\chi}_2 &= \chi_1-i\chi_2 = |\chi_1+i\chi_2|e^{-i\arg(\chi_1+i\chi_2)}. \end{aligned} \tag{3.16}$$

By means of Eqs. (3.12), (3.15), (3.16), the functions $\varphi(P, t)$ are found

$$\begin{aligned} \varphi_{1,2} &= |\chi_1+i\chi_2|e^{\pm i[k\sqrt{\varrho_R}P+\arg(\chi_1+i\chi_2)]}, \\ \varphi_{3,4} &= |\chi_1+i\chi_2|e^{\pm i[k\sqrt{\varrho_R}P-\arg(\chi_1+i\chi_2)]}. \end{aligned} \tag{3.17}$$

These relations represent a sinusoidal wave. The function $\arg(\chi_1+i\chi_2)$ depends on the time and the wave number k. The corresponding wave is then dispersive and propagates at the time-dependent velocity. The real-valued functions satisfying (3.11) are $\varphi_1+\varphi_2$, $\varphi_3+\varphi_4$, $(\varphi_1-\varphi_2)/i$, $(\varphi_3-\varphi_4)/i$. It should be born in mind functions χ_1 and χ_2 may be multiplied by arbitrary constants.

The expressions (3.17) can also be used to construct the function φ corresponding to a stationary wave. Such a wave is represented, for instance, by $\chi_1 \sin k\sqrt{\varrho_R}P$; its nodes are located at the same material points, though moving in space.

Similarly to (3.12), other particular forms of the function $\varphi(P, t)$ may be assumed — e.g.:

$$\varphi = \varphi\big(P-\alpha(t)\big), \qquad \varphi = \gamma(t)\beta\big(P-\alpha(t)\big). \tag{3.18}$$

They also lead to certain solutions, while the equations for $\alpha(t)$ are nonlinear. For the sake of brevity, we shall not investigate these cases in detail, particularly, since they are in part contained in the case previously considered.

Let us pass to a solution entirely different from (3.11). Consider the problem of plane vibrations $u_3 = 0$, $\partial/\partial X^3 = 0$, and seek a solution of the form

$$\begin{aligned} u_1 &= u = \alpha(t)e^{i(\mu X+\nu Y)}, \\ u_2 &= v = -\beta(t)e^{i(\mu X+\nu Y)}. \end{aligned} \tag{3.19}$$

Here μ and ν are fixed parameters. Since for some indices the functions $A_k{}^\alpha{}_m{}^\beta$ are identically zero, the system of Eqs. (1.19) is reduced to:

$$
(3.20)\qquad
\begin{aligned}
-(A_1{}^1{}_1{}^1\mu^2+A_1{}^2{}_1{}^2\nu^2)\alpha+2A_{12}^{(12)}\mu\nu\beta &= \varrho_R\ddot{\alpha},\\
2A_{12}^{(12)}\mu\nu\alpha-(A_1{}^2{}_1{}^2\mu^2+A_2{}^2{}_2{}^2\nu^2)\beta &= \varrho_R\ddot{\beta}.
\end{aligned}
$$

With $\mu = 0$ or $\nu = 0$, the solution reduces to that considered above. If $\mu \neq 0$ and $\nu \neq 0$, the system can be reduced to one differential equation for the function $\alpha(t)$

$$
(3.21)\qquad \left\{\left[\varrho_R\frac{d^2}{dt^2}+(\mu^2A_1{}^2{}_1{}^2+\nu^2A_2{}^2{}_2{}^2)\right]\frac{1}{2A_{12}^{(12)}}\times \right.
$$
$$
\left.\times\left[\varrho_R\frac{d^2}{dt^2}+(\mu^2A_1{}^1{}_1{}^1+\nu^2A_1{}^2{}_1{}^2)\right]-\mu^2\nu^2 2A_{12}^{(12)}\right\}\alpha = 0,
$$

$\beta(t)$ being determined by the relation:

$$
(3.22)\qquad \beta = \frac{1}{2\mu\nu A_{12}^{(12)}}\{\varrho_R\ddot{\alpha}+(A_1{}^1{}_1{}^1\mu^2+A_1{}^2{}_1{}^2\nu^2)\alpha\}.
$$

If the functions $A_i{}^\alpha{}_k{}^\beta$ are given, Eq. (3.21) can in principle be solved. To proceed with the analysis, let us assume $\alpha(t)$ to be the real solution of the equation and $\beta(t)$ — a function defined by (3.22). Replacement of (μ,ν) by $(-\mu, -\nu)$, $(-\mu,\nu)$, $(\mu, -\nu)$ does not change Eq. (3.21), and hence, also in these cases $\alpha(t)$ is a solution. In accordance with (3.22), only in the last two cases does $\beta(t)$ pass into $-\beta(t)$. Thus, we conclude that four solutions exist:

$$
(3.23)\qquad
\begin{aligned}
\overset{1}{u} &= \alpha e^{i(\mu X+\nu Y)}, \\
\overset{1}{v} &= -\beta e^{i(\mu X+\nu Y)}; \\[6pt]
\overset{2}{u} &= \alpha e^{-i(\mu X+\nu Y)}, \\
\overset{2}{v} &= -\beta e^{-i(\mu X+\nu Y)}; \\[6pt]
\overset{3}{u} &= \alpha e^{i(-\mu X+\nu Y)}, \\
\overset{3}{v} &= \beta e^{i(-\mu X+\nu Y)}; \\[6pt]
\overset{4}{u} &= \alpha e^{i(\mu X-\nu Y)}, \\
\overset{4}{v} &= \beta e^{i(\mu X+\nu Y)}.
\end{aligned}
$$

The system (1.19) being linear, each linear combination of the solutions (3.23) constitutes a solution. In particular, adding the first two solutions together, we obtain the solution

$$
(3.24)\qquad
\begin{aligned}
u &= \alpha\cos(\mu X+\nu Y),\\
v &= -\beta\cos(\mu X+\nu Y),
\end{aligned}
$$

while the third and fourth solutions added together yield the solution

$$
(3.25)\qquad
\begin{aligned}
u &= \alpha\cos(-\mu X+\nu Y),\\
v &= \beta\cos(-\mu X+\nu Y).
\end{aligned}
$$

The relations (3.24) and (3.25) represent a stationary wave with nodal points located on straight lines parallel to $\mu X+\nu Y = 0$ and $-\mu X+\nu Y = 0$. The further two solutions [(1)–(2)] [(3)–(4)] do not substantially differ from (3.24) and (3.25). Summing up all four solutions (3.23), we obtain

$$u = \alpha \cos \mu X \cos \nu Y,\quad v = \beta \sin \mu X \sin \nu Y, \tag{3.26}$$

which also represents a stationary wave. Three further solutions $[(3)+(4)-(1)-(2)]$, $[(3)-(4)+(1)-(2)]$, $[(3)-(4)-(1)+(2)]$ are not essentially different from (3.26).

The differential Eq. (3.21), as a fourth order equation with real coefficients, has four real, linearly independent solutions $\overset{1}{\alpha}(t)$, $\overset{2}{\alpha}(t)$, $\overset{3}{\alpha}(t)$, $\overset{4}{\alpha}(t)$. These solutions may be used to construct complex solutions, such as $\overset{1}{\alpha}(t)+\overset{4}{\alpha}(t)$. A typical solution of this type is denoted by $\tilde{\alpha}(t)$, and the corresponding $\beta(t)$ (3.22) — by $\tilde{\beta}(t)$.

On the basis of Eq. (3.19), we now write:

$$u = |\tilde{\alpha}(t)| e^{i(\arg \tilde{\alpha}(t)+\mu X+\nu Y)},\quad v = |\tilde{\beta}(t)| e^{i(\arg \tilde{\beta}(t)+\mu X+\nu Y)}. \tag{3.27}$$

Further solutions may be obtained from the above one by changing the signs of $\arg\tilde{\alpha}(t)$, $\arg\tilde{\beta}(t)$, ν and μ, bearing in mind the sign of $\tilde{\beta}(t)$ [cf. Eq. (3.23)]. These solutions represent propagating waves with phase planes parallel to $\mu X+\nu Y = 0$, $\mu X-\nu Y = 0$. Since $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ depend on the parameters μ and ν, the wave is dispersive and has a time-dependent velocity.

It should be stressed that vibrations of the form (3.19) are not possible at all in the case when the material coordinates are replaced by spatial coordinates x, y. In such case, x_i enters the equation for the function $\alpha(t)$ [analogous to (3.20)], which yields $\alpha \equiv 0$ and $\beta \equiv 0$.

In a similar manner, the three dimensional case $u_k = \alpha_k(t)e^{il n}$ may be considered, leading to a system of three ordinary differential equations which can be further reduced to a single eight order differential equation.

Let us consequently consider the displacement u_i of the form

$$u^i = l^i(t)\varphi(P). \tag{3.28}$$

This is a generalisation of Eqs. (3.19) to the case of vibrations in three directions. Equation (1.19) then yields the equation of motion:

$$B_{ik}(t)l^k(t)\varphi''(P) = \varrho_R \ddot{l}_i(t)\varphi(P), \tag{3.29}$$

B_{ik} being defined by Eq. (3.4). Separating the variables, we obtain:

$$\frac{\varphi''}{\varphi} = \frac{\varrho_R \ddot{l}_i}{B_{ik}l_k} = -k^2 \tag{3.30}$$

for each i, k being the coupling constant. This is a system of four equations for the functions $\varphi(P)$, $l_i(t)$, and its solution may be found in a manner analogous to the solution in the case of vibrations in two directions.

The existence of solution (3.1) suggests the possibility of existence of the solution:

$$u^i = l^i \varphi(p, t), \tag{3.31}$$

$$p = x^i n_i, \quad n^r n_r = 1, \quad l^i = \text{const}, \quad n_i = \text{const}. \tag{3.32}$$

In the general case, the vibrations (3.31) are not equivalent to (3.1). Taking into account Eq. (1.1), we have

$$\frac{\partial^2 u^m}{\partial X^\alpha \partial X^\beta} = l^m \frac{\partial^2 \varphi}{\partial p^2} \delta^j_\alpha \delta^k_\beta \lambda_j \lambda_k n_j n_k, \tag{3.33}$$

$$\frac{D^2 u^m}{Dt^2} = l^m \left[\frac{\partial^2 \varphi}{\partial p^2} \left(\sum_i c_i X^\alpha n_i \delta^i_\alpha \right)^2 + 2 \frac{\partial^2 \varphi}{\partial p \partial t} \sum_i c_i X^\alpha n_i \delta^i_\alpha + \frac{\partial^2 \varphi}{\partial t^2} \right]. \tag{3.34}$$

Substituting now (3.31) and (3.32) in (1.19), we obtain

$$C_{km} l^m \frac{\partial^2 \varphi}{\partial p^2} = \varrho_R l_k \left[\frac{\partial^2 \varphi}{\partial p^2} \left(\sum_i c_i X^\alpha n_i \delta^i_\alpha \right)^2 + 2 \frac{\partial^2 \varphi}{\partial p \partial t} \sum_i c_i X^\alpha n_i \delta^i_\alpha + \frac{\partial^2 \varphi}{\partial t^2} \right], \tag{3.35}$$

where

$$C_{km} = A_k{}^\alpha{}_m{}^\beta \delta_{\alpha r} \lambda_r n_r \delta_{\beta s} \lambda_s n_s. \tag{3.36}$$

The necessary condition for the existence of vibrations (3.31) is furnished by the requirement that C_{km} should have one time-independent eigenvector, but even if this condition is fulfilled, the vibrations (3.31) do not generally exist: the left-hand side of (3.35) is a function of p and t while the right-hand side is a function of p, t and the variables X^α.

An interesting particular case is encountered when $\mathbf{n}$ has the direction of one of the axes x_i — e.g. x^1, which means that $n_k = (1, 0, 0)$. Then

$$p = x^1, \quad \frac{D^2 u^i}{Dt^2} = \frac{\partial^2 \varphi}{\partial p^2} C_1^2 \left(\frac{x^1}{\lambda_1} \right)^2 + 2 \frac{\partial^2 \varphi}{\partial p \partial t} C_1 \frac{x^1}{\lambda_1} + \frac{\partial^2 \varphi}{\partial t^2}, \tag{3.37}$$

and only two independent variables p and t appear in Eq. (3.35). Another important particular case is obtained when $\partial^2\varphi/\partial p^2 = 0$. The left-hand side of Eq. (3.35) is then equal to zero and (3.31) represents a rigid translation.

4. Acoustical wave

Starting from the equations of compatibility on the surface at which the second derivatives of $x^i(X^\alpha, t)$ suffer a jump, the condition of propagation has been derived by C. TRUESDELL in the form

$$Q_{km} a^m = \varrho U^2 a_k, \tag{4.1}$$

where Q_{km} is the acoustical tensor corresponding to the normal n_k

$$Q_{km} = \frac{\varrho}{\varrho_R} A_k{}^\alpha{}_m{}^\beta x^p{}_{,\alpha} x^q{}_{,\beta} n_p n_q. \tag{4.2}$$

The scalar U is the propagation rpeed, and a_k is the vector connected with the jumps of derivatives of $x^i(X^\alpha, t)$ by means of the conditions

$$
\begin{aligned}
[x^k{}_{,\alpha;\beta}] &= a^k x^m{}_{,\alpha} x^p{}_{,\beta} n_m n_p, \\
[\dot{x}^k{}_{,m}] &= -U a^k n_m, \\
[\ddot{x}^k] &= U^2 a^k.
\end{aligned} \tag{4.3}
$$

ϱU^2 is the eigenvalue, and a_k the eigenvector of the acoustical tensor Q_{km}. According to (1.3) and (1.18) Q_{km} is a symmetric tensor.

Both a_k and U may be functions of time t. By contrast with Eq. (1.19), which was true for small displacements u_k, Eq. (4.1) is an exact equation.

It is easily verified that for the isotropic material considered n_i is not, in general, the eigenvector of the tensor $Q_{km}(n_i)$. It follows that in a nonlinear isotropic material the longitudinal elastic wave propagating in a prescribed direction n_i does not generally exist. If, however, n_i is assumed to be either $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$, then n_i is the eigenvector of the acoustical tensor $Q_{km}(n_i)$ and the longitudinal wave exists. Let us consider, for instance, the case in which $n_i = (1, 0, 0)$. Then, according to (1.3), we obtain:

$$
Q_{km} = \frac{1}{\lambda_1\lambda_2\lambda_3}\begin{bmatrix} A^1{}_1{}^1{}_1\lambda_1^2 & 0 & 0 \\ & A_2{}^1{}_2{}^1\lambda_1^2 & 0 \\ & & A_3{}^1{}_3{}^1\lambda_1^2 \end{bmatrix}, \tag{4.4}
$$

In addition to the eigendirection $(1, 0, 0)$ this tensor possessess the eigendirections $(0, 1, 0)$ and $(0, 0, 1)$. The longitudinal wave is accompanied by two transversal waves with amplitudes $a_k = (0, 1, 0)$ and $a_k = (0, 0, 1)$. Equations (4.1) yield the squares of propagation velocities corresponding to these waves

$$
U_{\parallel}^2 = \frac{1}{\varrho_R} A_1{}^1{}_1{}^1\lambda_1^2, \qquad U_{\perp(2)}^2 = \frac{1}{\varrho_R} A_2{}^1{}_2{}^1\lambda_1^2, \qquad U_{\perp(2)}^2 = \frac{1}{\varrho_R} A_3{}^1{}_3{}^1\lambda_1^2. \tag{4.5}
$$

Similar relations hold true for $n_i = (0, 1, 0)$ and $n_i = (0, 0, 1)$. For the direction of propagation $n_i = (n_1, n_2, 0)$ the eigendirection is $(0, 0, 1)$.

Thus, in an isotropic material three principal directions of propagation exist and they coincide with the principal directions of strain. Each principal direction of propagation corresponds to one longitudinal and two transversal waves. For other directions of propagation, the corresponding wave is neither longitudinal nor transversal. Each of the principal propagation velocities is defined by $A_j{}^\alpha{}_j{}^\alpha$. If all these quantities are positive, then all the principal propagation velocities are real. The expression for the velocity of propagation in an arbitrary direction contains, besides $A_j{}^\alpha{}_j{}^\alpha$, also the quantities $A_i{}^\alpha{}_j{}^\beta$; this explains why the condition for all the principal propagation velocities to be real does not ensure that the propagation velocity for a given direction n_i is real.

The tensor C_{km} (3.36), essential in the case of small vibrations, is, with accuracy to a constant multiplier, equal to Q_{km}, Eq. (4.2). Small vibrations of the form leading to the equations given above are, on the other hand, generally impossible, while the propagation of a wave defined by Q_{km} is always possible. This fact has been stressed by C. TRUESDELL for a material possessing a general symmetry. It follows from the considerations presented here that in the particular case of isotropic materials, no coincidence exists between small vibrations and the propagation.

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