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Additivity of mechanical power and the principle of stress

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THE PAPER is devoted to the relation between the principle of stress and the additivity of power of mechanical interactions. The considerations are based on a different from the classical system of axioms describing the mechanical interactions. The primitive concept is the power functional of the mechanical interactions. It is assumed that it is linear and continuous with respect to the velocity field. The force and moment are defined in terms of the above functional. The principle of stress is formulated as follows: the power consists of the powers of body and contact forces; the contact forces are identical on the common parts of the boundaries of each pair of subbodies, while the body forces are identical for every subbody. It is proved that the principle of stress constitutes a necessary condition of additivity of the power functional of mechanical interactions, but is not a sufficient condition.

W pracy rozpatrzono związek zasady naprężenia z addytywnością mocy oddziaływań mechanicznych. Rozważania oparto na odmiennym od klasycznego układzie aksjomatów charakteryzujących oddziaływania mechaniczne. Jako pojęcie pierwotne przyjęto funkcjonał mocy oddziaływań mechanicznych. Założono o nim, że jest liniowy i ciągły ze względu na pole prędkości. Pojęcia siły i momentu zdefiniowano przy jego pomocy. Zasadę naprężenia sformułowano następująco: moc składa się z mocy sił masowych i kontaktowych, siły kontaktowe są identyczne na częściach wspólnych brzegów każdej pary podciał, natomiast siły masowe są identyczne dla każdego podciała. Wykazano, że zasada naprężenia jest warunkiem koniecznym addytywności funkcjonału mocy oddziaływań mechanicznych, a nie jest warunkiem dostatecznym.

В работе исследована связь принципа напряжения с аддитивностью мощности механических взаимодействий. Исследование основано на системе аксиом, характеризующих механические взаимодействия, отличающейся от классической. В качестве основного понятия использован функционал мощности механических взаимодействий, который предполагается, линейным и непрерывным относительно поля скоростей. Понятия силы и момента сформулированы на его основе. Принцип напряжения выражен следующим образом: мощность состоит из мощности массовых и контактных сил; контактные силы тождественно равны на общих участках краевой поверхности произвольной пары подтел, тогда как массовые силы совпадают для произвольных подтел. Показано, что принцип напряжения является необходимым условием аддитивности функционала мощности механических взаимодействий, но не является достаточным условием.

1. Introduction

THE AIM of the paper is to analyse relations between the principle of stress and the additivity of the mechanical power. For principal reasons the analysis is based on a modified as compared with classical, axiomatics of mechanical interactions. A complete presentation of this axiomatics will be given elsewhere; in Sec. 6 we expose only the axioms of mechanical interactions necessary for further considerations. The primitive concept is the functional of mechanical interactions. It is only assumed that it is linear and continuous with respect to the velocity field. The remaining concepts, such as force and moment, are defined in terms of the above functional. The principle of stress is formulated in a somewhat more general manner than by W. NOLL and C. TRUESDELL [8]. It is assumed that the power consists of the powers of contact and body forces, the contact forces being identical on the common parts of the boundaries of each pair of subbodies, while the body forces are identical for each subbody.

Our axiomatics of the geometric structure of the body is modified as compared with that used by M. GURTIN and W. WILLIAMS [4]. The body and the subbodies are defined by means of topological concepts.

In Sec. 3 we present a few topological theorems and some theorems of the measure theory and functional analysis. The author is convinced that these theorems are not original, but has not been able to find them in literature.

2. Notations

$\bigwedge$, $\bigvee$	quantifiers, general and particular,
G, F, B	classes of subsets: open, closed and borelian,
o, $-$, ∂, c	operations: opening, closure, bounding and completion,
G_{-}	class of closures of open subsets,
F_0	class of openings of closed subsets,
K_c	class of completions of sets of class $\mathbf{K}$,
$\times$	symbol of Cartesian product,
p_i	projection operator from $X^1 \times \ldots X_n$ onto X_i, $i \in \{1, \ldots, n\}$,
C	class of continuous functions,
C^p	class of functions with continuous derivative of p-th order,
Df, $\mathfrak{a}f$	domain and image of the function f,
$\mid$	symbol of restriction of the function domain,
I	the identity function,
μ^+, μ^-, μ^*	variations: upper, lower and total of the function μ,
$\mathcal{R}$	the set of real numbers,
$\mathcal{R}_n$	n-dimensional Euclidean space,
tr	symbol of trace.

3. Mathematical preliminaries

Our definitions follow R. ENGELKING [3]. In what follows we shall frequently use the following, little known theorem on the properties of opening and closure operations.

THEOREM 1. *For an arbitrary topological space X and arbitrary sets A* and B we hawe

$$\text{(i)} \bigwedge_{A\subset X} A^{o-} = A^{o-o-}, \qquad \text{(ii)} \bigwedge_{A\subset X}, \bigwedge_{B\subset X} (A^- + B)^{o-} = A^{-o-} + B^{o-}.$$

Concerning this theorem see K. KURATOWSKI and A. MOSTOWSKI [7], theorems I.8(15) and I.8(18).

THEOREM 2. *For an arbitrary topological space X the class* C (*the class of closures of open subsets*) *is the greatest Boolean algebra with respect to the operations* $\bigvee$, $\bigwedge$ *and b defined by the formulae*

$$A \vee B = A + B, \quad A \wedge B = (AB)^{o-}, \quad A^b = A^{c-}. \tag{3.1}$$

For an arbitrary regular space X the class G_{-c} *constitutes a basis of this space.*

Prior to proving this theorem we shall prove a few Lemmas.

LEMMA 1. *The class G_- is the class of closed domains.*

P r o o f. For an arbitrary closed domain A, by definition we have $A = A^{0-}$. Hence $A \in G_-$, since $A^0 \in G$.

Let $A \in G_-$. Thus, there exists an open set B such that $A = B^-$ whence $A^{0-} = B^{0-0-}$. It follows from Theorem 1(i) that $A^{0-} = B^{0-} = A$; therefore A is an closed domain.

LEMMA 2. *For an arbitrary topological space X we have $G_{-c} = F_0$.*

P r o o f. The theorem follows from the following relation between the opening and closure operations: $A^{-c} = A^{c0}$.

P r o o f of Theorem 2. It can be proved that the class G_- is a Boolean algebra (R. SIKORSKI [9], § 1, Example B).

Let U be a Boolean algebra with respect to the operations $\bigvee, \bigwedge, {}^b$ and let A and B be its arbitrary elements. We have $A \bigvee A^b = X$ and $(A \bigvee A^b) \bigwedge B = B^{0-}$. In view of one of the axioms of Boolean algebras $(A \bigvee A^b) \bigwedge B = B$ whence $B^{0-} = B$ and on the basis of Lemma 1, $U \subset G_-$. Thus, class G_- is the greatest Boolean algebra with respect to the operations $\bigvee, \bigwedge, {}^b$.

The properties of the basis and Lemma 2 imply that to prove the second part of the theorem it is sufficient to prove that $\bigwedge_{\xi \in U \in G} \bigvee_{A \in F_0} \xi \in A \subset U$.

Let $\xi \in U \in G$. The one-point set $\{\xi\}$ and the closed set U^c are disjoint; hence, in view of the regularity of the space they have disjoint neighbourhoods U_1 and U_2. Thus, $\xi \in U_1 \subset \subset U_2^c \subset U$. Since the set U_2^c is closed, the closure operation implies that $U_1 \subset U_1^- \subset U_2^c$ whereas we have $U_1 \subset U_1^{-0} \subset U_2^c$ in accordance with the opening operation. Hence, $\xi \in U_1^{-0} \subset U$ and since $U_1^{-0} \in F_0$ ✳ the theorem is proved.

Our terminology of the measure theory and functional analysis follows that of A. ALEXIEWICZ [1]. The frequently employed concepts of the Radon measure and vector. Radon measure are the following: by the Radon measure we understand a real set function constituting a difference of two finite regular Borel measures. By the vectorial Radon measure we understand every set function a with values in the space $\mathscr{R}_n$, such that for every $i = 1, 2, \ldots, n$ the superposition $p_i \circ a$ is a Radon measure.

THEOREM 3. *Let A be open subset of a compact space X and a a vectorial Radon measure with values in $\mathscr{R}_n$, defined on Borel subsets of the compact space X. The integral $\int_A x \cdot da$ of every continuous function x; $X \to \mathscr{R}_n$ such that $x|A^c = 0$ vanishes, if and only if, the measure a vanishes on all Borel subsets of the set A.*

Prior to proving this theorem we present a few Lemmas.

LEMMA 3. *If μ is a Radon measure, then*

$$\bigwedge_{A \in B}, \bigwedge_{\varepsilon > 0} \bigvee_{U \in G}, \bigvee_{F \in F} U \supset A \supset F, [(E \subset U - F, E \in B) \Rightarrow |\mu(E)| \leqslant \varepsilon].$$

P r o o f. The definition of the Radon measure implies the existence of finite regular Borel measures μ_1 and μ_2, such that $\mu = \mu_1 - \mu_2$. Let ε be a positive number. If follows from the regularity of the measures μ_1 and μ_2 that the set A has neighbourhoods U_1 and U_2 and contains closed sets F_1 and F_2 such that

$$\bigwedge_{\substack{E \in B \\ E \subset U_1 - F_1}} \mu_1(E) \leqslant \varepsilon/2, \qquad \bigwedge_{\substack{E \in B \\ E \subset U_2 - F_2}} \mu_2(E) \leqslant \varepsilon/2.$$

The above conditions are simultaneously satisfied for every Borel set contained in the set $U-F$, where $U = U_1 U_2$ and $F = F_1+F_2$, since $U-F \subset U_1-F_1$ and $U-F \subset U_2-F_2$. Hence,

$$\bigwedge_{\substack{E\in\mathsf{B}\\ E\subset U-F}} |\mu(E)| = |\mu_1(E)-\mu_2(E)| \leqslant \mu_1(E)+\mu_2(E) \leqslant \varepsilon.$$

LEMMA 4. *If μ is a Radon measure, then*

$$\bigwedge_{A\in\mathsf{B}} \mu^+(A) = \sup\{\mu(F) : F\in\mathsf{F}, F\subset A\},$$
$$\mu^-(A) = \sup\{-\mu(F) : F\in\mathsf{F}, F\subset A\}.$$

P r o o f. Consider a Borel set A and a positive number ε. The properties of a bounded real set function constituting a Radon measure μ imply (N. DUNFORD and J. SCHWATRZ [2] III. 1.8) that there exists a Borel set B contained in A such that $\mu^+(A) \leqslant \mu(B)+\varepsilon/2$.

It follows from Lemma 3 that the set B has a neighbourhood U and contains a closed set F, such that for every Borel set E contained in the set $U-F$ we have $\mu(E) \leqslant \varepsilon/2$. The set $B-F$ is a Borel set, since it constitutes a difference of two Borel sets, and is contained in $U-F$ because $U \supset B$. Hence $\mu(B-F) \leqslant \varepsilon/2$. The measures μ_1 and μ_2 are subtractive (P. HALMOS [5] II.9.1), whence their difference is also subtractive; therefore $\mu(B) \leqslant \mu(F)+$ $+\varepsilon/2$ and $\mu^+(A) \leqslant \mu(F)+\varepsilon$. The inequality

$$\bigwedge_{\substack{F\in\mathsf{F}\\ F\subset A}} \mu^+(A) \geqslant \mu(F)$$

follows directly from the definition of the upper variation.

The proof of the second part of the Lemma is very similar.

LEMMA 5. *Let A be an open subset of a compact space X and let μ be a Radon measure. The integral $\int_A x d\mu$ of every continuous real function x such that $x/A^c = 0$ vanishes, if and only if, the Radon measure μ vanishes on all Borel subsets of the set A.*

P r o o f. The sufficiency of the condition $\mu|\mathsf{B}(A) = 0$ is obvious. Let F be a closed set contained in A. The upper and lower variations μ^+ and μ^-, respectively, are obviously regular Borel measures. Hence, for an arbitrary positive number ε there exist neighbourhoods U_1 and U_2 of the set F such that $\mu^+(U_1-F) \leqslant \varepsilon/2$ and $\mu^-(U_2-F) \leqslant \varepsilon/2$. The set $U = U_1U_2A$ constitutes a neighbourhood of the set F and is contained in the neighbourhoods U_1 and U_2. In view of the monotonicity of the variations $\mu^+(U-F) \leqslant \varepsilon/2$ and $\mu^-(U-F) \leqslant \varepsilon/2$.

Let us now make use of the fact that the compact space is normal. The sets F and U^c are closed and disjoint; consequently, the Urysohn lemma implies the existence of a continuous function x_0 defined on the space X with values in the closed interval [0, 1], such that $x_0|F = 1$ and $x_0|U^c = 0$. It can readily be verified that this function satisfies all conditions of the Theorem; hence $\int_A x_0 d\mu = 0$. Making use of the properties of the function x_0 and the integral, we obtain

$$|\mu(F)| = |\int_{U-F} x_0 d\mu| = |\int_{U-F} x_0 d\mu^+ - \int_{U-F} x_0 d\mu^-| \leqslant \int_{U-F} x_0 d\mu^+ + \int_{U-F} x_0 d\mu^- \leqslant$$
$$\leqslant \mu^+(U-F)+\mu^-(U-F) \leqslant \varepsilon.$$

Since ε is arbitrary, $\mu(F) = 0$ and in view of Lemma 4, for an arbitrary Borel subset B of the set A we have $\mu^+(B) = \mu^-(B) = 0$. Thus $\mu(B) = 0$.

Proof of Theorem 3. If x; $X \to \mathscr{R}$ is a continuous function, then the function f_i; $X \to \mathscr{R}_n$ given by $f_i = (\underbrace{0, \ldots, 0}_{(i-1)}, x, \underbrace{0, \ldots, 0}_{(n-i)})$ is also a continuous function. If $x|A^c = 0$, then $f_i|A^c = 0$. Making use of Lemma 5 we find now that for every $i = 1, \ldots, n$ we have $p_i \circ a \,|\mathsf{B}(A) = 0$, and, consequently, $a|\mathsf{B}(A) = 0$.

LEMMA 6. *A linear functional defined on the Cartesian product $X = X_1 \times \ldots \times X_n$ of linear normed spaces X_i is continuous, if and only if, for every $i = 1, \ldots, n$ there exists a continuous linear functional f_i defined on X_i, such that $f = \sum_{i=1} f_i \circ p_i$. If the norm of the space X is defined by the formula $\| \; \| = \left(\sum_{i=1}^{n} \|p_i \circ \|_i^2\right)^{1/2}$, then the norm of the functional f is $\|f\| = \left(\sum_{i=1}^{n} \|f_i\|^2\right)^{1/2}$.*

Proof. The first part of the Theorem is the same as in A. ALEXIEWICZ [1], III. 10.3. Let x be an element of the space X. The first part of the Theorem implies that $|f(x)| \leqslant \sum_{i=1}^{n} |f_i \circ p_i(x)|$. Making use of the properties of the norm of a functional and the Hölder inequality, we obtain the inequality

$$|f(x)| \leqslant \sum_{i=1}^{n} \|f_i\| \; \|p_i(x)\|_i \leqslant M\|x\|,$$

where $M = \left(\sum_{i=1}^{n} \|f_i\|^2\right)^{1/2}$. It follows from the properties of the norm of a functional that $M \geqslant \|f\|$.

Let ε be a positive number. The properties of the norm of a functional imply that for every $i = 1, \ldots, n$ there exists $x_i \in X_i$ such that $\|x_i\|_i \leqslant 1$ and

$$f_i(x_i) \geqslant \|f_i\| - \varepsilon M n^{-1} \|f_i\|^{-1}.$$

Let

$$x = M^{-1}(x_1\|f_1\|, \ldots, x_n\|f_n\|).$$

Then $x \in X$ and

$$\|x\| = M^{-1}\left(\sum_{i=1}^{n} \|x_i\|_i^2 \|f_i\|^2\right)^{1/2} \leqslant M^{-1}\left(\sum_{=1}^{n} \|f_i\|^2\right)^{1/2} = 1,$$

in view of the linearity of the functionals f_i,

$$f(x) = M^{-1} \sum_{i=1}^{n} \|f_i\| f_i(x_i),$$

whence

$$f(x) \geqslant M^{-1} \sum_{i=1}^{n} \|\mathrm{f}_i\|^2 - \varepsilon = M - \varepsilon,$$

thus M is the norm of the functional f.

THEOREM 4. *For every continuous linear functional f defined on the space C_n of all continuous functions defined on a compact space X with values in $\mathscr{R}_n$ there exists exactly one vectorial Radon measure a defined on the class of Borel subsets of the space X with values in $\mathscr{R}_n$, such that for every function $x \in C_n$, $f(x) = fx \cdot da$. The norm of this functional is*

$$\|f\| = \Big\{ \sum_{i=1}^{n} [(p_i \circ a)^*(X)]^2 \Big\}^{1/2}.$$

P r o o f. The theorem follows from Riesz theorem (A. ALEXIEWICZ [1], VIII, 2.2.), Lemma 6 and the fact that the space C_n is a Cartesian product of n spaces of continuous functions.

4. Continuous material body

The continuous material body or briefly *body* is a set $\mathscr{B}$ with the continuum cardinal number, consisting of elements ξ called *material particles of the body*, with the class of functions described by axioms BI–BV, defined on the Cartesian product $\mathscr{B} \times \mathscr{R}$ with values in the space $\mathscr{R}_3$ and called motions of the body.

BI. For every motion of the body χ and every real number τ, the function χ_τ; $\mathscr{R} \to \mathscr{R}_3$ called hereafter the *configuration of the body $\mathscr{B}$ at instant τ moving in accordance with χ*, or briefly the *configuration of the body*, defined by the formula $\bigwedge_{\xi \in \mathscr{B}} \chi_\tau(\xi) = \chi(\xi, \tau)$, is a one-to-one function, such that its image $\mathrm{G}\chi_\tau$ is a set of class $G_-(\mathscr{R}_3)$.

BII. For every pair of configurations of the body $\chi_{1\tau_1}$ and $\chi_{2\tau_2}$ the superposition $\chi_{1\tau_1} \circ (\chi_{2\tau_2}^{-1})$ is a diffeomorphism of class $C_3^p(\mathrm{G}\chi_{2\tau_2})$, $p \geqslant 1$ of the subspaces $\mathrm{G}\chi_{1\tau_1}$ and $\mathrm{G}\chi_{2\tau_2}$.

BIII. For every configuration of the body $\chi_{1\tau}$ and every motion χ_2 the function $\chi_2 \circ (\chi_{1\tau}^{-1}, \mathrm{I})$ is of class $C_3^p(\mathrm{D}\chi_1 \times \mathscr{R})$.

BIV. For every motion χ the function $\dot{\chi}_\tau$; $\mathrm{G}\chi_\tau \to \mathscr{R}_3$, called hereafter the velocity *field of the body at instant τ* moving in accordance with χ, or briefly the *velocity of the body*, defined by the formula $\bigwedge_{x \in \mathrm{G}\chi_2} \dot{x}_\tau(x) = \frac{\partial}{\partial \tau} x(\xi, \tau)|_{\xi = x_\tau^{-1}(x)}$ is bounded.

BV. For every motion χ at every instant $\tau \in \mathscr{R}$, for every vector $a \in \mathscr{R}_3$ and every antisymmetric tensor of rank 2 $A \in \mathscr{R}_3$, there exist motions χ_1, χ_2 and instants τ_1, τ_2 such that $\mathrm{D}\chi_\tau = \mathrm{D}\chi_{1\tau_1} = \mathrm{D}\chi_{2\tau_2}$, $\dot{\chi}_{1\tau_1} = \dot{\chi}_\tau + a$ and $\dot{\chi}_{2\tau_2} = \dot{\chi} + A$ (in the above formulae the symbol of the vector a is interpreted as the symbol of a constant function with the value a; the symbol of the tensor A is interpreted as the symbol of a linear function defined on the space $\mathscr{R}_3$ with values Ax).

The axioms BI–BIII constitute the mathematical statement of the concept of the continuity of the body. The special role of sets of the class $G_-(\mathscr{R}_3)$ in axiom BI follows from the fact that they are the most general subsets of the space $\mathscr{R}_3$ on which diffeomorphisms may be defined. The axiom BIV concerning the boundedness of the velocity field is of a physical nature. The axiom BV ensures the sufficient number of motions of the body.

THEOREM 5. *For every motion of the body χ and for every instant $\tau \in \mathscr{R}$ the image $\mathrm{G}\chi_\tau$ is a compact subspace of the space $\mathscr{R}_3$.*

P r o o f. Consider a configuration χ_τ and set

$$A_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}, \quad A_2 = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad A_3 = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

The axioms BIV–BV imply the existence of real numbers α_1, α_2 and α_3 such that for every $x \in \mathrm{C\!l}\,\chi_\tau$ we have $|A_i x| \leqslant \alpha_1$, $i = 1, 2, 3$. On the other hand it can easily be verified that for every $x \in \mathscr{R}_3$ we have $|x| = 2^{-1/2}\left(\sum_{i=1}^{3} |A_i x|^2\right)^{1/2}$. Hence, for every $x \in \mathrm{C\!l}\,\chi_\tau$, $|x| \leqslant 2^{-1/2} \times \left(\sum_{i=1}^{3} \alpha_i^2\right)^{1/2}$. Thus, the image $\mathrm{C\!l}\,\chi_\tau$ is contained in a sphere of radius $2^{-1/2}\left(\sum_{i=1}^{3} \alpha_i^2\right)^{1/2}$ and centre at zero. In view of the Bolzano-Weierstrass theorem this sphere is a compact subspace. In view of the heredity of the compactness with respect to closed subspaces, $\mathrm{C\!l}\,\chi_\tau$ is also a compact subspace.

THEOREM 6. *In a body there exists one and only one topology such that every configuration of the body χ_τ is a homeomorphism of the body and the image of the configuration $\mathrm{C\!l}\,\chi_\tau$. In this topology every function $\mathring{\chi}_\tau$; $\mathscr{B} \to \mathscr{R}_3$ defined by the formula $\bigwedge_{\xi \in \mathscr{B}} \mathring{\chi}_\tau(\xi) = \frac{\partial}{\partial \tau}\chi(\xi, \tau)$, where χ is a motion of the body, is continuous.*

P r o o f. This theorem follows from axioms BI-BIII and the properties of the diffeomorphism. This topology consists of the class of inverse images of open sets in the image of the configuration of the body; more precisely this topology consists of the class $\chi_\tau^{-1}[\mathrm{G}(\mathrm{C\!l}\,\chi_\tau)]$, where χ is an arbitrary motion and τ an arbitrary instant.

In our further considerations we shall regard the body as a topological space with the above topology.

5. Subbodies of a continuous material body

The class of subbodies of a continuous material body or briefly the *class of subbodies* is a class $\mathscr{S}$ of subsets of the body $\mathscr{B}$ satisfying the following axioms.

SI. For every motion of the body χ and every instant $\tau \in \mathscr{R}$ the class of complements (with respect to $\mathrm{C\!l}\,\chi_\tau$) of the images of the subbodies $[\chi_\tau(\mathscr{S})]_c$ is a basis of the topological subspace $\mathrm{C\!l}\,\chi_\tau$.

SII. For every motion of the body χ and every instant $\tau \in \mathscr{R}$ the class of images of the subbodies $\chi_\tau(\mathscr{S})$ is a Boolean algebra with respect to the operations $\bigvee$, $\bigwedge$ and ^{b} defined by the formulae (3.1) in the topology of the subspace $\mathrm{C\!l}\,\chi_\tau$.

The necessity of introducing the class of subbodies is due to the necessity of investigating additive set functions. The axiom SI ensures the sufficient number of the subbodies. The axiom SII is aimed at providing the subbodies with geometric properties of the body and making it possible to divide the body into two "disjoint" subbodies. The axioms SI and SII are not contradictory — this statement follows from Theorem 2.

Evidently $\mathscr{S} \subset \mathsf{G}_-(\mathscr{B})$, $\mathscr{S}_c$ is a basis of the body $\mathscr{B}$ regarded as a topological space and $\mathscr{S}$ is a Boolean algebra with respect to the operations $\bigvee, \bigwedge$ and ^{b} defined in the topology of the body.

The subbodies A and B will be *called disjoint* if for every motion x and every instant τ, $\chi_\tau(A) \bigwedge \chi_\tau(B) = 0$, where the operation $\bigwedge$ is defined in the topology of the subspace $\mathrm{C\!l}\,\chi_\tau$. It is obvious that these subbodies are disjoint, if and only if, $A \bigwedge B = 0$, the operation $\bigwedge$ being defined in the topology of the space $\mathscr{B}$.

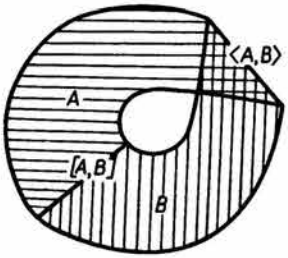
For every subbody A the pair of subbodies A and A^b is a division of the body $\mathscr{B}$ into disjoint subbodies.

Consider a pair of subbodies A, B and the sets $[A, B]$, $\langle A, B\rangle$ and $\{A, B\}$ defined as follows:

$$[A,B] = A^\partial B^\partial (A+B)^0, \quad \langle A,B\rangle = A^\partial B^\partial (A\bigwedge B + A^b \bigwedge B^b)^0,$$

$$\{A,B\} = (A+B)^\partial \langle A, A+B\rangle^c \langle B, A+B\rangle^c.$$

The set $[A, B]$ is this part of the boundaries of the subbodies A and B for which they are situated on the opposite sides (Fig. 1). The set $\{A, B\}$, called hereafter the common part of the boundaries of the pair of subbodies A, B, is the part of the boundaries of the subbodies A and B for which these subbodies are situated on the same side (Fig. 1). If the subbodies A and B are disjoint, then the set $\{A, B\}$ is the boundary of the sets $[A, B]$ and $\langle A, A+B\rangle$ (Fig. 2). The justification for the above "geometric" interpretation of the sets



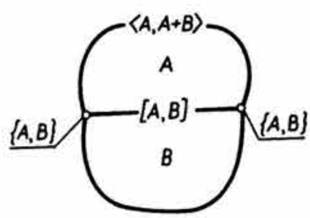


FIG. 2.

$[A, B]$, $\langle A, B\rangle$ and $\{A, B\}$ are the following two theorems.

THEOREM 7. *For arbitrary pair of subbodies A, B we have*

$$\text{(i)} \quad \langle A,B\rangle = A^\partial (A\bigwedge B + A^b \bigwedge B^b)^0,$$

$$\text{(ii)} \quad \langle AB\rangle [A,B] = 0,$$

$$\text{(iii)} \quad \langle A, A+B\rangle [A,B] = 0.$$

THEOREM 8. *For every pair of disjoint subbodies A, B,*

$$\text{(i)} \quad [A,B] = A^\partial (A+B)^0,$$

$$\text{(ii)} \quad \langle A, A+B\rangle \langle B, A+B\rangle = 0,$$

$$\text{(iii)} \quad [A,\mathrm{B}]\{A,B\} = 0,$$

(iv) $\langle A,B\rangle \{A,B\} = 0,$

(v) $(A+B)^{\partial} = \langle A,A+B\rangle + \langle B,A+B\rangle + \{A,B\},$

(vi) $A^{\partial} = \langle A,A+B\rangle + [A,B] + \{A,B\}.$

Proof of Theorem 7. (i) The properties of the closure operation imply that $(A^0B^0)^- \subset \subset A^{0-}B^{0-}$ and $(A^cB^c)^- \subset A^{c-}B^{c-}$. Consequently,

$$(5.1)\qquad (A^0B^0)^-(A^cB^c)^- \subset B^{\partial}.$$

For arbitrary subsets C, D of an arbitrary topological space we have the inclusion

$$(5.2)\qquad (C+D)^0 \subset C^0 + C^-D^- + D^0.$$

Making use of the inclusions (5.2) and (5.1) we obtain

$$[(A^0B^0)^- + (A^cB^c)^-]^0 \subset (A^0B^0)^{-0} + (A^cB^c)^{-0} + B^{\partial}.$$

Since, the sets A^0B^0 and A^cB^c are open domains

$$[(A^0B^0)^- + (A^cB^c)^-]^0 \subset A^0B^0 + A^cB^c + B^{\partial} \subset A^0 + A^c + B^{\partial}.$$

The above inclusion is equivalent to the following:

$$A^{\partial}(A\wedge B + A^b\wedge B^b)^0 \subset \langle A,B\rangle.$$

The inverse inclusion is obvious.

(ii) For arbitrary subsets C, D of an arbitrary topological space we have the identity $(C^0D)^- = (C^0D^-)^-$ (K. KURATOWSKI [6], 5.4) equivalent to the identity

$$(5.3)\qquad (C^- + D)^0 = (C^- + D^0)^0.$$

Hence

$$[(AB)^{0-} + (A+B)^{\partial}]^0 = (AB)^{0-0} = A^0B^0.$$

On the other hand

$$\langle A,B\rangle [A,B] = A^{\partial}B^{\partial}[(AB)^{0-} + (A+B)^{\partial}]^0,$$

therefore

$$\langle A,B\rangle [A,B] = A^{\partial}B^{\partial}A^0B^0 = 0.$$

(iii) The proof consists in applying the relation

$$\langle A,A+B\rangle [A,B] = A^{\partial}B^{\partial}[A + (A+B)^{\partial}]^0,$$

and the identity (5.3) to the set $[A+(A+B)^{\partial}]$.

Proof of Theorem 8. (i) For arbitrary subsets C, D of an arbitrary topological space we have the inclusion $(C+D)^0 \subset C^0 + D^-$. For subbodies A, B, therefore, $(A+B)^0 \subset \subset A^0 + B$. Hence $A^{\partial}(A+B)^0 \subset AB$. Since the subbodies A and B are disjoint $AB = A^{\partial}B^{\partial}$ and, consequently, $A^{\partial}(A+B)^0 \subset [A, B]$. The inverse inclusion is obvious.

(ii) From the properties of a Boolean algebra

$$[A\wedge(A+B) + A^b\wedge(A+B)^b] = A + B^b$$

while in view of the disjointness of the subbodies A and B $A+B^b = B^b$. Therefore

$$(5.4)\qquad \langle A,A+B\rangle = A^{\partial}B^c,$$

and

$$\langle A, A+B\rangle \langle B, A+B\rangle = (A^{\partial}A^{c})(B^{\partial}B^{c}) = 0.$$

The relations (iii), (iv) and (v) follow directly from the definitions of the sets $\langle A, B\rangle$, $[A, B]$ and $\{A, B\}$.

(vi) After simple transformations we obtain

$$\langle A, A+B\rangle + [A, B] + \{A, B\} = A^{\partial}\{[B(A+B)^{0c}]^{c} + B(A+B)^{0c}\} = A^{\partial}.$$

6. The functional of the mechanical interactions power

Consider a motion χ of the body $\mathscr{B}$. The history of the motion of the body $\mathscr{B}$ to the instant τ or briefly the history of motion is the function $\chi^{(\tau)}$; $\mathscr{B} \times [0, \infty) \rightarrow \mathscr{R}_3$ defined by the formula

$$\bigwedge_{(\xi,\sigma)\in\mathscr{B}\times[0,\infty)} x^{(\tau)}(\xi, o) = x(\xi, \tau-\sigma).$$

The concept of the functional of mechanical interactions power is an original concept of this paper. In the classical axiomatics of the continuum mechanics the concept of power is defined in terms of the concepts of the stress and body force; more precisely it is defined as a certain particular value of a functional defined by the stress and body forces. Evidently, the concepts of the stress and body force are subject to certain restrictions ensuring the possibility of definition of the considered functional. The concept of axiomatics of mechanical interactions presented here constitutes an inversion of the above logical chain of reasoning. As the fundamental concept we have here the functional of mechanical interactions power; in fact we introduce two functionals, namely the functional of mechanical interaction power of an arbitrary subbody on an arbitrary subbody disjoint from it and the functional of mechanical interaction power of the neighbourhood of the body on an arbitrary subbody. The postulates FI and FII constitute a precise formulation of the requirements concerning the functionals.

FI. For every pair of disjoint subbodies A, B and for every history of motion $\chi^{(\tau)}$ there exists a continuous linear functional $\pi^{I}_{A,B,\chi^{(\tau)}}$; $C_3[\chi_\tau(B)] \rightarrow \mathscr{R}$ called hereafter *the functional of mechanical interactions power of subbody A on subbody B at the instant τ the history of motion being $\chi^{(\tau)}$*, or briefly *the functional of mechanical interactions power of subbody A on subbody B.*

FII. For every subbody A and for every history of motion $\chi^{(\tau)}$ there exists a continuous linear functional $\pi^{II}_{A,\chi^{(\tau)}}$; $C_3[\chi_\tau(A)] \rightarrow \mathscr{R}$ called hereafter *the functional of mechanical interactions power of the neighbourhood of the body $\mathscr{B}$ on the subbody A at the instant τ the history of motion being $\chi^{(\tau)}$* (or briefly *the functional of mechanical interactions power of the neighbourhood of the body $\mathscr{B}$* on the subbody A.

The introduced concepts of the functionals of mechanical interactions power make it possible to define such concepts as force, moment, power. *The force of interaction of subbody A on subbody B at the instant τ the history of motion being $\chi^{(\tau)}$*, or briefly *the force of interaction of subbody A on subbody B* is the vector $f^{I}_{A,B\,\chi^{(\tau)}}$ satisfying the condition

$\bigwedge_{a \in \mathscr{R}_3} a \cdot f^{I}_{A,B,\chi^{(\tau)}} = \pi^{I}_{A,B,^{(\tau)}}(a)$. The definition of the force $f^{I}_{A,B,\chi^{(\tau)}}$ is logical and unique, since the constant a is a continuous function and in view of the linearity of the functional $\pi^{I}_{A,B,\chi^{(\tau)}}$ the existence and uniqueness of the vector $f^{I}_{A,B,\chi^{(\tau)}}$ is obvious.

On the other hand, the condition defining the force $f^{I}_{A,B,\chi^{(\tau)}}$ is satisfied in all existing theories of mechanical interactions.

The moment with respect to the point $0 \in \mathscr{R}_3$ *of the interaction of subbody A on subbody B at the instant* τ, *the history of motion being* $\chi^{(\tau)}$ or briefly *the moment of interaction of subbody A on subbody B*, is the antisymmetric tensor $M^{I}_{A,B,^{(\tau)}}$ of rank 2 such that for every antisymmetric tensor S we have $\mathrm{tr}(SM^{I}_{A,B,\chi^{(\tau)}}) = \pi^{I}_{A,B,\chi^{(\tau)}}(S)$. The justification of this definition of the moment is similar to that of the force $f^{I}_{A,B,\chi^{(\tau)}}$.

The power of interaction of subbody A on subbody B at the instant τ, *the history of motion being* $\chi^{(\tau)}$ or briefly *the power of interaction of subbody A on subbody* $\mathscr{B}$, *is the number* $\pi^{I}_{A,B,\chi^{(\tau)}}(\dot{\chi}_\tau)$.

In an analogous manner we can define *the force* $f^{II}_{A,\chi^{(\tau)}}$, *the moment* $M^{II}_{A,\chi^{\tau}}$ *and the interaction power of the neighbourhood of the body* $\mathscr{B}$ *on subbody A at the instant* τ, *the history of motion being* $\chi^{(\tau)}$. Moreover, we can define the concept of *the functional* $\tilde{\pi}_{A,\chi^{(\tau)}}$ *of the power of resultant mechanical interactions on subbody A at the instant* τ, *the history of motion being* $\chi^{(\tau)}$, as the sum of the functionals $\pi^{I}_{A^b,A,\chi^{(\tau)}}$ and $\pi^{II}_{A,\chi^{(\tau)}}$. The concept of the functional of power of resultant mechanical interactions may be used to define the resultant force, resultant moment and the power of resultant mechanical interactions.

The power of resultant mechanical interactions is called *additive* if for every pair A, B of disjoint subbodies, for every history of motion $\chi^{(\tau)}$ and for arbitrary function $\vartheta \in \mathrm{C}_3(\mathrm{Cl}\,\chi_\tau)$,

$$\tilde{\pi}_{A,x^{(\tau)}}[\vartheta|x_\tau(A)] + \tilde{\pi}_{B,x^{(\tau)}}[\vartheta|x_\tau(A)] = \tilde{\pi}_{(A\vee B),x^{(\tau)}}[\vartheta|x_\tau(A \vee B)].$$

In most of the existing theories of mechanical interactions in a continuous medium, the resultant mechanical interactions are described by means of the stress tensor, the vector of body forces, tensors of hyperstress and body hyperforces. The additivity of the powers of these interactions is then obvious. The aim of this paper is an analysis of the implications of the additivity of the power of the resultant mechanical interactions.

To simplify the notations we change the domain of the functional of power of resultant mechanical interactions, from the space $\mathrm{C}_3[\chi_\tau(A)]$ to the space $\mathrm{C}_3(A)$. This change is formal, in view of the homeomorphism of every subbody with its image. For every subbody A and every history of motion $\chi^{(\tau)}$ there exists therefore exactly one continuous linear functional $\pi_{A,\chi^{(\tau)}}$; $\mathrm{C}_3(A) \to \mathscr{R}$ such that for every function $\vartheta \in \mathrm{C}_3[\chi_\tau(A)]$ we have $\tilde{\pi}_{A,\chi^{(\tau)}}(\vartheta) = \pi_{A,\chi^{(\tau)}}(\vartheta_0\chi_\tau)$. The power of resultant mechanical interactions is additive, if and only if, for every pair A, B of disjoint subbodies, for every history of motion $\chi^{(\tau)}$ and every function $\vartheta \in \mathrm{C}_3(\mathscr{B})$ we have $\pi_{A,\chi^{(\tau)}}(\vartheta|A) + \pi_{B,\chi^{(\tau)}}(\vartheta|B) = \pi_{(A\vee B),\chi^{(\tau)}}(\vartheta|A \vee B)$. The axioms FI and FII and Theorem 4 imply that for every subbody A and every history of motion $\chi^{(\tau)}$there exists exactly one vectorial Radon measure $a_{A,\chi^{(\tau)}}$; $\mathrm{B}(A) \to \mathscr{R}_3$ such that

$$\bigwedge_{\vartheta \in \mathrm{C}_3(A)} \pi_{A,x^{(\tau)}}(\vartheta) = \int_A \vartheta \cdot da_{A,\chi^{(\tau)}}$$

and

$$\|\pi_{A,\chi^{(\tau)}}\| = \left\{\sum_{i=1}^{3} [(p_i \circ a_{A,\chi^{(\tau)}})^*(A)]^2\right\}^{1/2}.$$

7. Generalized principle of stress

We say that the generalized principle of stress is satisfied if for every subbody A and every history of motion $\chi^{(\tau)}$ there exist vectorial Radon measures $b_{\chi^{(\tau)}}$; $\mathrm{B}(\mathscr{B}) \to \mathscr{R}_3$ and $t_{A,\chi^{(\tau)}}$; $\mathrm{B}(A^\partial) \to \mathscr{R}_3$ such that:

(i) for every function $\vartheta \in C_3(\mathscr{B})$ and for every subbody A

$$\pi_{A,\chi^{(\tau)}}(\vartheta/A) = \int_{A^0} \vartheta \cdot db_{\chi^{(\tau)}} + \int_{A^\partial} \vartheta \cdot dt_{A,\chi^{(\tau)}},$$

(ii) for every pair of subbodies A, B,

$$t_{A,\chi^{(\tau)}}|\mathrm{B}(\langle A,B\rangle) = t_{B,\chi^{(\tau)}}|\mathrm{B}(\langle A,B\rangle).$$

The first condition of the generalized principle of stress means that the body forces formally separated from the mechanical interactions are characterised by one Radon measure defined on the Borel subsets of the whole body. If the remaining part of mechanical interactions is called contact force, then the second condition of the principle of stress requires that the Radon measures characterizing the contact forces be identical on the common parts of the boundaries of an arbitrary pair of subbodies.

THEOREM 9. *The generalized principle of stress constitutes the necessary condition of additivity of the power of resultant mechanical interactions.*

P r o o f. Let A be an arbitrary subbody. The subbodies A and A^b are disjoint. The additivity of the power of resultant mechanical interactions implies therefore that for every function $\vartheta \in \mathrm{C}_3(\mathscr{B})$ we have

$$\pi_{\mathscr{B},\chi^{(\tau)}}(\vartheta) = \pi_{A,\chi^{(\tau)}}(\vartheta|A) + \pi_{A^b,\chi^{(\tau)}}(\vartheta|A^b).$$

Making use of the Riesz theorem and assuming that $\vartheta|A^b = 0$, we obtain

$$\int_{A^0} \vartheta \cdot da_{A,\chi^{(\tau)}} = \int_{A^0} \vartheta \cdot da_{B,\chi^{(\tau)}}$$

or, equivalently,

$$\int_{A^0} \vartheta \cdot dc = 0, \tag{7.1}$$

where we have introduced the notation $c = (a_{A,\chi^{(\tau)}} - a_{B,\chi^{(\tau)}})|\mathrm{B}(A^0)$. The relation (7.1) is satisfied for every function $\vartheta \in \mathrm{C}_3(\mathscr{B})$ such that $\vartheta|A^b = 0$. It follows from Theorem 5 that the equation $c = 0$ is equivalent to Eq. (7.1). Therefore, $a_{A,\chi^{(\tau)}}|\mathrm{B}(A^0) = a_{\mathscr{B},\chi^{(\tau)}}$ $|\mathrm{B}(A^0)$. This completes the proof of the first part of the theorem.

Consider a pair of subbodies A, B. The subbodies $C = A \wedge B^b$, $D = B \wedge A^b$, $E = A \wedge B$ (Fig. 3) are disjoint and we have the identities $A = C+E$, $B = D+E$, $C+D = (A \wedge B + A^b \wedge B^b)^b$. Making use of Theorem 7(i) and the above proved part of

Theorem 9, the additivity of the power of resultant mechanical interactions for the pairs of subbodies C, E and D, E, and assuming that $\vartheta|(A \bigwedge B+A^b \bigwedge B^b)^b = 0$, we obtain

$$\int_{\langle A,B\rangle} \vartheta\cdot da_{A,\chi(\tau)} = \int_{\langle A,B\rangle} \vartheta\cdot da_{E,\chi(\tau)} = \int_{\langle A,B\rangle} \vartheta\cdot da_{B,\chi(\tau)}.$$

Exactly as in the first part of the theorem we can prove the relation

$$a_{A,\chi(\tau)}|\mathsf{B}(\langle A,B\rangle) = a_{B,\chi(\tau)}|\mathsf{B}(\langle A,B\rangle).$$

The generalized principle of stress is not sufficient for the additivity of the power of resultant mechanical interactions. For instance the power functional defined as follows:

$$\pi_{A,\chi(\tau)}(\vartheta) = \begin{cases} \vartheta(\xi)\cdot c & \text{for} \quad \mathsf{A}^0 \ni \xi, \\ 0 & \text{for} \quad \mathsf{A}^0 \not\ni \xi, \end{cases}$$

where c is a constant vector and ξ a fixed point of the body, satisfies the generalized principle of stress, but is not additive.

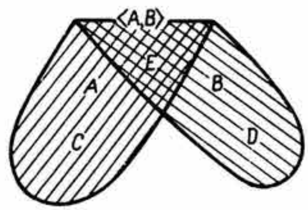


FIG. 3

THEOREM 10. *The functional of power of resultant mechanical interactions is additive, if and only if, the generalized principle of stress and the following two conditions are satisfied*:

(i) $\bigwedge_{A\in\mathscr{S}}, \bigwedge_{B\in\mathscr{S}} A\wedge B = 0 \Rightarrow (t_{A,\chi(\tau)} + t_{B,\chi(\tau)})|\mathsf{B}([A,B]) = b_{\chi(\tau)}|\mathsf{B}([A,B])$,

(ii) $\bigwedge_{A\in\mathscr{S}}, \bigwedge_{B\in\mathscr{S}} A\wedge B = 0 \Rightarrow (t_{A,\chi(\tau)} + t_{B,\chi(\tau)})\|\mathsf{B}(\{A,B\}) = t_{(A\vee B),\chi(\tau)}|\mathsf{B}(\{A,B\})$.

P r o o f. To prove the condition (i) it is sufficient to prove the additivity of the functional π for continuous functions ϑ such that $\vartheta|(A+B)^b = 0$. The condition (ii) can be proved making use of Theorems 7 and 8. The sufficiency becomes evident in the course of proving the necessity of the condition (i).

Under the assumption of the absolute continuity of the body forces b with respect to the volume, the condition (i) takes the form of the principle of reciprocity of the contact interactions. Under the assumption of the absolute continuity of the contact forces t with respect to the area of the boundary surface, the condition (ii) is trivially satisfied.

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