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Differential manifolds of the optimal solutions of a system of three ordinary equations

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OPTIMAL SYNTHESIS of an ordinary differential equation with control consists in ascribing the motion laws determined by control values to the domains of a decomposition of the state space, that is in establishment of a space of optimal solutions, or in rendering a definite structure to this space. In the paper are presented the considerations relating to the properties of the optimal structure of a three-dimensional space. Necessary conditions of optimality have been given, in geometrical form, of a three-dimensional differential equation.

1. Introduction

THE CONSIDERATIONS concern the differential equation of the following form

$$\dot{x} = Ax + Bu, \qquad x \in R^3 \backslash K, \qquad \mu K = 0, \qquad u \in U \subset R, \tag{1.1}$$

where U is a compact set. The aim of these considerations is the optimal synthesis of Eq. (1.1), that is determination of the function: $u^* : R^3 \backslash K \to U$ such, that every solution $x(x^0, t)$, $x^0 \in R^3$, $t \in \langle t_0, t_{\mathcal{O}} \rangle$, of Eq. (1.1) attains the origin $\mathcal{O}$ with $u = u^*$ at the shortest time $t_{\mathcal{O}} - t_0$.

The solution of the problem of synthesis consists in a decomposition of the state space into some geometrical manifolds of various dimensions with the assigned constant value controls, correspoding to apexes of rectangle controls U. Such decomposition of the space, together with assignation of the controls is connected with a problem of the optimal structure of the differential equation, which has been defined in Sec. 4, and which is considered as a correlate of the notion of entirety, built in some way from elements being manifolds of optimal solutions of Eq. (1.1).

The notion of a structure is generally valid for differential equations of the form

$$\dot{x} = f(x), \qquad x \in D \backslash K \subset R^n, \qquad \mu K = 0 \tag{1.2}$$

and it is connected with a decomposition $\mathcal{D}$ of set D into some n-dimensional domains D_i $(i = 1, 2, ...)$, to which some forms f_i $(i = 1, 2, ...)$ of the function f are referred, where f_i: $D_i \to R^n$ are continuous and Lipschitz functions. Definition of structure and optimal structure, as referred to Eq. (1.1), will be given below.

For the equation (1.1) the following assumptions have been made: A is a constant matrix 3×3 with negative real eigenvalues, B is a constant matrix 3×2, $u = (u_1, u_2)$, $U = \{u : c_l^1 \leq u_l \leq c_l^2\}$, $c_l^1 < 0$, $c_l^2 > 0$, $l = 1, 2$.

The optimal structure of Eq. (1.1) for the assumptions given above will be based upon a fact well known from the control theory [1]:

STATEMENT 1. The number of switchings on the optimal solution of Eq. (1.1) attains its minimum value.

Let $x(x^0, u, t)$ be the solution of Eq. (1.1), corresponding to control u with the initial condition $x(x^0, u, O) = x^0$, when it should be emphasized, that control on the solution is u, and let $x(x^0, t)$ be the solution of the equation (1.1) if such emphasizing is not necessary; let $T_u(x^0; (t_1, t_2))$ be a segment of a trajectory of the equation (1.1), corresponding to control u and interval (t_1, t_2) of parameter t; let $T_u(x^0, I^-)$ be a negative semi-trajectory of the equation (1.1): $\{x : x \in T(x^0; \langle 0, -\infty))\}$, and let, finally, $T_u(x^0)$ be a segment of trajectory corresponding to the interval $\langle 0, t_0 \rangle$.

It is known, [2], that Eq. (1.1) has the following properties: (1) – the domain of controllability coincides with R^3, (2) – the optimal control u^* is a piece-wise constant function with values $\mathrm{col}(c_1^j, c_2^k)$, $j, k \in \{1, 2\}$, such that each of its coordinates has in the interval $\langle t_0, t \rangle$ not more than 2 switchings.

We are interested in the optimal solutions only and for this reason it will be convenient, because of property (2), to use the form of Eq. (1.1) adequate to this property:

$$\dot{x} = Ax + Bu \quad (= f(x, u)), \qquad x \in R^n \backslash K, \qquad \mu K = 0, \tag{1.3}$$

$u \in \mathfrak{A}$, where $\mathfrak{A} = (a, b, c, d)$ and where $a, \ldots, d$ are control values u, corresponding to apexes of rectangle of controls U, that is two-element columns $\mathrm{col}(c_1^j, c_2^k)$, $j, k \in \{1, 2\}$. We will speak about the motion laws $(a), \ldots, (d)$, having in mind Eq. (1.3) with $u = \mathrm{col}(c_1^j, c_2^k)$, where (j, k) are the respective sequences. $\mathfrak{A}$ will mean everywhere a sequence of controls corresponding to a sequence of sucessive apexes of rectangle U in some ordering.

Concept of the optimal structure of a differential equation contains the notion of a set of semi-slides of solutions of this equation, [3].

2. Sets of semi-slides of solutions

Let V be a domain of R^3 and let almost everywhere on V vector field $f : V \backslash S \to R^3$, $\mu S = 0$, continuous and satisfying the Lipschitz conditions in domains W_i, $i = 1, \ldots, m$, $\bigcup_i \overline{W}_i = \overline{V}$ be determined. Let Γ ($\dim \Gamma \geq 1$) be a smooth boundary set in V such that $\Gamma \subset S$ (Fig. 1). Γ will be then a set belonging to the boundaries of m ($m \geq 2$) domains $W_i \subset V$ ($i = 1, \ldots, m$).

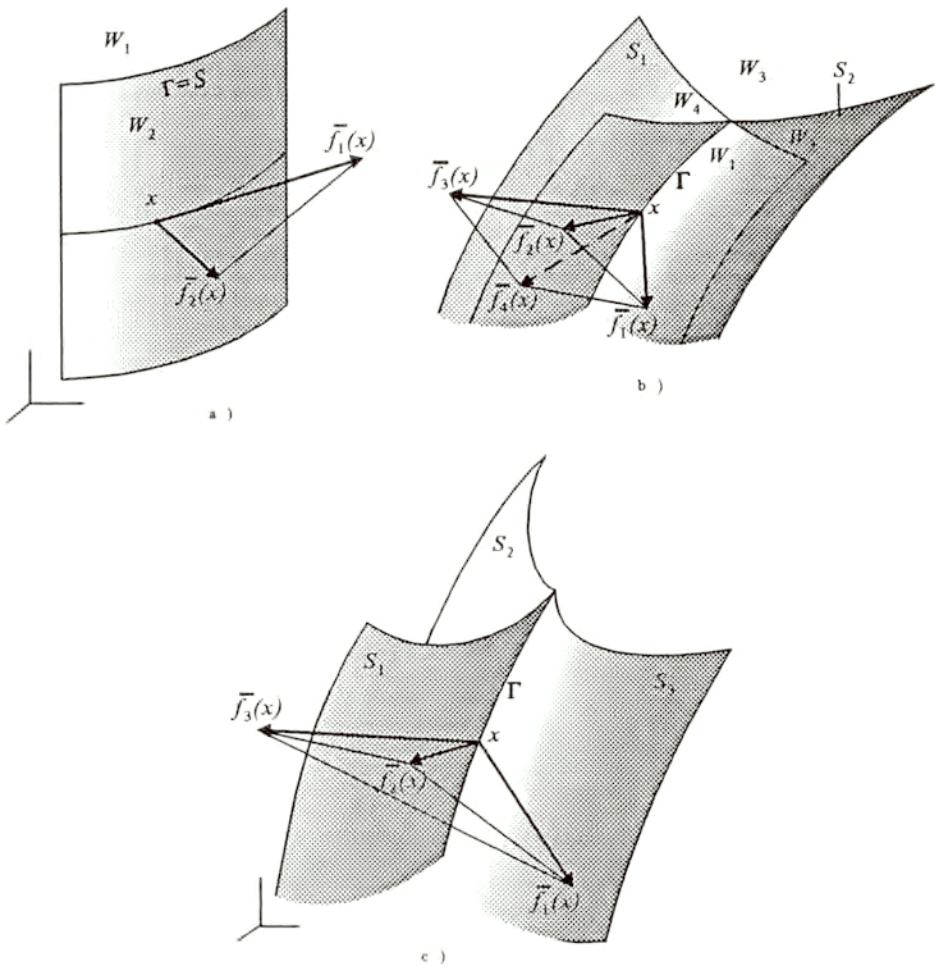

FIG. 1. a) dim $\Gamma = 2$, b) dim $\Gamma = 1$ ($S = S_1 \cap S_2$), c) dim $\Gamma = 1$ ($S = \overline{S}_1 \cap \overline{S}_2 \cap \overline{S}_3$).

If $x^0 \in \Gamma$, then there are m boundaries $\overline{f}_i(x^0)$ of the function f at point x^0:

$$\overline{f}_i(x) = \lim_{\substack{y \to x \\ y \in W_i}} f(y).$$

Let $\kappa(x)$ denote an arbitrary vector tangent to Γ at point x^0.

DEFINITION 1. *If for* $x \in \Gamma$

$$\exists\, i \in (1, \ldots, m)\ \exists\, k \in R\ \ (k \neq 0), \qquad k\overline{f}_i(x^0) = \kappa(x), \tag{2.1}$$

wherein $\overline{f}_i(x^0)$ *is the edge of the smallest convex pyramid built on vectors* $\overline{f}_i(x^0)$, $i = 1, \ldots, m$, *then we say that* x^0 *is a point of semi-slide of vector field* f *on set* Γ.

If the statement (2.1) *is true for any point* $x \in \Gamma \cap V$, *that is if*

$$(2.2) \qquad \exists\, i \in (1, ..., m) \quad \forall\, x \in \Gamma \cap V \quad \exists\, k \in R \quad (k \neq 0), \qquad k\overline{f}_i(x^0) = \kappa(x),$$

then we say that Γ *is a set of semi-slides of vector field* f *in set* V.

Let $W = V \backslash S$. In conformity with the assumption, $W = \bigcup_i W_i$. Consider the differential equation

$$(2.3) \qquad \dot{x} = f(x), \qquad x \in W,$$

where f is a function defined, continuous and satisfying the Lipschitz condition in each of domains W_i $(i = 1, ..., m)$. Let $x^0 \in \Gamma \subset S$. It is known, [4], that there exists a solution x: $O(t_0) \to R^3$ of the equation (2.3) at point x^0; $O(t_0)$ is the neighbourhood of t_0.

DEFINITION 2. *If for* x^0 *the statement* (2.1) *is true, then we say that the solution* x *is semi-sliding on* Γ *at point* x^0, *or that* x^0 *is a point of semi-slide of solutions of Eq.* (2.3) *on* Γ. *If the statement* (2.2) *is true, then solutions of Eq.* (2.3) *are semi-sliding on* Γ, *or* Γ *is a set of semi-slides of solutions of Eq.* (2.3) *in set* V.

The point $x^0 \in \Gamma$ such that

$$\exists\, y \in R^3 \quad \forall\, i \in (1, ..., m) \quad \exists\, k \in R \quad (k \neq 0), \qquad (f_i(x^0) = y, \quad ky = \kappa(x))$$

is, according to the definition 1, the point of a semi-slide of vector field f.

Point x^0, which is not a semi-slide point of vector field f, is either a semi-slide point, [4], or a point of a strict passage of the solution x of the equation (2.3) through Γ.

3. Basic decomposition, basic frame

Consider Eq. (1.3). Let (i) $(i \in \mathfrak{A})$ be a motion law, and T_i be a negative half-trajectory of Eq. (1.3), $T_i = T_i\langle \mathcal{O}, I^- \rangle$, corresponding to that motion law. T_i is the unique sub-set R^3 having the property:

$$\forall\, x^0 \in T_i \quad \exists\, t_0 \geq 0, \qquad x(x^0, i, t_0) = 0.$$

STATEMENT 2. If $x(x^0, t)$, $x^0 \in R^3 \backslash \mathcal{O}$ is the optimal solution of Eq. (1.3), then

$$\exists\, i \in \mathcal{A} \quad \exists\, \bar{t} \geq 0 \quad \forall\, t \in \langle \bar{t}, t_0 \rangle, \qquad x(x^0, t) \in T_i\,.$$

From the Statement 1 there follows

STATEMENT 3. Every solution $x(x^0, i, t)$, $x^0 \in T_i$, is optimal.

Let (i), (j) be two motion laws, $i, j \in \mathfrak{A}$, $i \neq j$, with i, j being the neighbouring elements of the sequence $\mathfrak{A}$ (cf. Sec. 1). Let $\overline{S}_{(i,j)}$ be a two-dimensional set:

$$\overline{S}_{(i,j)} = T_j(T_i, I^-) = \{x : \ x \in \{T_j(x^0, T^-), \quad x^0 \in T_i\}\}$$

having a bundle structure and generated by semi-trajectory T_i. It may be noted that each half-trajectory T_i, $i \in \mathfrak{A}$, generates two sets $\overline{S}_{(i,j)}$, j being the element of the sequence $\mathfrak{A}$ neighbouring i (Fig. 2). Each of the sets $S_{(i,j)} = \overline{S}_{(i,j)} \backslash T_j \backslash T_i$ is by definition a differential manifold, on which the motion law (j) is valid. We will use the notation S_{ij}, if we don't indicate which of the half-trajectories T_i, T_j generates the set S_{ij}, that is the set $S_{(ij)}$ or $S_{(ji)}$. We will use the notation $S_{ij}^{(i)}$, if the motion law (i) will be valid on set S_{ij}, that is

$$S_{ij}^{(i)} = S_{(j,i)}, \qquad S_{ij}^{(j)} = S_{(i,j)}.$$

S_{ij} are manifolds on half-trajectories T_i, T_j, $i, j \in \mathfrak{A}$, $i \neq j$.

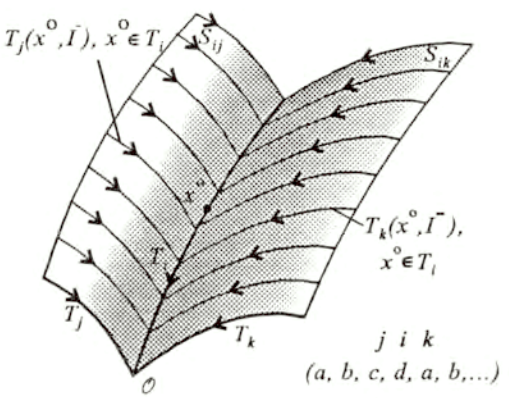

FIG. 2.

It may be easily noticed that no manifold S_{ij} contains singular points of the equation (1.3).

From definition of the sets S_{ij} it follows that, if $x(x^0, j, t)$ is the solution of Eq. (1.3), and if $x^0 \in S_{ij}$, then

$$(3.1) \qquad \exists\, t_1 > 0, \quad (\forall\, t \in \langle t_0, t_1 \rangle), \qquad x(x_0, j, t) \in S_{ij}, \qquad x(x^0, j, t_i) \in T_i).$$

Let $x(x^0, t)$ be the optimal solution of the equation (1.3). In this case the following statements hold.

STATEMENT 4, [5]. The sequence of controls on the optimal solution is a sequence with elements of $\mathfrak{A}$, neighbouring in $\mathfrak{A}$.

From definition of sets S_{ij} and from Statement 4 we will obtain the following

STATEMENT 5. If $x(x^0, t)$ is the optimal solution reaching the set T_i ($i \in \mathfrak{A}$) for $t = t_1 > 0$, then

$$\exists\, j \in \mathfrak{A} \quad (j \neq i, \quad i,j\text{– neighbouring in } \mathfrak{A})$$

$$\exists\, \varepsilon > 0 \quad \forall\, t \in \langle t_1 - \varepsilon, t_1), \qquad x(x^0, t) \in S_{(i,j)}\,.$$

LEMMA 1. All manifolds S_{ij}, $i,j \in \mathfrak{A}$, $i \neq j$, where i, j are neighbouring elements of the sequence $\mathfrak{A}$, are the sets of optimal solutions of Eq. (1.3).

P r o o f. Let $x^0 \in S_{(i,j)} \in \mathfrak{A}$, $i \neq j$, i,j being neighbours in $\mathfrak{A}$ and let $x(x^0, t)$ be the optimal solution of Eq. (1.3). Then

$$\exists\, t_1 > 0, \quad (\forall\, t \in \langle t_0, t_1), \qquad x(x^0, t) \in S_{(i,j)} \qquad \text{and} \qquad \forall\, t \in \langle t_1, t_{\mathcal{O}}), \qquad x(x^0, t) \in T_i), \tag{3.2}$$

that is there exists only one switching for $t = t_1$ and the sequence of controls on the solution is (j,i). Indeed, let us assume, to the contrary, that the sequence of controls on the optimal solution is different from (j,i), e.g. $(k, \ldots)$, $k \neq j$. The solution $x(x^0, k, t)$, $x^0 \in S_{(i,j)}$, in order to reach the origin $\mathcal{O}$, must previously reach some of the sets T_p, $p \in \mathfrak{A}$ (see Statement 2). Let us assume that it has reached this set for $t = t_2 > 0$. In order to reach T_p it must earlier reach some manifold $S_{(p,r)}$, $r \in \mathfrak{A}$, $r \neq p$, $r \neq j$, where p, r are neighbouring in the sequence $\mathfrak{A}$; x^0 belonging to manifold $S_{(i,j)}$ cannot belong to manifold $S_{(o,r)}$ (case $r \neq j$). Let us assume that $x(x^0, t)$ has reached $S_{(p,r)}$ for $t = t_1$, $0 < t_1 < t_2$. In such a case, on solution $x(x^0, t)$ there are two switchings: t_1 and t_2, and the sequence of controls will be (k, r, p). Hence a contradiction with the assumption on the optimality of the solutions follows: at the sequence of controls (j,i) on the solution $x(x^0, t)$ there is one switching only, cf. Statement 1.

Similarly it can be shown that the solution $x(x^0, t)$ with sequence of controls $(j, r, \ldots)$, $r \in \mathfrak{A}$, $r \neq i$ is not optimal with regard to the number of switchings.

The sequence of controls (j,i) on the solution $x(x^0, t)$ with the switching time different from t_1, causes that the point $\mathcal{O}$ is not being reached.

Satisfaction of (3.2) proves the Lemma.

■

Let K^* be a geometrical figure of zero measure in R^3:

$$K^* = \mathcal{O} \cup \bigcup_{i \in \mathfrak{A}} T_i \cup \bigcup_{i,j \in \mathfrak{A}} S_{i,j}\,, \quad i,j - \text{neighbouring in } \mathfrak{A}. \tag{3.3}$$

From the definition it follows that K^* has the properties given below:

i. K^* divides the space R^3 into six discontinuous three-dimensional domains Ω_i ($i = 1, \ldots 6$). The elements of the figure K^*, together with the domains Ω_i,

are elements of decomposition $\mathcal{D}^*$ of the space R^3. Such decomposition will be called the basic one, its elements being respectively: edges (T_i), walls $(S_{(i,j)})$ and cells (Ω_i), Fig. 3. Figure K^* is called the basic frame of decomposition $\mathcal{D}^*$. $\mathcal{D}^*$ is determined uniquely by Eq. (1.3).

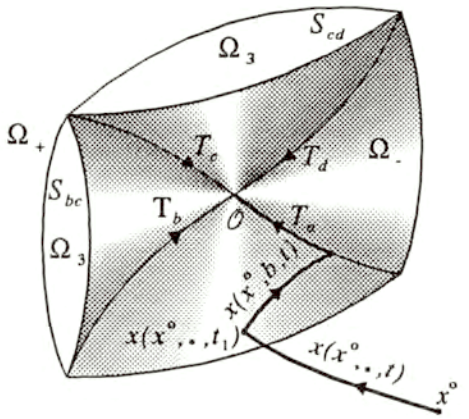

FIG. 3.

ii. To each element of the basic frame K^*, except the point $\mathcal{O}$, one of the motion laws: (a), (b), (c), (d) is uniquely ascribed, that is the following surjection is determined:

$$\gamma : K^* \to \mathfrak{A}. \tag{3.4}$$

iii. All elements K^*, except the point $\mathcal{O}$, are sets of optimal solutions of Eq. (1.3), cf. Statement 3 and Lemma 1.

LEMMA 2. If $x(x^0, t)$, $x^0 \neq \mathcal{O}$ is the optimal solution of the equation (1.3), then

$$\exists\, \bar{t} \geq 0 \quad \forall\, t \in \langle t, t_0 \rangle, \qquad x(x^0, t) \in K^*. \tag{3.5}$$

P r o o f. The sequence of controls on the optimal solution of Eq. (1.3) is composed of elements $\mathfrak{A}$.

If $x^0 \in K^*$, then in conformity with the property iii, (3.5) is true for $\bar{t} = 0$.

If $x^0 \notin K^*$, then in order for the solution $x(x^0, t)$ to reach the point $\mathcal{O}$ for $t = t_{\mathcal{O}}$ it is necessary:

$$\exists\, i \in \mathfrak{A} \quad \exists\, \bar{\bar{t}} > 0 \quad \forall\, t \in \langle \bar{\bar{t}}, t_{\mathcal{O}} \rangle, \qquad x(x^0, t) \in T_i.$$

Having assumed $\bar{\bar{t}} = \bar{t}$ we will have (3.5).

■

On the basis of Lemma 2 and Statement 5, the following statement is true.

STATEMENT 6. If $x(x^0, t)$, $x^0 \notin K^*$ is the optimal solution of the equation (1.3), then

$$\exists\,(i,j)\ \ (i,j \in \mathfrak{A}, \quad i,j \ - \ \text{neighbouring in } \mathfrak{A})$$
$$\exists\,\bar{t} > 0 \quad \exists\,\bar{\bar{t}} > \bar{t} \quad \forall\, t \in \langle \bar{t}, \bar{\bar{t}} \rangle, \qquad x(x^0, t) \in S_{ij}$$

that is the optimal solution beginning in a cell of decomposition $\mathcal{D}^*$ reaches the wall of that decomposition.

COROLLARY 1. The optimal solution of the equation (1.3), beginning in the cell of the decomposition $\mathcal{D}^*$, cannot reach its edge without having previously reached its wall.

K^* is an orientable figure. The cells of the decomposition $\mathcal{D}^*$ (see Fig. 3) are of two kinds: two-wall ones, bounded by walls $S_{(i,j)}$, $S_{(j,i)}$ $(i,j \in \mathfrak{A},\ i \neq j,\ i,j$ neighbouring in $\mathfrak{A})$, designated as Ω_{ij}, and four-wall ones, designated as Ω_-, Ω_+ bounded by four walls $S_{ab}^{(\cdot)}$, $S_{bc}^{(\cdot)}$, $S_{cd}^{(\cdot)}$ and $S_{ad}^{(\cdot)}$ of the basic frame. Let us recall that each element of decomposition $\mathcal{D}^*$, and hence also each cell, contains only its own elements: the elements of $\mathcal{D}^*$ are disjoint sets.

4. Structure of equation (1.3)

Let us assume that in Eq. (1.3) in which the set K has the following form:

$$K = K^* \cup L. \tag{4.1}$$

K is a frame of a new decomposition $\mathcal{D}$ of the space R^3 which has been built on the basic frame K^* by joining to it some new zero measure elements in such a way that these new elements do not intersect the basic frame:

$$L \cap K^* = \emptyset\,.$$

The assumption of a non-empty set L means a division of the cells of the basic decomposition $\mathcal{D}^*$ into the domains of definition (and continuity) of function f.

Let $\{\Omega\}$ denote a set of all cells of decomposition, and let δ be an injection

$$\delta : \{\Omega\} \to \mathfrak{A}\,. \tag{4.2}$$

DEFINITION 3. *Couple (K, δ) is called the structure of Eq. (3.1), if all elements of the basic frame K^* are sets of semi-slides of solutions of Eq. (1.3). The structure (K, δ) is called optimal, if all solutions $x(x^0, t)$, $x^0 \in R^3$ of this structure are optimal ones.*

DEFINITION 4. *We can say that two structures (K_1, δ_1), (K_2, δ_2) are equal, that is $(K_1, d_1) = (K_2, \delta_2)$, if:*

i) *their frames are isomorphic, that is if there exists isomorphism* $g : R^3 \to R^3$ *such that*

$$K_2 = \wp(K_1), \tag{4.3}$$

ii) $\delta_1 = \delta_2$.

If $\{K\}$ denotes a set of all frames of Eq. (1.3), then (4.3) is a relation of equivalence in set $\{K\}$. For a given number l of elements of frame K, the relation (4.3) divides the set $\{K\}$ into a finite number of equivalence classes $\{K\}_j$ ($j = 1, ..., J$, $J < \infty$). We will indentify in the next part of the paper the frame K, that is the representation of some equivalence class $\{K\}_j$ with that class. Indeed

$$\forall\ K \in \{K\} \quad \exists\ i \in (1, ..., J), \qquad K \in \{K\}_i,$$

that means that each frame is a representation of some class.

It is obvious that

$$\forall\ (i,j)\ \ (i,j \in (1, ...J),\ \ i \neq j)\ \ \forall\ K' \in \{K\}_i \quad \forall\ K'' \in \{K\}_j \quad \sim \exists\ \wp,\ \ K'' = \wp(K'). \tag{4.4}$$

DEFINITION 5. *Decomposition* $\mathcal{D}$ *of space* R^3 *is called possible, if it generates structure* (K, δ) *of Eq.* (1.3).

Let $\{\mathcal{D}\}$ be a set of all possible decompositions in space R^3, defined by (4.1). From Statement 1 we obtain:

STATEMENT 7. In the optimal structure (K, δ) of Eq. (1.3) the numbers of cells and elements of the set L reach their minima on set $\{\mathcal{D}\}$.

LEMMA 3. Couple (K^*, δ), where K^* is a basic frame of Eq. (1.3), is not the optimal structure of this equation, irrespective of the map δ.

P r o o f. Let there exist such mapping δ that (K^*, δ) is the optimal structure of Eq. (1.3). To fix the attention, let the control a be referred to the four-walled cell Ω_-: $\Omega_-^{(a)}$. Three different cases can be considered, concrening systems of elements of boundaries of the cell $\Omega_-^{(a)}$, on which the motion law (a) is valid and beyond which it is not valid (cf. Fig. 3): (1) – edge T_a, (2) – edge T_a and one of the walls $S_{ab}^{(a)}$ or $S_{ad}^{(a)}$, (3) – edge T_a and two walls S_{ab} and S_{ad}. These cases correspond to three different basic frames (as equivalence classes) K_1^*, K_2^* and K_3^*, respectively. Since, by the contrary assumption, every solution $x(x^0, t)$, $x^0 \in \Omega_-^{(a)}$, is the optimal one of Eq. (1.3), then, in conformity with the Lemma 2 and Corollary 1, it will reach the respective basic frame K_i^* ($i = 1, 2, 3$) for $t = \bar{t} < \infty$ in the subset (see Fig. 3):

$$S_{ab}^{(b)} \cup S_{bc}^{(.)} \cup S_{cd}^{(.)} \cup S_{ad}^{(d)} \quad \text{for } i = 1 \qquad \text{case 1;}$$

$$(S_{ab}^{(b)} \cup S_{ad}^{(d)}) \cup S_{bc}^{(.)} \cup S_{cd}^{(.)} \quad \text{for } i = 2 \qquad \text{case 2;}$$

$$S_{bc}^{(.)} \cup S_{cd}^{(.)} \quad \text{for } i = 3 \qquad \text{case 3.}$$

Since in both these sums a term $S_{bc}^{(\cdot)} \cup S_{cd}^{(\cdot)}$ appears that is the sum of two walls adjacent to edge T_c, then from the continuous dependence of initial conditions we have

$$\forall\, i \quad \exists\, x^0 \in \Omega_-^{(a)} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega_-^{(a)} \\ \text{and} \\ x(x^0, \bar{t}) \in T_c\,, \end{cases}$$

which contradicts: (1) with Corollary 1, and (2) Statement 4 – switching $a \to c$. Hence, the thesis of the lemma is true.

■

COROLLARY 2. The basic frame K^* is a subset of the frame K of the optimal structure of Eq. (1.3), that is $L \neq \emptyset$.

CONCLUSION 1. The optimal structure of the equation (1.3) does not contain four-wall cells: Ω_-, Ω_+.

5. Structure of four-wall cells

Let us make the simplest, in the sense of the number of elements, division of the cells Ω_-, Ω_+. Such a division of a four-wall cell consists in construction of two three-wall cells of it by introduction of one "diagonal" wall stretched on the edges corresponding to controls of diagonals of a rectangle of controls U, and which does not change the number of frame edges. Let us assume that the frame K has the following form:

$$K = K^* \cup S_- \cup S_+\,, \tag{5.1}$$

where S_- and S_+ are the walls of division of the cells Ω_-, Ω_+ accordingly.

Let us consider, to fix the attention, the cell Ω_-. On the grounds of the assumption that the numbers of edges of frames K and K^* are the same, we may conclude that a closure of the wall S_- contains the origin $\mathcal{O}$ and the pairs of edges T_a and T_c, or T_b and T_d, see Fig. 3. Let, again for fixing attention, this be the pair T_a, T_c. Hence, decomposition $\mathcal{D}$ contains two three-walled cells, which will be denoted by Ω_b and Ω_d from the edges T_b and T_d, respectively, which belong to their closures and do not belong to $\overline{S}_-$. Let us assume that in these cells the motion laws (j), (i), $i, j \in \mathfrak{A}$, $i \neq j$ are valid, respectively. We will show later that such decomposition is possible (see Definition 5).

Let us assume that the wall S_- is a set of points of a strict transition of the solutions of Eq. (3.1). Let us assume further, for fixing attention, that the solutions pass through S_- from the cell Ω_d to Ω_b: control on solutions of Eq. (1.3) changes its value from i to j.

LEMMA 4. If the couple $(\Omega_d, (i))$ is a cut-off $(K, \delta)\Big|_{\Omega_d}$ of the optimal structure (K, δ) to set Ω_d, then

(i) $i = d$,

(ii) $\overline{\Omega}_d \supset S_{ad}^{(d)}$, $\overline{\Omega}_d \supset S_{cd}^{(d)}$.

P r o o f. It may easily be noted that only in the case determined by the conditions of the Lemma, the following relation will hold

$$\forall\, x^0 \in \Omega_d^{(d)} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega_d^{(d)} \\ \text{and} \\ x(x^0, \bar{t}) \in S_- \,. \end{cases}$$

In all the remaining cases of combinations $(i, (S_{ad}^{(\cdot)}, S_{cd}^{(\cdot)}))$ we will have for the cell $\Omega_d^{(d)}$:

$$\exists\, k \in \{a, c, d\} \quad (k \neq i) \quad \exists\, x^0 \in \Omega_d^{(d)} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega_d^{(i)} \\ \text{and} \\ x(x^0, \bar{t}) \in T_k \,, \end{cases}$$

which is excluded, on the grounds of the assumption that $x(x^0, t)$ is the optimal solution of Eq. (3.1), see Conclusion 1.

■

Figure 4 shows an example of the couple (K, δ) such that

$$(K, \delta)\Big|_{\Omega_d} = (\Omega_d, (a)), \qquad \overline{\Omega}_d \supset S_{ad}^{(a)}, \qquad \overline{\Omega}_d \supset S_{cd}^{(c)}.$$

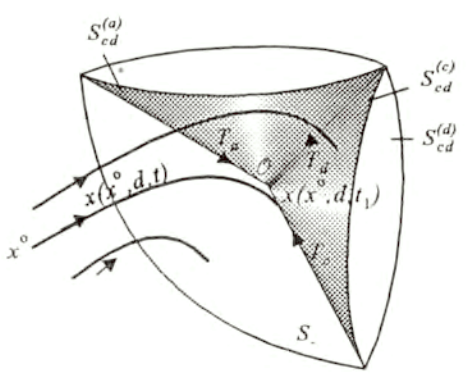

FIG. 4. Cell Ω_d.

LEMMA 5. If the couple $(\Omega_b, (i))$ is a cut-off $(K, \delta)\big|_{\Omega_b}$ of the optimal structure (K, δ) to the set Ω_b, then one of the following two conditions is satisfied:

$$1) \quad i = a, \qquad \overline{\Omega}_b \supset S_{ab}^{(a)}, \qquad \overline{\Omega}_b \supset S_{bc}^{(b)},$$

$$2) \quad i = c, \qquad \overline{\Omega}_b \supset S_{ab}^{(b)}, \qquad \overline{\Omega}_b \supset S_{bc}^{(c)}.$$

P r o o f. (i) is the motion law in cell Ω_b, wherein $i = a$, or $i = c$. Indeed, $i \neq d$, because (i) is, on the grounds of the assumption, different from the motion law valid for the cell $\Omega_d^{(d)}$ (see Lemma 4), and $i \neq b$, because switching $d \to b$ (Statement 4) at a passage of the solution $x(x^0, t)$ of the structure (K, δ) through wall S_- is excluded.

To fix the attention, assume $i = a$. In such a case the following combinations of the walls of cell Ω_b are excluded:

$$(S_{ab}^{(b)}, S_{bc}^{(b)}), \quad (S_{ab}^{(b)}, S_{bc}^{(c)}).$$

Obviously, would it be true, then in the cell $\Omega_b^{(a)}$

$$\exists\, x^0 \in \Omega_b \cup S_- \quad \exists\, \bar{t} > 0 \quad \begin{cases} \forall\, t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in T_b. \end{cases}$$

In view of Statement 4, combination of the walls $(S_{ab}^{(a)}, S_{bc}^{(c)})$ is also excluded because, if this were true, switching of controls $a \to c$ on the wall $S_{bc}^{(c)}$ would take place.

Hence a single possible wall combination is $(S_{ab}^{(a)}, S_{bc}^{(c)})$, for which (see Fig. 5):

$$\exists\, x^0 \in \Omega_b \cup S_- \quad \exists\, \bar{t} > 0 \quad \begin{cases} \forall\, t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in S_{bc}^{(b)}. \end{cases}$$

For $i = c$ the proof is identical as before: only one combination of walls of the cell Ω_{bc} is possible: $(S_{ab}^{(b)}, S_{bc}^{(c)})$, for which (see Fig. 6)

$$\exists\, x^0 \in \Omega_b \cup S_- \quad \exists\, \bar{t} > 0 \quad \begin{cases} \forall\, t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in S_{ab}^{(b)}. \end{cases}$$

■

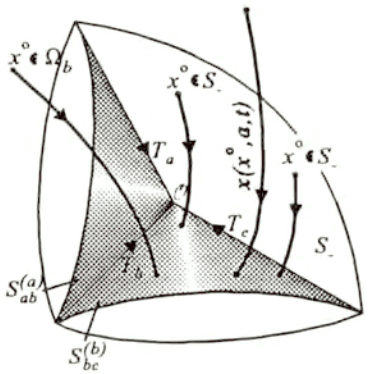

FIG. 5. Cell Ω_b.

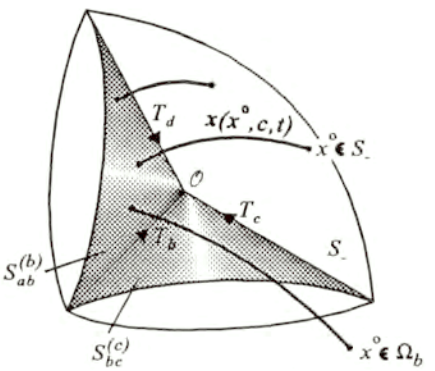

FIG. 6. Cell Ω_b.

Lemmas 4 and 5 can be generalized by formulation of the following

COROLLARY 3. If Ω_i $(i \in \mathfrak{A})$ is a three-wall cell defined above, whose one wall is S_k $(k \in \{-, +\})$, then the necessary condition for ensuring that the couple $(\Omega_i, (j))$ $(j \in \mathfrak{A})$ is a cut-off of the optimal structure (K, δ) to set Ω_i, is the satisfaction of one of the following two conditions:

1) $j = i$, when two walls adjacent to T_i have the motion law (i),

2) $j = l$, $l \neq i$, when there are two walls $S_{il}^{(l)}$ and $S_{im}^{(m)}$ $(l, m \in \mathfrak{A})$ adjacent to edge T_i, and the pairs (i, l) and (i, m) are sides of rectangle of controls U.

In the case 1) all solutions of the optimal structure (K, δ), which begin in cell Ω_i, will reach in a finite time the wall S_k; in the case 2) the cell Ω_i is a set of segments of trajectories of that structure with starting points belonging to S_k and end points belonging to $S_{im}^{(m)}$.

Let us recall that K is a frame built on the basic frame K^* by joining it with the walls S_-, S_+. From Lemmas 4 and 5 there follows

COROLLARY 4. K allows for two optimal structures (Fig. 7): (K_1, δ_1), (K_2, δ_2) with the frames:

$$K_1: \ \mathrm{Fr}\,\Omega_- \supset S_{ad}^{(d)} \cup S_{cd}^{(d)} \cup S_{ab}^{(a)} \cup S_{bc}^{(b)},$$
$$K_2: \ \mathrm{Fr}\,\Omega_- \supset S_{ad}^{(d)} \cup S_{cd}^{(d)} \cup S_{ab}^{(b)} \cup S_{bc}^{(c)},$$

and functions δ_1, δ_2, respectively:

$$(5.2) \qquad \begin{aligned} (K_1, \delta_1)\big|_{\Omega_d} &= (\Omega_d, (d)), & (K_1, \delta_1)\big|_{\Omega_b} &= (\Omega_b, (a)), \\ (K_2, \delta_2)\big|_{\Omega_d} &= (\Omega_d, (d)), & (K_2, \delta_2)\big|_{\Omega_b} &= (\Omega_b, (c)). \end{aligned}$$

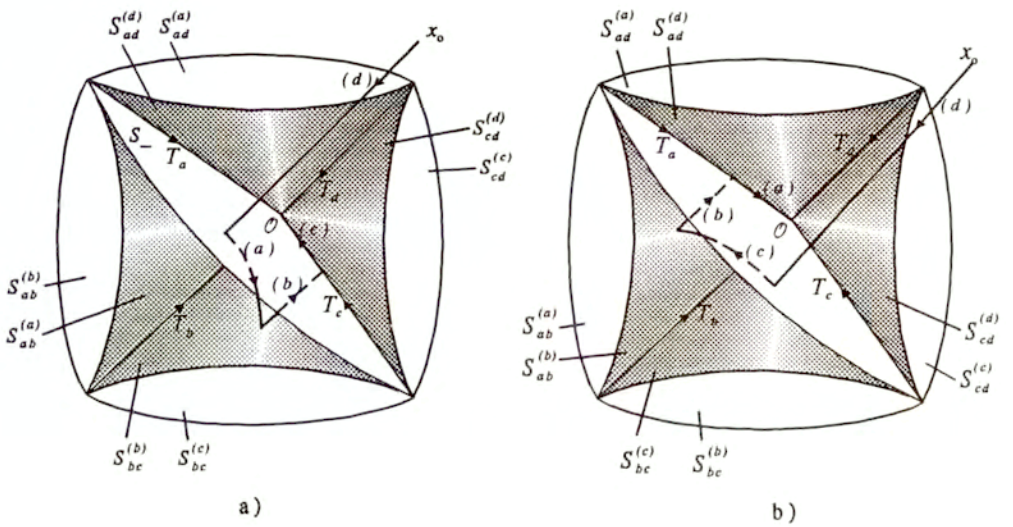

FIG. 7. a) Structure (K_1, δ_1), b) Structure (K_2, δ_2).

COROLLARY 5. Determining the boundary of the set Ω_-, the optimal structures (K_1, δ_1), (K_2, δ_2) lead to determination of the boundary of set Ω_+ (see Fig. 7):

$$(5.3) \qquad \begin{aligned} &\mathrm{Fr}\,\Omega_+ \supset S_{ad}^{(d)} \cup S_{cd}^{(d)} \cup S_{cb}^{(b)} \cup S_{bc}^{(c)} && \text{for struct. } (K_1, \delta_1), \\ &\mathrm{Fr}\,\Omega_+ \supset S_{ad}^{(d)} \cup S_{cd}^{(d)} \cup S_{ab}^{(a)} \cup S_{bc}^{(c)} && \text{for struct. } (K_2, \delta_2). \end{aligned}$$

We may note that for both structures, the system of the walls of set Ω_+ is analogous to the system of the walls of Ω_-: there are two walls with the same motion law: $S_{cd}^{(c)}$ and $S_{bc}^{(c)}$ for the structure (K_1, δ_1) and $S_{ad}^{(a)}$ and $S_{ab}^{(a)}$ for (K_2, δ_2). This, in turn, in view of Lemma 4 (after renumeration of the edges) leads to a method of division of the set Ω_- by wall S_+: similarly to the case of division of set Ω_-, the wall S_+ passes through the origin $\mathcal{O}$ and, in the case of both structures, it passes also through the edges T_b, T_d – Fig. 8.

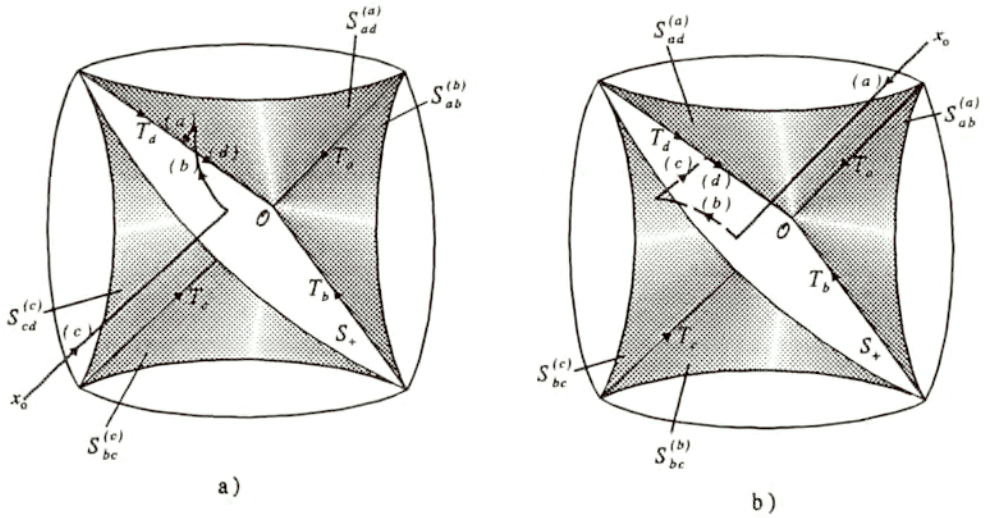

FIG. 8. a) Structure (K_1, δ_1), b) Structure (K_2, δ_2).

Hence, after the same analysis as that applied to the set Ω_-, we can draw the following conclusion.

COROLLARY 5. If (K_1, δ_1), (K_2, δ_2) are the optimal structures of the equation (1.3) then (see Fig. 8):

(1) conditions (5.3) must be satisfied, and

(2)

$$
\begin{aligned}
&(K_1, \delta_1)\big|_{\Omega_c} = (\Omega_c, (c)), \qquad (K_1, \delta_1)\big|_{\Omega_a} = (\Omega_c, (b)),\\
&(K_2, \delta_2)\big|_{\Omega_c} = (\Omega, (b)), \qquad (K_2, \delta_2)\big|_{\Omega_a} = (\Omega_c, (a)).
\end{aligned} \tag{5.4}
$$

Here a question may be posed whether there exist two walls S_-, S_+, which would ensure the switchings $d \to a$, $d \to c$, $c \to b$ and $a \to b$, respectively. A positive answer to this question can be given, if the following hypothesis is assumed.

HYPOTHESIS. If (K, δ) is the optimal structure of Eq. (1.3), then all switchings are effected on differential manifolds of that equation in the form $T_\cdot$ or $S_{\cdot\cdot}^{(\cdot)}$.

This leads to the following conclusion.

COROLLARY 6. Walls S_-, S_+ are differential manifolds of Eq. (1.3), that is of the form $S_{ac}^{(\cdot)}$, $S_{bd}^{(\cdot)}$, respectively, where (see Figs. 7 and 8):

$$
S_- = \begin{cases} S_{ac}^{(c)} & \text{for structure } (K_1, \delta_1), \\ S_{ac}^{(a)} & \text{for structure } (K_2, \delta_2), \end{cases}
$$

$$
S_- = S_{bd}^{(d)} \quad \text{for the both structures } (K_1, \delta_1),\ (K_2, \delta_2).
$$

These walls are sets of a strict passage of solutions of the optimal structure (K, δ), cf. Definition 3.

Three-wall cells Ω_b, Ω_d are thus formed by a division of four-wall cell Ω_- by wall S_- of the form $S_{ac}^{(i)}$ $(i \in \{a, c\})$; the cells Ω_a, Ω_c are formed by a division of the cell Ω_+ by wall S_+ of the form $S_{bd}^{(d)}$.

6. Variants of the optimal structure

Let us have the basic decomposition $\mathcal{D}^*$ of space R^3 defined by equation (1.3) on the basic frame K^*. Let us have a definite ordering (one of two) of apexes of rectangle of controls U without ascribing the definite apexes to controls a, b, c and d of U.

Let us assume that:

1. Among the walls of each four-wall cell there are exactly two walls with a common edge and with a common motion law for these walls.

2. Motion laws for the common edge of those walls correspond to controls being apexes of controls rectangle U, the apexes being the ends of one of its sides.

Let us distinguish (arbitrarily) one of the four-wall cells, and let us denote the edge of two walls having the same motion law by T_d, and the cell itself by Ω_- (the second four-wall cell will thus be denoted by Ω_+). The ordering controls have been determined and hence the sequence (a, b, c, d) is determined by control values u corresponding to the apexes of U. Let us note that, depending upon the ordering in the cell Ω_+, either T_c or T_a can then be the common edge of the wall pair with a common motion law.

Hence, for the assumptions given above and taking into account the considerations presented in Sec. 5, the following theorem is true.

STATEMENT 8. A necessary condition of optimality of the structure (K, δ), where K is a frame defined in Sec. 5, is that the walls of the four-wall cell Ω_- of basic frame K^* must form one of two systems given below:

$$(S_{ab}^{(a)}, S_{bc}^{(b)}, S_{cd}^{(d)}, S_{ad}^{(d)})_{\mathrm{I}} \quad \text{or} \quad (S_{ab}^{(b)}, S_{bc}^{(c)}, S_{cd}^{(d)}, S_{ad}^{(d)})_{\mathrm{II}}. \tag{6.1}$$

The resulting wall systems for the cell Ω_+ are, respectively

$$(S_{ab}^{(b)}, S_{bc}^{(c)}, S_{cd}^{(c)}, S_{ad}^{(a)})_{\mathrm{I}} \quad \text{or} \quad (S_{ab}^{(a)}, S_{bc}^{(b)}, S_{cd}^{(c)}, S_{ad}^{(a)})_{\mathrm{II}}. \tag{6.2}$$

Let us assume, further, that there exists a division of the cells Ω_-, Ω_+ by the walls S_- and S_+ of the following form:

$$S_- = \begin{cases} S_{ac}^{(c)} & \text{for system I, or} \\ S_{ac}^{(a)} & \text{for system II,} \end{cases}$$

$$S_+ = S_{bd}^{(d)} \quad \text{for both systems}$$

into three-wall cells: S_d, S_b, S_c, S_a.

Let K_1 and K_2 denote respectively the decomposition frames $\mathcal{D}_1$ and $\mathcal{D}_2$ formed from $\mathcal{D}^*$ (Sec. 5) and corresponding to the systems I and II of the walls S_- and S_+ (see (5.1)).

It may easily be noticed that by changing the order of notation of apexes of the rectangle of controls U, the frame K_2 can be transformed into frame K_1. It is sufficient to assume in Ω_+ as T_c the edge of the wall pair with the common motion law.

Let us assume that the frame K_1 has been determined by the differential equation (1.3). Then the results of analysis given in Secs. 5 and 6 can be presented in the form of the following lemma.

LEMMA 6. If (K, δ) is the optimal structure of the equation (1.3), the following conditions are satisfied:

1 $K = K_1$,

2 δ is any function of δ_1 defined by its free cuts to three-wall cells Ω_d, Ω_b, Ω_c, Ω_a according to (5.2) and (5.4).

7. Two-wall cells

Let Ω_{ij} $(i, j \in \mathfrak{A},\ i \neq j,\ i, j$ neighbouring in the sequence $\mathfrak{A})$, be a cell of the decomposition $\mathcal{D}$.

LEMMA 7. If (K, δ) is the optimal structure of Eq. (1.3), then

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (k)),$$

where

$$k = i \vee j.$$

P r o o f. Let (K, δ) be the optimal structure of Eq. (1.3). The walls of the cell Ω_{ij} (Sec. 3) have the form $S_{ij}^{(i)}$ and $S_{ij}^{(j)}$. Let (k) $(k \in \mathfrak{A})$ be the motion law valid in Ω_{ij}. In conformity with the Statement 6, the control k transfers an arbitrary solution $x(x^0, k, t)$, $x^0 \in \Omega_{ij}$, on one of the walls of that cell, that is

$$\forall\ x^0 \in \Omega_{ij} \quad \exists\ l \in \{i, j\} \quad \exists\ \bar{t} > 0 \quad \begin{cases} \forall\ t \in \langle 0, \bar{t}) \subset \Omega_{ij}, \\ x(x^0, k, \bar{t}) \in S_{ij}^{(l)}. \end{cases}$$

Let us assume that the thesis of the lemma is false, that is that

$$k \neq i \wedge j.$$

On the grounds of the assumption on the optimality of the structure (K, δ), both walls $S_{ij}^{(i)}$ and $S_{ij}^{(j)}$ are sets of semi-slides of solutions of Eq. (1.3), with the motion laws (i) and (j) $(i, j \neq k)$, accordingly. Thus both the walls are the sets of points such that

$$\forall\, y \in S_{ij}^{(i)} \cup S_{ij}^{(j)} \quad \exists\, x^0 \in \Omega_{ij} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}) \in \Omega_{ij}\,, \\ x(x^0, k, \bar{t}) = y, \end{cases}$$

and hence

$$\forall\, l \in \{i, j\} \quad \exists\, x^0 \in \Omega_{ij} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}) \in \Omega_{ij}\,, \\ x(x^0, k, \bar{t}) \in T_l\,, \end{cases}$$

which contradicts the Statement 6 and Corollary 1 and proves the lemma.

■

Let $\Omega_{ij}^{(k)}$ $(i, j, k \in \mathfrak{A},\ i \neq j,\ k \in \{i, j\},\ i, j$ – neighbouring in $\mathfrak{A})$ be a cell of the structure (K, δ).

LEMMA 8. If (K, δ) is the optimal structure of the equation (1.3) then at least one of the walls $S_{ij}^{(l)}$ $(l \in \{j, j\})$ of the cell $\Omega_{ij}^{(k)}$ belongs to the boundary of a three-wall cell $\Omega^{(k)}$.$(l \neq k)$.

P r o o f. If Ω_{ij} is a two-wall cell of decomposition $\mathcal{D}$, then, in conformity with Sec. 5, its walls are also the walls of the respective three-wall cells Ω (Fig. 9).

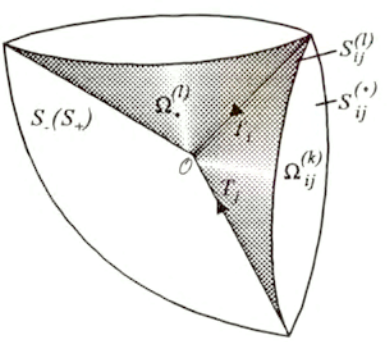

FIG. 9.

Let us assume that the thesis of the lemma is not true, that is let the motion laws in the three-wall cells adjacent to $\Omega_{ij}^{(k)}$ $(k = i \vee j$ – see Lemma 7) be different

from the motion laws valid for the walls $S_{ij}^{(i)}$, $S_{ij}^{(j)}$: $\Omega_{\cdot}^{(l)}$, $\Omega_{\cdot}^{(m)}$, $l, m \in \mathfrak{A}$, $l \neq i$, $m \neq j$. To fix the attention, assume $k = i$, that is $\Omega_{ij}^{(i)}$. Then the wall $S_{ij}^{(j)}$ will not be a set of semi-slides of solutions of Eq. (1.3). Indeed, in the cells with a common wall $S_{ij}^{(j)}$, the motion laws are valid: $(i) \neq (j)$ in the two-wall cell $\Omega_{ij}^{(i)}$ and $(m)(\neq (j))$ in the three-wall cell $\Omega_{\cdot}^{(m)}$ the motion law will be valid $(m) \neq (j)$, whereas the motion law on the wall $S_{ij}^{(j)}$ is (j) – Fig. 10. Hence the wall $S_{ij}^{(j)}$ is not a set of semi-slides of solutions of equation (1.3), what contradicts the assumption of optimality of the structure (K, δ).

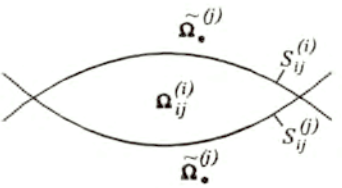

FIG. 10.

■

LEMMA 9. If Ω_{ij} and $\Omega_{\cdot}$. $(i, j, l \in \mathfrak{A}$, $i \neq j$, i, j – neighbouring in $\mathfrak{A})$ are, respectively, the two-walled and three-wall cells of the decomposition $\mathcal{D}$ adjacent to the wall $S_{ij}^{(i)}$, if $l \neq i$ and if (K, δ) is the optimal structure of Eq. (1.3), then

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (i)).$$

P r o o f. If $l \neq i$ and $S_{ij}^{(i)}$, on the grounds of the assumption on the optimality of the structure (K, δ), is a set of semi-slides of solutions of Eq. (1.3), then the motion law in the cell Ω_{ij} is (i), see Lemma 7. Indeed, assumption of the motion law (j) in Ω_{ij} according to Lemma 7 would be contradictory to the assumption of optimality of (K, δ) – see Definition 3.

■

LEMMA 10. Let Ω_{ij} be a two-wall cell of the decomposition $\mathcal{D}$, and $\Omega_{\cdot}^{(l)}(i)$, $\Omega_{\cdot}^{(m)}(j)$ $(l, m \in \mathfrak{A})$ be three-wall cells, adjacent to the respective walls $S_{ij}^{(i)}$, $S_{ij}^{(j)}$ of the cell Ω_{ij}. If (K, δ) is the optimal structure of Eq. (1.3) and

$$(l = i) \wedge (m = j),$$

then (Fig. 11)

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (i)) \vee (\Omega_{ij}, (j)).$$

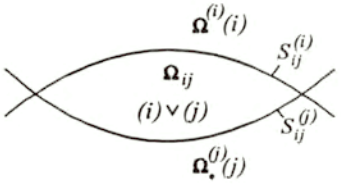

FIG. 11.

8. Necessary condition of optimality

Let there be a differential equation (1.3). Let K^* be the basic frame of this equation and K the frame of the form (5.1), where

$$S_- = S_{ac}^{(c)}, \qquad S_+ = S_{bd}^{(d)}$$

and where T_d, T_c have been defined as the edges of the respective wall pairs with a common motion law of three-wall cells of the decomposition $\mathcal{D}^*$. From the analysis given in Sec. 5 it follows that the frame K $(= K_1)$ exists – see Lemma 6. Decomposition $\mathcal{D}$ contains 8 cells: 4 three-wall and 4 two-wall ones.

On the grounds of the Statement 8 and Lemmas 7–10, the following necessary conditions of optimality of the structure (K, δ) may be given.

THEOREM 8.1. *The necessary conditions for* (K, δ) *to be the optimal structure of Eq.* (1.3) *are as follows:*

$$\left.\begin{aligned} (K,\delta)\big|_{\Omega_d} &= (\Omega_d, (d)) \\ (K,\delta)\big|_{\Omega_b} &= (\Omega_b, (a)) \end{aligned}\right\} \text{ in four-wall cell,}$$

$$\left.\begin{aligned} (K,\delta)\big|_{\Omega_c} &= (\Omega_c, (c)) \\ (K,\delta)\big|_{\Omega_a} &= (\Omega_a, (b)) \end{aligned}\right\} \text{ in four-wall cell,}$$

$$\begin{aligned} (K,\delta)\big|_{\Omega_{ab}} &= (\Omega_{ab}, (a)\vee(b)), \\ (K,\delta)\big|_{\Omega_{bc}} &= (\Omega_{bc}, (b)), \\ (K,\delta)\big|_{\Omega_{cd}} &= (\Omega_{cd}, (c)\vee(d)), \\ (K,\delta)\big|_{\Omega_{cd}} &= (\Omega_{cd}, (a)). \end{aligned} \tag{8.1}$$

It may be noted that in the case of two two-wall cells Ω_{ab}, Ω_{cd} the motion law valid for these cells has been formulated alternatively. Assuming this alternative we would obtain from (8.1) the necessary and sufficient conditions of existence of optimality of the structure (K, δ).

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Energy and equations of motion in a tentative theory of gravity with a privileged reference frame

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BASED ON A TENTATIVE interpretation of gravity as a pressure force, a scalar theory of gravity was previously investigated. It assumes gravitational contraction (dilation) of space (time) standards. In the static case, the same Newton law as that appearing in special relativity was expressed in terms of these distorted local standards, and it was found to imply geodesic motion. Here, the formulation of motion is reexamined in the most general situation. A consistent Newton law can still be defined, which accounts for the time variation of the space metric, but it is not compatible with geodesic motion for a time-dependent field. The energy of a test particle is defined: it is constant in the static case. Starting from "dust", a balance equation is then derived for the energy of matter. If the Newton law is assumed, the field equation of the theory allows to rewrite this as a true conservation equation, including the gravitational energy. The latter contains a Newtonian term, plus the square of the relative rate of the local velocity of gravitation waves (or that of light), the velocity being expressed in terms of absolute standards.

1. Introduction

AN ATTEMPT to deduce a consistent theory of gravity from the idea of a physically privileged reference frame or "ether" was proposed previously [1–3]. This work is a further development which is likely to close the theory. It is well-known that the concept of ether has been abandoned at the beginning of this century. It is less widely known that, in the meantime, the objective situation in physics has made it reasonable and interesting to reexamine this concept. In such a matter, it is necessary for the writer to appeal to authorities. Thus:

i. Due to the work of BUILDER [6–7], JANOSSY [11], PROKHOVNIK [21, 23], it has now been proved in detail that special relativity (SR) is, after all, *fully* compatible with the assumption of the ether as envisaged by Lorentz and Poincaré, i.e. with the ether being an inertial frame in which Maxwell's equations can be written and relative to which all material objects undergo a "true" Lorentz contraction. This has been emphasized by physicists as important as BELL [4], the author of theorems on "hidden variables" in quantum mechanics. In a recent review on the latter subject, MERMIN [17] writes: "What Bell's Theorem did suggest to Bell was *the need to reexamine our understanding of Lorentz invariance*" (italics ours). It is worth to recall that several basic results of SR, including the Lorentz transformation and the Lorentz invariance of Maxwell's equations, were found by Lorentz and Poincaré, prior to Einstein, *within a theory based on ether*. As stated by BELL [4], the main differences between the approaches by Lorentz (and Poincaré) and by Einstein are that: (a) "Since it is experimentally impossible to say which of two uniformly moving systems is *really* at rest, Einstein declares the notions "really resting" and "really moving" as meaningless." (b) "Instead

of inferring the experience of moving observers from known and conjectured laws of physics, Einstein starts from the *hypothesis* that the laws will look the same to all observers in uniform motion" (italics are Bell's). It is now accepted that the Lorentz-Poincaré opposite philosophy leads to the same physical theory, namely SR.

ii. It has become evident that quantum mechanics leads to attribute definite physical properties to the so-called "vacuum", and that these properties do affect our measuring instruments as predicted by quantum mechanics. As SCIAMA [25] states: "For example, the electric and magnetic fields in the electromagnetic vacuum are fluctuating quantities. This leads to a kind of reintroduction of the ether, since some physical systems interacting with the vacuum can detect the existence of its fluctuations. However, this ether is Lorentz-invariant, so there is no contradiction with special relativity."

iii. Finally, relative velocities of astronomical objects are being measured with increasing accuracy. We can associate a privileged reference frame with the "cosmic fluid", the velocity of which is the average velocity of matter. This reference frame appears clearly in the cosmological models because, in these simplified models, our universe is assumed homogeneous so that the average is already done. For example, the Robertson-Walker space-time metric of an expanding universe distinguishes a particular class of "comoving observers" – that is, comoving with the expansion commonly assumed to explain, as a Doppler effect, the observed red-shift of the spectra emitted by distant galaxies. Their set makes a privileged reference frame, as emphasized by PROKHOVNIK [21–22].

Thus, we have a privileged reference frame in which also the "empty" parts have physical properties. This may be called an "ether" for short, even though "physical vacuum" would be more appropriate. The use of the old name does not imply to forget modern physics: but that precisely today's physics allows to reconsider the assumption of an "ether", provides some justification for reconsidering also the *possibility* that, after all, the allegable existence of this "ether" *might manifest itself in certain laws of physics*. Since the privileged reference frame is kinematically defined from the average motion of matter at a very large scale, gravitation could be the range of such speculated manifestations. In other words, it seems allowable to investigate a theory of gravity with a privileged frame, the one which would not necessarily be neutral in the theory: a non-covariant theory. Some important astronomical facts at the scale of galaxies, such as the famous rotation curves in the disc galaxies, have not yet received any undebatable interpretation [19, 30]. Hence an explanation derived from the effect of motion through an "ether" might be considered.

However, Newtonian gravity (NG) gives an excellent description of astronomical motions at smaller scales, which is even refined by the corrections of general relativity (GR). Since neither NG nor GR does admit a preferred reference frame, it is undoubtedly a risky affair to attempt the construction of a theory of gravity

with such a frame. Such an attempt has yet been proposed [1–3]. The aim was to investigate whether the "logic of absolute motion", which furthers our understanding of *special* relativity (SR), as compared with the usual "space-time logic" [4, 6, 7, 11, 21], can be extended so as to build a sensible theory of *gravity*. The investigated theory is nonlinear, like GR, and coincides with NG at the lowest approximation, also like GR. As an argument in favour of giving some attention to this theory, we mention the analytical study of the gravitational collapse in "free fall" with spherical symmetry, i.e. the situation analytically studied within GR by OPPENHEIMER and SNYDER [18]. Since then, it has been shown, notably by PENROSE [20], that this situation contains all essential features of gravitational collapse for very massive objects in GR. According to the investigated theory, no singularity occurs during this collapse, neither for the metric nor even for the energy density, because the implosion would stop within finite time (for freely falling clocks and for remote clocks as well) and would be followed by an explosion [3]. On the other hand, this theory gives exactly the same predictions as GR for the motion of a test particle in the assumed static gravitation field of a spherical body, for in that case the Schwarzschild space-time metric of GR is obtained and the Newton law of the theory implies geodesic motion for any static field. The last result applies to mass particles [2] and to light-like particles as well [3], and provides a link between dynamics in NG and in GR, in the sense of MAZILU [16]. In this non-covariant theory, however, a gravitation field is static only if its source, the mass-energy density in the frame bound to "ether", is time-independent. Whereas in NG and GR, a uniform motion of the massive body can be eliminated by changing the reference frame, which is not the case here. Thus, in order to analyse either planetary motion in the solar system or stellar motion in a galaxy, one might have to reckon with velocities in the range 100 – 1000 km/s.

In the previous work, a Newton law has been defined only in the static case. Since it then implies geodesic motion, geodesic motion has been assumed in the general case. The purpose of this paper is to study in greater depth the formulation of motion in the theory, in connection with the problem of energy. We first recall (Sec. 2) the basic principles and equations of this theory, obtaining an alternative (equivalent) form of the field equations. For the equations of motion (Sec. 3), it is shown that in the general, non-static case, a consistent Newton law can still be defined, but is incompatible with the geodesic formulation of motion. Then the energy problem in the theory is analysed (Sec. 4). As is known, an exact energy can hardly be defined within GR, since the obtainment of a conservation equation for energy-momentum asks for a privileged class of reference frames (see e.g. STEPHANI [28]). After having recalled some aspects of the question of gravitational and total energy in NG and in GR, it is shown that a local energy balance equation, including the gravitational energy, is obtained by the studied theory if one assumes the (extended) Newton law of motion, but not under the geodesic formulation of motion.

2. Basic assumptions and equations

2.1. Absolute space and physical vacuum

Since astronomical observations and the success of Newtonian theory suggest that Galilean space-time is a very accurate description of our real world-theater, the notion of a physical vacuum has first to be conciled with *classical mechanics*. This, however, was not accomplished by ether theorists. For the ether must be the absolute space, hence rigid – otherwise, one would have *two* independent absolute spaces. But it also must offer no resistance against motion, and this in Newtonian mechanics can be true only of a perfect fluid. The proposed answer is that *it is the average motion of the hypothetical perfect fluid that defines the privileged inertial frame* [1–2], corresponding to the absolute space of Newtonian theory. Since any motion must be referred to some space (and time), it may seem that we are lead to a vicious circle. Thus we first postulate the absolute space or 3-D manifold M which, as a working assumption, we may take to have Euclidean structure ([1]). And we postulate the absolute time t, i.e. the world of events M^4 (time and position) is assumed to be the product $\mathbf{R}\times M$ ([2]). In Newtonian theory, the natural space metric $\mathbf{g}^0$ on M and the absolute time are assumed to be experimentally accessible, independently of motion and gravitation. In the studied nonlinear theory, the coincidence between "absolute" space (or time) metric and physical space (or time) measurements is found only in the first approximation. Then we may define the motion of a continuous medium, relative to M, by its velocity vector field $\mathbf{v}_0 = \mathbf{dx}/dt$. The particular metric structure we assume for M ensures that the integral of a vector field makes sense and is a vector (note that it is *not* the case for a general Riemannian space and thus for GR). Hence, the volume average of the velocity in any finite domain Ω occupied by the continuous medium is well defined, say $\overline{\mathbf{v}_0}^{\Omega}$. Now we may formulate the assumption according to which the "microscopic ether", that one which would be a perfect fluid, has in the average no motion relative to the absolute space M: we suppose that the velocity field of the micro-ether, $\mathbf{v}_{0e}$, is defined everywhere (it fills the space) and is such that $\overline{\mathbf{v}_{0e}}^{\Omega} \to 0$ as the size of the regular domain Ω (let us say it is a ball) infinitely increases, and this is true independently of the spatial position of Ω. We may call M the macro-ether.

Since the micro-ether must fill the space, any kind of matter, made of material particles, should be actually made of this universal fluid. Thus, material particles should be local, organized flows in the ether, such as the vortices which in a perfect fluid can be everlasting. We note that this concept is compatible with the

([1]) However, the whole theory can be written if M is more generally assumed to be equivalent (isometric) to either of Euclidean space E^3, (hyper)sphere S^3, or Lobatchevsky space L^3, i.e. M is a space with constant curvature, be it nil, positive or negative: in either case, M is equipped with a natural metric $\mathbf{g}^0$ which admits a 6-parameter group of isometries.

([2]) Formally speaking, one should start with M^4 and admit privileged space and time projections: $M^4 \to M$ and $M^4 \to \mathbf{R}$ making M^4 isomorphic to $\mathbf{R} \times M$ in a canonical way. This is the "Newton–Lorentz universe" considered by Soós [26].

lack of complete separability between particles, which is predicted by quantum mechanics and has received experimental confirmation. It would be an ambitious program, however, to attempt recovering micro-physics from this assumption of a "*constitutive ether*" (but not a hopeless program: cf. the hydrodynamic interpretation of Schrödinger's wave equation by MADELUNG [15]; more recently, ROMANI [24], WINTERBERG [31–32] and DMITRYIEV [9] obtained results in the same direction). Anyway, the gravitation theory needs almost only the macro-ether, which will be physically defined from the average motion of *matter*, i.e. the average motions of ether and matter are assumed to be identical. Recall that the notion of average used in this definition, is the asymptotic volume average. This means that no *a priori* bound needs to limit the size of local inhomogeneities, i.e. the size of astronomical structures.

2.2. Scalar field equation and space-time metric

A perfect fluid can exert only pressure forces. The gravitation force is tentatively interpreted as the pressure force or Archimedes' thrust, resulting from the gradient of the macroscopic pressure p_e in the ether. This gives the following expression for the gravity acceleration $\mathbf{g}$ [1]:

$$\mathbf{g} = -\frac{\operatorname{grad} p_e}{\varrho_e}, \tag{2.1}$$

where ϱ_e is the macroscopic density of the hypothetical fluid, assumed to be barotropic; the latter means that ϱ_e depends only on p_e, it is imposed to a true "perfect" fluid which should be continuous at any scale, so that no temperature and no entropy can be defined [24]. The interpretation leading to Eq. (2.1) must be qualified as tentative in so far as it is based on the assumption that a micro-ether (or physical vacuum, or subquantum medium) does exist as a perfect fluid, whereas a consistent theory of microphysics following this line still remains to be built. In this interpretation, the gravitation force can be seen as a correction, i.e. it is what would remain when the microscopic fluctuations of ether pressure, which should be responsible for the other three forces, would have been accounted for. Although it is derived from this tentative interpretation, we take Eq. (2.1) firmly, but as a *phenomenological* equation in which p_e plays the role devoted to the Newtonian potential U in NG; it is thus substituted for $\mathbf{g} = \operatorname{grad} U$. Just like for the latter, the macroscopic nature of Eq. (2.1) (the fact that the involved fields vary significantly over macroscopic distances only) does not prevent $\mathbf{g}$ from being felt at the scale of material particles.

The field equation for p_e, playing the role of Poisson's equation for U, is written as:

$$\Delta p_e - \frac{1}{c^2}\frac{\partial^2 p_e}{\partial t_{\mathbf{x}}^2} = 4\pi G \varrho \varrho_e, \tag{2.2}$$

with Δ the Laplace operator, ϱ the density of matter (the density of the conserved mass in the first, Newtonian approximation; in the nonlinear theory, ϱ is the mass-energy density in the frame $\mathcal{E}$ in which the points of M have no motion ([3])), $t_{\mathbf{x}}$ a *local time* to be precised hereafter, and G Newton's gravitation constant. It is obtained basically from four requirements:

i. Since NG propagates with infinite speed, it must correspond to the case where the fluid is incompressible, $\varrho_e = \text{const}$, and the field equation must become equivalent to Poisson's equation as the compressibility evanesces.

ii. In the compressible case, pressure waves must appear (except for static situations where p_e does not depend on time), the velocity c_e of which will be determined by the local ether compressibility, $K = 1/c_e^2 = d\varrho_e/dp_e$.

iii. Accounting for SR in the Lorentz–Poincaré interpretation thoroughly developed by PROKHOVNIK [21, 23], the velocity of light, c, becomes a limiting speed; so should be also the velocity of the pressure waves (the "sound" velocity) in the assumed constitutive ether, hence one must have $c_e = c$ everywhere (this implies that the barotropic relationship between p_e and ϱ_e is *linear*, $p_e = c^2\varrho_e$).

iv. In the theory, the equivalence principle arises naturally, as a correspondence between the absolute metric effects of motion and gravitation. As a consequence, *the space (time) standards are assumed to be contracted (dilated) in the gravitation field, in the ratio* $\beta = p_e/p_e^\infty$, where p_e^∞ is the value of p_e in regions which are remote from any matter and hence free from gravitation (see footnote 4, however); the space contraction occurs only in the direction of the gravity acceleration $\mathbf{g}$. Hence, the "physical" space metric $\mathbf{g}$ in the frame $\mathcal{E}$ becomes distorted, as compared with the natural metric $\mathbf{g}^0$, and *the grad operator in* Eq. (2.1), *as well as the Laplace operator in* Eq. (2.2), *are in terms of the curved physical metric* $\mathbf{g}$:

$$(\text{grad}\,\phi)^i = (\text{grad}_{\mathbf{g}}\phi)^i = g^{ij}\frac{\partial\phi}{\partial x^j}, \qquad (g^{ij}) \equiv \mathbf{g}^{-1}, \tag{2.3}$$

$$\Delta\phi = \Delta_{\mathbf{g}}\phi = \text{div}_{\mathbf{g}}\,\text{grad}_{\mathbf{g}}\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial\phi}{\partial x^j}\right), \qquad g = \det(g_{ij}). \tag{2.4}$$

Moreover, a local time $t_{\mathbf{x}}$ appears at any fixed point $\mathbf{x}$ in M; thus if the point $\mathbf{x}_0$, also bound to M, is far enough from any matter so that no gravitation field is felt at $\mathbf{x}_0$:

$$dt_{\mathbf{x}}/dt = dt_{\mathbf{x}}/dt_{\mathbf{x}_0} = \beta = p_e(t,\mathbf{x})/p_e(t,\mathbf{x}_0), \qquad \beta = p_e/p_e^\infty < 1, \tag{2.5}$$

whence the definition of the derivative with respect to local time, appearing in Eq. (2.2):

$$\frac{\partial\phi}{\partial t_{\mathbf{x}}} \equiv \frac{p_e^\infty}{p_e}\frac{\partial\phi}{\partial t} \equiv \frac{p_e(t,\mathbf{x}_0)}{p_e(t,\mathbf{x})}\frac{\partial\phi}{\partial t_{\mathbf{x}_0}}, \tag{2.6}$$

([3]) A (generally deformable) frame $\mathcal{F}$ can be defined as a time-dependent diffeomorphism ψ_t of M onto M, hence the frame $\mathcal{E}$ corresponds to the case where ψ_t is the identity mapping at any time t.

with $t_{\mathbf{x}_0} = t$ if is $\mathbf{x}_0$ is "far enough", i.e. if $p_e(t, \mathbf{x}_0) \equiv p_e^\infty$. Note, however, that Eq. (2.6) assigns a unique value to any derivative with respect to local time, $\partial\phi/\partial t_{\mathbf{x}}$, even if one changes the reference point $\mathbf{x}_0$ in M, and this occurs whether $\mathbf{x}_0$ is "far enough" or not.

The assumed contraction of the physical space standards (and thus the dilation of measured distances), with respect to the natural metric on M, occurs in the direction of the gravity acceleration $\mathbf{g}$. Hence the expression of $\mathbf{g}$, the physical space metric in the frame $\mathcal{E}$, is the simplest in an "isopotential" coordinate system. This is a space-time coordinate system (y^α) such that, at any given time t, $y^1 =$ const (in space) is equivalent to $p_e =$ const, and such that the natural metric $\mathbf{g}^0$ is diagonal: $(g^0{}_{ij}) = \mathrm{diag}\,(a^0{}_1, a^0{}_2, a^0{}_3)$. In such a system, the gravitational space contraction implies that $\mathbf{g}$ also is diagonal, and is written as

$$(2.7) \qquad (g_{ij}) = \mathrm{diag}\left(\frac{1}{\beta^2}a^0{}_1, a^0{}_2, a^0{}_3\right) \equiv \mathrm{diag}\,(a_i)_{1\le i\le 3} \text{ (isopotential system)}, \qquad \beta = p_e/p_e^\infty .$$

(Greek indices will vary from 0 to 3 and Latin ones from 1 to 3). Note that, in the general case where p_e depends on the time t, such a system will usually not be bound to the frame $\mathcal{E}$, i.e. a point $\mathbf{x}$ in M will in general have time-dependent coordinates $y^i(t, \mathbf{x})$. An important exception is the case where one has spherical symmetry around a fixed point $\mathbf{x}_0$ in M (this either implies that $(M, \mathbf{g}^0)$ is Euclidean, or must be understood locally). Then, the time t makes with the spherical coordinates r, θ, ϕ an isopotential coordinate system bound to $\mathcal{E}$.

The line element of the space-time metric γ, measuring the proper time $d\tau$ along an element of trajectory, follows straightforwardly from the combination of the slowing down of the mobile clock due to its absolute motion and to the gravitation field [2]. If dl is the line element of the space metric $\mathbf{g}$, i.e. if dl is the elementary distance covered by the mobile, as measured with rods of the momentarily coincident observer bound to $\mathcal{E}$, one has:

$$(2.8) \qquad ds^2 = \gamma_{\lambda\mu}dx^\lambda dx^\mu = c^2d\tau^2 = \beta^2(dx^0)^2 - dl^2, \quad x^0 = ct, \quad dl^2 = g_{ij}dx^i dx^j.$$

The validity of Eq. $(2.8)_3$ assumes that the coordinates (x^α) are bound to the frame $\mathcal{E}$. By Eq. (2.7), a diagonal expression is hence obtained also for γ in the particular case where an isopotential coordinate system bound to the frame $\mathcal{E}$ can be found. In that case:

$$(2.9) \qquad (\gamma_{\lambda\mu}) = \mathrm{diag}\left(\beta^2, -\frac{1}{\beta^2}a^0{}_1, -a^0{}_2, -a^0{}_3\right) \equiv \mathrm{diag}\,(b_\lambda)_{0\le\lambda\le 3}\,.$$

If general coordinates (z^α) are used, one writes instead of Eq. $(2.8)_3$:

$$(2.10) \qquad dl^2 = dt_{\mathbf{x}}^2 g_{ij}v^i v^j = dt_{\mathbf{x}}^2 v^2,$$

with v^i the components of $\mathbf{v} = \mathbf{dx}/dt_{\mathbf{x}} = (1/\beta)\mathbf{dx}/dt$ ($\mathbf{v}$ is the velocity vector with respect to $\mathcal{E}$ and using the local time of the momentarily coincident clock of $\mathcal{E}$) in the coordinates (z^i) (these being considered, at a given time t, as a space coordinate system, thus $v^i = \dfrac{\partial z^i}{\partial x^j}\dfrac{dx^j}{dt_{\mathbf{x}}}$ if the (x^α) are bound to $\mathcal{E}$). Of course, γ is a space-time tensor and may be written in any space-time coordinates (z^α). Also, a space metric $\mathbf{h}$ can be defined in the frame $\mathcal{F}$ bound to such coordinates, from the metric γ: the components of the inverse matrix $\mathbf{h}^{-1}$ are $h^{ij} = -\gamma^{ij}$, with $(\gamma^{\lambda\mu})$ the inverse matrix of $(\gamma_{\lambda\mu})$. As shown by LANDAU and LIFSHITZ [12], the metric $\mathbf{h}$ is that one which gives the distances between neighbouring points bound to $\mathcal{F}$ (i.e. having constant space coordinates z^i), as evaluated from the interval of their local time, $d\tau$, for a to-and-fro light path: $dl' = (h_{ij}\,dz^i\,dz^j)^{1/2} = c\,d\tau/2$. Since SR still holds locally in the presence of gravitation, the result of PROKHOVNIK [21] is valid, according to which dl' is indeed the physical distance which may be measured with rods of the frame $\mathcal{F}$.

2.3. Expression of the field equation in terms of natural metric and absolute time

The expressions of the "main" field equation (2.2) and the "auxiliary" one (2.1) are in terms of the physical space metric and local (physical) time, in the frame bound to the absolute space or macro-ether M (it is recalled that Eq. (2.2) is only valid in this frame $\mathcal{E}$). However, this physical space-time metric depends on the unknown, i.e. on the field p_e. In some cases, it is easier to handle Eq. (2.2) when it is expressed in terms of the natural metric $\mathbf{g}^0$ and absolute time t, as this turns out to be possible. Starting from Eq. (2.1) with $p_e = c^2\varrho_e$, we use isopotential coordinates (y^α) and Eqs. (2.7) and (2.3), and obtain, since p_e or β depends only on y^1:

$$\mathbf{g} = -c^2\frac{\mathrm{grad}_{\mathbf{g}}p_e}{p_e} = -c^2\frac{\mathrm{grad}_{\mathbf{g}}\beta}{\beta} = -\frac{c^2}{\beta}\frac{\beta^2}{a^0{}_1}\beta_{,1}\mathbf{e}_1 = -\frac{c^2}{2}\frac{1}{a^0{}_1}(\beta^2)_{,1}\mathbf{e}_1\,, \tag{2.11}$$

where $(\mathbf{e}_i)_{1\le i\le 3}$ is the natural basis associated with the coordinates (y^i). Thus:

$$\mathbf{g} = -\frac{c^2}{2}\mathrm{grad}_{\mathbf{g}^0}f = -\frac{c^2}{2}\mathrm{grad}_0 f, \qquad f \equiv \beta^2 = (\gamma_{00})_{\mathcal{E}} \tag{2.12}$$

(that $(\gamma_{00})_{\mathcal{E}} = \beta^2$ if one uses the absolute time coordinate t, is due to Eq. $(2.8)_1$). Turning to Eq. (2.2), we have $g \equiv \det(g_{ij}) = g^0/\beta^2$ from (2.7), thus we first obtain in the same way:

$$\Delta p_e \equiv \Delta_{\mathbf{g}}p_e = p_e^\infty\Delta_{\mathbf{g}}\beta = \frac{p_e^\infty}{\sqrt{g}}\left(\sqrt{g}\frac{\beta^2}{a^0{}_1}\beta_{,1}\right)_{,1} = p_e^\infty\frac{\beta}{\sqrt{g^0}}\left(\sqrt{g^0}\frac{\beta\beta_{,1}}{a^0{}_1}\right)_{,1}$$

(cf. Eq. (2.4)), which may be rewritten as:

$$\Delta p_e = p_e^\infty\frac{\beta}{2}\Delta_0(\beta^2), \qquad \Delta_0 \equiv \Delta_{\mathbf{g}^0}\,. \tag{2.13}$$

Now, let us assume that our (model of) universe, in which the studied gravitation field and matter are embedded, is "static at infinity", which is likely to be an extremely good approximation – except for cosmological problems. That is, assume that p_e^∞ is independent of the time t (4). In that case, the term with time derivatives in Eq. (2.2) becomes:

$$\frac{1}{c^2}\frac{\partial^2 p_e}{\partial t_{\mathbf{x}}^2} = \frac{1}{c^2}\frac{1}{\beta}\frac{\partial}{\partial t}\left(\frac{1}{\beta}\frac{\partial p_e}{\partial t}\right) = \frac{p_e^\infty}{\beta}\left(\frac{\beta_{,0}}{\beta}\right)_{,0} = p_e^\infty \frac{\beta}{2f}\left(\frac{f_{,0}}{f}\right)_{,0}. \tag{2.14}$$

Combining Eqs. (2.13) and (2.14) and multiplying by $2/(\beta p_e^\infty)$, we rewrite Eq. (2.2) as:

$$\Delta_0 f - \frac{1}{f}\left(\frac{f_{,0}}{f}\right)_{,0} = \frac{8\pi G}{c^2}\varrho\,. \tag{2.15}$$

Thus, the field equation reduces in the static case to the ordinary Poisson equation, which is *linear*, and indeed Eqs. (2.12) and (2.15) are equivalent, for time-independent f (and ϱ), to the Newtonian equations: exactly the Newtonian gravity acceleration $\mathbf{g}$ will be associated with any given density of mass-energy, $\varrho(\mathbf{x})$, by Eqs. (2.12) and (2.15). However, in NG, only the "invariable" (rest) mass is counted in ϱ; moreover, it remains to formulate the definition of ϱ, and it turns out that ϱ must depend on the gravitation field, i.e. on f itself (Sec. (4.2), point (iv)). Anyhow, not the same motion will be predicted for test particles if $\mathbf{g}$ is known, since in the studied theory Newton's second law is expressed in terms of space and time measurements with clocks and rods of the local observer in $\mathcal{E}$, which are affected by the gravitation field. In the static case with spherical symmetry, for example, Eq. (2.15) (plus the requirement that $\mathbf{g}$ remains bounded as $r \to 0$, or the boundary condition $f = 1$ at infinity), gives $f = 1 - 2Gm/(c^2 r)$ outside the body (r being the radial Euclidean distance, and with

$$m = \int_M \varrho\, dV^0. \tag{2.16}$$

where dV^0 is the volume element of the Euclidean metric). Then Eq. (2.9), with $a^0{}_1 = 1$, $a^0{}_2 = r^2$ and $a^0{}_3 = r^2 \sin^2\theta$ in spherical coordinates r, θ, ϕ, leads to Schwarzschild's exterior space-time metric. It is striking that, after natural account of SR and the equivalence principle, this theory (assumed to reduce to NG asymptotically as the "ether compressibility" evanesces), gives exactly and simultaneously the Newtonian attraction field $\mathbf{g}$ and the Schwarzschild metric in the spherical static case [2]. The new result is that the Newtonian $\mathbf{g}$-field is

(4) This assumption is in fact less restrictive than that of an "insular matter distribution embedded in a Galilean space-time", which is commonly set in GR (e.g. FOCK [10], LANDAU and LIFSHITZ [12]). In particular, p_e^∞ does not actually need to be reached somewhere, even asymptotically [3], and thus it can be constant even if the matter distribution has unbounded support.

predicted for any static situation, whether it is "spherical" or not. The form (2.15) of the field equation has important applications also in the non-static case, see Sec. 4.2 (energy balance).

3. The motion of a test particle: Newton law vs. space-time geodesics

To state Newton's second law demands to define the acceleration or rather (since SR must be taken into account) the time derivative of the momentum. Also because SR must hold at the local scale (and for sure with physical space-time metric), one has to use the physical, distorted space and time standards, i.e. the Riemannian space metric $\mathbf{g}$ and the local time $t_{\mathbf{x}}$.

3.1. The case of a time-independent spatial metric g (static gravitation field)

If on a manifold M a Riemannian metric is given (thus a *fixed* metric $\mathbf{g}$), one has a natural definition of the "time" derivative of a vector $\mathbf{u}(t)$ attached to a "trajectory", i.e. a differentiable mapping $t \mapsto X(t)$ from an open interval in $\mathbf{R}$ into M. The components η^i of the derivative $D\mathbf{u}/Dt$ in a local coordinate system (x^i) on M are given by:

$$\eta^i = \left(\frac{D\mathbf{u}}{Dt}\right)^i = \frac{d(u^i)}{dt} + \Gamma^i_{jk} u^j v_0{}^k , \tag{3.1}$$

with Γ^i_{jk} being the second-kind Christoffel symbols associated with metric $\mathbf{g}$ in coordinates (x^i), and $v_0^k = dx^k/dt = (\mathbf{dx}/dt)^k$ – the components of the "velocity vector" (with respect to the "time" t which in (3.1) may be an arbitrary parameter, although in the rest of the paper t denotes the absolute time). The definition (3.1) may be found in the literature, e.g. in BRILLOUIN [5] and in LICHNEROWICZ [13] where it is induced from that of a covariant derivative. However, a covariant derivative demands, strictly speaking, that $\mathbf{u}$ should be a vector field (defined in an open domain around the considered point X), whereas here $\mathbf{u}$ is only defined along the trajectory, as a function of the parameter t (and this, of course, is what is needed to use Eq. (3.1)). On requiring that the derivative $D\mathbf{u}/Dt$ should cancel for any parallel vector and be a true derivative, i.e. obey the Leibniz rule, one is lead uniquely to characterize $D\mathbf{u}/Dt$ by the following intrinsic property:

for any vector $\mathbf{w}$ in the tangent space TM_X at point $X \in M$,

$$\left(\frac{D\mathbf{u}}{Dt}\right) \cdot \mathbf{w} = \frac{d}{dt}(\mathbf{u} \cdot \mathbf{w}'), \tag{3.2}$$

where point means scalar product $\mathbf{g}$ and $\mathbf{w}'$ is the parallel transport (using $\mathbf{g}$) of vector $\mathbf{w}$ on the trajectory [2]; moreover, the components of the *vector* $D\mathbf{u}/Dt$ defined in this way, are indeed given by Eq. (3.1). Note that here M is *also* equipped with the Euclidean metric $\mathbf{g}^0$ which would allow to speak of "a vector $\mathbf{v}$ in M"

(i.e. not specifically attached to a point $X \in M$), and in fact to identify points in M and vectors (once an arbitrary origin point $O \in M$ has been selected) ([5]), whence follows our notation $\mathbf{x} \in M$ used in previous sections. Thus the Newton law of SR has been extended to this theory of gravity, in the form:

$$\mathbf{F} \equiv \mathbf{F}_0 + m(v)\mathbf{g} = \frac{D}{Dt_{\mathbf{x}}}(m(v)\mathbf{v}), \qquad \frac{D}{Dt_{\mathbf{x}}} \equiv \frac{1}{\beta}\frac{D}{Dt}, \tag{3.3}$$

where $\mathbf{F}_0$ is the non-gravitational (e.g. electromagnetic) force, $\mathbf{v} = \dfrac{\mathbf{dx}}{dt_{\mathbf{x}}}$ is the velocity of the test particle with respect to M, $v = \mathbf{g}(\mathbf{v},\mathbf{v})^{1/2} \equiv (g_{ij}v^i v^j)^{1/2}$ its modulus and $m(v) = m(0)/(1 - v^2/c^2)^{1/2}$ is the inertial mass, which is thus identical to the passive gravitational mass, as is also true in NG. Although the latter identity is often referred to in GR (under the name of "weak equivalence principle"), it does not make any exact sense in GR, because there is no Newton law there [27]. The Newton law (3.3) is also defined for *light-like* particles (photons, neutrinos?), in substituting the energy $e = h\nu$ (or rather e/c^2) for the inertial mass.

3.2. Extension to the time-dependent situation

In the general case, the metric $\mathbf{g}$ will yet depend on the time t (i.e. its components g_{ij} in coordinates bound to $\mathcal{E}$ will depend on t), hence it seems at first that Eq. (3.2), which is at the root of the definition of the time derivative of the momentum, used in Eq. (3.3), does not make sense any more: how can one define a parallel transport with respect to a metric that varies, i.e. with respect to a one-parameter family of metrics? Thus it seems at first that the Newton law (3.3) makes sense only for *static* gravitation fields, and so was it defined and used in the previous work [2–3]. However, it turns out that Eqs. (3.2)–(3.3) do still make sense, provided one "freezes" the space metric $\mathbf{g}_{t_0}$ *of the time* t_0 *where the derivative is to be calculated*. In order to calculate the derivative $D\mathbf{P}/Dt_{\mathbf{x}}$ of the momentum $\mathbf{P} = m(v)\mathbf{v}$, one thus might use Eq. (3.1) (with $\mathbf{P}$ in the place of $\mathbf{u}$ and the local time $t_{\mathbf{x}}$ in the place of t) where the Γ symbols refer to the metric $\mathbf{g}_{t_0}$. A natural definition of the parallel transport $\mathbf{w}'$ in Eq. (3.2) can indeed be given only if one considers the fixed metric $\mathbf{g}_{t_0}$. But this observation does not completely solve the question on the extension of Eq. (3.2) to the case with time-dependent metric $\mathbf{g}_t$: it remains to investigate whether the scalar product on the right of Eq. (3.2) should also refer to the fixed metric $\mathbf{g}_{t_0}$, or instead to the time-dependent metric $\mathbf{g}_t$. *If we take* $\mathbf{g}_{t_0}$, we define from (3.2) a derivative $D_0\mathbf{u}/Dt$ for which the Leibniz rule and Eq. (3.1) hold true, but clearly we are

([5]) In the case where $(M, \mathbf{g}^0)$ would be instead a space with constant curvature, one also might identify vectors in TM_X and in TM_O by the parallel transport along the geodesic of $\mathbf{g}^0$ joining O to X. Contrary to the Euclidean case, this identification procedure allowing to speak of "a vector $\mathbf{v}$ in M" would be dependent on the origin O, but in a way which should be harmless for the definition of a velocity field with *nil* macro-average (Sec. 2.1).

missing something, namely the time variation of the physical metric $\mathbf{g}_t$. *If we take* $\mathbf{g}_t$ and reexamine the derivation of Eq. (3.1) from Eq. (3.2) (Appendix 1 in [2]) we easily obtain, instead of (3.1):

$$(3.4)\qquad \left(\frac{D_1\mathbf{u}}{Dt}\right)^i \equiv \frac{d(u^i)}{dt} + \Gamma^i_{jk}u^j v_0{}^k + g^{ij}\frac{\partial g_{jk}}{\partial t}u^k \equiv \left(\frac{D_0\mathbf{u}}{Dt}\right)^i + \left(\mathbf{g}^{-1}\cdot\left(\frac{\partial \mathbf{g}}{\partial t}\cdot\mathbf{u}\right)\right)^i,$$

where, for a twice covariant second-order tensor $\mathbf{h}$ (here $\partial\mathbf{g}/\partial t$) and a vector $\mathbf{u}$, $\mathbf{h}\cdot\mathbf{u}$ is the covector with components $h_{ij}u^j$. And, for a twice contravariant second-order tensor $\mathbf{k}$ (here $\mathbf{g}^{-1}$) and a covector $\mathbf{u}^*$, $\mathbf{k}\cdot\mathbf{u}^*$ is the vector with components $k^{ij}u^*{}_j$. Thus, Eq. (3.4) also defines a vector $D_1\mathbf{u}/Dt$. Yet this time-derivative, which accounts for the time variation of the metric, does not cancel for a vector $\mathbf{u}$ that is transported parallel to itself (necessarily with respect to the fixed metric $\mathbf{g}_{t_0}$) along the trajectory. The use of Eq. (3.2) for the definition is hence questionable, since Eq. (3.2) was obtained (in the case of a fixed metric) under the former requirement, plus the condition that the Leibniz rule has to be verified. If we provisionally forget the Leibniz rule, we can therefore define a one-parameter family of vector time-derivatives as well:

$$(3.5)\qquad \left(\frac{D_\lambda\mathbf{u}}{Dt}\right)^i \equiv \frac{d(u^i)}{dt} + \Gamma^i_{jk}u^j v_0{}^k + \lambda g^{ij}\frac{\partial g_{jk}}{\partial t}u^k \equiv \left(\frac{D_0\mathbf{u}}{Dt} + \lambda\mathbf{g}^{-1}\cdot\left(\frac{\partial \mathbf{g}}{\partial t}\cdot\mathbf{u}\right)\right)^i,$$

(that $D_0\mathbf{u}/Dt$ enters the definition (3.5) with the coefficient 1 is enforced, since we want to recover Eqs. (3.1) and (3.3) for a time-independent metric). Now let us come to Leibniz rule. We obtain from Eq. (3.5) and the fact that, by construction, $D_0\mathbf{u}/Dt$ obeys the Leibniz rule with the fixed metric $\mathbf{g}_{t_0}$:

$$(3.6)\qquad \begin{aligned}\mathbf{u}\cdot\frac{D_\lambda\mathbf{v}}{Dt} + \mathbf{v}\cdot\frac{D_\lambda\mathbf{u}}{Dt} &= \mathbf{u}\cdot\frac{D_0\mathbf{v}}{Dt} + \mathbf{v}\cdot\frac{D_0\mathbf{u}}{Dt} + 2\lambda\frac{\partial\mathbf{g}}{\partial t}(\mathbf{u},\mathbf{v})\\ &= \frac{d}{dt}\left[\mathbf{g}_{t_0}(\mathbf{u}(t),\mathbf{v}(t))\right] + 2\lambda\frac{\partial\mathbf{g}}{\partial t}(\mathbf{u}(t_0),\mathbf{v}(t_0)).\end{aligned}$$

But, by direct calculation, we also have:

$$(3.7)\qquad \begin{aligned}\frac{d}{dt}[\mathbf{g}_t(\mathbf{u}(t),\mathbf{v}(t))] &= \frac{d}{dt}\left[g_{ij}(t,\mathbf{x}(t))u^i(t)v^j(t)\right]\\ &= \frac{d}{dt}\left[g_{ij}(t_0,\mathbf{x}(t))u^i(t)v^j(t)\right] + \frac{\partial g_{ij}}{\partial t}u^i(t_0)v^j(t_0),\\ \frac{d}{dt}[\mathbf{g}_t(\mathbf{u}(t),\mathbf{v}(t))] &= \frac{d}{dt}\left[\mathbf{g}_{t_0}(\mathbf{u}(t),\mathbf{v}(t))\right] + \frac{\partial\mathbf{g}}{\partial t}(\mathbf{u}(t_0),\mathbf{v}(t_0)).\end{aligned}$$

By comparing (3.6) and (3.7), we find that *Leibniz' rule holds (with the variable metric* $\mathbf{g}_t$*) if and only if* $\lambda = 1/2$ *in* Eq. (3.5). Nevertheless, with any value of λ, we may associate a particular Newton law, in the following way:

$$(3.8)\qquad \mathbf{F} \equiv \mathbf{F}_0 + m(v)\mathbf{g} = \frac{D_\lambda}{Dt_{\mathbf{x}}}(m(v)\mathbf{v}), \quad \frac{D_\lambda}{Dt_{\mathbf{x}}} \equiv \frac{1}{\beta}\frac{D_\lambda}{Dt}, \quad \mathbf{v} \equiv \frac{\mathbf{dx}}{dt_{\mathbf{x}}} \equiv \frac{1}{\beta(t,\mathbf{x})}\frac{\mathbf{dx}}{dt}.$$

Now, is $\lambda = 1/2$ really the correct value for the parameter λ? Another possible criterion for the choice could be the compatibility of Eq. (3.8) with the formulation of motion in GR. It has been proved that, *in the static case*, the Newton law (3.3) *implies* that any free test particle follow a geodesic line of the metric γ, and this for a mass particle [2] as well as for a light-like one [3]. It is not difficult (though a bit tedious) to follow the proof for a mass particle in the static case, the calculation method also applying to the case where one still has isopotential coordinates bound to $\mathcal{E}$. One then finds that, already in that particular case, any time dependence of the field p_e makes new terms appear in the geodesic equation for the values 0 and 1 of index α, corresponding to new non-zero Christoffel symbols of metric γ in isopotential coordinates; and these new terms imply that, for $\alpha = 0$ and $\alpha = 1$, this equation $G^\alpha = 0$ is verified if and only if $\lambda = 1$. The $\alpha = 0$ term will be examined indirectly in Sec. 4.2 (in connection with the energy balance). In summary, we presently cannot state for sure that the Newton law (3.8) with $\lambda = 1$ implies geodesic motion in the most general case, but we already know that *the law* (3.8) *with any* $\lambda \neq 1$, in particular with $\lambda = 1/2$ (as is needed to have the Leibniz rule) *is incompatible with geodesic motion*. Here is thus a crucial bifurcation in the studied ether theory: *one has to decide between Newton law (with Leibniz rule, thus* $\lambda = 1/2$*) and space-time geodesics*. In our previous work, the second choice had been made. This choice is consistent with the commonly accepted "logic of the space-time", but this was not the main reason for the choice: simply, no other possibility had been seen! From a rather philosophical point of view, the first choice would be in fact more consistent with the "logic of absolute motion" which is studied here (cf. PROKHOVNIK [21] for a detailed discussion of the two different "logics" in the frame of SR). The agreement with observation might be the true judge between the two possibilities, both of which are indeed compatible with the kernel of the studied theory (summarized in Secs. 2.1 and 2.2) from a purely physical viewpoint. A strong physical argument for the choice "Newton law" (with $\lambda = 1/2$) and against "Einstein geodesics" within the studied theory will, however, appear in Sec. 4.2.

4. The energy problem in the studied theory of gravitation

4.1. Some remarks on energy and conservation laws in Newtonian theory and in general relativity

i. Is the concept of energy a relative one?

The concept of energy is among the most important ones in today's physics, especially in classical physics (classical mechanics of mass points and continuous media, thermodynamics, classical electromagnetism) and in microphysics as well, but the notable exception is the gravitation theory (GR): there, this concept can have only an approximate status since it is not a covariant concept. Already in NG, the (kinetic plus potential) energy of a mass point, $e = (v^2/2 - U)m$, is not even a Galilean invariant. Indeed, v is obviously not invariant, whereas the potential is a

Galilean invariant, since both Poisson's equation and the merely spatial boundary conditions, e.g. rU and $r^2 \operatorname{grad} U$ bounded at infinity, are so (it is recalled that strict NG imposes a spatially bounded distribution of mass). That e is changed by a constant when changing the inertial frame makes, of course, no problem in NG since only the variations of e are relevant there. But since NG demands a bounded mass distribution, the energy in the frame of the mass-center *could* be preferred on theoretical grounds, and is indeed preferred in actual analyses: in classical mechanics, be it celestial or terrestrial, one considers an assumed isolated system and one refers the velocity, and hence the kinetic energy, to the mass center. Strictly speaking, one should consider an (approximately) isolated subsystem of the assumed bounded universe and, for elements of the subsystem, evaluate their velocity in the global mass-center frame. But this would differ by a *definite* constant (the velocity of the mass center of the subsystem) from the velocity in the frame bound to the mass center of the subsystem. Thus in NG (and in the same way in a good part of classical physics), an absolute concept of energy is allowable, probably favourable indeed (cf. the definition of temperature in statistical thermodynamics, as a mean kinetic energy). Apart from this, it seems that, in non-relativistic quantum mechanics, one would prefer to avoid discussing the effect, on the energy levels, of changing the reference frame. It may be that in quantum mechanics also, an absolute concept of energy would be favourable.

ii. Energy and conservation laws in Newtonian gravity

In addition to this (debatable) absolute character, a still more important (and undebatable) aspect of the energy concept is that, in classical physics, it gives rise to local *balance equations* for continuous media, which lead to global *conservation laws*. We take the example of NG, which is relevant here, and we assume elastic behaviour for simplicity (this includes the case of a perfect barotropic fluid). Thus one has, in NG, the following definition and conservation equation for the volume density w of the total energy, i.e. the energy of matter (including its potential energy in the gravitation field), w_m, plus the energy of the gravitation field itself, w_g:

$$w = w_m + w_g, \qquad w_m = \varrho(v^2/2 + \Pi - U), \qquad w_g = \frac{\mathbf{g}^2}{8\pi G}, \tag{4.1}$$

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(w_m \mathbf{v} - \boldsymbol{\sigma}\cdot\mathbf{v} - \frac{\partial U}{\partial t}\frac{\mathbf{g}}{4\pi G}\right) = 0, \tag{4.2}$$

with $\boldsymbol{\sigma}$ the stress tensor and Π the mass density of elastic energy; this is derived from the more usual energy balance (in which the energy w_g does not appear and a source term $\varrho\partial U/\partial t$ remains), by using Poisson's equation. Upon integrating Eq. (4.2) in the whole space and since the matter distribution is spatially bounded, one obtains the energy integral in two successive forms, by using the fact that the

field $\mathbf{g}$ is $O(1/r^2)$ at large r and $\partial U/\partial t$ tends towards 0 at large r:

$$(4.3) \qquad E_m + E_g \equiv \int_{\text{matter}} w_m \, dV + \int_{\text{space}} w_g \, dV = \text{const} = E,$$

$$(4.4) \qquad E_m + E_g = \int_{\text{matter}} \left(w_m + \frac{\varrho U}{2} \right) dV = \int_{\text{matter}} \varrho \left(\frac{v^2}{2} + \Pi - \frac{U}{2} \right) dV = \text{const} = E.$$

These equations allow a clear analysis of the energy transfer from matter to gravitation field and, inside the contribution of matter, from potential to kinetic and internal energy. Thus, starting from an "unbound" state in which both the (*positive*) gravitational energy and the (negative) potential energy $E_p = \int -\varrho U \, dV$ are closer to zero than in the final state, a gravitational concentration of matter will increase the gravitational energy E_g and thus decrease the total energy of matter, E_m, Eq. (4.3). But since w_m includes the potential energy, the decrease in E_m is only due to the decrease in E_p, δE_p, and the "pure" (internal plus kinetic) energy of matter is in fact increased by the amount $-\delta E_p/2$, Eq. (4.4). There is also a conservation equation for momentum in NG, namely:

$$(4.5) \qquad \frac{\partial(\varrho \mathbf{v})}{\partial t} + \mathbf{div}\,(\varrho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}) + \frac{1}{4\pi G} \mathbf{div} \left(\mathbf{g} \otimes \mathbf{g} - \frac{\mathbf{g}^2}{2} \mathbf{I} \right) = 0,$$

(here $\mathbf{u} \otimes \mathbf{v}$ is the tensor product, thus $(\mathbf{u} \otimes \mathbf{v})^{ij} = u^i v^j$; the vector operator **div** means the Euclidean divergence of a second-order space tensor $\mathbf{t}$, with $(\mathbf{div}\,\mathbf{t})^i = t^{ij}{}_{,j}$ in any coordinates deduced from Cartesian ones by a linear transformation; and $\mathbf{I}$ or $\boldsymbol{\delta}$ is the identity tensor, $I^{ij} = \delta^{ij}$). This is the mere rewriting, using the Poisson and continuity equations, of Newton's second law for the continuous medium: Poisson's equation implies that the last term in Eq. (4.5) is simply $-\varrho \mathbf{g}$. But, contrary to Eqs. (4.3)–(4.4) in which the energy of the gravitation field does influence the global balance, the integration of Eq. (4.5) in the whole space gives nothing else than the standard conservation of the total momentum of matter. In other words: *in NG, the gravitation field has energy and this is positive, but it has zero total momentum*. Equations (4.4) and (4.5) are given in CHANDRASEKHAR [8] though Eq. (4.5) is written there in a form which is similar to the "conservation equation" for the mass space-time tensor $\mathbf{T}$ of GR,

$$(4.6) \qquad [\mathrm{div}_\gamma \mathbf{T}]^\alpha \equiv T^{\alpha\beta}{}_{;\beta} = 0.$$

iii. General relativity

With Eq. (4.6), the problem of energy and conservation laws in GR appears. As explained by LANDAU and LIFSHITZ [12], it "does not in general express the conservation law of anything" (§101), i.e. it cannot be considered as a true conservation equation. The reason is that no Gauss theorem applies to the divergence

of a second-order tensor in a curved Riemannian space (see also STEPHANI [28]). In several relevant situations, including that of an asymptotically flat metric γ (i.e. a space-time that is Galilean at infinity), Eq. (4.6) nevertheless implies integral conservation laws. In order to obtain such laws, one rewrites Eq. (4.6) in the form of a divergence with respect to a flat space-time metric γ^0, by adding to $\mathbf{T}$ a so-called "energy-momentum pseudo-tensor of the gravitation field", $\mathbf{t}$ (the expression of which is not unique), thus:

$$(4.7) \qquad (\mathrm{div}_{\gamma^0}\boldsymbol{\theta})^\alpha \equiv \theta^{\alpha\beta}{}_{,\beta} = 0, \qquad \boldsymbol{\theta} \equiv (-\gamma)(\mathbf{T} + \mathbf{t})$$

(the factor $-\gamma$ is there for reasons which are bound to the form of the field equations in GR). That the space-time may be equipped with a global flat metric is, of course, a strong topological assumption; in general, the discussion here is valid in the domain Ω where the coordinate system (x^α) in Eq. $(4.7)_1$ is regular, and the metric γ^0 can be *defined* by: $\gamma^0_{\alpha\beta} = \eta_{\alpha\beta}$ $(= \varepsilon_\lambda \delta_{\alpha\lambda}\delta_{\beta\lambda}$ with $\varepsilon_0 = 1$, $\varepsilon_i = -1$ for $i = 1,2,3)$ in these coordinates. Anyway, in GR, the choice of the "background" metric γ^0 (local or global) cannot be imposed. The first equation remains valid in any coordinates deduced by a linear transformation from Galilean coordinates of the flat metric γ^0, and the definition of (any) $\mathbf{t}$ in GR makes $\mathbf{t}$ a tensor only under linear coordinate transformations. Due to the factor $-\gamma$, the definition of $\boldsymbol{\theta}$ and the whole set (4.7) are hence covariant only for those transformations from Galilean coordinates of the metric γ^0 that are both linear and "unimodular", thus essentially for Lorentz transformations of the space-time (or the domain Ω) equipped with the flat metric γ^0. In particular, the expression of (any) $\mathbf{t}$ gives zero in locally geodesic coordinates for the curved metric γ. These peculiarities allow to state that "it is doubtful that (one of such pseudotensors $\mathbf{t}$, namely that proposed by Einstein) can in general describe something which could be called energy" (TRAUTMAN [29]). If the values of $\theta^{\alpha\beta}$ decrease sufficiently fast at infinity ([6]) (this implies in particular that the metric γ asymptotically coincides with the flat one γ^0, thereby ensuring an "asymptotic uniqueness" to γ^0, in so far as the metric γ of GR may be considered unique), the integration of Eq. (4.7) over the whole "space" (spatial section $x^0 = \mathrm{const}$ of the space-time) is possible and gives, owing to the Gauss theorem:

$$(4.8) \qquad P^\alpha \equiv \frac{1}{c}\int_{\text{space}} \theta^{\alpha 0}\, dx^1\, dx^2\, dx^3 = \mathrm{const}.$$

This is sometimes interpreted in GR as the conservation law for energy ($\alpha = 0$) and momentum ($\alpha = 1,2,3$; see e.g. LANDAU and LIFSHITZ [12]). In contrast to Eqs. (4.3)–(4.4), however, the complex expression of pseudo-tensor $\mathbf{t}$ in terms of derivatives of the metric makes it difficult to draw definite conclusions from Eq. (4.8), regarding the energy transfer. Moreover, it does not seem to be

([6]) A sufficient condition is: $\theta^{\alpha 0} = o(1/r^3)$, $\theta^{\alpha 0}{}_{,0} = o(1/r^3)$ and $\theta^{\alpha i}x^i = o(1/r)$ at large $r \equiv (x^i x^i)^{1/2}$.

completely clear what should be the physically motivated conditions ensuring the sufficient decrease at infinity for θ. Thus if one assumes a "time-independent far field", one finds that $P^i = 0$ and that P^0 is the active mass entering the expression of the metric at large distance (cf. STEPHANI [28]). But, strictly speaking, only a constant gravitation field, thus without any motion of massive bodies, will give a time-independent field – even if this is the far field. According to the spirit of GR, a general (time-dependent) gravitation field should radiate gravitational energy, and here the discussion is less clear-cut. One often uses the linear approximation of GR and then, as observed by STEPHANI [28], "if one tries to calculate not the loss of energy but the total energy of the system emitting quadrupole waves, the corresponding integrals diverge for $r \to \infty$ if the system emits continuously (the whole space is filled by radiation)." Thus in relevant situations which exhibit qualitative differences with NG, the first-order expansions must be checked by complex expansions. A more severe difficulty is the contradiction between the principle of general relativity and the necessity of introducing a flat reference metric, i.e. some equivalent of the Galilean class of the inertial frames in NG, in order to define a concept of energy in GR. This difficulty, and some other reasons, led LOGUNOV *et al.* [14] to propose a "relativistic theory of gravitation" (RTG), in which the field equations of GR are supplemented by the De Donder–Fock harmonic condition (Eqs. (2.39)–(2.40) in LOGUNOV *et al.* [14]). The RTG is written in generally covariant form, hence the harmonic condition (2.40) is there an *additional field equation* instead of a coordinate condition. Yet the RTG does admit a privileged class of coordinates (hence also a class of reference frames), similar to the Galilean class of NG or rather to the Lorentz class of SR. These are, in the RTG, Galilean coordinates of the "base Minkowski space", i.e. the space-time equipped with a flat metric γ^0 ($(\gamma^{\mu\nu})$ in the notation of LOGUNOV *et al.*; in the RTG, the metric γ^0 is assumed to be global, and as a consequence of the field equations γ^0 is unique). In such coordinates, as also in coordinates deduced by a linear transformation, their generally covariant Eq. (2.40) becomes identical to Fock's harmonic *coordinate condition*. Hence, Galilean coordinates for γ^0 are also harmonic coordinates for the "effective Riemannian space", i.e. the space-time equipped with the curved metric γ ($(g^{\mu\nu})$ in their notations). A possible objection to the RTG is then, that the arguments leading to the field equations of GR depend strongly on the equal status enjoyed by all reference frames: if one has a privileged class of reference frames (exchanging by Lorentz transformations), one has less physical reasons to express gravity by generally covariant equations.

4.2. The energy conservation in the studied theory with Newton law

i. The energy of a free test particle, and its time evolution in non-static situations

As in NG, the energy of a mass point appears first as a natural conserved

quantity in the case of time-independent gravitation potential. We have in general from Eq. (2.1) with $p_e = \varrho_e c^2$ and $\beta = p_e/p_e^\infty$:

$$\mathbf{g} = -\frac{c^2}{\beta}\mathrm{grad}_{\mathbf{g}}\beta = \mathrm{grad}_{\mathbf{g}}U, \qquad U \equiv -c^2\mathrm{Log}\,\beta\,. \tag{4.9}$$

By the Newton law (3.8) with purely gravitational force ($\mathbf{F}_0 = 0$), we have also, using Eq. (3.5):

$$\mathbf{g} = (c^2 - v^2)\frac{D_0}{Ds}\left(\frac{\mathbf{dx}}{ds}\right) + \frac{\lambda}{\beta}\mathbf{g}^{-1}\cdot\left(\frac{\partial \mathbf{g}}{\partial t}\cdot\mathbf{v}\right) \tag{4.10}$$

where, by definition,

$$\begin{aligned} \frac{dt}{ds} &\equiv \frac{\gamma_v}{c\beta(t,\mathbf{x})}\,, \qquad & \gamma_v &\equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\,, \\ v &\equiv \mathbf{g}_t(\mathbf{v},\mathbf{v})^{1/2}, & \mathbf{v} &\equiv \frac{1}{\beta(t,\mathbf{x})}\frac{\mathbf{dx}}{dt}\,. \end{aligned} \tag{4.11}$$

To evaluate the rate of work per unit rest mass, $c\mathbf{g}\cdot(\mathbf{dx}/ds)$, with $\mathbf{g}$ from (4.10), we first observe that

$$\mathbf{g}_{t_0}\left(\frac{D_0}{Ds}\left(\frac{\mathbf{dx}}{ds}\right), \frac{\mathbf{dx}}{ds}\right) = \frac{1}{2}\frac{d}{ds}\left[\mathbf{g}_{t_0}\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right)\right] \tag{4.12}$$

and, as in Eq. (3.7), we find that

$$\frac{1}{2}\frac{d}{ds}\left[\mathbf{g}_{t_0}\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right)\right] = \frac{1}{2}\frac{d}{ds}\left[\mathbf{g}_t\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right)\right] - \frac{1}{2}\frac{\partial \mathbf{g}_t}{\partial s}\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right). \tag{4.13}$$

Now, just as in the Lemma in [2] (p. 127), one shows easily with (4.11) that

$$\frac{c^2 - v^2}{2}\frac{d}{ds}\left[\mathbf{g}_t\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right)\right] = \frac{d}{ds}\left(c^2\mathrm{Log}\,\gamma_v\right). \tag{4.14}$$

Accounting for Eqs. (4.12)–(4.14), we thus obtain with (4.10):

$$\mathbf{g}\cdot\frac{\mathbf{dx}}{ds} \equiv \mathbf{g}_{t_0}\left(\mathbf{g}, \frac{\mathbf{dx}}{ds}\right) = \frac{d}{ds}\left(c^2\mathrm{Log}\,\gamma_v\right) - \frac{c^2 - v^2}{2}\frac{\partial \mathbf{g}}{\partial s}\left(\frac{\mathbf{dx}}{ds}, \frac{\mathbf{dx}}{ds}\right) + \frac{\lambda}{\beta}\frac{\partial \mathbf{g}}{\partial t}\left(\mathbf{v}, \frac{\mathbf{dx}}{ds}\right),$$

or, using (4.11):

$$\mathbf{g}\cdot\frac{\mathbf{dx}}{ds} = \frac{d}{ds}\left(c^2\mathrm{Log}\,\gamma_v\right) + \left(\lambda - \frac{1}{2}\right)\frac{\partial \mathbf{g}}{\partial s}(\mathbf{v},\mathbf{v}). \tag{4.15}$$

Combining Eqs. (4.9) and (4.15), we get

$$\frac{d}{dt}\left(c^2 \mathrm{Log}\,\gamma_v\right) + \left(\lambda - \frac{1}{2}\right)\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v},\mathbf{v}) = \mathbf{g}\cdot\frac{\mathbf{dx}}{dt} = \left(\mathrm{grad}_{\mathbf{g}_{t_0}} U\right)\cdot\frac{\mathbf{dx}}{dt} = U_{,i}\frac{dx^i}{dt} \equiv \frac{dU}{dt} - \frac{\partial U}{\partial t}. \tag{4.16}$$

In the static case ($\beta_{,0} = 0$, whence $U_{,0} = 0$ and $\mathbf{g}_{,0} = 0$) we have thus, coming back to the expression $(4.9)_2$ of U:

$$\frac{d}{dt}\left(c^2 \mathrm{Log}\,\gamma_v + c^2 \mathrm{Log}\,\beta\right) = 0, \qquad \gamma_v\beta = \text{const}. \tag{4.17}$$

But the expression of the (internal plus kinetic) energy in SR is

$$e_{pm} = m(v)c^2 = m(0)\gamma_v c^2, \tag{4.18}$$

hence we have got the result that the total energy of the test particle, including its potential energy in the gravitation field, is simply

$$e_m = e_{pm}\beta \ (= \text{const if } \beta_{,0} = 0). \tag{4.19}$$

As in NG, this total energy is lower than the "pure" energy e_{pm}, in other words the gravitational (potential) energy of *matter*, $e_{gm} = e_m - e_{pm}$, is negative. The constancy of e_m in the static case is also a result of GR (cf. LANDAU and LIFSHITZ [12], § 88), since $\beta = (\gamma_{00})^{1/2}$ (Eq. (2.12)). Yet in GR, this result is not derived from a Newton law and therefore the evolution of e_m in the time-dependent case is complex (moreover, the definition of e_m, which involves only γ_{00} and is hence non-covariant, does not make sense in the general case in GR, nor in the RTG). Here the evolution of e_m is simple, especially if $\lambda = 1/2$; by Eq. (4.16) we get:

$$\frac{d}{dt}\left(\mathrm{Log}(\gamma_v\beta)\right) = \frac{\partial}{\partial t}(\mathrm{Log}\,\beta) + \frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v},\mathbf{v}), \tag{4.20}$$

$$\frac{d}{dt}(\beta\gamma_v) = \gamma_v\frac{\partial\beta}{\partial t} + \beta\gamma_v\frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v},\mathbf{v})$$

or

$$\frac{de_m}{dt} = \frac{e_m}{\beta}\frac{\partial\beta}{\partial t} + \frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v},\mathbf{v}). \tag{4.21}$$

For a light-like particle (photon) one defines $e_{pm} = h\nu$ with ν the frequency as measured with the local time $t_{\mathbf{x}}$ of the momentarily coincident clock of the frame $\mathcal{E}$, and one then assumes the Newton law (3.8) with e_{pm} (or rather e_{pm}/c^2) in the place of the inertial mass $m(v)$. Defining also $e_m \equiv \beta e_{pm}$ for a photon, a similar way of reasoning (7) shows that *exactly the same balance equation* $(4.21)_2$

(7) Express the Newton law in terms of $D_{1/2}(e_{pm}\mathbf{v})/Dt_{\mathbf{x}}$, allowing to differentiate $\mathbf{g}_t(\mathbf{v},\mathbf{v}) \equiv c^2$ using Leibniz formula with $\mathbf{g}_t$. This gives an expression of $e_{pm}\mathbf{g}.\mathbf{v}$. Equate this, as in Eq. (4.16), to that derived from Eq. (4.9).

is obtained. This extends the result of the static case [3], i.e. $e_m = \text{const}$. Finally, let us come back to the case of an ordinary free test particle, but assuming now the space-time geodesic equation $G^\alpha = 0$ of metric γ (as in GR). Using, in the special case where an isopotential coordinate system is bound to the privileged frame $\mathcal{E}$, the calculation method of ref. [3], one easily rewrites the "time part" $G^0 = 0$ as

$$\frac{d}{dt}(\beta\gamma_v) = \gamma_v \frac{\partial\beta}{\partial t} - \frac{\beta\gamma_v}{2c} g_{ij,0} v^i v^j . \tag{4.22}$$

This is the same as Eq. (4.21) with $\lambda = 1$.

In summary, if one assumes a Newton law (3.8), then the balance equation $(4.21)_2$ is derived for the total energy e_m of any free test particle (mass point or light-like particle), which includes its potential energy in the gravitation field. Unless $\lambda = 1/2$ is assumed, a term involving both e_m and the time-variation of the metric is obtained, which will be seen to be incompatible with the obtainment of a local conservation equation, i.e. a balance equation without source term. The value $\lambda = 1/2$ is the only one for which the Leibniz rule holds true for the vector time derivative (3.5) entering the Newton law (3.8), but it is incompatible with geodesic motion, which rather corresponds to $\lambda = 1$.

ii. The balance equation of energy for dust

We first consider dust, since we have an expression of the energy balance for a free test particle (Eq. (4.21)). Dust is a continuum made of non-interacting mass particles, each of which conserves its rest mass, so that we may use both Eq. (4.21) and the continuity equation for the density of rest mass, expressing the conservation of the rest mass. The latter means that the rest mass δm_0 contained within any given "substantial" domain $\delta\omega$ of the continuum is constant, and is thus a statement that can be expressed in terms of any consistent space and time metric (in so far as the very notion of rest mass is taken for granted). If we follow the motion of the continuum from the privileged frame, we can define its velocity $\mathbf{v}_0$ with the absolute time, and we can use the volume measure dV^0 associated with the natural metric $\mathbf{g}^0$ on M, thus

$$\mathbf{v}_0 = \mathbf{dx}/dt, \qquad dV^0 = \sqrt{g^0}\, dx^1\, dx^2\, dx^3.$$

We then have the usual continuity equation:

$$\frac{\partial \varrho_{00}}{\partial t} + \mathrm{div}_0(\varrho_{00}\mathbf{v}_0) = 0, \qquad \varrho_{00} \equiv \delta m_0/\delta V^0, \quad \mathrm{div}_0 \equiv \mathrm{div}_{\mathbf{g}^0}, \tag{4.23}$$

which leads to the following expression for a "substantial" derivative:

$$\begin{aligned} \frac{d\psi}{dt}\varrho_{00} &= \frac{\partial}{\partial t}(\psi\varrho_{00}) + \mathrm{div}_0(\psi\varrho_{00}\mathbf{v}_0), \\ \frac{d\psi}{dt} &= \frac{\partial\psi}{\partial t} + (\mathrm{grad}_0\psi)\cdot\mathbf{v}_0 = \frac{\partial\psi}{\partial t} + \psi_{,i} v_0{}^i \end{aligned} \tag{4.24}$$

(where the scalar product is of course $\mathbf{g}^0$; but it is true that $\mathbf{g}(\mathrm{grad}_{\mathbf{g}}\,\psi, \mathbf{w}) = \psi_{,i}w^i$ for any metric $\mathbf{g}$ and in any coordinates). We apply this to $\psi = \beta\gamma_v$. According to Eqs. (4.18) and (4.19), ψ is the (rest) mass density of the total energy e_m of dust matter:

$$\delta e_m = \beta\gamma_v \delta m_0 = \varrho_{00}\beta\gamma_v\, \delta V^0 \equiv \varepsilon_m\, \delta V^0\,,$$

hence ε_m is the volume density of the energy e_m in the frame $\mathcal{E}$, the volume δV^0 being expressed with "uncontracted" rods, i.e. in terms of the natural metric $\mathbf{g}^0$ which is not affected by gravity. Combining Eqs. $(4.21)_1$ and (4.24), we get the energy balance for the dust continuum:

$$\frac{\partial \varepsilon_m}{\partial t} + \mathrm{div}_0(\varepsilon_m \mathbf{v}_0) = \varrho_{00}\gamma_v \frac{\partial \beta}{\partial t} + \varepsilon_m \frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v}, \mathbf{v}). \tag{4.25}$$

It is necessary to make contact with the mass tensor $\mathbf{T}$, which may be defined for a perfect barotropic fluid as in SR (and as in GR also), thus (cf. FOCK [10]):

$$T^{\lambda}{}_{\mu} = (\mu^* + p/c^2)u^{\lambda}u_{\mu} - (p/c^2)\gamma^{\lambda}{}_{\mu} \qquad (\gamma^{\lambda}{}_{\mu} = \delta^{\lambda}{}_{\mu}), \tag{4.26}$$

$$\mu^* \equiv \varrho^*(1 + \Pi/c^2), \qquad \varrho^* = \varrho^*(p), \qquad \Pi \equiv \int_0^p \frac{dq}{\varrho^*(q)} - \frac{p}{\varrho^*(p)}\,,$$

$$u^{\lambda} \equiv dx^{\lambda}/ds, \qquad u^{\lambda}u_{\lambda} = 1 \tag{4.27}$$

$$\text{and,} \quad \text{if } x^0 = ct: \qquad u^0 = \gamma_v/\beta, \quad u^i = (u^0/c)dx^i/dt$$

and the same for the $T^{\lambda\mu}$ (or $T_{\lambda\mu}$) components, though with $u^{\lambda}u^{\mu}$ (or $u_{\lambda}u_{\mu}$) instead of $u^{\lambda}u_{\mu}$, and with $\gamma^{\lambda\mu}$ (or $\gamma_{\lambda\mu}$) instead of $\gamma^{\lambda}{}_{\mu}$, all indices being raised or lowered with metric γ. Here, ϱ^* is the proper density of rest mass, i.e. the density of rest mass, as measured in a frame $\mathcal{C}$ that is bound to the continuum, at least locally and momentarily (comoving frame). In order to find the connection between ϱ^* and the density of rest mass ϱ_0 which would be evaluated by observers in the frame $\mathcal{E}$, one just has to use SR which holds true locally. The local observers in $\mathcal{E}$ and in $\mathcal{C}$ relate their space and time measurements by local Lorentz transformation, hence the measures, in $\mathcal{E}$, of the "substantial" volume element in $\mathcal{C}$ and the mass density are

$$\delta V_{\mathcal{E}} = \delta V_{\mathcal{C}}/\gamma_v\,, \qquad \varrho_0 \equiv \frac{\delta m_0}{\delta V_{\mathcal{E}}} = \gamma_v \frac{\delta m_0}{\delta V_{\mathcal{C}}} = \gamma_v \varrho^* \tag{4.28}$$

(note that the observer in $\mathcal{E}$ uses the "true" simultaneity, hence he finds the same relation $\delta V_{\mathcal{E}} = \delta V_{\mathcal{C}}/\gamma_v$ if he simply remembers that the measuring rods of the observer in $\mathcal{C}$ are "truly" Lorentz-contracted in direction $\mathbf{v}$). Since the physical metric $\mathbf{g}$ in the frame $\mathcal{E}$ is affected by gravitational contraction of measuring rods, the corresponding volume $\delta V_{\mathcal{E}}$ is dilated with respect to the volume δV^0 expressed in terms of the natural metric (cf. Eq. (2.7)), thus from $(4.23)_2$ and (4.28):

$$\delta V^0 = \beta \delta V_{\mathcal{E}}\,, \qquad \varrho_{00} = \varrho_0/\beta = \gamma_v \varrho^*/\beta\,. \tag{4.29}$$

Now, from Eqs. (4.26) and (4.27), we have for dust ($p = 0$, $\Pi = 0$):

$$\varrho^* \gamma_v^2 = (T^0{}_0)\varepsilon\,, \qquad \varrho^* \gamma_v^2 v_0^i / c = (T^i{}_0)\varepsilon \qquad (x^0 = ct),$$

whatever space coordinates bound to $\mathcal{E}$ are used; indeed, one has always $\gamma_{0i} = 0$ in such coordinates (see Eq. (2.8)), hence $u^0 u_0 = \gamma_{00}(u^0)^2 = \gamma_v^2$. By (4.29), we have $\varrho^* \gamma_v^2 = \varrho_{00} \gamma_v \beta = \varepsilon_m$, hence we may rewrite the balance equation (4.25) as:

$$(4.30) \qquad cT^\alpha{}_{0,\alpha} \equiv c\left(\mathrm{div}_{\gamma^0}\mathbf{T}\right)_0 \equiv c\left(T^0{}_{0,0} + \mathrm{div}_0(T^i{}_0 \mathbf{e}_i)\right) = \frac{T^0{}_0}{\beta}\frac{\partial \beta}{\partial t} + T^0{}_0 \frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{g}}{\partial t}(\mathbf{v},\mathbf{v}),$$

where $x^0 = ct$, the space coordinates are bound to $\mathcal{E}$, and the space-time metric γ^0 is defined by $(ds^0)^2 = (dx^0)^2 - (dl^0)^2$ with dl^0 the line element of the natural space metric $\mathbf{g}^0$; the first identity is valid in any coordinates linearly bound to Galilean coordinates of the metric γ^0 (if $(M, \mathbf{g}^0)$ is Euclidean. Except for the first identity, Eq. (4.30) is yet valid also if $(M, \mathbf{g}^0)$ is a space with constant curvature).

Equation (4.30) (or (4.25)) still contains the two source terms on the right-hand side. If it transforms into a true conservation equation, this must include the energy of the gravitation field itself. As in NG, we have to use the field equation in order to replace the energy density by a combination of derivatives of the potential (here $f \equiv \beta^2$, with $\beta = p_e/p_e^\infty$, is the exact analogue of the Newtonian potential, Eq. (2.12)). However, there remains some ambiguity as regards the precise definition of the mass-energy density ϱ in the field equation (2.2) or (2.15): is that $T^0{}_0$, T_{00} or T^{00}? To answer this question, we will precisely impose the condition that a conservation equation must be obtained, and we will check the interpretation of ϱ by the physical consequences of this interpretation. In the case where the model of universe is static at infinity, in the rather wide sense (see after Eq. (2.13)) that p_e^∞ does not depend on the time t, we may use the field equation in the form (2.15). We obtain from (2.15) and (2.12), by using (locally) the Cartesian coordinates (i.e. $g^0{}_{ij} = \delta_{ij}$ and $g^0{}_{ij,k} = 0$ at the considered point):

$$(4.31) \qquad \frac{8\pi G}{c^2}\varrho f_{,0} = \left(f_{,0} f_{,i}\right)_{,i} - \frac{1}{2}\left[\left(\frac{f_{,0}}{f}\right)^2 + f_{,i} f_{,i}\right]_{,0} = -\frac{2}{c^2}\mathrm{div}_0(f_{,0}\mathbf{g}) - \frac{1}{2}\left[\left(\frac{f_{,0}}{f}\right)^2 + \frac{4}{c^4}\mathbf{g}^2\right]_{,0}$$

(where the scalar square $\mathbf{g}^2$ is in terms of the natural metric, $\mathbf{g}^2 = \mathbf{g}^0(\mathbf{g}, \mathbf{g})$). This strongly recalls the "pure gravitational" terms in the Newtonian Eq. (4.2), though with an additional term $(f_{,0}/f)^2$. Now we observe that the first source term on the right of Eq. (4.30) can be written as

$$(4.32) \qquad \frac{T^0{}_0}{\beta}\frac{\partial \beta}{\partial t} = \frac{T^0{}_0}{2f}\frac{\partial f}{\partial t} = \frac{T^{00}}{2}\frac{\partial f}{\partial t}\,.$$

Hence, if we interpret the mass-energy density ϱ as T^{00}, and if we demand that Leibniz rule applies to the definition of a vector time derivative, i.e. $\lambda = 1/2$ in Eqs. (3.5) and (3.8), then we obtain for dust, from Eqs. (4.30)–(4.32), the following *conservation equation for the energy*:

$$(4.33) \qquad \left(\mathrm{div}_{\gamma^0}\mathbf{T}\right)_0 + \frac{1}{8\pi G}\left\{\left[\frac{\mathbf{g}^2}{c^2} + \frac{c^2}{4}\left(\frac{f_{,0}}{f}\right)^2\right]_{,0} + \mathrm{div}_0(f_{,0}\mathbf{g})\right\} = 0.$$

As will be seen, the interpretation of ϱ as T^{00} means that the gravitation field definitely reinforces itself and is thus very plausible. Other possible interpretations: $\varrho = T^0{}_0$ or $\varrho = T_{00}$, certainly do not allow to rewrite the energy balance (4.30) as a true conservation equation if $\lambda = 1/2$ (because one then adds to Eq. (4.33) the term $f_{,0}(T^{00} - \varrho)/2$, which is clearly not a 4-divergence). Finally, if a different value is assumed for λ, not only Leibniz rule fails to apply, but the mixed source term (the last term in Eq. (4.30)) makes it more than unlikely that a conservation equation could be obtained for the energy. Thus *we assume* $\lambda = 1/2$ *and* $\varrho = T^{00}$ *from now on*.

iii. Extension of the energy conservation equation to general matter behaviour

For general matter behaviour, the constitutive material particles interact with each other and do not necessarily conserve their rest mass. Still, other conservation laws of microphysics hold true in rather general situations, in particular that for the baryon number which was proposed by CHANDRASEKHAR [8] as a substitute for the mass conservation. However, it is considered here that no conservation law is more fundamental than that for energy. Equation (4.33) applies in the absence of gravitation, i.e. in SR, with $\mathbf{T}$ the mass tensor of any kind of matter and field (incidentally, matter and non-gravitational fields would be of similar nature, i.e. all these would be microscopic flows in ether, according to the concept of constitutive ether (cf. ROMANI [24])). In the presence of gravity, Eq. (4.33) has been derived, for dust, from the assumed conservation of the rest mass. We therefore postulate that Eq. (4.33) holds true with $\mathbf{T}$ the (mixed) mass tensor of any kind of matter and non-gravitational field, in the presence of gravitation. Since the total energy, not the rest mass, is thereby conserved, this postulate contains the possibility that (depending on the constitutive equation) *matter can be created or destroyed by its interaction with a variable gravitation field*.

iv. Some local and global consequences of the conservation equation

The density of pure gravitational energy (or rather of its mass equivalent), with respect to the natural volume measure dV^0, appears clearly in Eq. (4.33); it is

$$(4.34) \qquad \varepsilon_g = \frac{1}{8\pi G}\left[\frac{\mathbf{g}^2}{c^2} + \frac{c^2}{4}\left(\frac{f_{,0}}{f}\right)^2\right] = \frac{1}{8\pi G}\left[\frac{\mathbf{g}^2}{c^2} + \frac{1}{4}\left(\frac{1}{\tilde{c}}\frac{\partial\tilde{c}}{\partial t}\right)^2\right],$$

where $\tilde{c} \equiv dl^0/dt = cf$ (cf. Eqs. (2.5) and (2.7)) is the "absolute" velocity of light (and that of waves of the pressure p_e, i.e. gravitation waves) in the direction **g**. Comparing with the Newtonian gravitational energy w_g (Eq. (4.1)), we see that the studied theory, in which the metric varies with time, says that this variation demands energy. On the other hand, it is recalled that the energy density of matter, $T^0{}_0$ with

$$T^0{}_0 = \varepsilon_m \ (= \varrho_{00}\gamma_v\beta \ \text{ for dust}), \tag{4.35}$$

which also is relative to dV^0, and which enters the balance equation (4.33), includes its "potential" energy in the gravitation field (cf. Eqs. (4.18)–(4.19)). Let us examine, in the static case, what follows from assuming that ϱ on the r.h.s. of the field equation (2.15) is $T^{00} = T^0{}_0/\beta^2$ (as is imposed if one admits that a conservation equation must exist for energy). Since the density of the "pure" energy of matter, i.e. without its potential energy, is $\varepsilon_{pm} = T^0{}_0/\beta$ (again with respect to dV^0), it means that $\varrho = \varepsilon_{pm}/\beta$. Due to the "Poisson equation" (2.15) with $f_{,0} = 0$, Eq. (2.16) determines in the general static case the "active mass" m, giving the expression (2.9) of the space-time metric: at large distance from the massive body (which is unique, since any other body would fall, and which is at rest in the ether), we have approximate spherical symmetry, hence

$$f = 1 - 2Gm/(c^2 r) + o(1/r) \tag{4.36}$$

(r being the Euclidean distance from the body), with

$$m = \int_{\text{body}} \varrho \, dV^0 = \int_{\text{body}} (\varepsilon_{pm}/\beta) \, dV^0. \tag{4.37}$$

This means that the density ε_{pm} of the "pure" (internal plus kinetic) energy is reinforced, as regards its contribution to the "active gravitational" mass m, in the very proportion of the local gravitation field $\beta < 1$, i.e. *the gravitation field reinforces itself*, as desired. If, instead of $\varrho = T^{00}$, we would assume $\varrho = T^0{}_0$ or $\varrho = T_{00}$, we would get $\varepsilon_{pm}\beta$ or $\varepsilon_{pm}\beta^3$ in the place of ε_{pm}/β in Eq. (4.37), i.e. the gravitation field would have a *weakening* effect on itself. But the notion of active mass, especially its comparison with the passive gravitational mass (which is also the inertial mass, and still coincides with the pure energy of matter, cf. the Newton law (3.8) and Eqs. (4.18)–(4.19)), implies a linearity of the acceleration field **g** with respect to the amount of matter. This amount is best represented by the pure energy of matter, thus its density ε_{pm} does not coincide with ϱ. The scalar f depends nonlinearly on the distribution of ε_{pm}, even in the static case (Eq. (2.15) with $\varrho = \varepsilon_{pm}/\beta$), hence the same is true for the vector **g** (Eq. (2.12)). One therefore cannot isolate the gravitation force exerted by a subdomain Ω on a mass point, i.e. the contribution of Ω to the field **g**. Hence, an action-reaction principle cannot even be defined for the gravitation force, even in the static

case. In summary, the studied theory implies (at least in the static case) that the gravitation field really reinforces itself, and this forbids to define the active gravitational mass for a subdomain of the system. As to the global active mass, it may be defined, at least in the static case, and it is then greater than the sum of the pure energy of matter, for the same reason.

In the more general situation of an isolated, but not necessarily static matter distribution, embedded in *Euclidean* space M, the integration of Eq. (4.33) over the whole space and its transformation is possible, under the sufficient condition that $(f_{,0})^2$ is $o(1/r^3)$ and $\mathbf{g}$ is $O(1/r^2)$ at large r (with r the current Euclidean distance from some point in the group of bodies, and using the fact that $f \approx 1$ at large r. In assuming that the behaviour at large r is as in NG, one indeed would expect this decrease of $\mathbf{g}$ like $1/r^2$; actually, this is not the general case, due to the gravitation waves). Using Gauss theorem on the sphere $r = R$ and making R tend towards ∞, we obtain then

$$W_m + W_g \equiv \int_{\text{matter}} \varepsilon_m dV^0 + \int_{\text{space}} \varepsilon_g dV^0 = \text{const} = W, \tag{4.38}$$

which is the same as Eq. (4.3), though here ε_m and ε_g are given by Eqs. (4.35) and (4.34). Due to the presence of the term $(f_{,0}/f)^2$ in ε_g, the reduction of the second integral to an integral restricted to matter seems possible only in the static case. We have then from (2.12) and (2.15):

$$\frac{\mathbf{g}^2}{c^2} = \frac{c^2}{4}(\text{grad}_0 f)^2 = \frac{c^2}{4}\Big(\text{div}_0(f\,\text{grad}_0 f) - f\Delta_0 f\Big) = -\frac{1}{4}\Big(2\,\text{div}_0(f\mathbf{g}) + 8\pi G\varrho f\Big). \tag{4.39}$$

At large r, we have Eq. (4.36) plus $\mathbf{g} \approx -(Gm/r^2)\mathbf{e}_r$ (with $\mathbf{e}_r$ being the unit radial vector, and where $\approx$ means "equivalent to"), whence from (4.34) and (4.39) by using the Gauss theorem on the sphere $r = R$ and making R tend towards ∞

$$W_g \equiv \int_{\text{space}} \varepsilon_g dV^0 = \left(m - \int_{\text{body}} \varrho f\, dV^0\right)/4.$$

Since $\varepsilon_m = \varrho f = \varepsilon_{pm}\beta (= T^0{}_0)$, we can thus rewrite the conserved energy, in the static situation (which may be the case for the initial, as well as for the final state), as

$$W = \frac{3}{4}\int_{\text{body}} \varepsilon_m dV^0 + \frac{1}{4}\int_{\text{body}} \frac{\varepsilon_m}{f} dV^0. \tag{4.40}$$

It is smaller than the active mass m (Eq.(4.37)), and the latter does not need to remain the same in any couple of initial and final states, both being assumed to be static. Furthermore, and in contrast with NG (see the discussion after Eq. (4.4)), the structure of Eq. (4.40) does not seem to imply that the pure energy of matter, $\int_{\text{matter}} \varepsilon_{pm} dV^0$, is necessarily increased as the pure gravitational energy W_g is increased, i.e. in the case of gravitational concentration of matter.

5. Conclusion

Although less immediately than in the case of a static gravitation field, a consistent Newton law can still be defined in the general situation within the present theory (and in fact in any theory which, in a reference frame, provides us with a space metric and a time metric). It is consistent in that it is based on a true vector derivative of the momentum, obeying Leibniz rule with the variable metric $\mathbf{g}$ (here $\mathbf{g}$ is the physical space metric in the privileged frame bound to ether); moreover, this time derivative coincides with the usual absolute derivative in the case of a time-independent metric $\mathbf{g}$, i.e. in the static case. The requirement that Leibniz rule must hold true permits to select the relevant vector derivative, i.e. $\lambda = 1/2$ in Eq. (3.5), from a one-parameter family of candidates. Among the possibilities which are thereby eliminated, the value $\lambda = 1$ is likely to correspond to Einstein's motion along space-time geodesics; anyhow, $\lambda = 1$ does correspond to geodesic motion in the important case where "isopotential" coordinates bound to ether still exist (see the definition near Eq. (2.7); this case includes spherical symmetry around a point bound to ether). A general Newton law is important for the status of the theory, since it means that the acceleration field $\mathbf{g}$ (Eq. (2.1)) keeps a direct physical meaning in the most general case. Thus the above result implies a crucial departure of the theory from the "logic of space-time". The latter is the commonly accepted *interpretation* of SR, even though the "logic of absolute motion" can be vindicated as well in SR [4, 6–7, 11, 21–23]. The logic of space-time is, however, *essential* in GR, and even also in its various formulations or modifications as a field theory in flat space-time (including the theory proposed by LOGUNOV *et al.* [14]).

It seems therefore natural to hesitate before taking the new direction at this bifurcation in the studied theory. But another argument leads to the same choice, and this is not quite a detail: if one wants to have a true conservation equation for energy in this theory, one also must assume the Newton law (with Leibniz rule, thus $\lambda = 1/2$) and hence forget space-time geodesics. One then indeed obtains for energy a local balance equation without source term, and this is in terms of a flat space-time metric. The gravitational energy is thereby unambiguously defined (in contrast with GR). Just like in NG, one must recognize that one part of gravitational energy is embedded in matter, as its (negative) potential energy in the gravitation field, while the other (positive) part is present in the whole

space and may be called the pure gravitational energy. The expression of the latter in the present theory (Eq. (4.34)) contains just the Newtonian expression, plus a term bound to the fact that any time-dependence of the gravitation field implies a time-dependence of the local space and time standards: according to the studied theory, such variation demands an energy supply. This feature of the theory will be worth to discuss in connection with the question of gravitation waves. When the local balance equation can be integrated (which is not the case if gravitation waves are filling the space), it implies the conservation of the global energy (Eq. (4.38)), which is the sum of the energy of matter (including its negative potential energy) and the pure gravitational energy. The gravitation field does reinforce itself in a direct sense (Eq. (4.37)).

According to the present ether theory, gravitation would have some rather concrete aspects. But it is once again emphasized that the theory is non-covariant, which is a risk as regards its application to celestial mechanics. One thus has to study what would be the effect, say on the motion of a planet considered as a test particle, of a *uniform* motion of the attracting body. Since this effect is at most in u^2/c^2 with u the corresponding constant velocity, the perturbation of the Newtonian analysis is likely to be small enough so that the theory, at least, is not *worse* than NG for mechanics of the solar system – except perhaps for time scales beyond the "horizon of predictibility" implied by chaotic behaviour of N-bodies problems. It *might* happen, however, that (due to the effect of the velocity u), the theory could fail to account for such very small effects as the advance of perihelion of Mercury, i.e. it could fail to improve NG in that respect (the motion of the perihelion is very sensitive to almost any kind of perturbation). As to the gravitational red shift, the deflection of light rays and the delay of radar signals, it is likely that the effect of the velocity u does not change significantly their magnitude, which would thus be correctly predicted. The author has already verified this point as regards the red shift. Anyhow, one testable difference with GR is already known: no Lense-Thirring effect, i.e. "gravimagnetic field" due to the self-rotation of a missive body appears in the present theory [3].

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On a meso-elastoplastic constitutive equation with application to deformation-induced anisotropy of a polycrystalline aggregate

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BASED ON A 3-D MESO-MATERIAL MODEL of a polycrystalline aggregate, an elastoplastic constitutive equation for finite deformation is derived. The emphasis is focused on discussing the anisotropy induced by the heterogeneity in the material properties due to a complex deformation history, as well as re-orientation of the crystals or the variation of the micro-structure due to finite deformation. The theoretical predictions are in good agreement with experimental results.

1. Introduction

POLYCRYSTALLINE MATERIALS, such as metals, rock or ice, display distinct anisotropy when they undergo complex deformation processes, like multistage rolling and stretching connected with heat treatment of the material during the manufacturing process or large shear under creep. At greater depths in the Antarctic or Greenland ice caps, glacier ice gives rise to re-crystallization due to extreme stresses consisting of a combination of pressure and shear so as to result in anisotropy of polycrystalline ice (THORSTEINSSON [1]). Therefore, much attention is devoted to the deformation or stress-induced anisotropy in polycrystalline materials. Because the anisotropy induced by plastic deformation history in a polycrystalline material is load-dependent and thus very complicated, SZCZEPIŃSKI [2] proposed a theoretical description of the deformation-induced anisotropy, which is treated as an existing property of the material, without connecting it with the previous deformation history, and discussed the corresponding experimental determination of the coefficients of plastic anisotropy (SOCHA and SZCZEPIŃSKI [3]). This description involves the introduction of structure tensors. DAFALIAS [4] applied the representation theorems for isotropic functions in conjunction with the concept of tensorial structure variables to provide explicit forms of constitutive relations for the plastic spin, and developed a general constitutive formulation employing multiple constitutive and plastic spins for the tensorial internal variables (DAFALIAS [5]). RANIECKI and MRÓZ [6] proposed that the texture orientation can be specified by a rigidly rotating triad; the plastic spin is then the difference of the material and texture spins.

Many factors can cause anisotropy of a polycrystalline material. Generally, in macro-experiments one measures the anisotropy induced by the heterogeneity in the material properties of a body undergoing some deformation history (SZCZEPIŃSKI [7]). With the development of micro-observation techniques, the

micro-experiment is focused on the re-orientation of crystals or the variation of micro-structure (STEINEMANN [8]). Consequently, theoretical descriptions of the orientation of crystals or the micro-structure of the material became feasible. RIBE [9]) VAN DER GIESSEN [10] and SVENDSEN and HUTTER [11] applied the concept of an orientation distribution function (ODF) to modelling induced anisotropy in polycrystalline materials. LIPIŃSKI *et al.* [12] had let 100 ellipsoidal inclusions replace FCC crystal grains as a polycrystalline aggregate and analysed the variation of grain orientation and their influence on the macro-stress distribution by a statistical approach.

In fact, a practical polycrystalline material consists of hundreds of thousands of irregular single crystals. Its micro-structure is so complicated that we can hardly describe it clearly, let alone its evolution during deformation. Since polycrystalline ice or metal are aggregates of randomly oriented grains of single crystals, they can be considered as initially isotropic materials. Generally, the bulk mechanical response of the polycrystalline aggregate is the average result of interactions between various micro-structures, in which some physical actions observed on the microscale have little influence on the macro-response of the material so that they can be neglected. Experimental results show that the sliding occurring between grains is mainly a plastic deformation mechanism of polycrystalline materials, while the deformation of the grains is very small. Based on these experimental facts, equivalent slip systems are introduced, which are basic components distributed homogeneously in the 3-D space. Thus a meso-material model is composed of the equivalent slip systems and elastic grains. With the aid of the model under small deformation, LIANG *et al.* [13] and LIU and LIANG [14] discussed the active hardening, latent hardening and Bauschinger effect, and predicted the evolution of subsequent yield surfaces and stress-strain responses under complex and cyclic loading, respectively. Here, a meso-elastoplastic constitutive equation is derived for finite deformation. Emphasis is laid on discussing the deformation-induced anisotropy and its evolution due to the variation of the properties, and re-orientation of the equivalent slip systems due to the deformation history of the polycrystalline material. Numerical results are presented and compared with the experimental data (SZCZEPIŃSKI [7]). The re-orientation or re-crystallization of polycrystalline ice induced by creep deformation will be discussed in later papers.

2. Fundamental assumptions of a material model

The polycrystalline material is assumed to consist of a vast number of micro-grains. Sliding is heterogeneously and randomly distributed in the 3-D space; it occurs principally between grains or on their boundaries, while the deformation of the grains is usually small. Basing on various experimental observations, we postulate several basic phenomenological features of polycrystalline aggregates

in deformation and attempt to build a material model that has characteristics consistent with the material and can predict the behavior of the material during complex loadings. Here, we assume that

1. *The sliding between grains is the only plastic deformation mechanism. Then, a basic component of a material model is an equivalent slip system*, which is composed of many intermittent micro-slidings in a direction $\mathbf{m}$ on slip planes of grains with a unit normal vector $\mathbf{n}$. *The sliding driving stress* τ *is the objective scalar equal to the macro resolved shear stress associated with directions* $\mathbf{m}$ *and* $\mathbf{n}$*:*

$$\tau = \mathbf{T} : \mathbf{m} \otimes \mathbf{n} = \mathbf{T} : \mathbf{P}, \tag{2.1}$$

where $\mathbf{T}$ is the Cauchy stress, and

$$\mathbf{P} = \frac{1}{2}(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) \tag{2.2}$$

is the symmetric orientation tensor of the slip system. The sliding rate is work conjugate with the resolved shear stress rate of the slip system. The relation between them is determined by the *slip hardening law* proposed by LIANG *et al.* [13]. Because polycrystalline aggregates can be considered as initially homogeneous and isotropic, *equivalent slip systems are homogeneously distributed and oriented in the* 3-D *space*.

2. The deformation of grains is small, so that *grains in the aggregate are supposed to form an isotropic elastic medium*. Thus, its constitutive equation may be stated in the form

$$\overset{\circ}{\mathbf{T}} = \mathbf{IK}_g : \mathbf{D}_g, \tag{2.3}$$

where $\overset{\circ}{\mathbf{T}}$ is the Jaumann rate of the stress $\mathbf{T}$, i.e.

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{W} + \mathrm{tr}(\mathbf{D})\mathbf{T}, \tag{2.4}$$

$\mathbf{D}$ the overall strain rate, $\mathbf{D}_g$ the strain rate of the grains and $\mathbf{IK}_g$ the elastic stiffness tensor of the grains.

3. *The overall strain rate of the aggregate is the sum of the strain rates of the grains* $\mathbf{D}_g$ *and the slip system,* $\mathbf{D}_s$, i.e.,

$$\mathbf{D} = \mathbf{D}_g + \mathbf{D}_s, \tag{2.5}$$

where $\mathbf{D}_s$ is the strain rate produced by sliding.

4. *The slip systems and their orientations deform as material lines and surfaces, respectively*. Therefore, the rates of unit sliding direction $\mathbf{m}$ and normal vector $\mathbf{n}$ are derived, respectively, as

$$\begin{aligned} \dot{\mathbf{m}} &= \mathbf{L} \cdot \mathbf{m} - (\mathbf{m} \otimes \mathbf{m} : \mathbf{D})\mathbf{m}, \\ \dot{\mathbf{n}} &= -\mathbf{n} \cdot \mathbf{L} + (\mathbf{n} \otimes \mathbf{n} : \mathbf{D})\mathbf{n}, \end{aligned} \tag{2.6}$$

where the deformation velocity gradient $\mathbf{L} = \operatorname{grad} \mathbf{v}$ of the aggregate is

$$\mathbf{L} = \mathbf{D} + \mathbf{W}. \tag{2.7}$$

With these, the rate of orientation tensor of a slip system is expressible as

$$\dot{\mathbf{P}} = \mathbf{D} \cdot \mathbf{R} - \mathbf{R} \cdot \mathbf{D} - \mathbf{P} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{P} + \mathbf{P} \otimes (\mathbf{n} \otimes \mathbf{n} - \mathbf{m} \otimes \mathbf{m}) : \mathbf{D}, \tag{2.8}$$

where $\mathbf{W} = \operatorname{skw} \mathbf{L}$ is the spin tensor and $\mathbf{R} = \frac{1}{2}(\mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m})$ the antisymmetric orientation tensor of the slip system.

It is noteworthy that two forms of the tensor products, $\otimes$ and $\bowtie$, defined, respectively, as

$$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = (A)_{ij}\,(B)_{kl} \qquad \text{and} \qquad (\mathbf{A} \bowtie \mathbf{B})_{ijkl} = A_{ik} B_{jl} \tag{2.9}$$

are used in this paper.

3. Elastoplastic constitutive relation of the aggregate

In the current configuration, the overall rate of work dissipated in a unit volumetric element is equal to the sum of the powers dissipated by all active slip systems

$$\dot{w}_s = \mathbf{T} : \mathbf{D}_s = \mathbf{T} : \frac{1}{J} \int_{\Omega} \int_{\Psi} \dot{\gamma}\ \mathbf{P}\, d\Omega\, d\Psi, \tag{3.1}$$

where $J = \det \mathbf{F}$ is the Jacobian of the deformation gradient $\mathbf{F}$ and $\dot{\gamma}$ is the equivalent sliding rate with normal vector $\mathbf{n}$ within a solid angle $d\Omega$, and slip direction $\mathbf{m}$ within a plane angle $d\Psi$. Therefore the strain rate $\mathbf{D}_s$ produced by sliding is obtained as

$$\mathbf{D}_s = \frac{1}{J} \int_{\Omega} \int_{\Psi} \dot{\gamma}\ \mathbf{P}\, d\Omega\, d\Psi. \tag{3.2}$$

According to the slip hardening law proposed by LIANG *et al.* [13], the sliding rate is contributed by *active hardening* and *latent hardening*, respectively, i.e.,

$$\dot{\gamma} = \frac{1}{h}(\dot{\tau} - \chi \mathbf{P} : \mathbf{D}_s), \tag{3.3}$$

where h is the active hardening modulus, while χ is the latent hardening modulus. Using Eqs. (2.4) and (2.8), the resolved shear stress rate Eq. (2.1) on the current configuration becomes

$$\begin{aligned} \dot{\tau} &= \mathbf{P} : \dot{\mathbf{T}} + \dot{\mathbf{P}} : \mathbf{T} \\ &= \mathbf{P} : \mathring{\mathbf{T}} + [\mathbf{R} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{R} + (\mathbf{T} : \mathbf{P})(\mathbf{n} \otimes \mathbf{n} - \mathbf{m} \otimes \mathbf{m} - \mathbf{I})] : \mathbf{D}. \end{aligned} \tag{3.4}$$

Substituting Eqs. (3.3) and (3.4) into Eq. (3.2) and using Eq. (2.5) yields

$$\begin{aligned}(3.5)\qquad \mathbf{D}-\mathbf{D}_{\mathrm{g}} = {} & \left[\frac{1}{J}\int_{\Omega}\int_{\Psi}\frac{1}{h}(\mathbf{P}\otimes\mathbf{P})\,d\Omega\,d\Psi\right] : \mathring{\mathbf{T}} \\ & - \left[\frac{1}{J}\int_{\Omega}\int_{\Psi}\frac{1}{h}\chi\mathbf{P}\otimes\mathbf{P}\,d\Omega\,d\Psi\right] : (\mathbf{D}-\mathbf{D}_{\mathrm{g}}) \\ & + \left[\frac{1}{J}\int_{\Omega}\int_{\Psi}\frac{1}{h}\{\mathbf{P}\otimes[\mathbf{R}\cdot\mathbf{T}-\mathbf{T}\cdot\mathbf{R}+(\mathbf{T}:\mathbf{P})(\mathbf{n}\otimes\mathbf{n}-\mathbf{m}\otimes\mathbf{m}-\mathbf{I})]\}\,d\Omega\,d\Psi\right] : \mathbf{D}.\end{aligned}$$

Rewriting the above equation, thereby using Eq. (2.3) in the form $\mathbf{D}_{\mathrm{g}} = \mathbf{I}\mathbf{K}_{\mathrm{g}}^{-1} : \mathring{\mathbf{T}}$, the explicit form of an elastoplastic constitutive equation for finite deformation takes the form

$$\begin{aligned}(3.6)\qquad & \left[\mathbf{I}\bowtie\mathbf{I}+\frac{1}{J}\int_{\Omega}\int_{\Psi}\frac{1}{h}[\chi\mathbf{P}\otimes\mathbf{P}-\mathbf{P}\otimes(\mathbf{R}\cdot\mathbf{T}-\mathbf{T}\cdot\mathbf{R}) \right. \\ & \left. -(\mathbf{P}:\mathbf{T})\mathbf{P}\otimes(\mathbf{n}\otimes\mathbf{n}-\mathbf{m}\otimes\mathbf{m}-\mathbf{I})]d\Omega\,d\Psi\right] : \mathbf{D} \\ & = \left[\mathbf{K}_{\mathrm{g}}^{-1}+\frac{1}{J}\int_{\Omega}\int_{\Psi}\frac{1}{h}(\mathbf{P}\otimes\mathbf{P}+\chi\mathbf{P}\otimes\mathbf{P}:\mathbf{K}_{\mathrm{g}}^{-1})\,d\Omega\,d\Psi\right] : \mathring{\mathbf{T}}.\end{aligned}$$

Under small deformation, this simplifies [13] to

$$\begin{aligned}(3.7)\qquad & \left[\mathbf{I}\bowtie\mathbf{I}+\int_{\Omega}\int_{\Psi}\frac{\chi}{h}\mathbf{P}\otimes\mathbf{P}\,d\Omega\,d\Psi\right] : \mathbf{D} \\ & = \left[\mathbf{K}_{\mathrm{g}}^{-1}+\int_{\Omega}\int_{\Psi}\frac{1}{h}(\mathbf{P}\otimes\mathbf{P}+\chi\mathbf{P}\otimes\mathbf{P}:\mathbf{K}_{\mathrm{g}}^{-1})\,d\Omega\,d\Psi\right] : \dot{\mathbf{T}}.\end{aligned}$$

Equations (3.6) and (3.7) give two variants of the explicit form of the constitutive equation with two kinds of material parameters h and χ. The Bauschinger effect $\beta = (\tau_{\mathrm{cr}} - \tau_{-\mathrm{cr}})/2\tau_{\mathrm{cr0}}$ is reflected in the activation rule of slip systems (LIANG *et*

al. [13]),

$$\text{if } \tau = \tau_{cr} \text{ and } \dot{\tau} > \dot{\tau}_{\text{latent}}, \text{ then } \begin{cases} \dot{\gamma} > 0, \\ \dot{\tau}_{cr} = \dot{\tau} = h\dot{\gamma} + \chi \mathbf{P} : \mathbf{D}^p \\ \text{and} \\ \dot{\tau}_{-cr} = \dot{\tau}_{cr} - 2\dot{\beta}\tau_{cr0}; \end{cases}$$

$$\text{if } \tau = \tau_{-cr} \text{ and } \dot{\tau} < \dot{\tau}_{\text{latent}}, \text{ then } \begin{cases} \dot{\gamma} < 0, \\ \dot{\tau}_{-cr} = \dot{\tau} = h\dot{\gamma} + \chi \mathbf{P} : \mathbf{D}^p \\ \text{and} \\ \dot{\tau}_{cr} = \dot{\tau}_{-cr} + 2\dot{\beta}\tau_{cr0}; \end{cases} \tag{3.8}$$

$$\text{otherwise } \dot{\gamma} = 0 \begin{cases} \text{if } \dot{\tau}_{\text{latent}} \geq 0, \text{ then } \begin{cases} \dot{\tau}_{cr} = \chi \mathbf{P} : \mathbf{D}^p \\ \text{and} \\ \dot{\tau}_{-cr} = \dot{\tau}_{cr} - 2\dot{\beta}\tau_{cr0}, \end{cases} \\ \text{if } \dot{\tau}_{\text{latent}} < 0, \text{ then } \begin{cases} \dot{\tau}_{-cr} = \chi \mathbf{P} : \mathbf{D}^p \\ \text{and} \\ \dot{\tau}_{cr} = \dot{\tau}_{-cr} + 2\dot{\beta}\tau_{cr0}, \end{cases} \end{cases}$$

where τ_{cr} and τ_{-cr} are the critical resolved shear stresses corresponding to positive and negative sliding directions, τ_{cr0} is the initial critical resolved shear stress.

It should be noticed that it is not always convenient to apply Eq. (3.6) or (3.7) directly for numerical analysis, because material parameters depend on the deformation history and the activation states of the slip systems. Usually, using the slip hardening law, Eqs. (3.3) and (3.2), yield

$$\dot{\gamma} = \frac{\dot{\tau}}{h} - \frac{\chi}{h} \int_{\Omega} \int_{\Psi} (\mathbf{P} : \tilde{\mathbf{P}}) \tilde{\dot{\gamma}} \, d\Omega \, d\Psi, \tag{3.9}$$

where the tildas mean that the corresponding variable is a function of the integral variable. Under small deformation, the above equation is a standard Fredholm integral equation with symmetric integral kernel $\mathbf{P} : \tilde{\mathbf{P}} = \tilde{\mathbf{P}} : \mathbf{P}$. If the coefficient $h > 0$, there exists one and only one solution. By dividing half of the spherical angle, orientation range of $\mathbf{n}$, into N equal parts and half of the plane angle, orientation range of $\mathbf{m}$, into M equal parts, a discrete model with MN slip systems will be obtained for numerical analysis. In this case, Eq. (3.9) becomes

$$\dot{\gamma}^{(j)} + \frac{2\pi^2}{MN} \frac{\chi}{h^{(j)}} \sum_{i=1}^{MN} [\mathbf{P}^{(j)} : \mathbf{P}^{(i)}] \dot{\gamma}^{(i)} = \frac{1}{h^{(j)}} \dot{\tau}^{(j)} \tag{3.10}$$

(for all activated slip systems).

As far as all activated slip systems are concerned, Eq. (3.10) is a set of equations with a symmetric and positive definite coefficient matrix. However, the determination of activation state of slip systems must be combined with the activation rule of slip systems. Therefore, it is generally solved by iteration. For finite deformation, terms $\overset{\circ}{\tau}{}^{(j)}$ of the above equations are functions of **D**, so that the iteration is more complicated than that for small deformation. When all slidings are determined, the macro-stress or strain and the stiffness or compliance tensors can completely be determined.

4. Discussion of deformation-induced anisotropy

The initially isotropic assumption of a polycrystalline aggregate can be tested by macroscopic experiments as well as microscopic observations. The anisotropy that follows is then deformation-induced. Based on the material model proposed in this paper, the material element is divided into MN= 876 discrete slip systems.

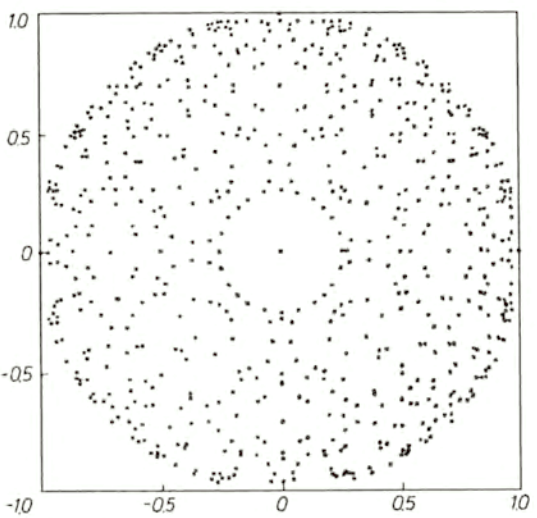

FIG. 1. The initial distribution of the discrete slip systems in the model material, here projected on a plane.

Figure 1 gives the initial distribution of the discrete slip systems in the model material, here projected on a plane. The evolution laws of the three kinds of material parameters h, χ and β included in the constitutive relation are, respectively (LIU and LIANG [13]),

$$h = (h_0 - h_{\text{sat}})\frac{\tau_{\max 0} - \tau}{\tau_{\max 0} - \tau_{\text{cr}0}} + h_{\text{sat}}, \tag{4.1}$$

$$\chi = \chi_0 \frac{\sum |\dot{\gamma}^{(i)}| h^{(i)}}{\sum |\dot{\gamma}^{(i)}|} \qquad \text{(for all activated slip systems)}, \tag{4.2}$$

$$\beta = (1 - \lambda_\beta)e^{-\alpha_\beta \Lambda^2} + \lambda_\beta , \tag{4.3}$$

where $\Lambda = \int_0^t \frac{|\max(\dot{\tau}, \dot{\tau}_\chi)|}{\tau_{cr0}}\, dt$ and $\dot{\tau}_\chi = \chi \mathbf{P} : \mathbf{D}^p$. The material constants are listed in Table 1.

Table 1. Material constants.

Material constant	Symbol	Value
Initial critical shear stress	τ_{cr0}	665 kg/cm^2
Latent hardening coefficient	χ_0	0.7
Initial hardening modulus	h_0	6916 kg/cm^2
Saturation hardening modulus	h_{sat}	5320 kg/cm^2
Nominal maximum shear stress	$\tau_{max\,0}$	798 kg/cm^2
Attendant Bauschinger coefficient	λ_β	0.8
Saturation-rate coefficient of λ_β	α_β	10

For initially isotropic materials, SZCZEPIŃSKI's [7] experiment can illustrate well the deformation-induced anisotropy on the macroscale. A sheet of an Al-2%Mg aluminium alloy, initially isotropic, was prestressed under small uniaxial tension in the x-direction until 1.92% of permanent deformation was reached. Small specimens were cut out in different directions making various angles α with the

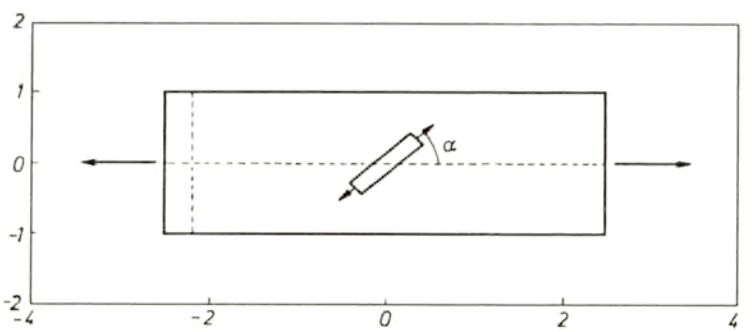

FIG. 2. The diagrammatic sketch of original specimen and small specimens.

x-axis (Fig. 2). Then all these specimens were loaded by uniaxial tensions. The experimental results show that the material evidently exhibits the Bauschinger effect, and that the consecutive yield surfaces at the prestress point possess larger curvature. The basic characteristics are the same as for other polycrystalline materials.

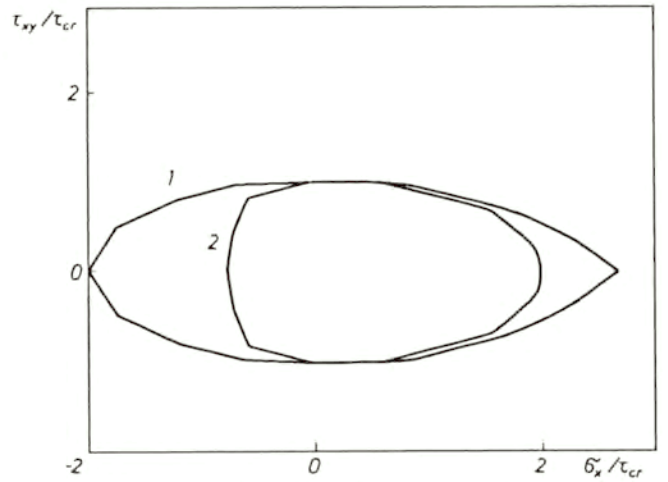

FIG. 3. Development of consecutive yield surfaces in a specimen under uniaxial tension in the x-direction. Curve (*1*) is the initial yield surface and (*2*) a subsequent yield surface.

Figure 3 shows two consecutive yield surfaces of the material measured by the proportional limit of the material. The initial yield surface is just the Tresca ellipse. The latent hardening and evident Bauschinger effect are embodied in the subsequent yield surface of the material. The change of shape of the subsequent yield surface indicates that the material behavior is far from isotropic. Figure 4

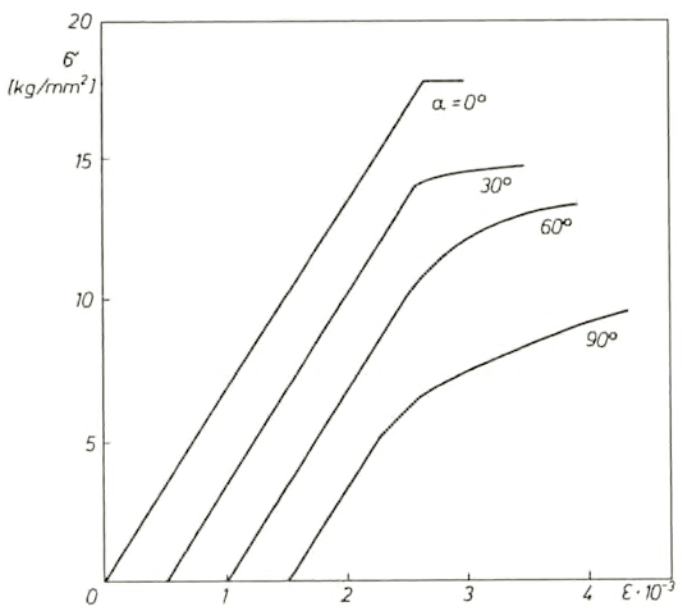

FIG. 4. Stress-strain curves in α-direction for specimens which were subjected to a permanent prestrain of 1.92% in the x-direction. α indicates the direction of the uniaxial tension relative to the x-direction.

presents tensile curves along different directions after the material underwent the 1.92% of permanent deformation. The results demonstrate that the proportional limits of the material are distinctly different in various directions. The proportional limits near $\alpha = 90°$ (perpendicular to x-axis) are mainly affected by the Bauschinger effect, and the ones near $\alpha = 0°$ mainly by the latent hardening. On the meso-scale, the change is mainly reflected in the variation of the critical resolved shear stress of the slip systems. As is well known, the critical resolved shear stress of a slip system not only depends on its activation state and deformation history, but is also affected by the interaction between the slip systems. Because

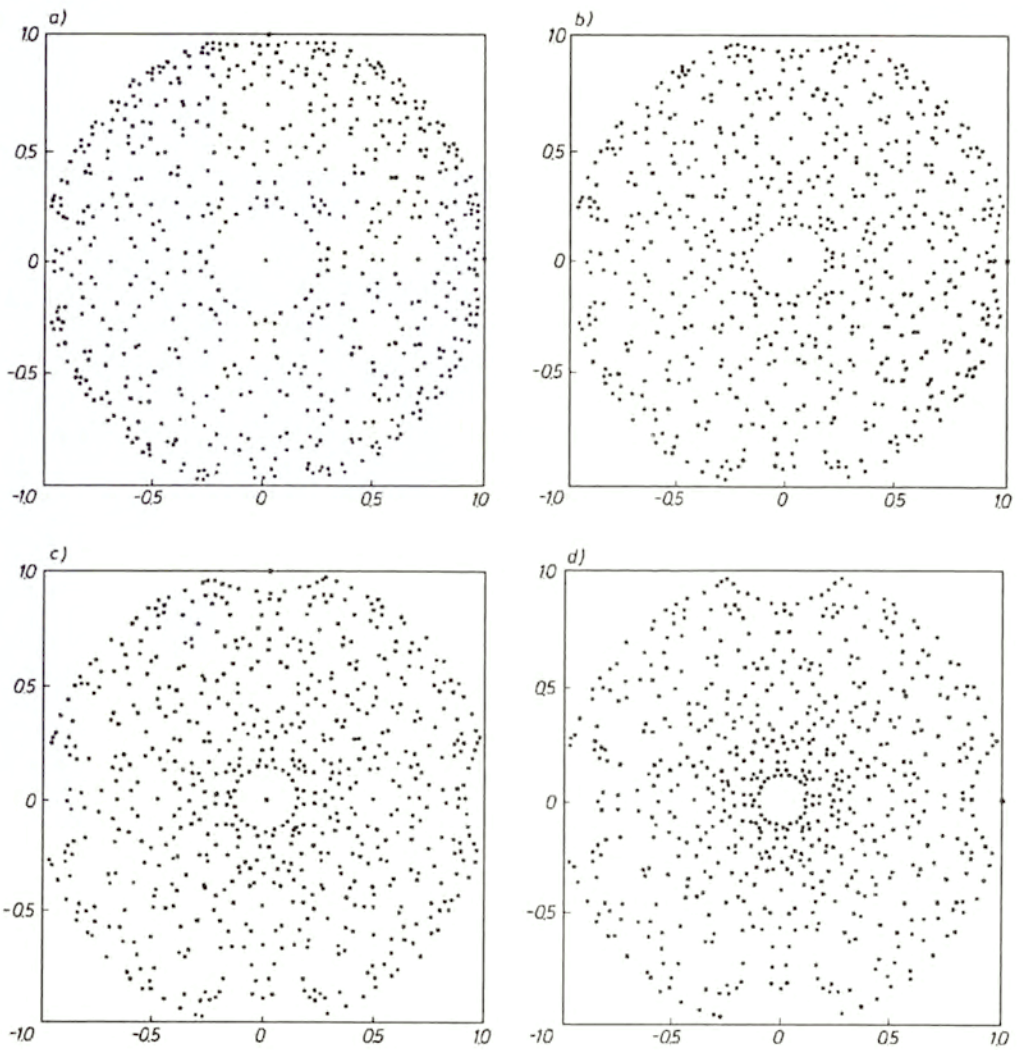

FIG. 5. Evolution of the orientation of slip systems under uniaxial tension, a) is the distribution of orientations of slip systems with a permanent prestrain of 1.92%, b), c) and d) are the distributions corresponding to 31.8%, 47.9% and 71.8% of tensile deformation, respectively.

the activation state and the deformation history are different from one slip system to another, the distinction between critical resolved shear stresses naturally results in the anisotropy measured by the macro-experiment. The evolution of consecutive yield surfaces and variation of the critical resolved shear stress under complex deformation histories have been given in reference [13] and will not be repeated here.

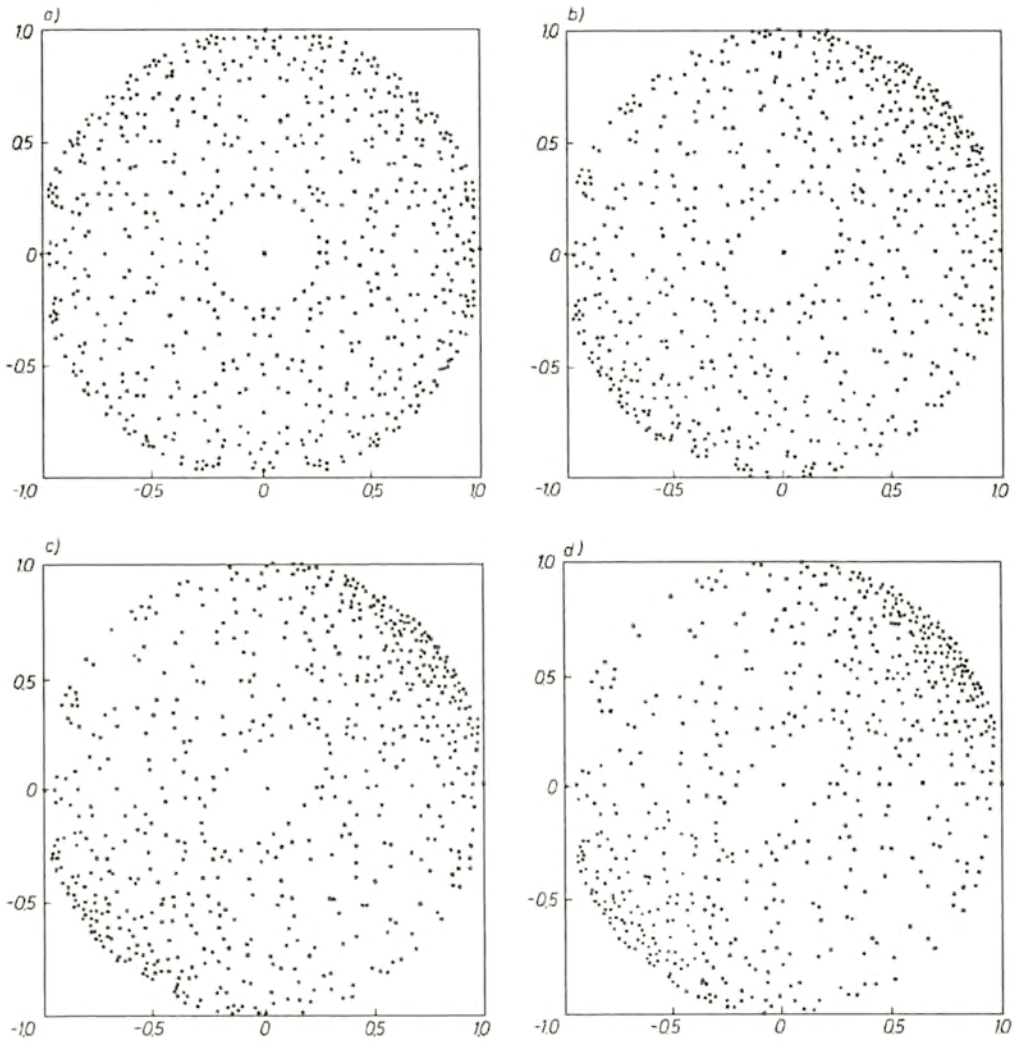

FIG. 6. Evolution of the orientations of slip systems under pure shear. (a)–(d) are the distributions corresponding to the initial distribution, 26%, 44% and 63% of shear deformation, respectively.

On the other hand, the deformation-induced anisotropy can be analyzed by the density of the orientation distribution of slip systems. Generally, the tips of

the unit sliding direction vectors **m** distributed in the 3-D space are points on the unit hemisphere, and for isotropy this distribution on this hemisphere is uniform. Figure 5a shows the distribution of the sliding orientations projected on the plane perpendicular to the tensile axis after the material underwent a 1.92% permanent tensile deformation corresponding to the above example. The variation of the sliding orientation can hardly be seen. Therefore the anisotropy discussed above is induced by the heterogeneity in the material property. However, under finite deformation, the variation of the orientation of the slip systems plays an important role in the deformation-induced anisotropy. Figure 5 b, c, d demonstrates the evolution of the orientation of the slip systems corresponding to different tensile deformation. With increasing deformation, the slip system orientations accumulate around the tensile axis. The variations of the sliding orientations under pure shear are shown in Fig. 6. With the increase of deformation, the slip system orientations accumulate around the 45° direction.

To sum up, the deformation-induced anisotropy of polycrystalline material can be divided into two kinds: one is induced by the heterogeneity of the material property due to complex deformation history, the other by the re-orientation of the slip systems or the variation of the micro-structure due to finite deformation. The meso-elastoplastic constitutive equation for finite deformation is able to predict the anisotropy mentioned above and the predicted results are in good agreement with experiments. This analysis also indicates that Szczepiński's experiment may be used as a simple and feasible method to calibrate material constants of the model with the active hardening, latent hardening and Bauschinger effect.

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Nondestructive evaluation of cracks in conducting magnetoelastic materials

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THEORETICAL ANALYSIS of the alternating current flow around the crack appearing in the loaded material is given. The effective conductivity near the crack has been found to be nonhomogeneous, so the phenomenon of accumulation of the charge appears there. The charge density of accumulated charge near the crack is given. The potential drop has been calculated taking into account the effect of load (stress intensity factor) of the material for three basic modes of a cracked body: opening, sliding, and tearing. The nonhomogeneous conductivity tensors are reported for all modes. The analysis is based on general rotationally invariant nonlinear magnetoelastic equations (energy balance equation, Clausius–Duhem formula, and Gibbs function expansion). Bilinear constitutive relations for large quantities of the bias (loading) have been given and linear, but parametric constitutive formulas for small-field variables (superposed over the bias, for example current) have been derived in the reference or intermediate frame. The obtained effective material tensors of the biased magnetoelastic media, such as effective elastic, piezomagnetic, permeability tensors, as well as effective conductivity, allow to treat any other small-field phenomena in the intermediate frame in the same manner as the linear phenomena for the free (unbiased) media with the only change of material parameters.

1. Introduction

THERE ARE MANY METHODS of nondestructive testing (NDT) and evaluation (NDE) of materials. One of them is the ultrasonic technique, when acoustic waves are applied to the material under testing to get information about possible cracks or defects. The other can be magnetic technique, which is based on the distribution of magnetic field around the crack. Recently however, the potential drop technique has gained much interest of many researchers [15, 18, 27, 28]. Briefly, the technique involves application of a constant current to a cracked specimen or structure and the measurement of the potential drop across the crack. As the crack increases the effective electrical resistance of the material, the potential drop between two points on both sides of the crack rises. By monitoring the potential drop due to the crack and comparing it with a reference value of potential drop measured elsewhere far from the crack, the crack depth can be determined. The method can be used with either a direct current (d.c.) or an alternating current (a.c.). The older of these two techniques is the d.c. method, which is simple and relatively cheap. Some studies associated with development of the a.c. potential drop (ACPD) technique for detection and measurement of surface cracks in metals have been reported, for example, in [15, 16, 27]. A method for evaluation of the stress intensity factor for a 3-D surface crack by means of the a.c. potential drop technique was proposed in [25]. An advanced a.c. potential drop technique has been developed in [26] for nondestructive inspection of a crack.

All of these works are based on experimental techniques of evaluation of the crack size. The purpose of this paper is to give the theoretical derivation of the potential drop on the crack when the a.c. current is applied to the loaded, cracked magnetoelastic body in general. It is well known that the region very near to the crack or defect tip, regardless their origin, exhibits extremely large deformation and mechanical stress. Both fields are singular (tend to infinity, theoretically) at the end of the wedge, which can be the model of the defect. This explains why the linear magnetoelasticity for ferromagnetic conducting materials is not valid and can not be used near defects. Therefore, this work is based on nonlinear thermodynamics for magnetoelastic bodies [1 – 15] to derive constitutive relations for the material under large bias. This is the subject of the first few sections. Next, the proper relations derived are applied to the ACPD NDE problem for three basic modes of loading of the cracked body.

2. Basic definitions

What follows here, we start from general phenomenological, thermodynamic, nonlinear and rotationally invariant formulation. The theory is given for nonpolarizable, but conducting magnetoelastic solid, taking into account magnetostriction and piezomagnetic effects for wider use. The material is assumed to be anisotropic but homogeneous. Nonlinearity is reduced to quadratic terms only, as they play the most important role in nonlinear phenomena of coupling between predeformation and small-field vibration (bilinear model).

Two basic configurations are used. The reference frame is associated with material Cartesian coordinates, X_I, $I = 1,2,3$, which are denoted always by capital letters and indices as well as every quantity given in this frame. The second is the actual configuration associated with spatial Cartesian coordinates, x_i, $i = 1,2,3$, which are denoted by small letters and indices as well as every quantity given in this frame. Full advantage of well-known relations and tensor variables will be taken in the paper [2, 14, 20, 21, 22]:

- mapping of the material point

$$x_i = x_i(X_K, t); \tag{2.1}$$

- motion gradient

$$F_{iK} = x_{i,K} = \frac{\partial x_i}{\partial X_K}, \qquad J = \det F \neq 0; \tag{2.2}$$

- displacement gradient tensor

$$H = \nabla U = F - I, \tag{2.3}$$

where I is the identity tensor, and $U_K = \delta_{Ki} x_i - X_K$;

• Cauchy strain tensor

$$C = F^T F; \tag{2.4}$$

• Lagrange strain tensor

$$S = \frac{1}{2}(C - I) = \frac{1}{2}(H + H^T + H^T H); \tag{2.5}$$

• velocity gradient tensor

$$L = (\nabla V)^T; \tag{2.6}$$

• rate of strain tensor

$$D = \frac{1}{2}(L + L^T). \tag{2.7}$$

Also, the electric strength in the actual frame will be denoted by $e_i = -\phi_{,i}$ (ϕ is electric scalar potential), conduction current by j_i, magnetic induction vector by $b_i = \epsilon_{ijk} a_{k,j}$ (a_k is the magnetic vector potential), and magnetization vector by m_i. Then, the Maxwell stress tensor in the actual frame can be written as [10, 20, 21, 22, 31]

$$\begin{aligned} t_{ij}^M &= t_{ij}^{Me} + t_{ij}^{Mm}, \\ t_{ij}^{Me} &= \varepsilon_0 e_i e_j - 0.5\varepsilon_0 e_k e_k \delta_{ij}, \\ t_{ij}^{Mm} &= \frac{1}{\mu_0} b_i b_j - m_i b_j - \frac{1}{2}\left(\frac{1}{\mu_0} b_k b_k \delta_{ij} - 2 m_k b_k \delta_{ij}\right), \end{aligned} \tag{2.8}$$

which is split above into two parts. Electric permittivity is denoted by ε_0 and magnetic permeability is μ_0. The electric part t_{ij}^{Me} is symmetric, because of absence of polarization, but the magnetic part t_{ij}^{Mm} is not symmetric as well as the total Maxwell stress tensor t_{ij}^M. The term $-m_i b_j$, which causes the asymmetry, can be added to the mechanical Cauchy stress tensor, t_{ij}^m (nonsymmetric), and then they both will become symmetric as well as the total stress tensor

$$t_{ij} = t_{ij}^m + t_{ij}^M = t_{ij}^{ms} + t_{ij}^{Ms}, \tag{2.9}$$

where

$$\begin{aligned} t_{ij}^{Ms} &= \varepsilon_0 e_i e_j + \frac{1}{\mu_0} b_i b_j - \frac{1}{2}\left(\varepsilon_0 e_k e_k + \frac{1}{\mu_0} b_k b_k - 2 m_k b_k\right)\delta_{ij}, \\ t_{ij}^{ms} &= t_{ij}^m - m_i b_j. \end{aligned} \tag{2.10}$$

The constitutive equations will be derived not in the actual frame, but in the reference frame like in [1, 5, 6, 7, 11, 13, 20, 21, 24, 29]. Next, the straightforward substitution of the sum of the bias and small-field quantities will be done to get a small-field variables, parametric constitutive equations with coefficients depending on large quantities of the bias. At the end, we will get new effective material constants for magnetoelastic conducting body upon initial external stress or magnetic field. It can be said that these effective tensors describe the nonlinear coupling between the solid itself and predeformation or initial magnetic field.

3. Nonlinear thermodynamic formulation

We will start our considerations from the first and second law of thermodynamics, which are reproduced here for completeness. The global balance of energy states that the change in time of the kinetic energy $\mathcal{K}$ and the internal energy $\mathcal{E}$ must be equal to the work $\mathcal{W}$ done upon the body by external forces and heat $\mathcal{Q}$ delivered to the body [8, 9, 10, 14, 20, 21, 22]:

$$\dot{\mathcal{K}} + \dot{\mathcal{E}} = \mathcal{W} + \mathcal{Q}. \tag{3.1}$$

In the isothermal conditions, the local balance equation, equivalent to the above, for magnetoelastic conducting body can be written as [20, 21]

$$\varrho \dot{\epsilon} = \operatorname{tr}\left(t^m L^T\right) + j \cdot e - m \cdot \dot{b} \tag{3.2}$$

(notation remark: vector z corresponds to its Cartesian coordinates z_i, where $z = j, e, m, b...$).

For purposes of this paper it is more convenient to use instead of the internal energy density ϵ, the scalar state function known as a Gibbs function, which can be obtained from ϵ by the Legendre transformation

$$\psi = \epsilon + \frac{1}{\varrho} m \cdot b, \tag{3.3}$$

so we can use local energy balance in the form

$$\varrho \dot{\psi} = \operatorname{tr}\left(t^m L^T\right) + j \cdot e + b \cdot \dot{m}. \tag{3.4}$$

The axiom of entropy (second law of thermodynamics) states that the time rate of the total entropy of the density η is never less than the entropy supply in the body (for the case of no influx through the boundary). In the isothermal conditions (or neglecting the change of temperature) for the simple and admissible process of magnetoelastic conducting body, the local law can be expressed in the form of Clausius - Duhem inequality written below for the Helmholtz free energy function $\Psi = \psi - \eta\theta$, where θ is the temperature [8, 12, 20]

$$-\varrho \dot{\Psi} + \operatorname{tr}\left(t^m L^T\right) + j \cdot e + b \cdot \dot{m} \geq 0. \tag{3.5}$$

The internal energy ψ will be rotationally invariant if it is expressed in terms of material measures of strain, magnetization, and electric field. We choose Lagrange strain tensor S, Eq. (2.5), and axial magnetization vector in the reference configuration [20]

$$M_I = J F^{-1}_{jI} m_j, \tag{3.6}$$

as well as electric field in the reference configuration [20]

$$E_I = e_j F_{jI} \tag{3.7}$$

as independent variables. Then, the magnetic induction and conduction current in the reference frame can be expressed for completeness as follows

$$B_I = b_j F_{jI}\,, \qquad \mathcal{J}_I = J F^{-1}_{jI} j_j\,. \tag{3.8}$$

To get the Gibbs function in a form $\psi = \psi(S, M, E)$, Eq. (3.4) must be rearranged to involve new variables. Differentiation of S and M in time gives

$$\dot{S} = F^T D F, \qquad \dot{m} = \dot{M} F^{-1} - mL + m\nabla \cdot L. \tag{3.9}$$

Then

$$\begin{aligned} \varrho \dot{\psi} &= \operatorname{tr}\left(t^m L^T\right) + J\mathcal{J} \cdot E + b \cdot \dot{M} F^{-1} - b \cdot mL + b \cdot m\nabla \cdot L \\ &= \operatorname{tr}\left((t^m - bm + b \cdot mI)L^T\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right) \\ &\operatorname{tr}\left((t^{ms} + b \cdot mI)L^T\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right). \end{aligned} \tag{3.10}$$

L^T can be now decomposed into $L^T = D - \Omega$, where $\Omega = 0.5(L - L^T)$ is the rate of rotation tensor always skewsymmetric, so the product of $(t^{ms} + b \cdot mI)$ and Ω is always zero. We get then

$$\varrho \dot{\psi} = \operatorname{tr}\left((t^{ms} + b \cdot mI)D\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right), \tag{3.11}$$

and with help of the inverse of the first of Eqs. (3.9)

$$\varrho \dot{\psi} = \operatorname{tr}\left((t^m - mb + b \cdot mI)F^{-1}F^{-1^T}\dot{S}\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right). \tag{3.12}$$

The last equation rewritten in components is

$$\begin{aligned} \varrho \dot{\psi} &= t^m_{ij} F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} - m_i b_j F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} + m_l b_l \delta_{ij} F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} \\ &\quad + J\mathcal{J}_I E_I + b_i \dot{M}_K F^{-1}_{iK}\,. \end{aligned} \tag{3.13}$$

4. Reference frame constitutive equations

To find constitutive relations we decompose the total derivative $\dot{\psi}$ into terms associated with independent variables:

$$\dot{\psi} = \frac{\partial \psi}{\partial S_{KL}} \frac{dS_{KL}}{dt} + \frac{\partial \psi}{\partial M_K} \frac{dM_K}{dt} + \frac{\partial \psi}{\partial E_K} \frac{dE_K}{dt}\,. \tag{4.1}$$

Multiplying Eq. (4.1) by $-\varrho$ and adding to Eq. (3.13) side by side one can obtain

$$\left(t_{ij}^m F_{iK}^{-1} F_{jL}^{-1} - m_i b_j F_{iK}^{-1} F_{jL}^{-1} + m_l b_l \delta_{ij} F_{iK}^{-1} F_{jL}^{-1} - \varrho \frac{\partial \psi}{\partial S_{KL}}\right) \dot{S}_{KL} + \left(b_i F_{iK}^{-1} - \varrho \frac{\partial \psi}{\partial M_K}\right) \dot{M}_K - \varrho \frac{\partial \psi}{\partial E_K} \dot{E}_K + J \mathcal{J}_I E_I = 0. \tag{4.2}$$

Additionally, the Clausius-Duhem inequality (3.5) gives

$$\left(t_{ij}^m F_{iK}^{-1} F_{jL}^{-1} - m_i b_j F_{iK}^{-1} F_{jL}^{-1} + m_l b_l \delta_{ij} F_{iK}^{-1} F_{jL}^{-1} - \varrho \frac{\partial \psi}{\partial S_{KL}}\right) \dot{S}_{KL} + \left(b_i F_{iK}^{-1} - \varrho \frac{\partial \psi}{\partial M_K}\right) \dot{M}_K - \varrho \frac{\partial \psi}{\partial E_K} \dot{E}_K + J \mathcal{J}_I E_I \geq 0. \tag{4.3}$$

Formulas (4.2) and (4.3) must hold for arbitrary nonzero independent time variations $\dot{S}_{KL}$, $\dot{M}_K$, and $\dot{E}_K$, so both can be written separately as

$$\frac{\partial \psi}{\partial E_K} = 0, \qquad \mathcal{J}_I E_I \geq 0, \tag{4.4}$$

and

$$\begin{aligned} t_{ij}^m &= \varrho F_{iK} F_{jL} \frac{\partial \psi}{\partial S_{KL}} + m_i b_j - m_l b_l \delta_{ij}, \\ b_i &= \varrho F_{iK} \frac{\partial \psi}{\partial M_K}, \end{aligned} \tag{4.5}$$

or

$$\begin{aligned} t_{ij}^m &= \varrho F_{iK} F_{jL} \frac{\partial \psi}{\partial S_{KL}} + \varrho m_i F_{jK} \frac{\partial \psi}{\partial M_K} - \varrho \delta_{ij} m_l F_{lK} \frac{\partial \psi}{\partial M_K}, \\ b_i &= \varrho F_{iK} \frac{\partial \psi}{\partial M_K}. \end{aligned} \tag{4.6}$$

The remaining constitutive relation, on the basis of (4.4), reads

$$\mathcal{J} = \mathcal{J}(S, M, E), \tag{4.7}$$

as the Thomson effect was neglected (homogeneous temperature distribution was assumed). Because the form $\mathcal{J}E$ must be positive definite (see inequality (4.4)), the continuity condition at the thermodynamic equilibrium must be fulfilled

$$\mathcal{J}(S, M, E) = 0 \quad \text{for} \quad E_I = 0, \tag{4.8}$$

what gives the linear representation in terms of E_I

$$\mathcal{J}_I = \Sigma_{IJ}(S, M) E_J, \tag{4.9}$$

where Σ must be the second order symmetric material tensor. Now, neglecting the Hall effect in the above relation and involving the isotropic representation of Σ due to the only independent variable S, one can find

$$\Sigma = \sigma I + \sigma_1 S + \sigma_2 S^2, \tag{4.10}$$

where three fundamental invariants of tensor S have been used. The first term is a natural (unbiased) media conduction term, whereas two next terms are due to the deformation of the solid. Coefficients σ, σ_1, σ_2 have to be found experimentally.

There are still three quantities t_{ij}^m, b_i, m_i in constitutive equations (4.5), (4.6) given in the actual frame (spatial coordinates). It is obvious that if we want to apply constitutive relations to practical problems, where we study the dynamical behaviour of a nonlinear elastic body of finite extent, it is much simpler to have these quantities convected to the reference frame (material coordinates). Then it will be easier to write the boundary conditions concerning either the mechanical displacement or the stress, and the magnetic induction on the fixed surface in the reference frame instead of the deforming surface in the actual configuration.

So, it is worth to introduce briefly the reference frame quantities [20, 21]:

- mechanical Piola – Kirchhoff stress tensor

$$T_{Ij}^m = JF_{kI}^{-1} t_{kj}^m; \tag{4.11}$$

- Maxwell Piola – Kirchhoff stress tensor

$$T_{Ij}^M = T_{Ij}^{Mm} + T_{Ij}^{Me} = JF_{kI}^{-1} t_{kj}^M; \tag{4.12}$$

- total Piola – Kirchhoff stress tensor

$$T_{Ij} = T_{Ij}^m + T_{Ij}^M = JF_{kI}^{-1} t_{kj}. \tag{4.13}$$

After substituting the second equation of (4.5) into the first of Eqs. (3.8) and using the conservation of mass equation in the form

$$\varrho^F = J\varrho \tag{4.14}$$

(where ϱ^F is the mass density in a natural state, when the body is acted upon neither by force nor by magnetic field), one can obtain

$$B_I = \varrho^F \frac{\partial \psi}{\partial M_I}. \tag{4.15}$$

With the help of the first equation of (4.5) and Eq. (4.11), one can express the mechanical Piola-Kirchhoff stress tensor as

$$T_{Ij}^m = \delta_{jJ} T_{IJ}^m = \varrho^F F_{jL} \frac{\partial \psi}{\partial S_{IL}} + \varrho^F F_{jK} M_I \frac{\partial \psi}{\partial M_K} - \varrho^F F_{jI} M_K \frac{\partial \psi}{\partial M_K}. \tag{4.16}$$

Equations (4.9) with (4.10), and (4.15) and (4.16) stand for the constitutive relations in which every quantity is related to the reference frame.

5. Bilinear expansion of Gibbs function

To obtain explicit forms (not with partial differentiation) of constitutive relations, we should expand the Gibbs thermodynamic function in terms of its independent variables in the reference frame. The expansion is cut after the third order terms to get constitutive equations in a bilinear form. Linear and quadratic terms of constitutive relations play the most important role in describing nonlinear phenomena in magnetoelastic solids [20, 21, 23]. The other, higher order terms can be neglected.

Let us introduce the commonly accepted expansion [20]

$$
\begin{aligned}
(5.1)\qquad \varrho^F\psi = {} & \frac{1}{2}c_{IJKL}S_{IJ}S_{KL} + \frac{1}{6}c_{IJKLMN}S_{IJ}S_{KL}S_{MN} \\
& + \frac{1}{2}f_{MIJKL}M_M S_{IJ}S_{KL} + f_{MIJ}M_M S_{IJ} + B_{IJKL}S_{IJ}M_K M_L \\
& + \frac{1}{2}\mu_{MN}M_M M_N + \frac{1}{6}\mu_{MNK}M_M M_N M_K ,
\end{aligned}
$$

where c_{IJKL} is the elastic tensor of the second order, c_{IJKLMN} is the elastic tensor of the third order, f_{MIJ} is the piezomagnetic tensor of the second order, f_{MIJKL} is the magnetoelastic tensor of the third order, μ_{MN} is the permeability tensor of the second order, μ_{MNP} is the magnetic anisotropy tensor of the third order, and B_{MNIJ} is the magnetostriction tensor of the third order (the contribution of exchange energy is neglected as it is an effect by one order higher than that taken into account).

Of course, in general the following symmetry conditions must be fulfilled [23]

$$
\begin{aligned}
& c_{IJKL} = c_{JIKL} = c_{IJLK} = c_{KLIJ}, \\
& c_{IJKLMN} = c_{JIKLMN} = c_{IJLKMN} = c_{IJKLNM} = c_{KLIJMN} = c_{MNKLIJ}, \\
& f_{MIJ} = f_{MJI}, \\
& f_{MIJKL} = f_{MJIKL} = f_{MIJLK} = f_{MKLIJ}, \\
& \mu_{MN} = \mu_{NM}, \\
& \mu_{MNP} = \mu_{NMP} = \mu_{PNM}, \\
& B_{MNIJ} = B_{NMIJ} = B_{MNJI},
\end{aligned}
\qquad (5.2)
$$

so we have only 21 independent coefficients of c_{IJKL}, 56 of $c_{IJKJLMN}$, 18 of f_{MIJ}, 63 of f_{MIJKL}, 6 of μ_{MN}, 10 of μ_{MNP}, and 36 of B_{MNIJ}, what together gives 210 independent material constants of magnetoelastic media in the bilinear theory (45 in a linear one). But if the solid has some symmetry points or axis or planes, what always happens, the number of material constants to be given is reduced further and usually is much less than 210.

Moreover, we want to get constitutive expressions in the form like the form below:

$$
T = T(H, M, E), \qquad B = B(H, M), \qquad (5.3)
$$

where M, H, E are material quantities in the reference frame, so the following is now substituted to (4.15) and (4.16)

$$S = \frac{1}{2}\left(H + H^T + H^T H\right), \qquad F = H + I, \tag{5.4}$$

and [20, 21]

$$\begin{aligned} F^{-1} &= I - H + H^T H + \mathcal{O}(H^3), \\ J &= 1 + \operatorname{tr} H + \frac{1}{2}(\operatorname{tr} H)^2 - \frac{1}{2}\operatorname{tr}(H^T H) + \mathcal{O}(H^3), \end{aligned} \tag{5.5}$$

where $\mathcal{O}(H^n)$ denotes the remaining terms of order higher or equal than n, which are to be neglected next. After substitution of Eqs. (5.4) and (5.1) into Eq. (4.15) one can obtain

$$\begin{aligned} B_I = {} & f_{IKL}S_{KL} + \frac{1}{2}f_{IKLMN}S_{KL}S_{MN} + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K \\ & + 2B_{KLIJ}M_J S_{KL} = f_{IKL}H_{KL} + \frac{1}{2}f_{IKL}H_{ML}H_{MK} + \frac{1}{2}f_{IKLMN}H_{KL}H_{MN} \\ & + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL} + \mathcal{O}(H^3) + \mathcal{O}(MH^2). \end{aligned} \tag{5.6}$$

After neglecting higher order terms and changing indices we have:

$$\begin{aligned} B_I = {} & f_{IKL}H_{KL} + \mu_{IJ}M_J + \frac{1}{2}\left(f_{IKLMN} + f_{IKN}\delta_{ML}\right)H_{KL}H_{MN} \\ & + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL}. \end{aligned} \tag{5.7}$$

After substitution of Eqs. (5.4) and (5.1) into Eq. (4.16) one can obtain

$$\begin{aligned} T^m_{IJ} = {} & (\delta_{JL} + H_{JL})\left(c_{ILMN}S_{MN} + \frac{1}{2}c_{ILMNPR}S_{MN}S_{PR}\right. \\ & \left. + f_{MIL}M_M + f_{MILPR}M_M S_{PR} + B_{ILMN}M_M M_N\right) \\ & + \left[(\delta_{JK} + H_{JK})M_I - (\delta_{IJ} + H_{IJ})M_K\right]\left(2B_{KLMN}S_{MN}M_L + f_{KMN}S_{MN}\right. \\ & \left. + \frac{1}{2}f_{KMNPR}S_{MN}S_{PR} + \mu_{KL}M_L + \frac{1}{2}\mu_{KLM}M_L M_M\right). \end{aligned} \tag{5.8}$$

Using the first of equations (5.4), the above formula can be simplified as follows

$$\begin{aligned} T^m_{IJ} = {} & c_{IJMN}H_{MN} + \frac{1}{2}c_{IJMN}H_{PM}H_{PN} + \frac{1}{2}c_{IJMNPR}H_{MN}H_{PR} \\ & + B_{IJMN}M_M M_N + f_{MIJ}M_M + f_{MIJPR}M_M H_{PR} + c_{ILMN}H_{MN}H_{JL} \\ & + f_{MIL}M_M H_{JL} + \mathcal{O}(H^3) + \mathcal{O}(HM^2) + f_{JMN}M_I H_{MN} + \mu_{JL}M_L M_I \\ & - \delta_{IJ}(f_{KMN}M_K H_{MN} - \mu_{KL}M_L M_K) + \mathcal{O}(MH^2) + \mathcal{O}(HM^2). \end{aligned} \tag{5.9}$$

Next, on the basis of Eqs. (2.8), (4.15), (5.4) and (5.6), the magnetic part of Maxwell Piola – Kirchhoff stress tensor can be found as:

$$
\begin{aligned}
(5.10)\qquad T_{IJ}^{Mm} &= -[(\delta_{JK}+H_{JK})M_I-(\delta_{IJ}+H_{IJ})M_K]\Big(2B_{KLMN}S_{MN}M_L \\
&\quad +f_{KMN}S_{MN}+\frac{1}{2}f_{KMNPR}S_{MN}S_{PR}+\mu_{KL}M_L+\frac{1}{2}\mu_{KLM}M_LM_M\Big) \\
&= -f_{JMN}M_IH_{MN}-\mu_{JL}M_LM_I+\delta_{IJ}(f_{KMN}M_KH_{MN}+\mu_{KL}M_LM_K) \\
&\quad +\mathcal{O}(MH^2)+\mathcal{O}(HM^2),
\end{aligned}
$$

and on the basis of Eqs. (2.8), (3.7), (4.12), and (5.6), the electric part of Maxwell Piola – Kirchhoff stress tensor can be written as:

$$
\begin{aligned}
(5.11)\qquad T_{IJ}^{Me} = \delta_{jJ}T_{Ij}^{Me} &= J\delta_{jJ}F_{kI}^{-1}\Big(\varepsilon_0F_{kJ}^{-1}E_JF_{jK}^{-1}E_K-\frac{1}{2}\varepsilon_0F_{lL}^{-1}E_LF_{lL}^{-1}E_L\delta_{kj}\Big) \\
&= J\delta_{jJ}\varepsilon_0E_JE_K\Big(F_{kI}^{-1}F_{kJ}^{-1}F_{jK}^{-1}-\frac{1}{2}F_{jI}^{-1}F_{kJ}^{-1}F_{kK}^{-1}\Big) \\
&= \delta_{jJ}\varepsilon_0E_LE_K\Big(\delta_{kI}\delta_{kL}\delta_{jK}-\frac{1}{2}\delta_{jI}\delta_{kL}\delta_{kK}\Big)+\mathcal{O}(H^2E) \\
&= \varepsilon_0E_LE_K\Big(\delta_{IL}\delta_{JK}-\frac{1}{2}\delta_{LI}\delta_{JK}\Big)=\frac{1}{2}\epsilon_0E_LE_K(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{IJ}\delta_{KL}).
\end{aligned}
$$

So, the total Piola – Kirchhoff stress tensor $T_{IJ}=T_{IJ}^M+T_{IJ}^m$ after disregarding higher order terms and changing indices, is now

$$
\begin{aligned}
(5.12)\qquad T_{IJ} &= c_{IJKL}H_{KL}+\Big(c_{ILMN}\delta_{JK}+\frac{1}{2}c_{IJNL}\delta_{MK}+c_{IJKLMN}\Big)H_{KL}H_{MN} \\
&\quad +(f_{MIL}\delta_{KJ}+f_{MIJKL})M_MH_{KL}+f_{MIJ}M_M+B_{IJMN}M_MM_N \\
&\quad +\frac{1}{2}\epsilon_0(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{IJ}\delta_{KL})E_LE_K.
\end{aligned}
$$

Additionally, substitution of Eq. (5.4) into Eqs. (4.9) with (4.10) gives the coincident formula for the conduction current,

$$
(5.13)\qquad \mathcal{J}_I=(\sigma\delta_{IJ}+\sigma^hH_{IJ})E_J,\qquad \sigma^h=\sigma_1.
$$

Two constitutive relations (5.7) and (5.12) can be simplified more by the introduction of new tensors

$$
\begin{aligned}
\tilde{c}_{IJKLMN} &= c_{ILMN}\delta_{JK}+\frac{1}{2}c_{IJNL}\delta_{MK}+\frac{1}{2}c_{IJKLMN}, \\
(5.14)\qquad \tilde{f}_{MIJKL} &= f_{MIL}\delta_{JK}+f_{MIJKL}, \\
\tilde{l}_{IJKL} &= 0.5\varepsilon_0(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{KL}\delta_{IJ}).
\end{aligned}
$$

Then

$$
\begin{aligned}
T_{IJ} &= c_{IJKL}H_{KL} + \tilde{c}_{IJKLMN}H_{KL}H_{MN} + \tilde{f}_{MIJKL}M_M H_{KL} \\
&\quad + f_{MIJ}M_M + B_{IJMN}M_M M_N + \tilde{l}_{IJKL}E_K E_L, \\
B_I &= \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL} \\
&\quad + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}H_{MN} + f_{IKL}H_{KL}.
\end{aligned}
\tag{5.15}
$$

There are less symmetries for the new third order tensors, because they contain the geometrical contribution to elasticity, piezomagnetism and electrostriction, respectively ($\tilde{l}$ is only the geometrical electrostriction, since the material one was neglected for magnetoelastic solids in (5.1)). It is as follows:

$$
\begin{aligned}
\tilde{c}_{IJKLMN} &= \tilde{c}_{JIKLMN} = \tilde{c}_{IJMNKL}, \\
\tilde{f}_{MIJKL} &= \tilde{f}_{MJIKL} = \tilde{f}_{MKLIJ}, \\
\tilde{l}_{IJKL} &= \tilde{l}_{JIKL} = \tilde{l}_{IJLK}.
\end{aligned}
\tag{5.16}
$$

The relations (5.13) and (5.15) stand for bilinear constitutive equations for magnetoelastic solids even if the deformation or magnetization in the solid is large.

6. Small-field constitutive equations

Constitutive formulas obtained in the previous section are written in the reference frame. Let us now suppose that the initial nonzero strain field, as a result of external force or magnetic field, exists in the body at first. A small field is considered to be superposed on this state next. It means that we have now the third, intermediate configuration of the body, in addition to the undisturbed, reference configuration and the actual one. The material point identified by the material coordinate X_I first moves to intermediate coordinate $X_k^i(X_I)$ by a large initial deformation, and next to $x_k(X_I)$ by a small alternating vibration, for example. So, every field quantity in the reference frame can be decomposed into two parts. The first associated with the bias and the second, small one – with the small deformation in the body. Let the sign tilde denote the quantities associated with the initial state of large deformation. We have then, in the reference frame,

$$
Z^t = \tilde{Z} + Z, \qquad \text{where} \qquad Z = T, H, M, B. \tag{6.1}
$$

Additionally, small-field electric strength and conduction current is supposed to exist in the biased body, so

$$
E^t = E, \qquad \mathcal{J}^t = \mathcal{J}. \tag{6.2}
$$

These total quantities must fulfill the constitutive equations (5.13) and (5.15), so

$$
\begin{aligned}
(6.3)\qquad \tilde{T}_{IJ} + T_{IJ} = {} & \Big\{ c_{IJKL}\tilde{H}_{KL} + \tilde{c}_{IJKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_M\tilde{H}_{KL} \\
& + f_{MIJ}\widetilde{M}_M + B_{-IJMN}\widetilde{M}_M\widetilde{M}_N \Big\} + c_{IJKL}H_{KL} + \tilde{c}_{IJKLMN}H_{KL}\tilde{H}_{MN} \\
& + \tilde{c}_{IJMNKL}H_{KL}\tilde{H}_{MN} + f_{MIJ}M_M + \tilde{f}_{MIJKL}M_M\tilde{H}_{KL} + \tilde{f}_{MIJKL}H_{KL}\widetilde{M}_M \\
& + B_{IJMN}M_M\widetilde{M}_N + B_{IJMN}M_N\widetilde{M}_M + [\tilde{c}_{IJKLMN}H_{KL}H_{MN} \\
& + \tilde{f}_{MIJKL}M_MH_{KL} + B_{IJMN}M_MM_N + \tilde{l}_{IJKL}E_KE_L] ,
\end{aligned}
$$

and

$$
\begin{aligned}
(6.4)\qquad \tilde{B}_I + B_I = {} & \Big\{ \mu_{IJ}\widetilde{M}_J + \frac{1}{2}\mu_{IJK}\widetilde{M}_J\widetilde{M}_K + 2B_{KLIJ}\widetilde{M}_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + f_{IKL}\tilde{H}_{KL} \Big\} + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J\widetilde{M}_K \\
& + \frac{1}{2}\mu_{IJK}M_J\widetilde{M}_K + 2B_{KLIJ}H_{KL}\widetilde{M}_J + 2B_{KLIJ}M_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}\tilde{H}_{MN} + \frac{1}{2}\tilde{f}_{IMNKL}H_{KL}\tilde{H}_{MN} + f_{IKL}H_{KL} \\
& + \left[\frac{1}{2}\mu_{IJK}M_JM_K + 2B_{KLIJ}M_JH_{KL} + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}H_{MN}\right] ,
\end{aligned}
$$

and

$$
(6.5)\qquad \mathcal{J}_I = \sigma\delta_{IJ}E_J + \sigma^h\tilde{H}_{IJ}E_J + \left[\sigma^hH_{IJ}E_J\right] .
$$

The terms in the last three formulas have been regrouped so it can be said that expressions in the first brackets are equal to the biasing quantities $\tilde{T}$ and $\tilde{B}$, respectively, while terms in the square brackets can be neglected because they are second order terms of small-field quantities. After separation one can obtain nonlinear constitutive equations for large initial state in the form

$$
\begin{aligned}
\tilde{T}_{IJ} = {} & c_{IJKL}\tilde{H}_{KL} + \tilde{c}_{IJKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_M\tilde{H}_{KL} \\
& + f_{MIJ}\widetilde{M}_M + B_{IJMN}\widetilde{M}_M\widetilde{M}_N ,
\end{aligned}
$$

$$
(6.6)\qquad
\begin{aligned}
\tilde{B}_I = {} & \mu_{IJ}\widetilde{M}_J + \frac{1}{2}\mu_{IJK}\widetilde{M}_J\widetilde{M}_K + 2B_{KLIJ}\widetilde{M}_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + f_{IKL}\tilde{H}_{KL} ,
\end{aligned}
$$

and the linear ones for a small-field superposed on this biasing state

$$
(6.7)\qquad
\begin{aligned}
T_{IJ} &= c^{\text{eff}}_{IJKL}H_{KL} + f^{\text{eff}}_{MIJ}M_M , \\
B_I &= \mu^{\text{eff}}_{IJ}M_J + f^{\text{eff}}_{IKL}H_{KL} , \\
\mathcal{J}_I &= \sigma^{\text{eff}}_{IJ}E_J ,
\end{aligned}
$$

where effective tensors are of course dependent on the biasing magnetization and displacement gradient (parameters)

$$
\begin{aligned}
c^{\mathrm{eff}}_{IJKL} &= c_{IJKL} + 2\tilde{c}_{IJKLMN}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_{M}\,,\\
f^{\mathrm{eff}}_{IKL} &= f_{IKL} + \tilde{f}_{IKLMN}\tilde{H}_{MN} + 2B_{KLIJ}\widetilde{M}_{J}\,,\\
\mu^{\mathrm{eff}}_{IJ} &= \mu_{IJ} + \mu_{IJK}\widetilde{M}_{K} + 2B_{KLIJ}\tilde{H}_{KL}\,,\\
\sigma^{\mathrm{eff}}_{IJ} &= \sigma\delta_{IJ} + \sigma^{h}\tilde{H}_{IJ}\,.
\end{aligned}
\tag{6.8}
$$

7. Loading of the cracked specimen

It is well known, that aside from ideally brittle materials, any loading of cracked body is accompanied by inelastic deformation in the neighbourhood of the crack tip due to stress concentration there. Usually three basic modes of loading are distinguished [19]: Mode I, called opening mode, Mode II, named sliding mode, and Mode III, described as tearing mode. All of them with the Cartesian coordinate system are depicted in Fig. 1. The stress concentrations around the crack

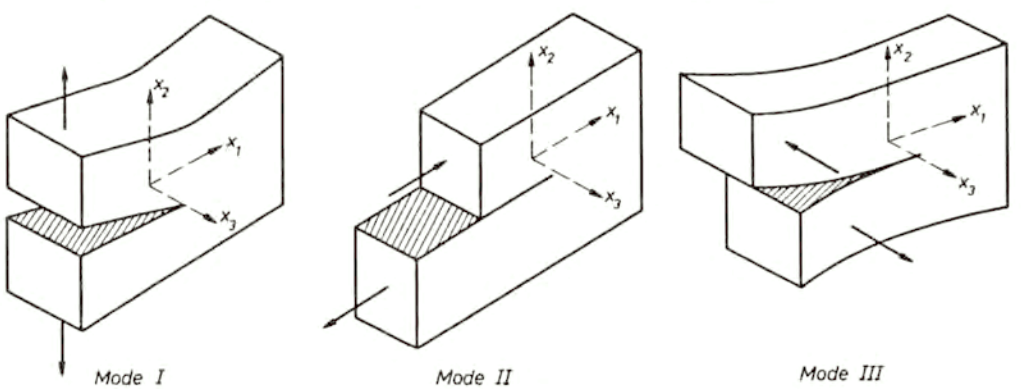

FIG. 1. Basic loading modes for a cracked body.

for different modes are given in [19, pages 138–147] in the formulas, following below, written in a polar coordinate system with the origin at the crack tip ($X_1 = r\cos(\theta)$, $X_2 = r\sin(\theta)$). For Mode I the nonzero stress components are

$$
\begin{Bmatrix} \tilde{T}_{11} \\ \tilde{T}_{12} \\ \tilde{T}_{22} \end{Bmatrix} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\cos(\theta/2)\begin{Bmatrix} 1-\sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1+\sin(\theta/2)\sin(3\theta/2) \end{Bmatrix}.
\tag{7.1}
$$

For Mode II and Mode III they are, respectively

$$\begin{Bmatrix} \tilde{T}_{11} \\ \tilde{T}_{12} \\ \tilde{T}_{22} \end{Bmatrix} = \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2)[2+\cos(\theta/2)\cos(2\theta/2)] \\ \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix}, \tag{7.2}$$

$$\begin{Bmatrix} \tilde{T}_{31} \\ \tilde{T}_{32} \end{Bmatrix} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix}. \tag{7.3}$$

It can be seen that for $r = 0$ the stress in all cases is singular, and approaches infinity, but the stress intensity factors K_{I}, K_{II}, K_{III} are defined in such a way that they are finite

$$K_{\mathrm{I}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{22}\Big|_{\theta=0} \right\}, \tag{7.4}$$

$$K_{\mathrm{II}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{12}\Big|_{\theta=0} \right\}, \tag{7.5}$$

$$K_{\mathrm{III}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{32}\Big|_{\theta=0} \right\}. \tag{7.6}$$

For the simple isotropic, nonmagnetostrictive material ($B_{KLIJ} = 0$, $f_{IKL} = 0$) possessing the elastic tensor in the form [14]

$$c_{IJKL} = 2\mu\delta_{KI}\delta_{LJ} + \lambda\delta_{KL}\delta_{IJ}, \tag{7.7}$$

where μ and λ are Lamé constants, the corresponding nonzero displacements near the crack tip can be written for Modes I, II, and III as [19]

$$\begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{Bmatrix} = \frac{K_{\mathrm{I}}}{2\mu}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos(\theta/2)[\kappa - 1 + 2\sin^2(\theta/2)] \\ \sin(\theta/2)[\kappa + 1 - 2\cos^2(\theta/2)] \end{Bmatrix}, \tag{7.8}$$

$$\begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{Bmatrix} = \frac{K_{\mathrm{II}}}{2\mu}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin(\theta/2)[\kappa + 1 + 2\cos^2(\theta/2)] \\ -\cos(\theta/2)[\kappa - 1 - 2\sin^2(\theta/2)] \end{Bmatrix}, \tag{7.9}$$

$$\{\tilde{U}_3\} = \frac{2K_{\mathrm{III}}}{\mu}\sqrt{\frac{r}{2\pi}}\{\sin(\theta/2)\}, \tag{7.10}$$

where for Mode I and II either $\kappa = 3 - 4\nu$ for plane strain or $\kappa = (3-\nu)/(1+\nu)$ for plane stress case, and ν is Poisson's constant. Taking into account the gradient description in polar coordinates (two-dimensional) of the form

$$\frac{\partial}{\partial X_1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial X_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial X_3} = 0, \tag{7.11}$$

the displacement gradients can be expressed for Mode I, II, III, respectively, as:

$$\tilde{H} = \frac{K_{\mathrm{I}}}{4\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} \cos(\theta/2)(\kappa - \cos\theta - 2\sin^2\theta) & \sin(\theta/2)(\kappa + \cos\theta + 2\cos^2\theta) & 0 \\ -\sin(\theta/2)(\kappa - \cos\theta + 2\sin^2\theta) & \cos(\theta/2)(\kappa + \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.12}$$

$$\tilde{H} = \frac{K_{\mathrm{II}}}{4\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} -\sin(\theta/2)(\kappa + 2 + \cos\theta - 2\sin^2\theta) & \cos(\theta/2)(\kappa + 2 - \cos\theta + 2\cos^2\theta) & 0 \\ \cos(\theta/2)(\kappa - 2 + \cos\theta - 2\sin^2\theta) & -\sin(\theta/2)(\kappa - 2 - \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.13}$$

$$\tilde{H} = \frac{K_{\mathrm{III}}}{\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin(\theta/2) & \cos(\theta/2) & 0 \end{bmatrix}. \tag{7.14}$$

8. Current flow around the crack

Nondestructive testing (NDT) of materials is a group of methods used to distinguish if the material given has some defects (cracks, dislocations) or not. Nondestructive evaluation (NDE) is more advanced since it should give more information about defects (size, type, etc.). There are many experimental techniques which can be used to get proper information: ultrasonic or magnetic methods, and the potential drop technique (d.c. or a.c.), which is now gaining much interests of researchers [15, 16, 25, 26, 28]. In these kinds of methods the specimen tested is subjected to the current flow, which is assumed to form around the crack. It is especially the property of the a.c. current which occurs mainly very near to the surface. For the values of frequency f and conductivity large enough, the penetration depth is small,

$$\delta = \frac{1}{\sqrt{\pi\mu\sigma f}}. \tag{8.1}$$

The path of the a.c. current is shown in Fig. 2.

If the specimen with crack is loaded, the large gradient of displacement (Sec. 7, Eqs. (7.12), (7.13), (7.14)) which exists near the crack can be regarded as the

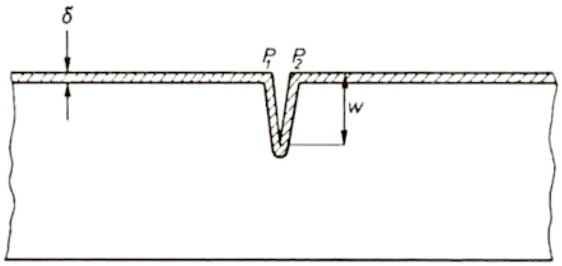

FIG. 2. A.c. current flow path around the crack.

mechanical bias for the current flow around the crack. Therefore, the conductivity near the crack is different from the natural (for unbiased material) one, and should be calculated with respect to predeformation as it is given in Sec. 6, Eq. (6.8). Substituting the gradient of displacement for different modes to the definition of effective conductivity, it is easy to obtain σ^{eff} for Mode I, II, III, respectively

$$(8.2)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma + \gamma_{\text{I}}\cos(\theta/2)(\kappa - \cos\theta - 2\sin^2\theta) & \gamma_{\text{I}}\sin(\theta/2)(\kappa + \cos\theta + 2\cos^2\theta) & 0 \\ -\gamma_{\text{I}}\sin(\theta/2)(\kappa - \cos\theta + 2\sin^2\theta) & \sigma + \gamma_{\text{I}}\cos(\theta/2)(\kappa + \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & \sigma \end{bmatrix},$$

$$(8.3)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma - \gamma_{\text{II}}\sin(\theta/2)(\kappa + 2 + \cos\theta - 2\sin^2\theta) & \gamma_{\text{II}}\cos(\theta/2)(\kappa + 2 - \cos\theta + 2\cos^2\theta) & 0 \\ \gamma_{\text{II}}\cos(\theta/2)(\kappa - 2 + \cos\theta - 2\sin^2\theta) & \sigma - \gamma_{\text{II}}\sin(\theta/2)(\kappa - 2 - \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & \sigma \end{bmatrix},$$

$$(8.4)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ -\gamma_{\text{III}}\sin(\theta/2) & \gamma_{\text{III}}\cos(\theta/2) & \sigma \end{bmatrix},$$

where

$$(8.5)\qquad \gamma_{\text{I}} = \frac{K_{\text{I}}\sigma^h}{4\mu\sqrt{2\pi r}}, \qquad \gamma_{\text{II}} = \frac{K_{\text{II}}\sigma^h}{4\mu\sqrt{2\pi r}}, \qquad \gamma_{\text{III}} = \frac{K_{\text{III}}\sigma^h}{\mu\sqrt{2\pi r}}.$$

It is clear that the conductivity is a function of θ and r, so the material near the crack tip can be modeled as nonhomogeneous. In such a case the flow of the current is no longer described by the formula $\mathcal{J}_{I,I} = 0$. To get the proper equation, it is necessary to consider

$$\mathcal{J}_{I,I} = \sigma^{\text{eff}}_{IJ} E_{J,I} + \sigma^{\text{eff}}_{IJ,I} E_J , \tag{8.6}$$

$$\mathcal{D}_{I,I} = \delta_{IJ}(\varepsilon E_{J,I} + \varepsilon_{,I} E_J), \tag{8.7}$$

and the Maxwell formula

$$\nabla \times \mathcal{H} = \mathcal{J} + j\omega \mathcal{D}, \tag{8.8}$$

where $\mathcal{H}$ is the magnetic intensity field, $\mathcal{D}$ is electric induction field, and $\omega = 2\pi f$ is angular frequency of the a.c. current applied. Calculating the divergence of both sides of the last formula, one can get the following equation

$$\mathcal{J}_{I,I} = -j\omega \varrho, \tag{8.9}$$

where ϱ is the charge density, and combining it with equations (8.6) and (8.7), the following formula is obtained

$$j\omega \varepsilon_0 \delta_{IJ} + \sigma^{\text{eff}}_{IJ} \varrho = \sigma^{\text{eff}}_{IJ} \varepsilon_0 \mathcal{J}_K \sigma^{\text{eff}^{-1}}_{KL,L} . \tag{8.10}$$

The density of the charge, which is accumulated during the flow process near the crack tip, can be expressed for large conductivity or not too high frequency approximately as

$$\varrho \approx \varepsilon_0 \mathcal{J}_I \sigma^{\text{eff}^{-1}}_{IJ,J} . \tag{8.11}$$

9. Alternating current potential drop

When the a.c. current is flowing on the surface of the specimen (see Fig. 2), the potential drop far from the crack, which can be observed by the probe on the surface, can be simply calculated as

$$\Delta U_0 = \frac{Il}{\sigma S} , \tag{9.1}$$

where I is total current applied to the specimen, l is the distance between the probe pins along X_2, $S = \delta d$ is the cross-section of a.c. current path (d is the width of the specimen along X_3), and σ is the conductivity (scalar for the natural, not predeformed state of the material).

When the probe pins enclose the crack underneath, then the potential drop is composed of two parts

$$\Delta U = \Delta U_0 + \Delta U_c . \tag{9.2}$$

The first component is the same as (9.1) because it comes from the separation of pins in the probe, and the second one is the potential drop along the crack. It is assumed here that the crack is modeled by the wedge in the X_1 direction with the gap tending to zero ($P_1 \to P_2$). This means that the current is flowing approximately along X_1 with the constant density $\mathcal{J}_1 = I/S$, so the electric field is $E_1 = \sigma_{11}^{\text{eff}^{-1}} \mathcal{J}_1$, and it differs from point to point since the effective conductivity is nonhomogeneous. The total potential drop around the crack can be calculated as

$$\Delta U_c = \int_{P_1}^{P_2} E_1 \, dX_1 = \frac{I}{S} \int_{P_1}^{P_2} \sigma_{11}^{\text{eff}^{-1}} \, dX_1 . \tag{9.3}$$

The integration variable can be changed to polar, remembering that first it goes forward and second time back along X_1

$$\Delta U_c = \frac{I}{S} \left[\int_0^w \sigma_{11}^{\text{eff}^{-1}} \, dr \Big|_{\theta=-\pi} + \int_0^w \sigma_{11}^{\text{eff}^{-1}} \, dr \Big|_{\theta=\pi} \right], \tag{9.4}$$

where w is the depth of the crack under the surface. To perform the integration it is necessary to find $\sigma_{11}^{\text{eff}^{-1}}$ for $\theta = \pm\pi$.

On the left-hand side of the crack, $\theta = -\pi$, the effective conductivity is simplified as compared with Eqs. (8.2), (8.3), (8.4) for Mode I, II, III, respectively

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma & -\gamma_{\text{I}}(\kappa+1) & 0 \\ \gamma_{\text{I}}(\kappa-1) & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.5}$$

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma + \gamma_{\text{II}}(\kappa+1) & 0 & 0 \\ 0 & \sigma + \gamma_{\text{I}}(\kappa-3) & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.6}$$

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ \gamma_{\text{III}} & 0 & \sigma \end{bmatrix}. \tag{9.7}$$

On the right-hand side of the crack, $\theta = \pi$, the effective conductivity is for Mode I, II, III, respectively

$$\sigma_{IJ}^{\text{eff}}(r, \pi) = \begin{bmatrix} \sigma & \gamma_I(\kappa+1) & 0 \\ -\gamma_{\text{I}}(\kappa-1) & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.8}$$

$$\sigma_{IJ}^{\text{eff}}(r, \pi) = \begin{bmatrix} \sigma - \gamma_{\text{II}}(\kappa+1) & 0 & 0 \\ 0 & \sigma - \gamma_{\text{I}}(\kappa-3) & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.9}$$

$$\sigma_{IJ}^{\text{eff}}(r,\pi) = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ -\gamma_{\text{III}} & 0 & \sigma \end{bmatrix}. \tag{9.10}$$

Now, $\sigma_{11}^{\text{eff}^{-1}}$ for $\theta = \pm\pi$ can be obtained for three different modes as follows:

$$\sigma_{11}^{\text{eff}^{-1}} = \frac{1}{\sigma} \tag{9.11}$$

for Mode I and Mode III as well, whereas for Mode II it is different

$$\sigma_{11}^{\text{eff}^{-1}} = \frac{1}{\sigma \mp \gamma_{\text{II}}(\kappa+1)}. \tag{9.12}$$

For Mode I and III, $\sigma_{11}^{\text{eff}^{-1}}$ is constant, independent of r, thus the integration in Eq. (9.4) is very simple and gives

$$\Delta U_c = \frac{2Iw}{\sigma S}, \tag{9.13}$$

just like the potential drop on the straight path of the length $2w$. For the Mode II, however, $\sigma_{11}^{\text{eff}^{-1}}$ is dependent on r through γ_{II}, see formula (8.5), what can be written explicitly for $\theta = \pm\pi$ as

$$\sigma_{11}^{\text{eff}^{-1}} = \sigma \mp \frac{\gamma}{\sqrt{r}}, \tag{9.14}$$

where

$$\gamma = \frac{K_{\text{II}}\sigma^h(\kappa+1)}{4\mu\sqrt{2\pi}}. \tag{9.15}$$

Then, the integral in Eq. (9.4) takes the form

$$\Delta U_c = \frac{I}{S}\left[\int_0^w \frac{\sqrt{r}}{\sigma\sqrt{r}+\gamma}dr + \int_0^w \frac{\sqrt{r}}{\sigma\sqrt{r}-\gamma}dr\right], \tag{9.16}$$

what after integration gives

$$\Delta U_c = \frac{2I}{\sigma S}\left[w + \frac{\gamma^2}{\sigma^2}\ln\left|\frac{\sigma^2}{\gamma^2}w - 1\right|\right]. \tag{9.17}$$

It can be seen from the above that the potential drop consists of the term resulting from the flow around the crack of the path length $2w$, and the second one is the result of the stress concentration near the crack tip, and is dependent on the ratio of squared conductivity and its deformation sensitivity.

10. Conclusions

Since for Mode I, and II no influence of predeformation was found, so it can be said that Eq. (9.17) stands for the potential drop on the crack for any kind of load in general, which always can be decomposed into three basic modes. The second term of the formula (9.17) is the correction of the potential drop due to the load through stress intensity factor K_{II} for different material described by stress sensitivity to conductivity σ^h, conductivity itself σ, and elastic parameters μ, κ. The derivation of the potential drop was done for the isotropic elastic material, its conductivity being sensitive to deformation. The coupling of the magnetic field was neglected as well as the change of the penetration depth due to the change of conductivity.

Following the form of the relations (6.8), the influence of the bias on the magnetoelastic solids can have the important implications:

- The properties of the biased material can be changed so far as the natural crystallographic symmetries are not kept the same for the biased state of material.
- The medium can become strongly piezomagnetic even if its piezomagnetic natural state tensor is zero (or very small, what is common); the reason can be the third order magnetoelastic term for very high strain, but mainly the magnetostriction in the magnetically biased solid (this is a stronger effect) – this is called biased piezomagnetism.
- On the other hand the material having small permeability can become ferromagnetic, when large initial strain is assumed due to the same magnetostriction effect.
- The conductivity can become even a nonsymmetric tensor (in general) instead of its natural state scalar value.

Equations (6.6), (6.7) and (6.8) seem to be very useful to make a numerical application. As we compare to the natural state case, we can see that the sole change of material tensors from second order tensors given in tables to effective tensors (6.8), which are dependent on the second and the third order tensors as well as on the biasing quantities, is sufficient to consider the biased solid as an unbiased one. We have to keep in mind to write the equations of motion and the boundary conditions also in the reference frame. It is easy to find effective tensors and use well-known methods or even numerical algorithms (developed earlier for linear problems), and to solve also the nonlinear problems. These effective parameters describe the nonlinear coupling between a crystal and predeformation of a mechanical or magnetic origin.

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Biaxial creep study of copper on the basis of isochronous creep surfaces

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A METHODOLOGY for determination of isochronous creep rupture surfaces is proposed on the basis of creep investigations carried out on pure copper under complex stress state at 523 K. This procedure is also used to determine the curves of the same time to reach the minimum creep rate and curves representing the same time to tertiary creep period. Analysis of these curves shows that for the stress level taken into account as a reference point in the plane $(\sigma_{11}, \sqrt{3}\sigma_{12})$, the material damage degree increases proportionally until rupture is achieved. It means practically that such isochronous curves coincide. Verification of the selected fundamental creep rupture criteria is carried out on the basis of the experimental data. It is shown that the Sdobyrev creep rupture criterion gives the most suitable tool to describe the degradation process of copper subjected to long-term constant loading conditions at elevated temperature.

1. Introduction

IN STRUCTURAL COMPONENTS subjected to long-term constant loading, which corresponds to the creep conditions, the material damage process develops. Two essential periods of such a process can be distinguished. During the first one, the damage process develops without microscopically visible cracks due to the nucleation process and growth of the microvoids. At the end of this stage, the macroscopically observed crack appears in the form of one or several fissures. In the second stage of the rupture process, the dominant fissure propagates reducing, as a consequence, the admissible loading capacity of a construction element and leading finally to its failure. In most cases the duration of the second stage of damage process is negligibly short in comparison to the exploitation time of an element [1]. Experiments concerning the processes of microcrack nucleation and growth, which are responsible for the failure of materials during creep, show that failure mechanisms can be divided into the three following types [2–7]:

1. Brittle failure mechanism, in which the microdefects are created and developed on the grain boundaries perpendicular to the maximum principal tensile stress.

2. Ductile failure mechanism, in which the microdefects are created at the grain boundaries, and they are developing due to grain boundary slides.

3. Mixed failure mechanism being a combination of the simultaneously developing mechanisms mentioned above.

The rupture character and time to rupture depends mainly on the material and the type and level of loading. Brittle failure mechanism is dominant in the case of polycrystalline materials tested at low levels of the uni-axial stress states. Material degradation during this mechanism has an intergranular character. At high stress levels the rupture takes place mainly due to the ductile failure mechanism, for

which the damages have a transgranular character and develop due to the slides passing through the grains. All mechanisms mentioned above may influence the form of a failure criterion determining time to rupture. In the paper it is one of the issues discussed in detail.

However, the main goal of the paper is to present a methodology of determination of the same time-to-rupture surfaces on the basis of creep tests carried out on pure copper under plane stress state at elevated temperature. It is proposed to adopt this technique to determine the surface of the same time to obtain the minimum creep rate (also called the surface of the same duration of primary creep) and surfaces of the same time to tertiary creep period.

2. Phenomenological creep rupture equations

At stress levels lower than the yield limit of a material subjected to creep, KACHANOV [8] has proposed to ignore the current necking of a specimen and to assume that the load is carried only by the effective part of the specimen cross-section, being the difference between the initial cross-section and the damage area, i.e. the area being the sum of all voids or fissures areas. He has introduced a scalar measure of damage in the form of the parameter of cross-section continuity which is 1 at the beginning of the deformation process and 0 at a localized failure of the material. It can be expressed by

$$\psi = \frac{A}{A_0}, \qquad 1 \geq \psi \geq 0, \tag{1}$$

where A denotes the effective area of specimen cross-section, and A_0 denotes the initial area of the cross-section.

Instead of this parameter it is more convenient to use a damage measure being the parameter complementary to that proposed by Kachanov. Such parameter represents the Rabotnov damage measure [9] and it can be given by

$$\omega = 1 - \psi = \frac{A_0 - A}{A_0}, \qquad 0 \leq \omega \leq 1. \tag{2}$$

It can be physically interpreted as the area of all defects referred to the undamaged initial cross-sectional area. Using this parameter, RABOTNOV [9] proposed modifications of the constitutive equations suggested by KACHANOV [8]. For uniaxial tensile stress tests, the strain rate and damage evolution equations are assumed to have the following simple normalised form:

$$\begin{aligned} \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} &= \frac{1}{(1-\omega)^m}\left(\frac{\sigma}{\sigma_0}\right)^n, \\ \frac{\dot{\omega}}{\dot{\omega}_0} &= \frac{1}{(1-\omega)^\eta}\left(\frac{\sigma}{\sigma_0}\right)^\nu, \end{aligned} \tag{3}$$

where n, m, ν, η, $\dot{\varepsilon}_0$, $\dot{\omega}_0$, σ_0 are material constants.

At the beginning of the test, when the material is undamaged, the strain rate equation in set (3) reduces to Norton's law describing the steady creep rate, given by

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} = \left(\frac{\sigma}{\sigma_0}\right)^n. \tag{4}$$

For constant stress level it is easy to integrate the equations in set (3) to obtain the time variations of strain and damage. By applying the rupture condition $\omega = 1$, it is possible to determine time to rupture t_R. Taking into account these conditions, the following relations [11] are obtained:

$$\begin{aligned}
\frac{\varepsilon}{\varepsilon_0} &= \lambda\left[1 - \{1 - (t/t_R)\}^{1/\lambda}\right], \\
\omega &= 1 - (1 - t/t_R)^{1/(\eta+1)}, \\
t_R &= \frac{1}{(\eta+1)\dot{\omega}_0(\sigma/\sigma_0)^\nu}, \\
\varepsilon &= \dot{\varepsilon}_0(\sigma/\sigma_0)^n t_R = \frac{\dot{\varepsilon}_0}{(\eta+1)\dot{\omega}_0}\left(\frac{\sigma}{\sigma_0}\right)^{n-\nu}, \\
\lambda &= (\eta+1)(\eta+1-m).
\end{aligned} \tag{5}$$

The creep theory proposed by Kachanov and Rabotnov was an inspiration to develop many new damage models, and it is considered as the beginning of the Continuum Damage Mechanics. Since it described only uni-axial creep behaviour, LECKIE and HAYHURST [11] have attempted to generalize the simple Rabotnov-Kachanov equations to multiaxial states of stress. The generalisation of Eqs. (3) to multiaxial stresses has been achieved by making the assumption that the influence of continuum damage on the deformation rate process is of a scalar character, and by the introduction of the homogeneous stress function which reflects the stress state effects on the time to rupture. The form of the generalisations was also influenced by the experimental results of JOHNSON *et al.* [12, 13] according to which the strain rates depend on the effective stress, and the components of strain rate tensor are proportional to the components of deviatoric stress. Leckie and Hayhurst also respected the fact, known from Johnson's tests, that when deterioration is taking place in the tertiary region, the ratio of the strain rate components remains practically constant and equals the value achieved in the steady state portion. Equations (3) can then be written as:

$$\begin{aligned}
\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_0} &= \frac{3}{2}\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\left(\frac{S_{ij}}{\sigma_0}\right)\left(\frac{1}{(1-\omega)^\mu}\right), \\
\frac{\dot{\omega}}{\dot{\omega}_0} &= \Delta^\nu \frac{1}{(1+\eta)(1-\omega)^\eta},
\end{aligned} \tag{6}$$

where $\Delta = \Delta(\sigma_{ij}/\sigma_0) = \sigma_{\max}/\sigma_0$ for copper, and $\Delta = \Delta(\sigma_{ij}/\sigma_0) = \sigma_e/\sigma_0$ for aluminium alloys. Integration of the damage evolution equation $(6)_2$ for the following boundary conditions: $\omega = 0$, $t = 0$ and $\omega = 1$, $t = t_R$, yields, after normalisation, the relation describing time to rupture in the form

$$\frac{t_R}{t_0} = \frac{1}{\Delta^\nu}. \tag{7}$$

Substitution of $t_R = t_0$ in (7) gives the equation of isochronous surface.

As mentioned above, the experimental results of JOHNSON *et al.* [12, 13] had an important role in developing new constitutive creep models, since they had carried out wide investigations for several different metals under complex stress states. They tried to find a correlation between time to rupture and such stress state parameters as:

1) maximum principal stress $\sigma_{\max}$, which controls the damage according to the Galileo hypothesis,

2) effective stress σ_e, which controls the damage according to the Huber – Mises criterion,

3) first invariant of the stress tensor J_1.

They have shown that the dependence of the rupture time upon the nature of the applied stress system for an aluminium alloy can be described by the octahedral shear stress criterion, whereas for pure copper – by the maximum principal stress criterion.

It has been found convenient to present the rupture results in terms of an isochronous surface which is that curve connecting the stress states with the same rupture times. For the plane stress conditions the isochronous surfaces obtained for different rupture criteria are illustrated in Fig. 1. According to JOHNSON [12, 13], the rupture criteria for aluminium alloy and pure copper represent the extremes of material behaviour, since the isochronous surface for many metals lies somewhere between these criteria. In spite of the fact that these observations have been made on the basis of a relatively limited amount of experimental data, and in certain cases they did not give precise description of rupture, they are still influencing the process of developing new creep damage models. Typical examples of such situation are: the rupture criterion applied by DYSON and MCLEAN [14] in which they assumed that the creep damage of Nimonic 80A tested at 1023 K was governed by the criterion being a product of the effective stress and maximum principal stress criteria, and the creep rupture criterion proposed by SDOBYREV [15] in the following form:

$$t_R^{(\mathrm{Sdob})} = A\{\beta\sigma_{\max} + (1-\beta)\sigma_e\}^{-\nu}, \tag{8}$$

where σ_e represents the effective stress defined by the second invariant of the stress deviator, $\sigma_{\max}$ denotes the maximum principal stress, β denotes the experimentally determined coefficient reflecting the relation between time to rupture and σ_e, $\sigma_{\max}$.

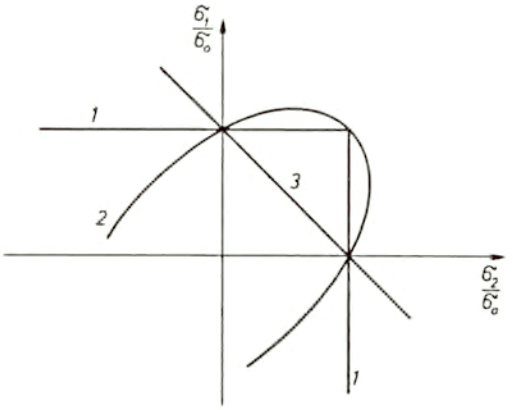

FIG. 1. Graphical representation of the creep rupture criteria (*1* – maximum principal stress criterion, *2* – Huber–Mises effective stress criterion, *3* – hydrostatic loading criterion).

On the basis of the experimental results obtained from tests made on cruciform specimens [2] and Johnson's investigations [12, 13], who used thin-walled tubular testpieces, HAYHURST [2], in attempts to describe the multi-axial behaviour of several materials, suggested a general relationship

$$t_R^{(\mathrm{Hay1})} = f(J_1, J_2, J_3), \tag{9}$$

where f is a homogeneous algebraic function of stress invariants. It has been found that both the relationships given by Eqs. (8) and (9) can be used to represent accurately the behaviour of copper alloys in the tension-compression quadrant of the diagram in Fig. 1, but predictions of the rupture time are in error for bi-axial tensile stresses. The bi-axial tension stress levels predicted by Eqs. (8) and (9) are too high and too low, respectively. On the basis of these remarks, HAYHURST [2] proposed a criterion which is capable of expressing the different types of behaviour represented by Eqs. (8) and (9) in the form

$$t_R^{(\mathrm{Hay2})} = A\left\{a\sigma_{\max} + bJ_1 + cJ_2'^{\frac{1}{2}}\right\}^{-\nu}, \tag{10}$$

being a linear combination of the maximum principal tensile stress and the first and second stress invariants; here A, ν – material constants which are independent of stress; a, b, c – constants; and J_1, J_2' represent the first invariant of the stress tensor and the second invariant of the stress deviator, respectively, which in terms of principal stresses are given by

$$\begin{aligned} J_1 &= \sigma_1 + \sigma_2 + \sigma_3\,, \\ J_2' &= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \end{aligned} \tag{11}$$

In expression (10) a, b, c are constants with $a + b + c = 1$, what limits the influence of particular terms in the criterion (10) on the time to rupture. By selecting appropriate values of these constants, all the simple rupture criteria previously discussed can be represented, i.e.

I) the maximum principal stress rupture criterion, if $a = 1$, $b = c = 0$; its graphical interpretation for the biaxial stress state is represented by the two straight lines (*1*), Fig. 1;

II) the Huber–Mises effective stress criterion, if $c = 1$, $a = b = 0$, represented by the ellipse (*2*) in Fig. 1;

III) the maximum hydrostatic stress criterion defined by the first invariant of the stress tensor; if $b = 1$, $a = c = 0$, it is represented by the straight line (*3*) in Fig. 1.

3. Programme of investigations

3.1. Material, specimen and testing device

The material investigated was electrolytic copper (M1E) of 99.9% purity. It was annealed for two hours at 673 K and furnace-cooled to achieve a uniform grain size, and next it was aged at constant room temperature for five years. Creep investigations were carried out on thin-walled tubular testpieces shown in Fig. 2 with the use of the biaxial creep testing machine, enabling satisfaction of the plane stress conditions by simultaneous loading of the testpieces by an axial force and twisting moment at elevated temperature (523 K). Details of the

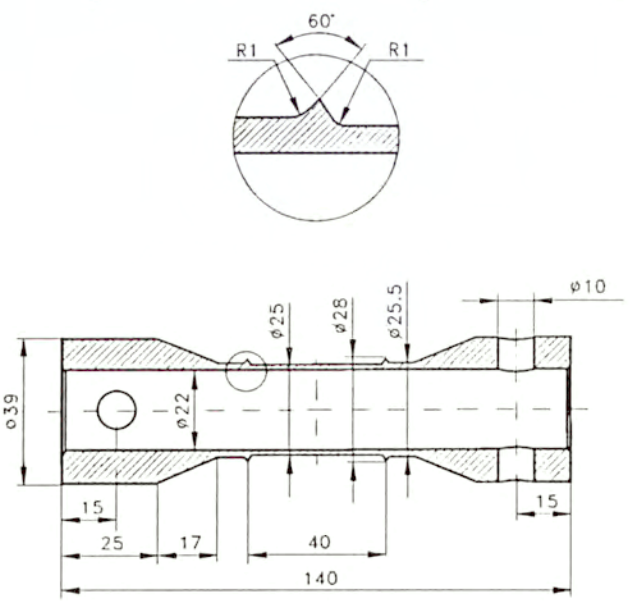

FIG. 2. Engineering drawing of the specimen.

experimental device are presented in [16, 17], whereas information concerning the measurement technique applied may be found in [18].

3.2. Experimental procedure

The experimental programme contained the creep tests up to rupture for copper specimens subjected to biaxial stress state, obtained by various combinations of an axial force and twisting moment leading to three different values of effective stress (σ_e = 70.0; 72.5; 75.0 [MPa]) defined by the following relationship:

$$\sigma_e = \left(\frac{3}{2} S_{ij} S_{ij}\right)^{1/2} = (\sigma_{11}^2 + 3\sigma_{12}^2)^{1/2}, \tag{12}$$

where S_{ij} – stress deviator, σ_{11} – axial stress, σ_{12} – shear stress. Diagram of the experimental programme is shown in Fig. 3. Creep tests under tension, torsion and combination of these loadings were carried out. The complex stress states in

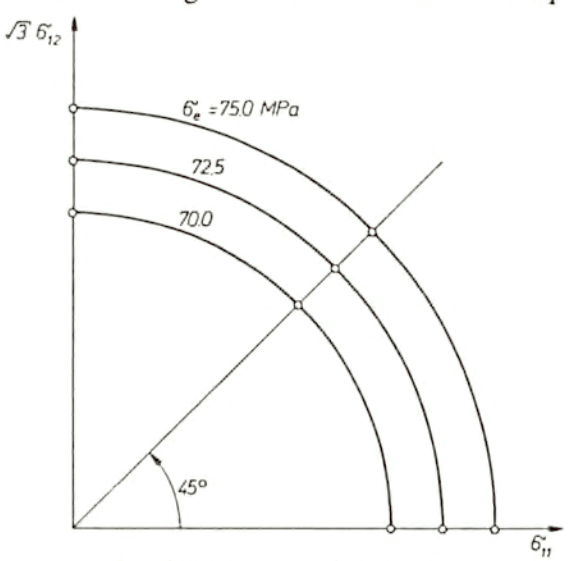

FIG. 3. Programme of creep tests.

the two-dimensional stress space (σ_{11}, $\sqrt{3}\sigma_{12}$) correspond to the points located on the rectilinear path of slope $\Theta_\sigma = 45°$ with respect to the σ_{11} stress axis. The angle Θ_σ was determined from the relation

$$\Theta_\sigma = \arctan\left(\frac{\sqrt{3}\sigma_{12}}{\sigma_{11}}\right). \tag{13}$$

Prior to creep test, each specimen was heated uniformly at the test temperature (523 K) for 24 hours. Creep investigations were carried out until rupture of the testpieces was achieved giving, as a consequence, the whole creep characteristics.

All creep tests were performed at stress levels which were lower than the value of yield point of the material at 523 K ($R_{0.2}$ = 76 [MPa]). Thus, the total strain has been expressed as a sum of the elastic and creep strain components in the form:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(e)} + \varepsilon_{ij}^{(c)}, \tag{14}$$

where strain components with the superscripts e and c denote elastic and creep strain, respectively.

4. Creep tests results of copper

The creep curves up to rupture for copper are presented in Figs. 4, 5, 6 for uni-axial tension, combination of tension and torsion, and pure torsion, respectively. The horizontal axes in Figs. 4, 5, 6 correspond to the time, while the vertical ones correspond to the effective creep strain defined by the relation in the following form:

$$\varepsilon_e^{(c)} = \left(\frac{2}{3}\varepsilon_{ij}^{(c)}\varepsilon_{ij}^{(c)}\right)^{1/2} = \sqrt{\varepsilon_{11}^2 + \left(\frac{4}{3}\right)\varepsilon_{12}^2}, \tag{15}$$

where ε_{11} and ε_{12} denote axial and shear strain, respectively.

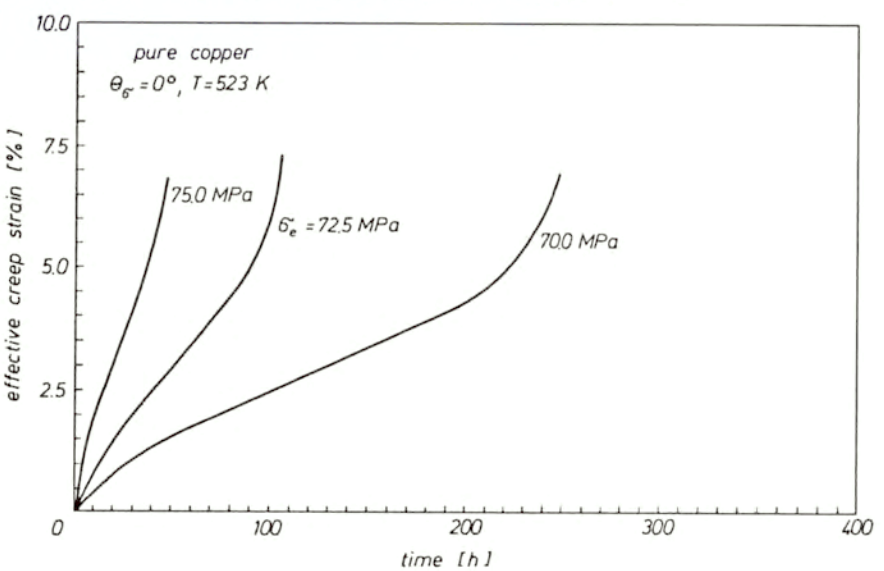

FIG. 4. Creep curves for copper tested under uni-axial tension.

The creep characteristics obtained at the same effective stress but under different stress states exhibit drastic differences for all the stress levels considered.

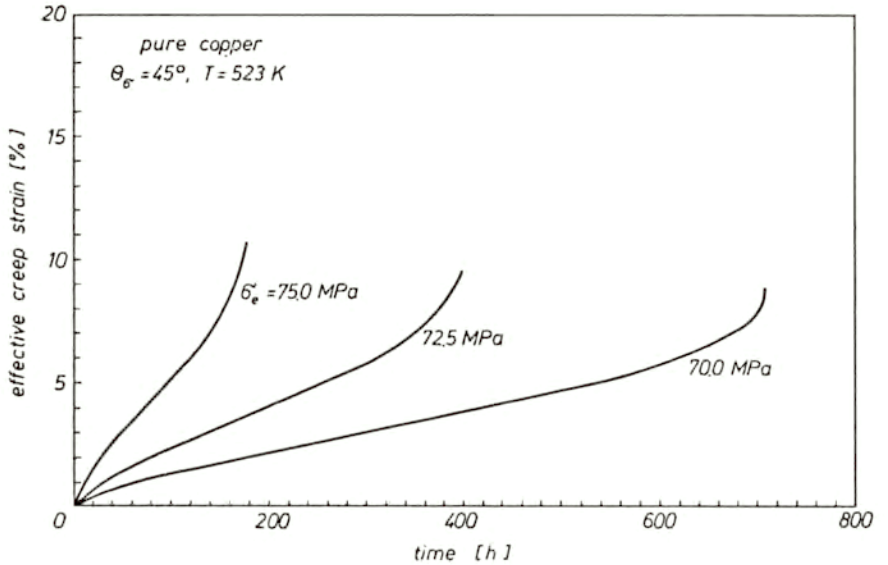

FIG. 5. Creep curves for copper tested under combination of uni-axial tension and pure torsion.

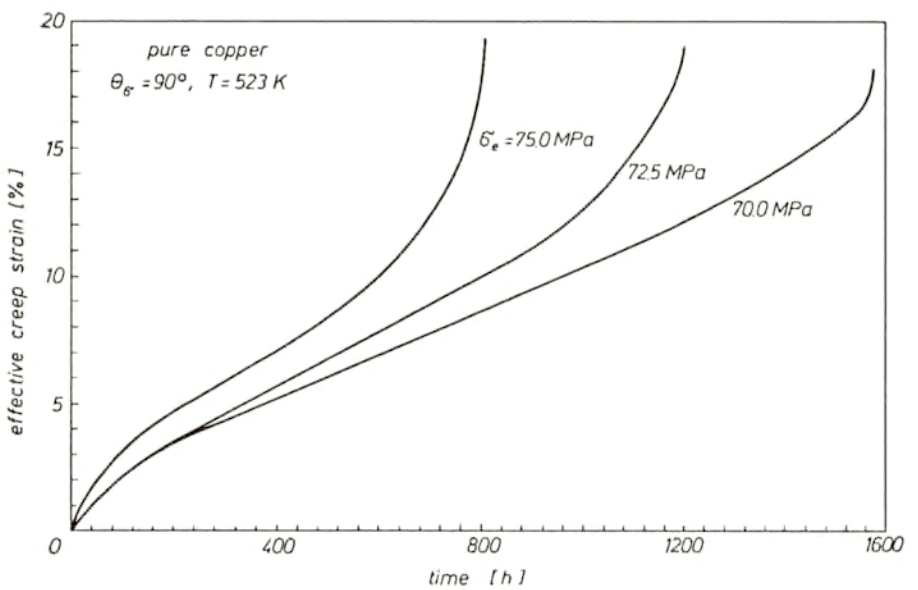

FIG. 6. Creep curves for copper tested under pure torsion.

In all cases the shortest lifetimes and, moreover, the lowest ductility have been achieved for tensioned testpieces. The opposite effect was observed for testpieces subjected to pure torsion. It has to be emphasized that differences in creep curves due to different loading types applied are considerable, and they are reflected by

variations of the basic creep parameters. All of them have been discussed in detail elsewhere [19]. In this paper they are only synthetically summarized in Table 1.

Table 1. Creep tests results for copper.

	σ_e = 70.0 [MPa]			σ_e = 72.5 [MPa]			σ_e = 75.0 [MPa]		
	$\Theta_\sigma=0°$	$\Theta_\sigma=45°$	$\Theta_\sigma=90°$	$\Theta_\sigma=0°$	$\Theta_\sigma=45°$	$\Theta_\sigma=90°$	$\Theta_\sigma=0°$	$\Theta_\sigma=45°$	$\Theta_\sigma=90°$
$\dot{\varepsilon}_e^{(c)}\times10^{-5}$ [1/h]	18.4	8.5	8.0	51.0	16.6	10.2	105.0	41.2	13.0
$\dot{\varepsilon}_{11}\times10^{-5}$ [1/h]	18.4	6.1	–	51.0	12.7	–	105.0	31.0	–
$2\dot{\varepsilon}_{12}\times10^{-5}$ [1/h]	–	10.2	13.9	–	18.5	17.7	–	47.0	22.5
t_R [h]	254	700	1560	101	391	1187	47	175	799
t_I [h]	50	90	260	25	65	200	15	45	160
t_{II} [h]	170	480	1000	70	255	760	35	125	450
$\varepsilon_e^{(c)}$ [%]	6.8	8.6	17.5	7.3	9.3	18.1	6.7	10.6	19.0

Notations in the Table 1: $\dot{\varepsilon}_e^{(c)}$ – minimum effective creep strain rate, $\dot{\varepsilon}_{11}$ – minimum axial creep strain rate, $\dot{\varepsilon}_{12}$ – minimum shear creep strain rate, t_R – time to rupture, t_I – duration of primary creep period, t_{II} – time to tertiary creep, $\varepsilon_e^{(c)}$ – effective creep strain at rupture.

5. Experimental data analysis

All the creep parameters, given in Table 1, which characterise macroscopically the creep behaviour prove that the process is stress-state-sensitive, in spite of the initial isotropy of the material in the sense of such parameters as Young's modulus, yield limit or ultimate tensile stress. Such material behaviour can be interpreted, on the one hand, as the material deformation due to different deformation mechanisms, activation of which is connected with the stress state type; on the other hand, however, it is known [2, 3, 14] that the majority of microcracks appearing at the grain boundaries are observed at those grain boundaries which are perpendicular to the maximum principal stress. It seems that for copper, the last conclusion can be confirmed by the shapes of the specimen cross-section in places where rupture has occurred. In case of creep tension tests the failure line was perpendicular to the main specimen axis, in case of complex stress states this line was inclined at angles equal typically to 15 – 20°, measured with respect to the line perpendicular to the main specimen axis, whereas for testpieces subjected to pure torsion these angles were approximately equal to 30 – 45°, Fig. 7. Thus, it can be concluded that for copper, the basic deformation mechanism seems to be the same for all stress state types considered, while the resulting variations in lifetimes for the same effective stress follow from the differences in the value of the maximum principal stress.

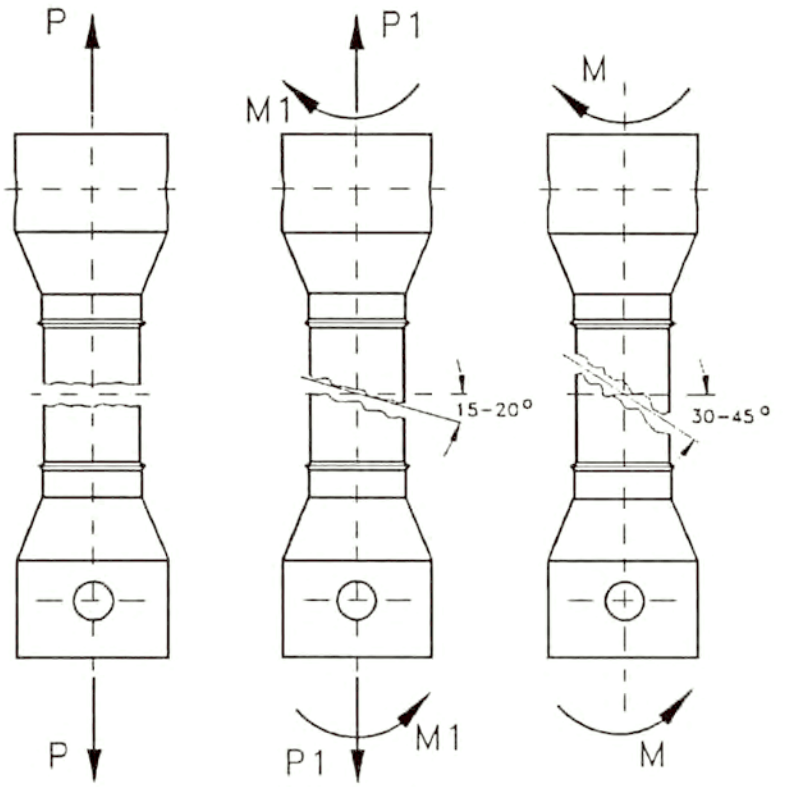

FIG. 7. Variations of the specimen failure line during creep at different types of stress states.

Clear presentation and comparison of the experimental data obtained from tests performed at complex stress states produce many difficulties, particularly for stress states being a combination of tension and torsion. In these cases, a comparison of results is usually carried out for the effective strains defined in the form of a function of the second invariant of the strain tensor, since the effects of the first as well as the third invariants are relatively small and they can be often neglected. Although creep curves in diagrams representing effective strains versus time can be compared, it is difficult to evaluate precisely all differences in material response due to the action of different stress state types. To overcome this deficiency, creep rupture results are commonly presented in the form of isochronous surfaces [1, 2, 11, 20–24], which in the stress space correspond to the same time to rupture. Experimental determination of the isochronous surface requires many creep tests at various stress states and at several stress levels. In the case of material anisotropy, there is practically no possibility to determine experimentally such a surface with sufficient accuracy. However, it can be done with a relatively good approximation on the basis of the relationship $\log(\sigma_e) = f[\log(t_R)]$, Fig. 8. In this figure results for the three types of stress states are presented. Data points

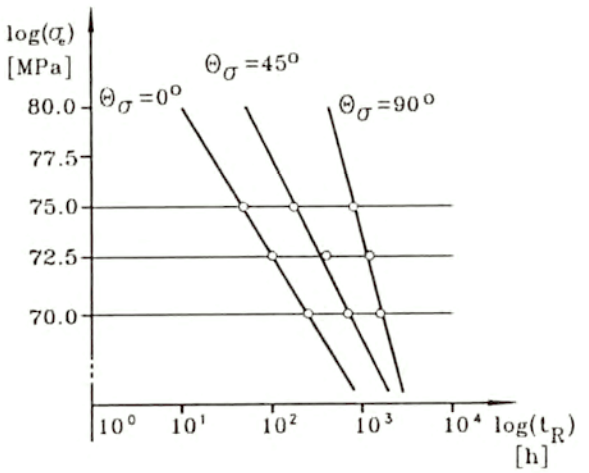

FIG. 8. Logarithmic diagram of the relation between effective stress and time to creep rupture for all stress states considered.

for the chosen type of stress state (with a relatively high accuracy) are located on the straight lines which have different mutual positions. These lines have been obtained by means of the least squares technique. Taking, as a reference point, the line representing pure torsion results, the remaining straight lines are shifted and rotated. Analysis of the mutual position of these lines indicates that the material subjected to creep process under uni-axial tension is significantly more sensitive to the stress variations, in comparison to the same material tested either at pure torsion or a combination of tension and torsion. On the basis of the diagrams shown in Fig. 8, drawing straight lines which are parallel to the stress axis and intersect the approximation lines representing the experimental data, it is easy to determine the points connecting the same times to rupture. The points for the rupture time taken into account provide the values of effective stresses necessary to achieve the rupture at the particular type of stress state considered. These values can be approximated on the stress plane $(\sigma_{11}, \sqrt{3}\sigma_{12})$ giving, as a consequence, the isochronous curves. Surfaces of the same lifetime determined in this way are shown in Fig. 9 for the rupture times equal to 200; 400, and 1000 [h]. It can be seen that the isochronous locus corresponding to the rupture time of 200 [h] is obtained for stress levels which are greater than those applied in experiments. Thus, a question arises in what range the relationship $\log(\sigma_e) = f[\log(t_R)]$ can be approximated by the straight lines for all stress types considered in the experimental programme. From the preliminary uniaxial tension creep tests made on copper it was known that in a logarithmic scale, the stress level was a linear function of lifetime in the limits considered, i.e. 60 – 90 [MPa]. Since the same behaviour has been also observed for this material by other researchers, e.g. ABO EL ATA and FINNIE [5], HAYHURST [3], in much wider stress limits and, moreover, taking into account the linearity of the relationship $\log(\sigma_e) = f[\log(t_R)]$ observed

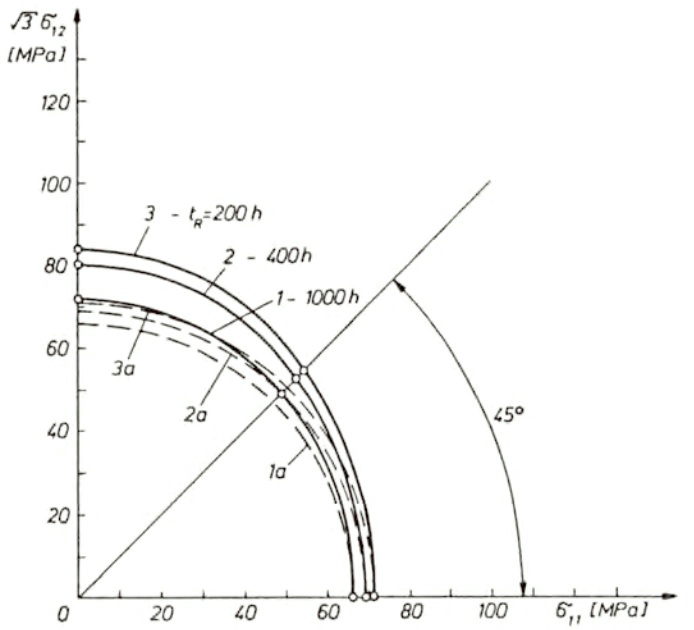

FIG. 9. Comparison of the isochronous creep rupture surfaces, t_R = 200; 400; 1000 [h], determined on the basis of experimental results (continuous lines) with the theoretical surfaces calculated for the Huber–Mises effective stress rupture criterion (dashed lines).

for the remaining loading cases ($\Theta_\sigma = 45°$, $\Theta_\sigma = 90°$) at stress levels within the range of 70 – 75 [MPa], it seems to be reasonable to extrapolate linearly this relationship for these loading combinations within the wider range representing ±20% of the average stress level applied in biaxial creep experiments.

The surface corresponding to the rupture time equal 400 [h] has been selected for further comparative studies on the isochronous surfaces predicted by the commonly used creep rupture criteria. Results of these considerations are presented in Sec. 6.

6. Verification of the fundamental creep rupture hypotheses

Applicability of the creep rupture criteria most frequently used is studied for the experimentally examined pure cooper. The curves of the same time to rupture, determined on the basis of the experimental programme realized, are compared with theoretical predictions of the following creep rupture hypotheses:

• the maximum principal stress rupture criterion, which is defined in the stress coordinate system $(\sigma_{11}, \sigma_{12})$ corresponding to the experimental programme by the following relationship

$$\sigma_R = \sigma_{\max} = \frac{1}{2}\left(\sigma_{11} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2}\right), \tag{16}$$

• the Huber-Mises effective stress rupture criterion which, for the combinations of stresses used in experiments, takes the form

$$\sigma_R = \sigma_e = \sqrt{\sigma_{11}^2 + 3\sigma_{12}^2}\,, \tag{17}$$

• the Sdobyrev creep rupture criterion (8), which will be used in further considerations in the form

$$\sigma_R = \beta\sigma_{\max} + (1-\beta)\sigma_e\,. \tag{18}$$

The isochronous surfaces resulting from these rupture criteria are compared with the surface determined on the basis of experimental results in Fig. 10. All curves presented in the normalized coordinate system are referred to the rupture

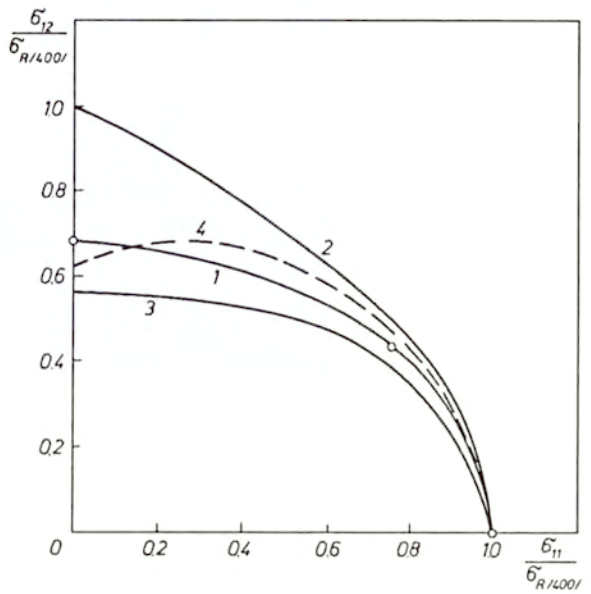

FIG. 10. Comparison of the isochronous creep rupture surfaces determined on the basis of experimental results (*1*) for t_R = 400 [h] with surfaces calculated according to: – the maximum principal stress rupture criterion (*2*), – the Huber–Mises effective stress rupture criterion (*3*), and – the Sdobyrev creep rupture criterion (*4*). Normalization of the coordinate system has been carried out with the use of the reference tension stress which gives the value of time to rupture 400 [h].

time equal to 400 [h]. Tensile stress corresponding to the lifetime of 400 [h] has been selected as the normalization value ($\sigma_{R/400/}$). As it is clearly seen, the best description of the experimental data has been achieved for the Sdobyrev creep rupture criterion taken with the coefficient β = 0.9, calculated on the basis of creep tests carried out. The value of β indicates that the damage mechanism governed by the maximum principal stress played a considerable role in the creep

rupture of the copper tested. Thus, this result can be treated as a confirmation of the earlier observed creep behaviour of copper having similar chemical composition [2, 3, 12, 13].

7. Method of determination of surfaces representing the same duration of primary creep and the same time to tertiary creep period

Typical creep process can be divided into three stages. In the first period the strain rate gradually decreases until minimum creep rate is achieved, the value of which is further maintained at the whole secondary creep stage. An increase of strain rate characterises the tertiary period of creep process, at the end of which rupture of the specimen tested occurs at time t_R. Such development of the phenomenon is connected with the increase of structure degradation velocity of a material which is manifested by the growth of voids and microcracks existing mainly at the grain boundaries. The material continuum is disturbed due to the microcracks propagation and due to development of other structural defects, what in consequence leads to the reduction of stiffness, and finally to rupture.

In the previous section it has been shown how to present clearly the experimental creep rupture results. Since the creep damage process already appears at the primary creep [25], it seems to be reasonable to adopt the isochronous surface concept at other characteristic points of the creep curve, such as the end of the primary or of secondary creep periods. Obviously, it is possible only if the conditions of the process and the material tested ensure the typical creep curve, i.e. a curve having all creep stages. In practice, the loading conditions and materials of structural components satisfy these conditions.

7.1. Isochronous curves of the end of primary creep period

Determination procedure of the curves representing the same duration of primary creep is analogous to that used to obtain surfaces of the same time to rupture. Thus, in the first step, the diagrams $\log(\sigma_e) = f[\log(t_1)]$ have been prepared for all stress states types considered in the plane $(\sigma_{11}, \sqrt{3}\sigma_{12})$. Similarly to the rupture times considerations, data points for the chosen type of stress state, with relatively high accuracy, are located on the straight lines which have different mutual positions, Fig. 11. These lines have been obtained by the least squares method. Taking as a reference point the line representing pure torsion results, the remaining straight lines are shifted and rotated. Analysis of the mutual location of these lines indicate that the material subjected to creep process under uni-axial tension is significantly more sensitive to the stress variations than the same material tested either at pure torsion or at a combination of tension and torsion. On the basis of diagrams shown in Fig. 11, it is easy to determine points connecting the same times to achieve stabilization of the creep strain rate. The points, for a chosen duration of primary creep period, provide values of effective

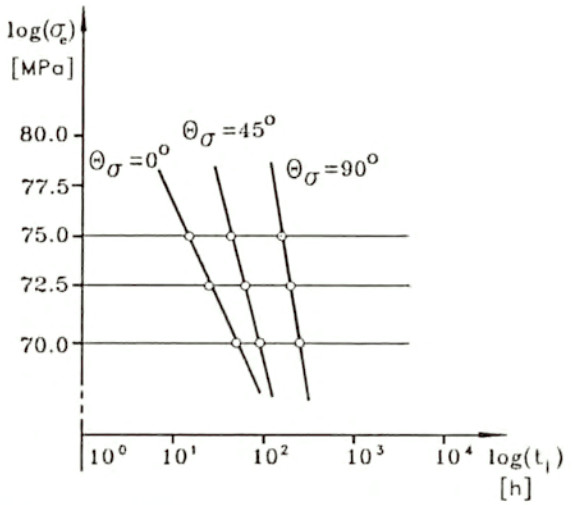

FIG. 11. Logarithmic diagram of the relation between effective stress and duration of primary creep for all stress states considered.

stresses necessary to achieve the end of primary creep at a particular type of the stress state considered. These values can be approximated on the stress plane $(\sigma_{11}, \sqrt{3}\sigma_{12})$ giving, as a consequence, the isochronous curves which describe the stress states corresponding to the end of primary creep. In Fig. 12 the surfaces of the same duration of primary creep period are shown for the time t_I equal to 50; 100, and 200 [h].

It can be seen that the shape and dimensions of the isochronous surfaces representing the same time to rupture (Fig. 9) and the same duration of primary creep period (Fig. 12) are similar. However, in order to compare them accurately it is necessary to provide clearly defined reference point. In order to assess mutual correlation of both types of isochronous surfaces, the tension stress equal 68.5 [MPa] has been selected as a reference point, which corresponds to the lifetime of 400 [h]. The cross-section of the isochronous surface for this rupture time is represented in Fig. 9 by the curve *2* and in Fig. 10 by the curve *1*. Taking into account the reference tension stress level and using diagrams $\log(\sigma_e) = f[\log(t_I)]$, shown in Fig. 11, it is easy to find the time to creep strain rate stabilization (t_I = 68 [h]). Knowing this value, all stress levels for the considered stress states which correspond to the same duration of primary creep can be determined, Fig. 11, giving the data necessary to obtain the isochronous surface. Comparison of the surfaces representing the same lifetime and the same duration of primary creep period is presented in Fig. 13. It is easy to see a great similarity of these surfaces. Thus, it can be stated for the tested copper that, on the basis of the degree of creep process development at the end of primary creep, represented by the surface of the same duration of primary creep, we can deduce the shape

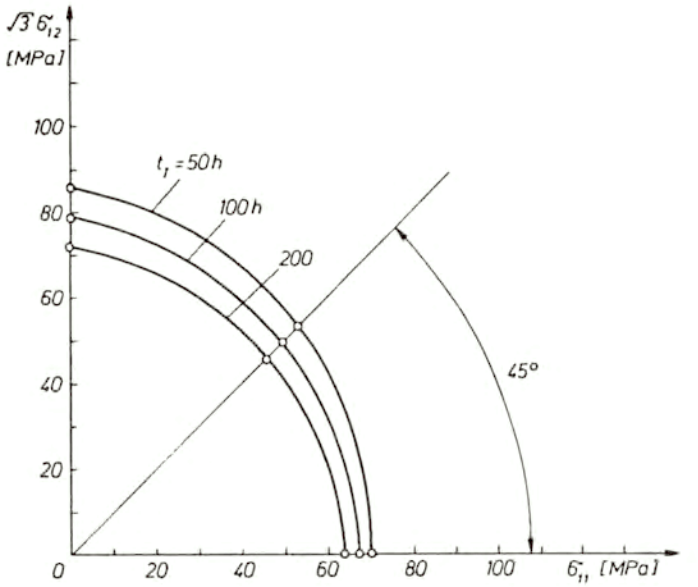

FIG. 12. Isochronous surfaces representing the same duration of primary creep period, t_1 = 50; 100; 200 [h], determined on the basis of experimental results.

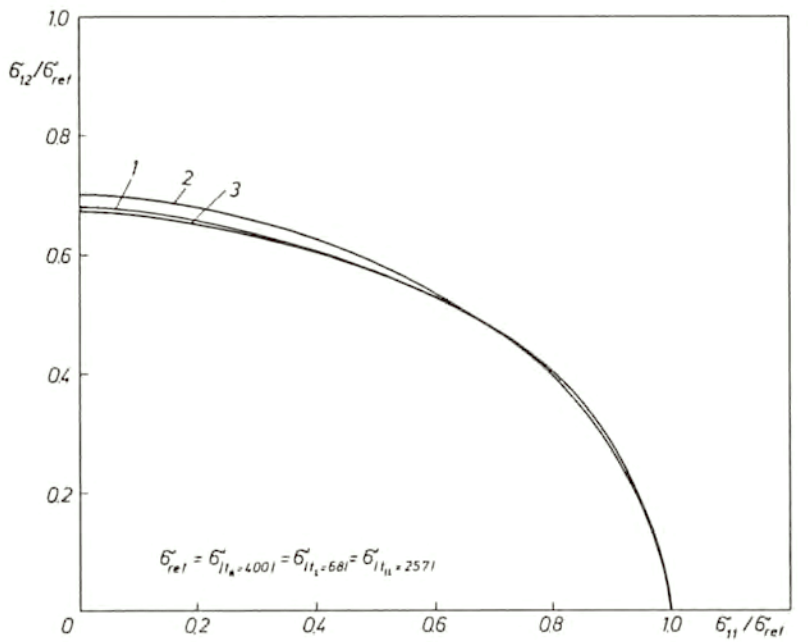

FIG. 13. Comparison of the experimentally determined surfaces representing the same time to rupture (*1*), the same duration of primary creep period (*2*), and the same time to tertiary creep period (*3*). The stress level at uni-axial tension which corresponds to: the time to rupture 400 [h], duration of primary creep 68 [h], and time to tertiary creep 257 [h] is used as a normalization factor.

of isochronous surface describing the same time to rupture. In other words, the degree of damage evolution in copper for each loading type in the experimental programme maintained a constant value.

In order to confirm this suggestion, the other characteristic point of the creep curve, i.e. time to tertiary creep, will be discussed in the next section.

7.2. Isochronous curves of the end of secondary creep period

The curves representing the same time to tertiary creep are determined by the same method as that used to obtain surfaces of the same time to rupture and the same duration of primary creep. First of all, the diagrams $\log(\sigma_e) = f[\log(t_{II})]$ have been prepared for all stress states types considered in the plane $(\sigma_{11}, \sqrt{3}\sigma_{12})$. Similarly to the rupture times considerations, data points for the chosen type of stress state are again located, with relatively high accuracy, on the straight lines which have different mutual positions, Fig. 14. As previously, these lines have been obtained by means of the least squares technique. On the basis of the diagrams

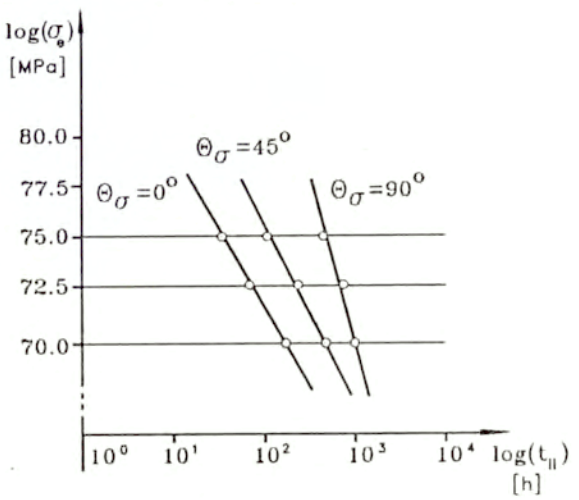

FIG. 14. Logarithmic diagram of the relation between effective stress and time to tertiary creep for all stress states considered.

shown in Fig. 14, the points connecting the points of the same times to tertiary creep can be determined. For the chosen time to tertiary creep period, these points provide the values of effective stresses necessary to achieve the end of secondary creep at a particular type of the stress state considered. These values can be approximated on the $(\sigma_{11}, \sqrt{3}\sigma_{12})$ stress plane, giving the isochronous curves which describe the stress states corresponding to the initiation of tertiary creep. In Fig. 15 the surfaces of the same time to tertiary creep period are shown for the time t_{II} equal to 100; 200, and 400 [h].

The shape and dimensions of the isochronous surfaces representing the same time to tertiary creep (Fig. 15), the same duration of primary creep period (Fig. 12),

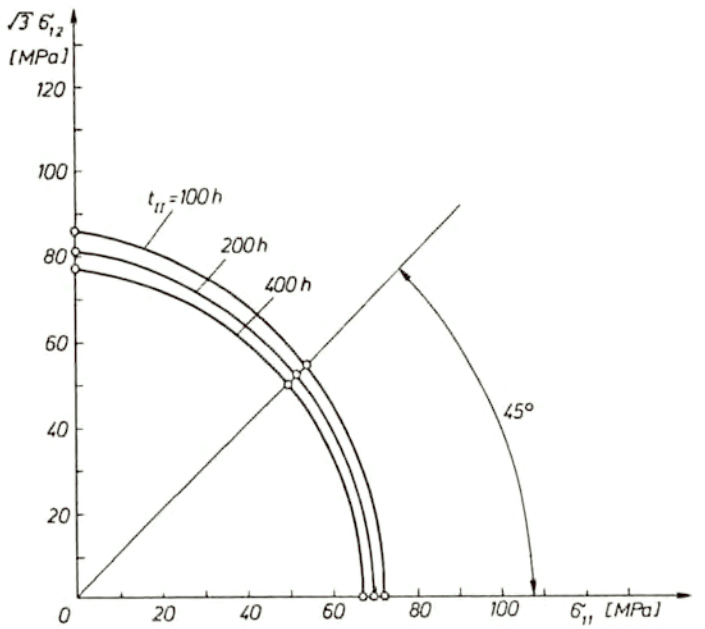

FIG. 15. Isochronous surfaces representing the same time to tertiary creep period, t_{II} = 100; 200; 400 [h], determined on the basis of experimental results.

and the same time to rupture are similar. However, in order to compare them, it is necessary to provide clearly defined reference point, selected in the same manner as in the previous section. Taking into account the tension stress level (68.5 [MPa]) as the reference point and using diagrams $\log(\sigma_e) = f[\log(t_{II})]$ shown in Fig. 14, it is easy to find the time to tertiary creep period (t_{II} = 257 [h]). Knowing this value, all stress levels corresponding to the same time to tertiary creep can be determined for the types of stress states considered, Fig. 14. It gives the data necessary to obtain the isochronous surface which reflects the same time to tertiary creep in the two-dimensional stress space $(\sigma_{11}, \sqrt{3}\sigma_{12})$. Comparison of the surfaces representing: (i) the same lifetimes; (ii) the same time to stabilization of the creep rate, and (iii) the same time to tertiary creep period, is presented in Fig. 13. It is easy to note that these surfaces are almost coincident. Such mutual location of the surfaces confirms the previously suggested thesis about the same degree of damage development in copper, measured as the ratio of the effective stress to the selected reference tension stress for the chosen direction in the $(\sigma_{11}, \sqrt{3}\sigma_{12})$ plane.

The knowledge of such material behaviour during creep makes it possible to simplify the procedure of determination of the isochronous creep rupture surfaces, since in such a case it is sufficient to carry out the creep tests with duration limited to the secondary creep period. Only one of them should be continued until rupture is achieved. On the basis of the isochronous surface

representing the same time to stabilization of a creep rate, and knowing the time to rupture for an arbitrary direction chosen in the $(\sigma_{11}, \sqrt{3}\sigma_{12})$ plane, it is easy to determine the isochronous creep rupture surface at the entire stress range considered.

8. Conclusions

The results of creep experiments made on pure copper under complex stress state at elevated temperature have been presented.

For the tested material such creep parameters analyzed as: duration of primary creep period, minimum creep rate, time to rupture and ductility were the functions of the type of stress state.

The procedure of determination of the isochronous creep rupture surfaces is proposed. In spite of the fact that this method can be applied to a limited range of stresses, i.e. stress levels close to those used in experiments, it gives a promising tool in multiaxial data analysis.

The procedure used to determine the isochronous creep rupture surfaces can be also applied to obtain both surfaces representing the same time to stabilized creep rate and surfaces of the same time to attain tertiary creep period. Analysis of all types of the isochronous surfaces for copper proved that the damage evolution develops proportionally until the creep rupture is achieved for different stress types taken into account in the $(\sigma_{11}, \sqrt{3}\sigma_{12})$ stress space.

The brittle failure was observed for copper under the considered testing conditions. Verification of the fundamental creep rupture criteria has shown that the Sdobyrev creep rupture criterion gives a promising tool to describe the damage process of the material subjected to the long-term constant loading conditions at elevated temperature. However, it was shown that the mechanisms depending on the maximum principal stress essentially influence creep rupture of the tested copper, what is in agreement with earlier observations.

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The application of the boundary element method to the study of 2D subsonic lifting flow past smooth profiles

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IN THIS PAPER, using the Imai–Lamla–Iacob method based on the asymptotic expansion of the complex velocity potential with respect to Chaplygin's number M_0, we study the subsonic steady 2D lifting flow around arbitrary smooth obstacles. Reducing the investigation of the compressible flow to a sequence of boundary value problems for the Laplace equation, we use the boundary element method in order to calculate the distribution of fluid speed and pressure coefficients on the obstacle. A comparison between the results obtained numerically and analytically for the circular obstacle shows a very good agreement.

Notations

$\mathbf{v} = (u, v)$	velocity,
V	local flow speed,
V_0	flow speed at infinity,
c_0	speed of the sound corresponding to null flow speed,
$M_0 = V_0/c_0$	Chaplygin's number,
γ	ratio of specific heats,
ϱ	density of fluid,
ϱ_0	density corresponding to null flow speed,
φ	potential of the velocity,
ψ	stream function,
Ψ	perturbation stream function,
$f = \varphi + i\psi$	complex potential,
f_0, f_1	terms of the asymptotic expansion of the complex potential,
Ψ_0	perturbation stream function for the incompressible approximation,
(x, y)	Cartesian coordinates,
$z = x + iy$	complex variable,
α	angle of attack,
$\mathbf{n} = (n_x, n_y)$	unit inward normal on the obstacle,
$\mathbf{s}$	unit tangent on the obstacle,
s	arc length on the obstacle,
Γ	obstacle,
ℓ	length of Γ,
Γ_j	panels on the obstacle,
ℓ_j	length of Γ_j,
(x_j, y_j)	control points (nodes) on Γ,
(ξ, η)	current variable on Γ,

$\mathbf{v}_0 = (u_0, v_0)$ velocity for the incompressible approximation,
k_0 circulation corresponding to the incompressible approximation,
C_p pressure coefficient,
C_L lift coefficient,
$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$,
$r_i = \sqrt{(x_i-\xi)^2 + (y_i-\eta)^2}$.

1. Introduction

THE PAPER is devoted to the numerical study of 2d potential subsonic steady flow of ideal fluids past smooth obstacles. We utilise herein the boundary element method which is more economical from the computational point of view than the domain-type methods like the finite-element method, finite-difference method etc.

The boundary element (or panel) method is widely and currently utilised for studying the potential incompressible flow past obstacles, because the equations governing the flow are linear and one can obtain representations of the solutions involving only integrals on the boundary of the domain. For the nonlinear equations governing the compressible flow such integral representations are not known.

In order to avoid this inconvenience, we shall use an approximate method conceived by I. IMAI [8], E. LAMLA [9] and improved by C. IACOB [7]. In the framework of this method, we consider the asymptotic expansion of the complex potential (and implicitly of the complex velocity) with respect to Chaplygin's number M_0.

For the first approximation corresponding to the incompressible flow, one has to solve, using the boundary element technique, the Neumann problem for Laplace's equation. For the second approximation, one utilises again the integral representation for the harmonic functions, but the boundary conditions depend on the results of the previous approximation and so on. In this way the nonlinear boundary value problem was replaced by a sequence of linear problems.

Until now the Imai - Lamla - Iacob approximate method was utilised especially for obtaining analytical results concerning the flow past obstacles [2, 5, 10, 11].

In order to establish if this method is satisfactory, some comparisons with other methods were performed. G. VOICULESCU–PLESI [11] compared the analytical results obtained for the circulation-free flow past the elliptical obstacle with the results obtained by I. FILIMON [4], who used Chaplygin's approximate hodograph method. A. DUCARU–DRAGA [2] studied the subsonic flow with circulation around the circular obstacle, both by means of the Imai - Lamla - Iacob method and the finite-element method. In all cases it is observed that the results obtained by means of Imai - Lamla - Iacob method and the results obtained by means of other methods are very close to each other.

In the present paper, using the first and second approximations, we investigate the subsonic flow with circulation past arbitrary smooth obstacles. Numerical results (tangential velocity and pressure coefficients in the control points)are obtained for the circular obstacle.

In the framework of the second asymptotic approximation, we know the analytical expression of the tangential velocity for the subsonic compressible flow with circulation past the circular obstacle [1]. The comparisons between the values of the pressure coefficients calculated analytically and by means of the boundary element method show a very good agreement, as we can observe from Fig. 1.

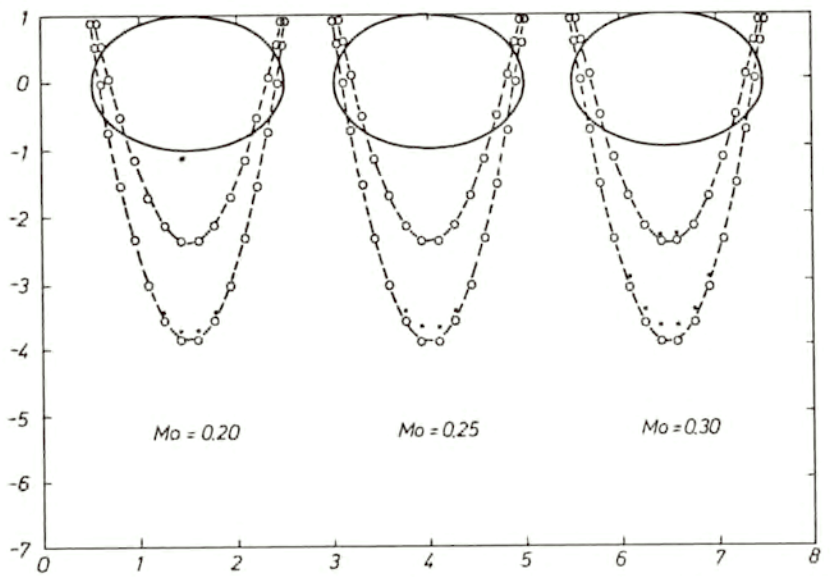

FIG. 1. Calculated (o) and analytical (– –) chordwise coefficient. • • • pressure coefficient for incompressible flow.

2. Imai–Lamla–Iacob approximate method

Following C. IACOB [7] we shall present the method conceived by I. IMAI [8] and E. LAMLA [8] for investigating the subsonic circulation-free flow and adjusted by C. Iacob for the study of the flow with circulation past obstacles.

From the equation of continuity

$$\operatorname{div}(\varrho \mathbf{v}) = 0 \tag{2.1}$$

it follows that there exists a function $\psi(x, y)$ (the stream function) so that

$$u = \frac{\varrho_0}{\varrho}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{\varrho_0}{\varrho}\frac{\partial \psi}{\partial x}. \tag{2.2}$$

We consider the irrotational flow, i.e. there exists a function $\varphi(x, y)$ (the velocity potential) such that

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}. \tag{2.3}$$

Introducing the operators,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.4}$$

and the complex potential

$$f = \varphi + i\psi \tag{2.5}$$

we get from (2.2)–(2.5)

$$\frac{\partial f}{\partial \overline{z}} = \frac{\varrho_0 - \varrho}{\varrho_0 + \varrho}\frac{\partial \overline{f}}{\partial \overline{z}}. \tag{2.6}$$

Taking into account that for the isentropic flow,

$$\varrho = \varrho_0\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{V_0^2}M_0^2\right)^{1/(\gamma-1)},$$

it results

$$\frac{\varrho_0 - \varrho}{\varrho_0 + \varrho} = \frac{M_0^2}{4}\frac{V^2}{V_0^2} + M_0^4(\ldots). \tag{2.7}$$

Expressing the local speed of the fluid as follows

$$V^2 = \left(\frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}} + \frac{\partial \overline{f}}{\partial \overline{z}}\right), \tag{2.8}$$

we deduce from (2.6)–(2.8)

$$\frac{\partial f}{\partial \overline{z}} = \left[\frac{M_0^2}{4V_0^2}\left(\frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}} + \frac{\partial \overline{f}}{\partial \overline{z}}\right) + M_0^4(\ldots)\right]\frac{\partial \overline{f}}{\partial \overline{z}}. \tag{2.9}$$

Considering the asymptotic expansion of f with respect to M_0^2

$$f(z, \overline{z}) = f_0(z, \overline{z}) + M_0^2 f_1(z, \overline{z}) + M_0^4 f_2(z, \overline{z}) + \ldots \tag{2.10}$$

we obtain from (2.9), (2.10), equating the coefficients of the same powers of M_0

$$\frac{\partial f_0}{\partial \bar{z}} = 0, \tag{2.11}$$

$$\frac{\partial f_1}{\partial \bar{z}} = \frac{1}{4V_0^2}\left(\frac{\partial f_0}{\partial z} + \frac{\partial \bar{f}_0}{\partial z}\right)\left(\frac{\partial f_0}{\partial \bar{z}} + \frac{\partial \bar{f}_0}{\partial \bar{z}}\right)\frac{\partial \bar{f}_0}{\partial \bar{z}}, \tag{2.12}$$

and so on.

From (2.4), (2.5) and (2.11) it follows that f_0 is an analytical function depending only on z. We shall consider $f_0(z)$ as the complex potential of the incompressible flow past the given obstacle.

From (2.11) and (2.12) it results

$$\frac{\partial f_1}{\partial \bar{z}} = \frac{1}{4V_0^2}\frac{df_0}{dz}\left(\frac{d\bar{f}_0}{d\bar{z}}\right)^2 \tag{2.13}$$

whence, integrating with respect to $\bar{z}$, we get

$$f_1(z,\bar{z}) = \varphi_1 + i\psi_1 = \frac{1}{4V_0^2}\frac{df_0}{dz}\overline{\int_{z_1}^{z}\left(\frac{df_0}{dz}\right)^2 dz} + \frac{1}{4}g(z), \tag{2.14}$$

where $g(z)$ is an analytic function. Similarly, we can calculate $f_2(z,\bar{z})$ and so on, but in this paper we shall use only the second approximation (i.e. we deal only with f_0 and f_1). From the relation

$$u - iv = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial y} = \frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z} \tag{2.15}$$

and from (2.9)–(2.14), we deduce

$$u - iv = \frac{df_0}{dz} + \frac{M_0^2}{4}\left[\frac{1}{V_0^2}\frac{d^2 f_0}{dz^2}\overline{\int_{z_1}^{z}\left(\frac{df_0}{dz}\right)^2 dz} + \frac{1}{V_0^2}\left(\frac{df_0}{dz}\right)^2\frac{df_0}{d\bar{z}} + \frac{dg}{dz}\right] + M_0^4(\ldots). \tag{2.16}$$

In the sequel we shall neglect the terms of order M_0^4.

The conditions that we impose are:

- at infinity,

$$\lim_{\infty}(u - iv) = V_0 e^{-i\alpha}, \qquad \lim_{\infty}(u_0 - iv_0) = V_0 e^{-i\alpha}, \tag{2.17}$$

• on the solid boundaries,

$$\psi|_\Gamma = q_0 + \frac{M_0^2}{4} q_1 , \qquad \psi_0|_\Gamma = q_0 , \tag{2.18}$$

(i.e. the solid boundaries are streamlines) and

$$u n_x + v n_y|_\Gamma = 0, \qquad u_0 n_x + v_0 n_y|_\Gamma = 0 \tag{2.19}$$

(the slip condition).

3. A boundary element approach to investigation of the incompressible lifting flow past smooth obstacles

Considering the flow uniform at infinity and incompressible, the behaviour of the stream function is

$$\psi_0 = V_0(y \cos\alpha - x \sin\alpha) + \frac{k_0}{2\pi i} \ln \sqrt{x^2 + y^2} + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \tag{3.1}$$

since $\sqrt{x^2 + y^2} \Rightarrow \infty$.

The perturbation stream function for the incompressible flow

$$\Psi_0(x, y) = \psi_0(x, y) - V_0(y \cos\alpha - x \sin\alpha) \tag{3.2}$$

behaves at infinity as follows:

$$\Psi_0 = \frac{k_0}{2\pi i} \ln \sqrt{x^2 + y^2} + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \qquad \sqrt{x^2 + y^2} \Rightarrow \infty . \tag{3.3}$$

From (3.3) it results, via Plemelj's formula, that the harmonic function $\Psi_0(x, y)$ has for $(x, y) \in \Gamma$ the integral representation

$$\frac{1}{2}\Psi_0(x, y) = \frac{1}{2\pi} \int_\Gamma \left(\frac{\partial \Psi_0}{\partial n}(\xi, \eta) \ln \frac{1}{r} - \Psi_0(\xi, \eta) \frac{\partial}{\partial n} \ln \frac{1}{r}\right) ds, \tag{3.4}$$

where (ξ, η) is the current variable on Γ, ds is the element of arc length, $\partial/\partial n$ is the inward normal derivative and $r^2 = (x - \xi)^2 + (y - \eta)^2$.

In the boundary element approach, the airfoil is approximated by a piecewise linear curve consisting of N panels Γ_j, $j = 1, N$; the extremes (nodes or control points) are found on the actual airfoil. For the i-th node Eq. (3.4) becomes

$$\frac{1}{2}\Psi_0(x_i, y_i) = \frac{1}{2\pi} \int_\Gamma \left(\frac{\partial \Psi_0}{\partial n}(\xi, \eta) \ln \frac{1}{r_i} - \Psi_0(\xi, \eta) \frac{\partial}{\partial n} \ln \frac{1}{r_i}\right) ds \tag{3.5}$$

with $r_i = \sqrt{(x_i - \xi)^2 + (y_i - \eta)^2}$.

Denoting by (x_j, y_j) and (x_{j+1}, y_{j+1}) the extremes of the panel Γ_j, the current point $(\xi, \eta) \in \Gamma_j$ may be represented as follows

$$(3.6) \qquad (\xi, \eta) = \frac{1-\sigma}{2}(x_j, y_j) + \frac{1+\sigma}{2}(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1].$$

For Ψ_0 and $\partial\Psi_0/\partial n$ on Γ_j we consider the linear interpolation

$$(3.7) \qquad \Psi_0(\xi, \eta) = \frac{1-\sigma}{2}\Psi_0(x_j, y_j) + \frac{1+\sigma}{2}\Psi_0(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1],$$

$$(3.8) \qquad \frac{\partial \Psi_0}{\partial n}(\xi, \eta) = \frac{1-\sigma}{2}\frac{\partial \Psi_0}{\partial n}(x_j, y_j) + \frac{1+\sigma}{2}\frac{\partial \Psi_0}{\partial n}(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1].$$

For smooth obstacles, the approximation of Ψ_0 and $\partial\Psi_0/\partial n$ by piecewise constant functions gives also good results; we prefer, however, the linear interpolation because it is more general and it allows for the implementation of the Kutta–Joukovsky condition in the case of airfoils with sharp trailing edge [6].

The linear element on Γ_j is

$$(3.9) \qquad ds = \sqrt{dx^2 + dy^2} = \frac{\ell_j}{2} d\alpha,$$

where

$$(3.10) \qquad \ell_j = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}$$

is the length of Γ_j.

On Γ_j we have also,

$$(3.11) \qquad \frac{\partial}{\partial n} \ln \frac{1}{r_i} = \frac{-\left(\frac{1+\sigma}{2}x_{j+1} + \frac{1-\sigma}{2}x_j - x_i\right) n_x^j}{r_{ji}^2} + \frac{-\left(\frac{1+\sigma}{2}y_{j+1} + \frac{1-\sigma}{2}y_j - y_i\right) n_y^j}{r_{ji}^2}$$

with

$$(3.12) \qquad n_x^j = \frac{y_j - y_{j+1}}{\ell_j},$$

$$(3.13) \qquad n_y^j = \frac{x_{j+1} - x_j}{\ell_j},$$

$$(3.14) \qquad r_{ji}^2 = \left(\frac{1+\sigma}{2}x_{j+1} + \frac{1-\sigma}{2}x_j - x_i\right)^2 + \left(\frac{1+\sigma}{2}y_{j+1} + \frac{1-\sigma}{2}y_j - y_i\right)^2.$$

The contour Γ_j being closed, the subscript $N+1$ is identified with 1 and the subscript 0 is identified with N.

From (3.6)–(3.14) we deduce that equation (3.5) may be written as follows:

$$\sum_{j=1}^{N} \frac{\partial \Psi_0}{\partial n}(x_j, y_j) G_{ji} - \sum_{j=1}^{N} \Psi_0(x_j, y_j) H_{ji} = 0, \qquad i = 1, N, \tag{3.15}$$

where

$$G_{ji} = g_{ji}^{(1)} + g_{ji}^{(2)}, \qquad i, j = 1, N, \tag{3.16}$$

$$g_{ji}^{(1)} = \frac{-\ell_j}{8\pi} \int_{-1}^{1} (1 - \sigma) \ln r_{ji}\, d\sigma, \tag{3.17}$$

$$g_{ji}^{(2)} = \frac{-\ell_{j-1}}{8\pi} \int_{-1}^{1} (1 + \sigma) \ln r_{(j-1)i}\, d\sigma, \tag{3.18}$$

$$H_{ji} = h_{ji}^{(1)} + h_{ji}^{(2)} + \frac{1}{2}\delta_{ji}, \tag{3.19}$$

$$h_{ji}^{(1)} = \frac{-\ell_j}{8\pi} \int_{-1}^{1} (1 - \sigma) \left[\frac{\left(\frac{1+\sigma}{2} x_{j+1} + \frac{1-\sigma}{2} x_j - x_i \right) n_x^j}{r_{ji}^2} + \frac{\left(\frac{1+\sigma}{2} y_{j+1} + \frac{1-\sigma}{2} y_j - y_i \right) n_y^j}{r_{ji}^2} \right] d\alpha, \tag{3.20}$$

$$h_{ji}^{(2)} = \frac{-l_{j-1}}{8\pi} \int_{-1}^{1} (1 - \sigma) \left[\frac{\left(\frac{1+\sigma}{2} x_{j+1} + \frac{1-\sigma}{2} x_j - x_i \right) n_x^j}{r_{(j-1)i}^2} + \frac{\left(\frac{1+\sigma}{2} y_{j+1} + \frac{1-\sigma}{2} y_j - y_i \right) n_y^j}{r_{(j-1)i}^2} \right] d\alpha. \tag{3.21}$$

The integrals (3.20), (3.21) may be computed analytically using the relations

$$I_1 = \int_{-1}^{1} \frac{d\sigma}{a\sigma^2 + b\sigma + c} = \frac{2}{\sqrt{-\Delta}} \operatorname{atan} \frac{\sqrt{-\Delta}}{c - a}, \qquad \Delta = b^2 - 4ac, \tag{3.22}$$

$$I_2 = \int_{-1}^{1} \frac{\sigma\, d\sigma}{a\sigma^2 + b\sigma + c} = \frac{-b}{a\sqrt{-\Delta}} \operatorname{atan} \frac{\sqrt{-\Delta}}{c - a} + \frac{1}{2a} \ln \frac{a + b + c}{a - b + c}. \tag{3.23}$$

For calculating (3.17), (3.18) we use the formulas

$$I_3 = \int_{-1}^{1} \ln(a\sigma^2 + b\sigma + c)\, d\sigma = \ln[(a+b+c)(a-b+c)] - 4 + bI_2 + 2cI_1 , \tag{3.24}$$

$$I_4 = \int_{-1}^{1} \sigma \ln(a\sigma^2 + b\sigma + c)\, d\sigma \tag{3.25}$$

$$= \frac{1}{2a}[(a+b+c)\ln(a+b+c) - (a-b+c)\ln(a-b+c) - 2b] - \frac{b}{2a} I_3 .$$

For $i = j-1,\ j,\ j+1$ we get for the singular integrals occurring in (3.17), (3.18)

$$g_{jj}^{(1)} = \frac{\ell_j}{8\pi}(3 - 2\ln \ell_j), \tag{3.26}$$

$$g_{j(j+1)}^{(1)} = \frac{\ell_j}{8\pi}(1 - 2\ln \ell_j), \tag{3.27}$$

$$g_{jj}^{(2)} = \frac{\ell_{j-1}}{8\pi}(3 - 2\ln \ell_{j-1}), \tag{3.28}$$

$$g_{j(j-1)}^{(2)} = \frac{\ell_{j-1}}{8\pi}(1 - 2\ln \ell_{j-1}). \tag{3.29}$$

We notice also that the singular integrals occurring in (3.20), (3.21) vanish. From (3.2) and the streamline condition (2.18) it follows that

$$\Psi_0|_\Gamma = q_0 - V_0(y\cos\alpha - x\sin\alpha). \tag{3.30}$$

From (3.15), (3.30) and from the relations

$$\sum_{j=1}^{N} H_{ji} = 1, \qquad i = 1, N \tag{3.31}$$

we obtain the algebraic system of equations

$$-q_0 + \sum_{j=1}^{N} \frac{\partial \Psi_0}{\partial n}(x_j, y_j) G_{ji} = -V_0 \sum_{j=1}^{N} (y_j \cos\alpha - x_j \sin\alpha) H_{ji} , \qquad i = 1, N. \tag{3.32}$$

The system (3.32) consists of N linear equations for $N+1$ unknows q_0 and $\frac{\partial \Psi_0}{\partial n}(x_j, y_j)$, $j = 1, N$.

We may establish the $N+1$-th equation imposing a prescribed value to the circulation

$$\int_\Gamma \frac{\partial \Psi_0}{\partial n} ds = k_0 . \tag{3.33}$$

Equation (3.33) may be transformed by discretization into

$$\sum_{j=1}^{N} \frac{\partial \Psi_0}{\partial n}(x_j, y_j)(\ell_j + \ell_{j+1}) = 2k_0 . \tag{3.34}$$

It is more convenient to reduce the number of unknowns imposing a prescribed value to the velocity in a certain point on the airfoil; it is usual to impose the zero value to the velocity in the vicinity of the trailing edge, i.e.

$$\frac{\partial \Psi_0}{\partial n}(x_1, y_1) = -V_0(n_y^1 \cos\alpha - n_x^1 \sin\alpha). \tag{3.35}$$

In the last case the circulation is calculated *a posteriori* using (3.34).

The tangential velocity in the nodes (x_j, y_j), $j = 1, N$ may be computed by means of the relation

$$\mathbf{v}_0 \cdot \mathbf{s} = \frac{\partial \psi_0}{\partial n}(x_j, y_j) = \frac{\partial \Psi_0}{\partial n}(x_j, y_j) + V_0(n_y^j \cos\alpha - n_x^j \sin\alpha), \qquad j = 1, N. \tag{3.36}$$

After computing $\dfrac{\partial \Psi_0}{\partial n}(x_j, y_j)$, the components of the velocity are obtained using the relations

$$u_0(x_j, y_j) = \frac{\partial \psi_0}{\partial n}(x_j, y_j) n_y^j , \tag{3.37}$$

$$v_0(x_j, y_j) = -\frac{\partial \psi_0}{\partial n}(x_j, y_j) n_x^j . \tag{3.38}$$

4. The study of the second approximation of the compressible lifting flow past a smooth obstacle by the boundary element method

Taking into account that the harmonic functions $u_0(x, y)$ and $v_0(x, y)$ behave at infinity as follows:

$$u_0(x, y) = V_0 \cos\alpha + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \qquad x^2 + y^2 \Rightarrow \infty, \tag{4.1}$$

$$v_0(x, y) = V_0 \sin\alpha + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \qquad x^2 + y^2 \Rightarrow \infty, \tag{4.2}$$

we obtain the integral representations (for $(x, y) \in \Gamma$):

$$\frac{1}{2}(u_0(x, y) - V_0 \cos\alpha) = \frac{1}{2\pi} \int_\Gamma \left(\frac{\partial u_0}{\partial n}(\xi, \eta) \ln \frac{1}{r} - (u_0(\xi, \eta) - V_0 \cos\alpha) \frac{\partial}{\partial n} \ln \frac{1}{r} \right) ds, \tag{4.3}$$

$$\int_\Gamma \frac{\partial u_0}{\partial n}(\xi, \eta)\, ds = 0;$$

$$\frac{1}{2}(v_0(x,y) - V_0 \sin\alpha)$$

$$= \frac{1}{2\pi}\int_\Gamma \left(\frac{\partial v_0}{\partial n}(\xi,\eta)\ln\frac{1}{r} - (v_0(\xi,\eta) - V_0\sin\alpha)\frac{\partial}{\partial n}\ln\frac{1}{r}\right) ds, \tag{4.4}$$

$$\int_\Gamma \frac{\partial v_0}{\partial n}(\xi,\eta)ds = 0.$$

From (4.3) we obtain by discretization the algebraic system

$$a + \sum_{j=1}^{N}\frac{\partial u_0}{\partial n}(x_j,y_j)G_{ji} = \sum_{j=1}^{N}[u_0(x_j,y_j) - V_0\cos\alpha]\,H_{ji}, \qquad i = 1,N,$$

$$\sum_{j=1}^{N}\frac{\partial u_0}{\partial n}(x_j,y_j)(\ell_j + \ell_{j+1}) = 0. \tag{4.5}$$

The unknowns are $\frac{\partial u_0}{\partial n}(x_j,y_j)$ and the control variable a, which must be zero.

Similarly, from (4.4) we obtain the system

$$b + \sum_{j=1}^{N}\frac{\partial v_0}{\partial n}(x_j,y_j)G_{ji} = \sum_{j=1}^{N}[v_0(x_j,y_j) - V_0\sin\alpha]\,H_{ji}, \qquad i = 1,N,$$

$$\sum_{j=1}^{N}\frac{\partial v_0}{\partial n}(x_j,y_j)(\ell_j + \ell_{j+1}) = 0, \tag{4.6}$$

whose unknowns are $\frac{\partial v_0}{\partial n}(x_j,y_j)$, $j = 1,N$ and the control variable b which must be zero.

After obtaining $\frac{\partial v_0}{\partial n}(x_j,y_j)$ and $\frac{\partial u_0}{\partial n}(x_j,y_j)$ we may calculate

$$\frac{\partial u_0}{\partial x}(x_j,y_j) = \frac{\partial u_0}{\partial n}(x_j,y_j)n_y^j + \frac{\partial v_0}{\partial n}(x_j,y_j)n_x^j, \tag{4.7}$$

$$\frac{\partial v_0}{\partial x}(x_j,y_j) = \frac{\partial u_0}{\partial n}(x_j,y_j)n_x^j - \frac{\partial v_0}{\partial n}(x_j,y_j)n_y^j, \tag{4.8}$$

whence we may obtain

$$\frac{d^2 f_0}{dz^2}(z_j) = \frac{\partial u_0}{\partial x}(x_j,y_j) - i\frac{\partial v_0}{\partial x}(x_j,y_j), \qquad z_j = x_j + iy_j, \quad j = 1,N. \tag{4.9}$$

To calculate the velocity distribution over the obstacle by means of Eq. (3.16), we have to find out the analytic function $g(z)$. First of all we notice that, according

to the behaviour of $f_0(z)$ at infinity,

$$f_0(z) = V_0 e^{-i\alpha} z + \frac{k_0}{2\pi i} \ln z + \frac{a_1}{z} + \dots , \tag{4.10}$$

it follows that $\int_{z_1}^{z} \left(\frac{df_0}{dz}\right)^2 dz$ is a multi-valued function and

$$\int_{\Gamma} \left(\frac{df_0}{dz}\right)^2 dz = 2k_0 V_0 e^{-i\alpha} \tag{4.11}$$

(we consider the integration along Γ in the positive sense).

Since $u - iv$ and df_0/dz are single-valued functions, it follows from (2.16) that dg/dz is a multi-valued function. We shall choose for $g(z)$ the expression

$$g(z) = -\frac{e^{i\alpha} k_0}{V_0 \pi i} \ln \frac{z}{z_1} \frac{df_0}{dz} + h(z). \tag{4.12}$$

The function

$$P(z, \overline{z}) = \overline{\int_{z_1}^{z} \left(\frac{df_0}{dz}\right)^2 dz} - \frac{V_0 e^{i\alpha} k_0}{\pi i} \ln \frac{z}{z_1} \tag{4.13}$$

is single-valued.

Using the function $P(z, \overline{z})$ and the relation (3.12), we deduce from (2.14) and (2.16) that

$$f_1(z, \overline{z}) = \frac{1}{4} \left(\frac{1}{V_0^2} \frac{df_0}{dz} P(z, \overline{z}) + h(z) \right), \tag{4.14}$$

$$u - iv = \frac{df_0}{dz} + \frac{M_0^2}{4} \left[\frac{1}{V_0^2} \frac{d^2 f_0}{dz^2} P(z, \overline{z}) + \frac{1}{V_0^2} \left(\frac{df_0}{dz}\right)^2 \frac{d\overline{f}_0}{d\overline{z}} - \frac{e^{i\alpha} k_0}{V_0 \pi i} \frac{1}{z} \frac{df_0}{dz} + \frac{dh}{dz} \right]. \tag{4.15}$$

From the relations (2.17), (4.1), (4.2) concerning the behaviour at infinity of $u - iv$ and df_0/dz, we deduce that

$$\lim_{z \to \infty} \frac{dh}{dz} = -V_0 e^{-i\alpha}, \tag{4.16}$$

whence we obtain (for the analytic function $h(z)$) the expansion

$$h(z) = -V_0 e^{-i\alpha} z + \frac{k_1}{2\pi i} \ln z + O\left(\frac{1}{z}\right). \tag{4.17}$$

Introducing the function

$$\tilde{h}(z) = J(x, y) + iI(x, y) = h(z) + V_0 e^{-i\alpha} z \tag{4.18}$$

we may represent the harmonic function $I(x, y)$ on Γ as follows

$$\frac{1}{2} I(x, y) = \int_{\Gamma} \left(\frac{\partial I}{\partial n}(\xi, \eta) \ln \frac{1}{r} - I(\xi, \eta) \frac{\partial}{\partial n} \ln \frac{1}{r} \right) ds. \tag{4.19}$$

From (4.14), (4.18) and the streamline condition (2.18) we deduce

$$I(x, y) = q_1 + \beta(x, y), \tag{4.20}$$

with

$$\beta(x, y) = -\operatorname{Im} \left[\frac{1}{V_0^2} \frac{df_0}{dz} P(z, \bar{z}) \right] + V_0(y \cos \alpha - x \sin \alpha). \tag{4.21}$$

Discretizing (4.19) we obtain the algebraic system

$$-q_1 + \sum_{j=1}^{N} \frac{\partial I}{\partial n}(x_j, y_j) G_{ji} = \sum_{j=1}^{N} \beta(x_j, y_j) H_{ji}, \qquad i = 1, N. \tag{4.22}$$

$\beta(x_j, y_j)$ is computed numerically using the relation

$$\beta(x_j, y_j) = V_0(y_j \cos \alpha - x_j \sin \alpha) - \frac{1}{V_0^2} \operatorname{Im} \left[(u_0(x_j, y_j) - i v_0(x_j, y_j)) P(z_j, \bar{z}_j) \right]; \tag{4.23}$$

$$\begin{aligned} P(z_j, \bar{z}_j) = \frac{1}{3} \sum_{l=1}^{j-1} & [(x_{l-1} - x_l + i(y_{l-1} - y_l)] \\ & \cdot \left[(u_0(x_{l+1}, y_{l+1}) - i v_0(x_{l+1}, y_{l+1}))^2 + (u_0(x_l, y_l) - i v_0(x_l, y_l))^2 \right. \\ & \left. + (u_0(x_{l+1}, y_{l+1}) - i v_0(x_{l+1}, y_{l+1}))(u_0(x_l, y_l) - i v_0(x_l, y_l)) \right] \\ & - \frac{V_0 e^{i\alpha} k_0}{\pi i} \ln \frac{z_j}{z_1}, \qquad j = 2, N, \end{aligned} \tag{4.24}$$

$$P(z_1, \bar{z}_1) = 0.$$

Relation $(4.24)_1$ was deduced from (4.13) and from the linear interpolation of $u_0(x, y)$ and $v_0(x, y)$ on Γ.

The system (4.22) consists of N equations for $N + 1$ unknowns q_1 and $\frac{\partial I}{\partial n}(x_j, y_j)$, $j = 1, N$. The most usual way to reduce the number of unknowns is to consider a prescribed value of the velocity in a certain point. It is natural to impose the zero value for the velocity at the same point (x_1, y_1) where $u_0 - iv_0$ vanishes. We have therefore $\frac{dh}{dz}(z_1) = 0$, whence by virtue of (4.18)

$$\frac{\partial I}{\partial n}(x_1, y_1) = \operatorname{Im} \left[(n_x^1 + i n_y^1) \frac{\partial \tilde{h}}{\partial z} \right] - V_0(n_y^1 \cos \alpha - n_x^1 \sin \alpha). \tag{4.25}$$

From (4.15), (4.18) and the slip condition (2.19) we deduce on Γ:

$$
\begin{aligned}
(4.26)\qquad 0 &= \operatorname{Re}\left[(u - iv)(n_x + in_y)\right] \\
&= -\frac{M_0^2}{4}\operatorname{Re}\left[(n_x + in_y)\left(\frac{1}{V_0^2}\frac{d^2 f_0}{dz^2}P(z,\bar{z}) + \frac{1}{V_0^2}\left(\frac{df_0}{dz}\right)^2\frac{d\bar{f}_0}{d\bar{z}}\right.\right. \\
&\qquad \left.\left. - \frac{k_0 e^{i\alpha}}{V_0 \pi i z}\frac{df_0}{dz} + \frac{d\tilde{h}}{dz} - V_0 e^{-i\alpha}\right)\right]
\end{aligned}
$$

whence, since

$$
(4.27)\qquad \operatorname{Re}\left[(n_x + in_y)\frac{d\tilde{h}}{dz}\right] = \frac{dJ}{dn},
$$

it follows that

$$
\begin{aligned}
(4.28)\qquad &\frac{\partial J}{\partial n}(x_j, y_j) \\
&= -\operatorname{Re}\left\{(n_x^j + in_y^j)\left[\frac{1}{V_0^2}\left(\frac{\partial u_0}{\partial x}(x_j, y_j) - i\frac{\partial v_0}{\partial x}(x_j, y_j)\right)P(z_j, \bar{z}_j)\right.\right. \\
&\quad + \frac{1}{V_0^2}(u_0(x_j, y_j) - iv_0(x_j, y_j))^2(u_0(x_j, y_j) + iv_0(x_j, y_j)) \\
&\quad \left.\left. - \frac{k_0(u_0(x_j, y_j) - iv_0(x_j, y_j))e^{i\alpha}}{V_0 \pi i z_j} + V_0 e^{-i\alpha}\right]\right\}.
\end{aligned}
$$

From (4.25) and (4.28) we obtain

$$
\begin{aligned}
(4.29)\qquad \frac{d\tilde{h}}{dz}(z_j) &= \frac{\partial J}{\partial x}(x_j, y_j) + i\frac{\partial I}{\partial x}(x_j, y_j) \\
&= \frac{\partial J}{\partial n}(x_j, y_j)n_x^j + \frac{\partial I}{\partial n}(x_j, y_j)n_y^j + i\left(\frac{\partial J}{\partial n}(x_j, y_j)n_y^j - \frac{\partial I}{\partial n}(x_j, y_j)n_x^j\right).
\end{aligned}
$$

From (4.15), (4.18) we may determine the complex velocity at the control points

$$
\begin{aligned}
(4.30)\qquad u(x_j, y_j) - iv(x_j, y_j) &= u_0(x_j, y_j) - iv_0(x_j, y_j) \\
&\quad + \frac{M_0^2}{4}\left[\frac{1}{V_0^2}(u_0(x_j, y_j) - iv_0(x_j, y_j))^2(u_0(x_j, y_j) + iv_0(x_j, y_j))\right. \\
&\quad + \frac{d\tilde{h}}{dz}(z_j) + V_0 e^{-i\alpha} - \frac{k_0 e^{i\alpha}}{V_0 \pi i z_j}(u_0(x_j, y_j) - iv_0(x_j, y_j)) \\
&\quad \left. + \frac{1}{V_0^2}\left(\frac{\partial u_0}{\partial x}(x_j, y_j) - i\frac{\partial v_0}{\partial x}(x_j, y_j)\right)P(z_j, \bar{z}_j)\right],
\end{aligned}
$$

and afterwards the speed of the fluid over the obstacle

$$V(x_j, y_j) = \sqrt{u(x_j, y_j)^2 + v(x_j, y_j)^2}. \tag{4.31}$$

5. Physical validity of the results

Since the Imai-Lamla-Iacob method is based on the asymptotic expansion with respect to M_0, the accuracy of the results increases when $M_0 \Rightarrow 0$. For investigating the compressibility effects, we are interested in working with great values of M_0, but the actual method imposes some restrictions on M_0.

As we could see, the study of the compressible flow past an obstacle was reduced to a series of boundary value problems for Laplace's equation. This method is not suitable for the supersonic compressible flow which is governed by hyperbolic partial differential equations; we have therefore to request the local speed of the fluid not to exceed the value of the speed of sound. For the isentropic flow, the speed of the sound depends on the speed of the fluid as follows:

$$c^2 = c_0^2 - \frac{\gamma - 1}{2} V^2 \tag{5.1}$$

or equivalently,

$$c^2 \frac{M_0^2}{V_0^2} = 1 - \frac{\gamma - 1}{2} M_0^2 \frac{V^2}{V_0^2}. \tag{5.2}$$

Imposing that $V \leq c$, we obtain from (5.2)

$$\frac{V^2}{V_0^2} M_0^2 \frac{\gamma + 1}{2} \leq 1 \tag{5.3}$$

and particularly,

$$\frac{V_{\max}^2}{V_0^2} M_0^2 \frac{\gamma + 1}{2} \leq 1. \tag{5.4}$$

The relation (5.4) is *a posteriori* condition which has to be checked after calculating the distribution of the velocity around the obstacle by the Imai-Lamla-Iacob method.

We shall assign to the ratio of specific heats the value $\gamma = 1.4$.

6. Numerical results

We shall investigate the flow past the circular obstacle

$$z = e^{i\theta}, \qquad \theta \in [0, 2\pi]. \tag{6.1}$$

We approximate the circle by the poligonal contour $\Gamma = \bigcup_{j=1}^{36} \Gamma_j$. The extremes of the panel Γ_j are the points

$$(x_j, y_j) = (\cos\theta_j, \sin\theta_j), \qquad (x_{j+1}, y_{j+1}) = (\cos\theta_{j+1}, \sin\theta_{j+1}) \tag{6.2}$$

with

$$\theta_j = \frac{(2j-3)\pi}{36}, \qquad j = 1, N. \tag{6.3}$$

We impose the zero value to the velocity at the point

$$z_1 = e^{i\theta_1} \tag{6.4}$$

and we consider the angle of attack $\alpha = 0$.

Using the boundary element method we compute the flow speed in the control points, and then the pressure coefficients

$$C_p = \frac{2\left(1 - \frac{\gamma-1}{2}M_0^2\right)}{\gamma M_0^2}\left[\frac{\left(1 - \frac{\gamma-1}{2}M_0^2\frac{V^2}{V_0^2}\right)^{\gamma/(\gamma-1)}}{\left(1 - \frac{\gamma-1}{2}M_0^2\right)^{\gamma/(\gamma-1)}} - 1\right]. \tag{6.5}$$

Expanding the pressure coefficient with respect to Chaplygin's number, we get

$$C_p = 1 - \frac{V^2}{V_0^2} + \frac{M_0^2}{4}\left(1 - \frac{V^2}{V_0^2}\right)^2 + M_0^4(\ldots). \tag{6.6}$$

On the circular obstacle, the analytical expression of the flow speed [1] is known,

$$V = 2V_0|\sin\theta - \sin\theta_1|\left[1 + \frac{M_0^2}{12}(1 - 6\cos 2\theta - 20\sin\theta\sin\theta_1 + 4\sin^2\theta_1)\right] \tag{6.7}$$

whence it follows that

$$C_p = 1 - 4(\sin\theta - \sin\theta_1)^2\left[1 + \frac{M_0^2}{6}(1 - 6\cos 2\theta - 20\sin\theta\sin\theta_1 + 4\sin^2\theta_1)\right] + \frac{M_0^2}{4}\left[(1 - 4(\sin\theta - \sin\theta_1)^2\right]^2. \tag{6.8}$$

Comparisons between the pressure coefficients in the control points, obtained by means of the boundary element method and by means of the analytical formula (6.8) are performed in Fig. 1 for various values of Chaplygin's number.

We introduce the lift coefficient

$$C_L = \frac{1}{\ell} \int_\Gamma C_p n_y \, ds, \tag{6.9}$$

where ℓ is the length of Γ. We can compute the lift coefficient numerically using the formulas

$$\ell = \sum_{j=1}^{N} \ell_j , \tag{6.10}$$

$$C_L = \frac{1}{2\ell} \sum_{j=1}^{N} (C_p(x_j, y_j) + C_p(x_{j+1}, y_{j+1}) n_y^j \ell_j . \tag{6.11}$$

For the circular obstacle, from (6.9) and (6.8) we get

$$C_L = -4 \sin\theta_1 - M_0^2 \sin\theta_1 \left(\frac{8}{3} + \frac{4}{3} \sin^2 \theta_1 \right) . \tag{6.12}$$

To conclude, we give some values for the lift coefficient obtained by means of formulas (6.12) and (6.11) for various values of M_0:

M_0	0	0.20	0.25	0.30
C_L (numerical)	0.3584	0.3611	0.3660	0.3703
C_L (analytical)	0.3486	0.3580	0.3632	0.3696

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On the elastic orthotropy

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EXPRESSIONS for Kelvin moduli of orthotropic linearly elastic bodies in terms of Young's moduli and Poisson's ratios are presented. The check of the orthotropy criterion is posed as a variational problem.

1. Introduction

MOST ELASTIC bodies which are considered in physics and in engineering either are, or can be, with acceptable accuracy, considered as orthotropic. Most of metal crystals (due to the symmetries of the lattice) and composites (because of the production techniques reasons) show these properties rigorously, while most of the polycrystalline textured materials, biological tissues, rock structures etc. can be, without significant errors, also considered as orthotropic. Moreover, all the second rank tensors exhibit the orthotropic symmetries, thus one can expect that, according to the generalized Curie law [11], the fourth order tensor-valued functions of e.g. one second rank tensor and arbitrary number of scalars, or of several coaxial second rank tensors etc., would preserve at least this symmetry. Reduction of the description of elastic properties of the anisotropic bodies to the orthotropic ones gives rise to a significant gain in the effectiveness and comprehensibility of the description, reducing the number of independent elastic constants from 18 to 9 ([1]) and the characteristic axes (eigenvectors of the proper elastic states) from 18 to 9 (to three, in fact).

In recent years some papers presenting the invariant properties, structure and transformation rules of plane fourth rank two-dimensional compliance and stiffness tensors (plane Hooke's tensors – according to [12]) were published [16, 17, 12, 4]. These papers exhausted the problem to some extent. Such a complete study of general three-dimensional case seems to be too complex to be useful, some helpful interesting facts, however, especially in the cases of higher symmetries, still can be established [15]. In the present paper the authors will touch only some aspects of the problem for the case of general orthotropy, with no claim for the completeness and generality of the considerations.

For the sake of brevity in this paper we shall use the term "*at least orthotropic*" for any class of symmetry having three mutually orthogonal planes of symmetry, i.e. we shall understand materials of higher symmetries, such as cubic, tetragonal transversely isotropic and isotropic as particular cases of orthotropic materials.

([1]) The problem: 18 or 21 constants? will be discussed later in this paper, compare also [9].

We shall discuss here the properties of elastic compliance and elastic stiffness tensors, but almost all the results remain valid for any fourth order tensors of the same symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}\,. \tag{1}$$

Throughout the present paper we shall use both the tensors and their representations and the matrices, which may not represent any tensors, thus to avoid any confusion we shall denote tensorial quantities using **boldface** symbols, while for matrices we shall use sanserif characters.

2. Modified Voigt matrices, Kelvin moduli, Young's moduli and Poisson's ratios

It is widely used to represent Hooke's symmetric operators mapping the set of second order symmetric strain tensors onto the set of the same type stress tensors (or *vice versa*) using 6×6 matrices. Thus instead of

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{2}$$

one writes

$$\varepsilon_K = C_{KL}\sigma_L\,, \tag{3}$$

where [2]

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{31} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}, \qquad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{31} \\ \sqrt{2}\sigma_{12} \end{bmatrix}. \tag{4}$$

Note please, that this notation differs from the classical Voigt one by the factor $\sqrt{2}$ (in the case when $i \neq j$). This modification makes it possible to operate with the corresponding matrices of representations using "tensorial" rules (cf. [6, 2]). The following correspondence between the representations of the compliance

[2] I.e. we take, in fact, the projections of the tensors onto the six-dimensional subspace defined by the following orthonormal basis $\mathbf{t}_i$: $\mathbf{t}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{t}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{t}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3$, $\mathbf{t}_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2)$, $\mathbf{t}_5 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1)$, $\mathbf{t}_6 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$.

tensor takes place:

$$
\begin{aligned}
&C_{11} = C_{1111}, && C_{22} = C_{2222}, && C_{33} = C_{3333},\\
&C_{44} = 2C_{2323}, && C_{55} = 2C_{1313}, && C_{66} = 2C_{1212},\\
&C_{12} = C_{1122}, && C_{13} = C_{1133}, && C_{32} = C_{3322},\\
&C_{45} = C_{2313}, && C_{46} = C_{2312}, && C_{56} = C_{1312},\\
&C_{14} = \sqrt{2}C_{1123}, && C_{15} = \sqrt{2}C_{1113}, && C_{16} = \sqrt{2}C_{1112},\\
&C_{24} = \sqrt{2}C_{2223}, && C_{25} = \sqrt{2}C_{2213}, && C_{26} = \sqrt{2}C_{2212},\\
&C_{34} = \sqrt{2}C_{3323}, && C_{35} = \sqrt{2}C_{3313}, && C_{36} = \sqrt{2}C_{3312}.
\end{aligned} \tag{5}
$$

For general considerations, as well as for the sake of illustrativeness, another representation, which can be traced back to Lord Kelvin and was intensively developed during last decade by J. RYCHLEWSKI [9] and some other authors (for references see [13, 8]), can be very fruitful. The elastic compliance tensor, as any fourth order tensor preserving symmetries (1) can be represented in the following form:

$$
\mathbf{C} = \frac{1}{\lambda_1}\,\boldsymbol{\kappa}_1 \otimes \boldsymbol{\kappa}_1 + \cdots + \frac{1}{\lambda_6}\boldsymbol{\kappa}_6 \otimes \boldsymbol{\kappa}_6, \tag{6}
$$

where $\boldsymbol{\kappa}_I$ (I = 1 to 6) are the second order tensors, mutually orthogonal in the sense of the scalar product in the corresponding linear space: $\boldsymbol{\kappa}_I \cdot \boldsymbol{\kappa}_J = \delta_{IJ}$; J. Rychlewski proposed to call them *proper elastic states*, while the non-negative scalars λ_I he proposed to call the *Kelvin moduli* [9]. Using (6) one can write the (reversed) Hooke's law in the following form:

$$
\boldsymbol{\varepsilon} = \frac{1}{\lambda_1}(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\sigma})\boldsymbol{\kappa}_1 + \cdots + \frac{1}{\lambda_6}(\boldsymbol{\kappa}_6 \cdot \boldsymbol{\sigma})\boldsymbol{\kappa}_6. \tag{7}
$$

In the case of orthotropy (or higher symmetry), a triplet of proper elastic states represent three pure shears in the mutually orthogonal planes; in appropriate basis their representations take the following form:

$$
\boldsymbol{\kappa}_4 \propto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_5 \propto \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_6 \propto \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}
$$

It is not difficult to notice that the other three proper elastic states have *in this basis* diagonal representations. Moreover, in view of mutual orthogonality, the triads of diagonal terms form rows of some orthogonal 3×3 matrix. Among

many possible representations, the following one seems to be convenient:

$$\boldsymbol{\kappa}_1 \propto \begin{bmatrix} \cos\theta\cos\varphi & 0 & 0 \\ 0 & \cos\theta\sin\varphi & 0 \\ 0 & 0 & \sin\theta \end{bmatrix},$$

$$\boldsymbol{\kappa}_2 \propto \begin{bmatrix} -\cos\gamma\sin\varphi \\ +\sin\gamma\sin\theta\cos\varphi & 0 & 0 \\ 0 & \cos\gamma\cos\varphi \\ & +\sin\gamma\sin\theta\sin\varphi & 0 \\ 0 & 0 & -\sin\gamma\cos\theta \end{bmatrix}, \tag{9}$$

$$\boldsymbol{\kappa}_3 \propto \begin{bmatrix} \sin\gamma\sin\varphi \\ +\cos\gamma\sin\theta\cos\varphi & 0 & 0 \\ 0 & -\sin\gamma\cos\varphi \\ & +\cos\gamma\sin\theta\sin\varphi & 0 \\ 0 & 0 & -\cos\gamma\cos\theta \end{bmatrix}.$$

Parameters φ, θ, γ can assume any values, they do not represent any angles in "physical" space ([3]).

Assuming particular values for these parameters and taking some Kelvin moduli equal to each other, one obtains all the particular cases of higher symmetries:

- isotropy

$$\sin\theta = \frac{1}{\sqrt{3}}, \qquad \cos\varphi = \frac{1}{\sqrt{2}}, \qquad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6,$$

- cubic symmetry

$$\sin\theta = \frac{1}{\sqrt{3}}, \qquad \cos\varphi = \frac{1}{\sqrt{2}}, \qquad \lambda_2 = \lambda_3, \qquad \lambda_4 = \lambda_5 = \lambda_6,$$

- transversal isotropy in $\{x_1, x_2\}$-plane

$$\sin\gamma = 0, \qquad \cos\varphi = \frac{1}{\sqrt{2}}, \qquad \lambda_2 = \lambda_6, \qquad \lambda_4 = \lambda_5,$$

- tetragonal symmetry ([4]) (about the x_3-axis)

$$\sin\gamma = 0, \qquad \cos\varphi = \frac{1}{\sqrt{2}}, \qquad \lambda_4 = \lambda_5.$$

([3]) The presented *form* of the representation of the orthogonal matrix corresponds nevertheless to the "rotational" interpretation of the matrix Q as a representation of the tensor $\mathbf{Q} = \mathbf{n}\otimes\mathbf{n} + (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\cos\gamma + \mathbf{n}\cdot\boldsymbol{\epsilon}\sin\gamma$ (cf. [3]).

([4]) We mean here the same case as that considered e.g in [14], we are not considering here the other case pointed out in [7], which in fact can be reduced to the previous one; for details see [4].

Let us assume that the preferred orthotropy axes are known. One can notice at once that for $i \geq 4$ Kelvin moduli λ_i are equal to the corresponding doubled Kirchhoff moduli:

$$\lambda_4 = 2G_{23}, \qquad \lambda_5 = 2G_{13}, \qquad \lambda_6 = 2G_{12}. \tag{10}$$

For the determination of the other three Kelvin moduli we shall consider uniaxial tension (compression) along each of the preferred axes x_i. Denoting:

$$Q_{ij} \equiv (\boldsymbol{\kappa}_i)_{jj} \qquad \text{(no summation!)} \tag{11}$$

one obtains, for the case of tension along x_1, the following components of $\boldsymbol{\varepsilon}$

$$\varepsilon_{ii} = \frac{1}{\lambda_1} Q_{1i} \boldsymbol{\kappa}_1 \cdot \boldsymbol{\sigma} + \frac{1}{\lambda_2} Q_{2i} \boldsymbol{\kappa}_2 \cdot \boldsymbol{\sigma} + \frac{1}{\lambda_3} Q_{3i} \boldsymbol{\kappa}_3 \cdot \boldsymbol{\sigma} = \sigma_{11} \sum_{j=1}^{3} \frac{Q_{ji} Q_{j1}}{\lambda_j} \tag{12}$$

(no summation! Compare (7)).

Denoting

$$E_i \equiv \frac{\sigma_{ii}}{\varepsilon_{ii}}, \qquad \nu_{ij} \equiv -\frac{\varepsilon_{jj}}{\varepsilon_{ii}} \qquad \text{(no summation!)} \tag{13}$$

one can write

$$\begin{aligned} \frac{1}{E_1} &= \frac{Q_{11}^2}{\lambda_1} + \frac{Q_{21}^2}{\lambda_2} + \frac{Q_{31}^2}{\lambda_3}, \\ -\frac{\nu_{12}}{E_1} &= \frac{Q_{12} Q_{11}}{\lambda_1} + \frac{Q_{22} Q_{21}}{\lambda_2} + \frac{Q_{32} Q_{31}}{\lambda_3}, \\ -\frac{\nu_{13}}{E_1} &= \frac{Q_{13} Q_{11}}{\lambda_1} + \frac{Q_{23} Q_{21}}{\lambda_2} + \frac{Q_{33} Q_{31}}{\lambda_3}. \end{aligned} \tag{14}$$

Similar relations are valid for the uniaxial tension along the other two axes. Thus, introducing new symbols

$$\nu_{ii} = -1, \qquad A_{ij} = -\frac{\nu_{ij}}{E_i} \qquad \text{(no summation!)} \tag{15}$$

we obtain the following relation

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad \text{(no summation!)} \tag{16}$$

and we are able to write

$$\mathsf{A} = \mathsf{Q}^{\mathsf{T}} \hat{\mathsf{L}}^{-1} \mathsf{Q}, \tag{17}$$

where $\hat{\mathsf{L}}$ is the diagonal matrix of the first three Kelvin moduli λ_i. Thus we have reduced the problem of expressing the Kelvin moduli in terms of Young's moduli and Poisson's ratios to the eigenvalue problem for a symmetric 3×3 matrix A.

Among many known ways of solving this problem, a particular one, giving rise to relatively simple analytic formulae can be pointed out.

Note that, due to orthogonality of Q, the invariants of the matrices A and $\hat{L}^{-1}$ are equal to each other; similar relation (such as (17)) holds true for the normalised deviatoric parts of these tensors, therefore also their corresponding invariants are equal to each other.

Thus, denoting

$$(18)\qquad \mathsf{A}' \equiv \mathsf{A} - \frac{1}{3}\mathrm{tr}\mathsf{A}\,\mathsf{I}, \qquad z_i \equiv \frac{1}{\sqrt{\mathrm{tr}(\mathsf{A}'^2)}}\left(\frac{1}{\lambda_i} - \frac{1}{3}\mathrm{tr}\mathsf{A}\right), \qquad b \equiv \sqrt{54}\frac{\det \mathsf{A}'}{\left(\sqrt{\mathrm{tr}(\mathsf{A}'^2)}\right)^3},$$

where I is the unit matrix, one can express the invariants of A′ by its eigenvalues z_i writing:

$$(19)\qquad \begin{aligned} z_1 + z_2 + z_3 &= 0,\\ z_1^2 + z_2^2 + z_3^2 &= 1,\\ z_1 z_2 z_3 &= \frac{b}{\sqrt{54}}. \end{aligned}$$

Solutions of the system (19) are equal to the real parts of the three branches of the following complex expression:([5])

$$(20)\qquad z_{1,2,3} = -\frac{1}{\sqrt{6}}\left[(b + i\sqrt{1-b^2})^{1/3} + (b - i\sqrt{1-b^2})^{1/3}\right].$$

Consequently, one obtains

$$(21)\qquad \frac{1}{\lambda_i} = z_i\sqrt{\mathrm{tr}\,\mathsf{A}^2 - \frac{1}{3}(\mathrm{tr}\mathsf{A})^2} + \frac{1}{3}\mathrm{tr}\mathsf{A},$$

while the *compliance distributors* Q_{ij} ([6]) can be easily obtained as the normalised solutions of the following linear system:

$$(22)\qquad A_{ij}Q_{Kj} = \frac{1}{\lambda_K}Q_{Ki} \qquad \text{(no summation over capital index } K\text{!)}.$$

3. Orthotropy criterion

Throughout the previous section it was assumed, that the distinguished axes of orthotropy are known, as they often really *are* in physics and in the engineering applications. This cannot be considered as a rule, however. Such practical

([5]) Note that always $|b| \leq 1$, thus substituting $b = \cos 3\alpha$ and using the Moivre formulae, one can easily verify that expressions (20) satisfy Eqs. (19), cf. [1].

([6]) They are simultaneously the *stiffness distributors*.

situations can be pointed out, when not only the axes are not known, but there is even no information if the material is at least orthotropic, or not.

If the material is orthotropic and the basis of unit vectors along the orthotropy axes is adopted, then the matrix of the six-dimensional representation of the elastic compliance tensor (cf. [7]) has the following form:

$$\mathsf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \qquad (C_{ij} = C_{ji}). \tag{23}$$

If the compliance tensor is given in an *arbitrary basis*, then the answer for the question if the material is at least orthotropic or not, is not immediate. Even if one solves (numerically) the six-dimensional eigenvalue problem, finds the Kelvin moduli and then calculates the proper elastic states and finds out, that the three of them are pure shears, there is still no certainty, that we have to do with the case of orthotropy. There exist such five orthogonal (in the sense of scalar product of second rank tensors) pure shears among which there are no three of them acting in mutually orthogonal planes in the "physical" space [5]. To find out the answer, one has to analyze the proper axes of almost all the proper elastic states.

Another possible approach to finding the answer to the orthotropy question can be as follows:

1. Take a general three-parameter expression for the rotation tensor **R** in 3-dimensional "physical" space.

2. For a known representation (in a chosen basis) of elastic compliance tensor, find a general form of the representation in a rotated basis

$$C'_{ijkl} = R_{ip} R_{jq} R_{kr} R_{ls} C_{pqrs}.$$

3. Find out if there exist such values of the parameters (e.g $\{\varphi_1, \varphi_2, \varphi_3\}$), defining rotation **R**, which lead to vanishing of all these components of C which vanish for the case of the representation (23) in the "proper" orthotropy basis.

This way can be fairly laborious as regards the computer programming, and it forces us to leave quite convenient and familiar six-dimensional space and to return to the space of fourth rank tensors.

The present authors in their recent paper written jointly with J. RYCHLEWSKI [4] have shown – in the case of plane elasticity – an effective criterion of orthotropy, expressed by the relation between the components of the compliance tensor in an arbitrary basis. It is not clear if such a criterion of reasonable complexity can be formulated for the 3-dimensional case. In the present paper the authors are

going to propose a way, which, being in fact equivalent to the one presented above, makes it possible to remain all the time in the six-dimensional space, to verify the intermediate results (which can solely be useful for certain other considerations) and to use standard computer procedures.

The one who wants to rotate the bases in a six-dimensional space should first establish the set of allowed orthogonal matrices, having their counterparts in the sets of rotations in three-dimensional space. The complete set of rotation matrices in six-dimensional space depends on 15 parameters ([7]). The set of rotations in 3-dimensional space is 3-parametric; this fact has some important implications: first – the 21 independent parameters describing general compliance tensor can be chosen in such a way, that 18 of them can be considered as material constants, the remaining three fixing the orientation of the axes of proper elastic states in a 3-dimensional space (the same considerations reduce the number of material constants for an orthotropic material from 12 to 9), and second – the set of the physically meaningful rotations in a six-dimensional space must be also three-parametric.

We shall find a general representation of the rotation tensor in a six-dimensional space, generated by the rotation in the "physical" 3-dimensional space. It is not difficult to verify (compare [4]), that the rotation $\mathbf{R}_{(1)}$ by the angle φ_1 around the axis x_1 generates the following orthogonal 6×6 matrix $\mathsf{R}_{(1)}$ describing the same rotation in a six-dimensional space:

$$\mathsf{R}_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\varphi_1 & \sin^2\varphi_1 & -\frac{1}{\sqrt{2}}\sin 2\varphi_1 & 0 & 0 \\ 0 & \sin^2\varphi_1 & \cos^2\varphi_1 & \frac{1}{\sqrt{2}}\sin 2\varphi_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\sin 2\varphi_1 & -\frac{1}{\sqrt{2}}\sin 2\varphi_1 & \cos 2\varphi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & 0 & 0 & 0 & -\sin\varphi_1 & \cos\varphi_1 \end{bmatrix}. \tag{24}$$

The corresponding expressions for the rotations around the other axes are straightforward. Any rotation R can be achieved as the superposition of the rotation around three non-coplanar, for example orthogonal, axes, thus a general expression for rotation matrix in a six-dimensional space can be obtained as a product of matrices describing rotations around the coordinate axes x_i by the angles φ_i,

$$\mathsf{R} = \mathsf{R}_{(1)}\mathsf{R}_{(2)}\mathsf{R}_{(3)}. \tag{25}$$

([7]) Every matrix of the type $\mathsf{A}\mathsf{A}^{\mathsf{T}}$ is symmetric, thus the orthogonality condition $\mathsf{R}\mathsf{R}^{\mathsf{T}} = \mathsf{I}$ imposes only 21 constrains on the 36 components of the matrix R.

The complete expression for such R is quoted in the Appendix ([8]).

Let us denote for convenience the "vector" composed of the elements of the k-th column of the matrix R by $\mathsf{r}_{(K)}$:

$$\mathsf{r}_{(K)} \equiv [R_{1K}, R_{2K}, R_{3K}, R_{4K}, R_{5K}, R_{6K}]. \tag{26}$$

It is not difficult to notice, that the "vectors" $\mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3$ represent in a six-dimensional space three pure shears in mutually orthogonal planes; indeed, denoting by $\mathsf{k}_{(i)}$ the six-dimensional representation of the tensors $\boldsymbol{\kappa}_i$ listed in (8), one obtains:

$$\mathsf{r}_{(i)} = \mathsf{R}\mathsf{k}_{(i)} \qquad (i = 4,5,6). \tag{27}$$

Thus:

The elastic compliance 6×6 *matrix* C *describes at least an orthotropic material if and only if there exists such a rotation* **R** *in 3-dimensional space, that the components of the last three columns of its six-dimensional representation* R *represent the three proper elastic states.*

If some symmetric tensor $\boldsymbol{\sigma}$, (certain vector in a six-dimensional space) represents a proper state, then obviously it is *energetically orthogonal* [10] to any symmetric tensor (six-dimensional vector) from its orthogonal complement; the reverse is equally true, i.e.:

$\boldsymbol{\sigma}$ *is a proper state if and only if for any* $\mathbf{a}$, $\mathbf{a}\cdot\boldsymbol{\sigma} = 0$ *implies* $\mathbf{a}\cdot\mathbf{C}\cdot\boldsymbol{\sigma} = 0$.

Hence:

The elastic compliance matrix C *describes at least an orthotropic material if and only if there exists such an orthogonal* 6×6 *matrix* R, *describing the rotation in three-dimensional space and consisting of such columns* $\mathsf{r}_{(K)}$, *that* $\mathsf{r}^{\mathsf{T}}_{(K)}\mathsf{C}\,\mathsf{r}_{(J)} = 0$ ([9]) *for* $K = 1,2,3,4,5,6$, $J = 4,5,6$, $K \neq J$.

Let us define, for a given compliance matrix C, the following function F of the three parameters $\{\varphi_1, \varphi_2, \varphi_3\}$ determining the "physical" rotation in six-dimensional space:

$$F(\varphi_1, \varphi_2, \varphi_3) = \sum_{K=1}^{6} \sum_{\substack{J=4 \\ J \neq K}}^{6} |\mathsf{r}^{\mathsf{T}}_{(K)}\mathsf{C}\,\mathsf{r}_{(J)}|. \tag{28}$$

([8]) It is evident that the set of such matrices is a subgroup of the group of orthogonal 6×6 matrices. Some interesting properties of the matrices of this subgroup, connected with the preservation of the tensorial invariants of the six-dimensional "vectors", can be pointed out. For example, sum of the first three terms in the case of 1-st, 2-nd and 3-rd column is equal to 1, while for the other columns it is equal to zero, and the same applies to the rows (the columns of the transposed matrix).

([9]) By $\mathsf{r}^{\mathsf{T}}_{(K)}$ we denote the row matrix having the same components as the column matrix $\mathsf{r}_{(K)}$, thus $\mathsf{r}^{\mathsf{T}}_{(K)}\mathsf{C}\,\mathsf{r}_{(J)}$ is a number.

The orthotropy condition can then be expressed as follows

$$\begin{aligned} &\min_{\varphi_1,\varphi_2,\varphi_3} F(\varphi_1,\varphi_2,\varphi_3) = 0, \\ &-\pi/2 \leq \varphi_1 \leq \pi/2, \\ &-\pi/2 \leq \varphi_2 \leq \pi/2, \\ &-\pi/2 \leq \varphi_3 \leq \pi/2. \end{aligned} \tag{29}$$

The ratio of the minimal value of the function $F(\varphi_1,\varphi_2,\varphi_3)$ to the norm of the elastic compliance tensor

$$\chi = \frac{\min_{\varphi_1,\varphi_2,\varphi_3} F(\varphi_1,\varphi_2,\varphi_3)}{||\mathbf{C}||} \tag{30}$$

can be considered as a good measure of obliqueness (departure from orthotropy) of the elastic properties.

Observe that χ is an invariant of the compliance tensor **C**. Indeed: all the matrices of its six-dimensional representations are mutually connected by the relation: $\mathsf{C}' = \mathsf{R}^{\mathsf{T}}\mathsf{C}\mathsf{R}$, while for any orthogonal 6×6 matrix R, generated by rotation in a 3-dimensional space and any "vector" $\mathsf{r}_{(J)}$, the product $\mathsf{R}\mathsf{r}_{(J)} \equiv \mathsf{r}'_{(J)}$ is again a "vector" of the same type, its "tensorial" invariants being preserved, thus:

$$\sum_{K=1}^{6}\sum_{\substack{J=4\\J\neq K}}^{6} |\mathsf{r}^{\mathsf{T}}_{(K)}\mathsf{C}'\,\mathsf{r}_{(J)}| = \sum_{K=1}^{6}\sum_{\substack{J=4\\J\neq K}}^{6} |\mathsf{r}'^{\mathsf{T}}_{(K)}\mathsf{C}\,\mathsf{r}'_{(J)}| = F(\varphi'_1,\varphi'_2,\varphi'_3). \tag{31}$$

This means that changing the representation of the compliance tensor **C** (and consequently the components of the matrix C), one merely renames variables in (29), what, evidently, can not affect the result of the minimization procedure. This completes the proof.

An entirely different approach to the problem under consideration can be also proposed: any compliance 6×6 matrix C can be represented as

$$\mathsf{C} = \mathsf{P}^{\mathsf{T}}\mathsf{L}^{-1}\mathsf{P}, \tag{32}$$

where P denotes a 6×6 orthogonal matrix and L is the diagonal matrix of Kelvin moduli. For example, in the case of orthotropy one can take

$$\mathsf{P} = \mathsf{B}\mathsf{R}, \qquad \text{where} \qquad B \equiv \left[\begin{array}{c|c} \mathsf{Q} & 0 \\ \hline 0 & \mathsf{I} \end{array}\right], \tag{33}$$

Q and R denote the same matrices as those in (11) and (25). Thus one can reduce the considerations on the obliqueness of the elastic properties to the problem of invariants of the matrix P; we shall not pursue this line in the present paper however.

This completes for the time being our considerations on the orthotropy condition.

Appendix

For reference, a general form of a 6×6 orthogonal three-parameter matrix describing the rotation in the three-dimensional space should be quoted. We shall omit here the elementary calculations and expose only the final result of multiplying matrices $R_{(1)}$, $R_{(2)}$ and $R_{(3)}$, where the first of them has already been shown (Eq. (24)), and the other two can be easily obtained by renaming the variables and reshuffling the columns and rows. For better comprehension we shall split the matrix R into four submatrices which will be specified separately:

$$R = \left[\begin{array}{c|c} R^{(1)} & R^{(2)} \\ \hline R^{(3)} & R^{(4)} \end{array}\right],$$

where

$$R^{(1)} = \begin{bmatrix} \cos^2\varphi_2\cos^2\varphi_3 & \cos^2\varphi_2\sin^2\varphi_3 & \sin^2\varphi_2 \\ \sin^2\varphi_1\sin^2\varphi_2\cos^2\varphi_3 + \cos^2\varphi_1\sin^2\varphi_3 - \frac{1}{2}\sin 2\varphi_1\sin\varphi_2\sin 2\varphi_3 & \sin^2\varphi_1\sin^2\varphi_2\sin^2\varphi_3 + \cos^2\varphi_1\cos^2\varphi_3 + \frac{1}{2}\sin 2\varphi_1\sin\varphi_2\sin 2\varphi_3 & \sin^2\varphi_1\cos^2\varphi_2 \\ \cos^2\varphi_1\sin^2\varphi_2\cos^2\varphi_3 + \sin^2\varphi_1\sin^2\varphi_3 + \frac{1}{2}\sin 2\varphi_1\sin\varphi_2\sin 2\varphi_3 & \cos^2\varphi_1\sin^2\varphi_2\sin^2\varphi_3 + \sin^2\varphi_1\cos^2\varphi_3 - \frac{1}{2}\sin 2\varphi_1\sin\varphi_2\sin 2\varphi_3 & \cos^2\varphi_1\cos^2\varphi_2 \end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix} \frac{\sin 2\varphi_2}{\sqrt{2}}\sin\varphi_3 & -\frac{\sin 2\varphi_2}{\sqrt{2}}\cos\varphi_3 & -\cos^2\varphi_2\frac{\sin 2\varphi_3}{\sqrt{2}} \\ -\frac{\sin 2\varphi_1}{\sqrt{2}}\cos\varphi_2\cos\varphi_3 - \sin^2\varphi_1\frac{\sin 2\varphi_2}{\sqrt{2}}\sin\varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}}\cos\varphi_2\sin\varphi_3 + \sin^2\varphi_1\frac{\sin 2\varphi_2}{\sqrt{2}}\cos\varphi_3 & -\sin^2\varphi_1\sin^2\varphi_2\frac{\sin 2\varphi_3}{\sqrt{2}} + \cos^2\varphi_1\frac{\sin 2\varphi_3}{\sqrt{2}} - \frac{\sin 2\varphi_1}{\sqrt{2}}\sin\varphi_2\cos 2\varphi_3 \\ \frac{\sin 2\varphi_1}{\sqrt{2}}\cos\varphi_2\cos\varphi_3 - \cos^2\varphi_1\frac{\sin 2\varphi_2}{\sqrt{2}}\sin\varphi_3 & \frac{\sin 2\varphi_1}{\sqrt{2}}\cos\varphi_2\sin\varphi_3 + \cos^2\varphi_1\frac{\sin 2\varphi_2}{\sqrt{2}}\cos\varphi_3 & -\cos^2\varphi_1\sin^2\varphi_2\frac{\sin 2\varphi_3}{\sqrt{2}} + \sin^2\varphi_1\frac{\sin 2\varphi_3}{\sqrt{2}} + \frac{\sin 2\varphi_1}{\sqrt{2}}\sin\varphi_2\cos 2\varphi_3 \end{bmatrix},$$

$$\mathsf{R}^{(3)} = \begin{bmatrix} -\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2\varphi_2 \cos^2\varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2\varphi_2 \sin^2\varphi_3 & \\ +\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2\varphi_3 & +\frac{\sin 2\varphi_1}{\sqrt{2}} \cos^2\varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \cos^2\varphi_2 \\ +\cos 2\varphi_1 \sin\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\cos 2\varphi_1 \sin\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \\ \cos\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos^2\varphi_3 & \cos\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin^2\varphi_3 & -\cos\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \\ +\sin\varphi_1 \cos\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\sin\varphi_1 \cos\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \\ -\sin\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos^2\varphi_3 & -\sin\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin^2\varphi_3 & \sin\varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \\ +\cos\varphi_1 \cos\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\cos\varphi_1 \cos\varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \end{bmatrix},$$

$$\mathsf{R}^{(4)} = \begin{bmatrix} \cos 2\varphi_1 \cos\varphi_2 \cos\varphi_3 & \cos 2\varphi_1 \cos\varphi_2 \sin\varphi_3 & \frac{1}{2} \sin 2\varphi_1 \sin^2\varphi_2 \sin 2\varphi_3 \\ +\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_2 \sin\varphi_3 & -\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_2 \cos\varphi_3 & +\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_3 \\ & & +\cos 2\varphi_1 \sin\varphi_2 \cos 2\varphi_3 \\ -\sin\varphi_1 \sin\varphi_2 \cos\varphi_3 & -\sin\varphi_1 \sin\varphi_2 \sin\varphi_3 & -\frac{1}{2} \sin 2\varphi_2 \sin 2\varphi_3 \\ -\cos\varphi_1 \cos 2\varphi_2 \sin\varphi_3 & +\cos\varphi_1 \cos 2\varphi_2 \cos\varphi_3 & +\sin\varphi_1 \cos\varphi_2 \cos\varphi_2 \\ -\cos\varphi_1 \sin\varphi_2 \cos\varphi_3 & -\cos\varphi_1 \sin\varphi_2 \sin\varphi_3 & \frac{1}{2} \sin\varphi_1 \sin 2\varphi_2 \sin 2\varphi_3 \\ +\sin\varphi_1 \cos 2\varphi_2 \sin\varphi_3 & -\sin\varphi_1 \cos 2\varphi_2 \cos\varphi_3 & +\cos\varphi_1 \cos\varphi_2 \cos 2\varphi_3 \end{bmatrix}.$$

As it has already been pointed out, summation of the elements of the rows and columns of $\mathsf{R}^{(1)}$ gives 1, while the sums over the columns of $\mathsf{R}^{(2)}$ and rows of $\mathsf{R}^{(3)}$ vanish.

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Stiffness loss in laminates with intralaminar cracks Part I. Two-dimensional modelling

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BY IMPOSING STRESS constraints of Hashin type and corresponding kinematic constraints on the Reissner two-field functional, a new two-dimensional model for the three-layer symmetric laminate is derived. This model is capable of taking into account transverse cracks in the internal layer. These cracks behave according to Signorini's conditions. In Part I the basic properties of the model without and with transverse crack in the internal layer are investigated. The equilibrium problem is governed by a variational inequality involving five kinematic unknowns.

1. Introduction

OWING TO A DISCREPANCY between the values of thermal expansions and elastic moduli of fibres and the matrix, the composite laminates incur transverse microcracking even under relatively low in-plane loadings. The appearing intralaminar cracks go usually across the whole thickness of the layer and are almost equally spaced, cf. GARRETT and BAILEY's [10, Fig. 2] experiments with glass fibre-reinforced polyester; graphite-epoxy patterns in GROVES *et al.* [11, Fig. 3] and cracks in glass-epoxy laminates in HIGHSMITH and REIFSNIDER [19]. At a certain level of loading, the crack patterns attain a saturation state in which the layout of cracks is also nearly uniform, called CDS-characteristic damage state.

The loss of stiffnesses of a laminate can be viewed as a characteristic of the degree of its damage. The main characteristics of the laminate interrelate its stiffnesses with density of the transverse microcracks defined as parameters inversely proportional to the crack spacing. Apparent regularity of cracking patterns warrants the derivation of the effective macro-properties of the laminate from the properties of microcracking. Such approach stipulates formation of special micromechanics models, the point of departure of which is the analysis of stresses in the vicinity of transverse cracks. A review of such models can be found in LEE *et al.* [25], YANG and BOEHLER [58], MCCARTNEY [39] and in TSAI and DANIEL [56], cf. also ABOUDI and BENVENISTE [2], ABRATE [4], THIONNET [55], TENG [54], GAMBAROTTA and LAGOMARSINO [9], KATTAN and VOYIADJIS [20], GUDMUNDSON and ÖSTLUND [12, 13], GUDMUNDSON and ZANG [14].

A successful method, albeit based on simple stress assumptions, of local analysis in the stretching and shearing problems is due to HASHIN [17, 18]. Hashin's approach was followed by new displacement-based approaches of HAN *et al.* [16], HAN and HAHN [15], ABOUDI [1], ABOUDI *et al.* [3] and TSAI and DANIEL [56]. The development of the theory resembles here the progress in the theory of moderately thick plates which was started by stress-based approach of REISSNER [42, 43] as developed by Hencky, Bollé and Mindlin into its displacement-based

formulations, cf. REISSNER [45]. Hashin's approach has recently been reconsidered in MCCARTNEY [37, 38], where two new relevant models have been discussed: a "generalized plane strain" model (called further GPS) and an "approximate 3-D solution". The range of validity of the former model is similar to that of HASHIN [17], but, as it has been reported in MCCARTNEY [38], this model results in the unique formulae for all the effective stiffnesses relevant to the tension problem. The latter model is applicable for tension of laminates with finite width. Hashin's approach is also a basis for NAIRN [40] to define a strain energy release rate due to microcracking. By putting forward a hypothesis that its value is a material constant, Nairn arrived at a relationship between the given load and the crack density, cf. also YALVAÇ *at al.* [57].

The modern tools of homogenization make it possible to perform the process of smearing-out the cracks rigorously. The aim of the present paper is, by using these new tools, to put forward an alternative model of cracked laminates by basing it on HASHIN's [17, 18] – type stress assumptions. The model is constructed by:

i) relaxing HASHIN's [18] stress assumptions by introducing new stress resultant fields, none of them being viewed as directly dependent upon the edge loading;

ii) augmenting them with assumptions on displacements;

iii) constructing a new two-dimensional model of a three-layer symmetric laminate by using a two-field variational principle of REISSNER [43]. Hashin adopted a method of REISSNER [42] but without introducing Lagrange multipliers;

iv) performing the process of smearing-out the transverse cracks in the internal layer by the method of homogenization.

A short outline of this modelling has been announced in LEWIŃSKI and TELEGA [32, 33, 34] and TELEGA and LEWIŃSKI [53].

The hypotheses assumed make it possible to reduce the transverse dimension. In spite of it, they enable us to figure out transverse cracks in the internal layer. Reduction of the transverse dimension implies neglecting the singularities of the transverse distribution of stresses but does not suppress the in-plane distributed singularities of the stress resultants.

The modelling carried out in the present paper is neither a generalization of HASHIN's [17, 18] approach nor that of MCCARTNEY's [37, 38] analysis. In the homogenization approach used in the second part [35], the periodicity cell is subjected to macrodeformations $\varepsilon^h_{\alpha\beta}$. These macrofields control deformations in the cell of periodicity. On the contrary, Hashin's approach does not distinguish between problems in macro- and microscale. This author considers a state of stress and deformation in a finite domain of the cracked laminate and the control variables are boundary loads. This does not mean that such an approach is equivalent to the stress method of homogenization, since in the latter case the deformation state in the periodicity cell are controlled by effective stress resultants $N_h^{\alpha\beta}(x)$, $x \in \Omega$ and not by boundary forces. Moreover, in contrast to Hashin's

approach, in the modelling proposed the stresses are not directly related to the edge loading. The in-plane stresses $\sigma^{\alpha\beta}$ are expressed in terms of two independent and *a priori* unknown tensors of stress resultants $N^{\alpha\beta}$, $L^{\alpha\beta}$ (see (2.7)), whilst in the Hashin-type models mentioned above N^{11} and N^{12} are viewed as known. Consequently, in the approach presented the number of Lagrangian multipliers (generalized displacements) is greater by three. Only in this manner one can make the model capable of describing a general family of in-plane edge loadings.

Despite these substantial differences between the Hashin-type models and the model presented, some comparisons will be made in [36] inasmuch as appropriate reinterpretation of Hashin's or McCartney's quantities in terms of notions of the homogenization approach is possible.

Regular crack patterns observed in experiments justify the assumption of periodicity of the crack distribution, which enables us to use homogenization methods effectively. A version of this method, developed primarily by SANCHEZ–PALENCIA [47], encompassing Signorini-type cracks that can open or close (and then extended by the present authors [29, 30, 31], TELEGA [51], TELEGA and LEWIŃSKI [52]) makes it possible to describe the opening and closure of the transverse cracks in the internal layer.

In 3D problems of averaging properties of periodic elastic composites, a uniquely determined construction of the homogenized model is implied by solution of the basic cell problem. All other averaging methods can only furnish approximations and their accuracy should be measured by the deviation from the homogenization results, cf. SUQUET [48]. Less clear situation occurs in problems of averaging stiffnesses of plates with a periodic structure. The results of KOHN and VOGELIUS [22, 23, 24] and CAILLERIE [7] prove that a correct starting point should be the three-dimensional model; the method of homogenization results in the Kirchhoff thin plate model whose stiffnesses are determined by the properties of 3D periodicity cells. Similar approach for the case of cracked plates is due to CHACHA and SANCHEZ–PALENCIA [8]. However, if the starting point is two-dimensional, then the following two methods of averaging are useful:

a) the method based on an in-plane scaling of the longitudinal dimensions of the periodicity cells,

b) the method based on simultaneous scaling of all dimensions of cells.

Consequences of both averaging methods for moderately thick plates have been examined in LEWIŃSKI [27, 28] and TELEGA [50]. The (a) method leads to moderately thick plate effective model and applies only when in-plane dimensions of the periodicity cells are much smaller than plate thickness. The (b) method leads to thin plate effective model and does not impose any conditions on the dimensions of periodicity cells. Thus the applicability ranges of both methods are not identical.

In the second part of the paper [35] the consequences of using (a) and (b) scalings applied to a new laminate model will be examined.

Part I constitutes the basis for such considerations. Particularly, the two-dimensional model of an undamaged three-layer laminate will be derived in Sec. 2. In Sec. 3 we assume that the internal layer is weakened by a transverse crack. The variational inequality of type (3.11) will be used as a starting point for the homogenization procedures developed in the second part of the paper [35].

Throughout the whole paper (Part I and II), the following conventions are adopted: small Greek indices (except for ε) assume values 1, 2 while Latin ones (except for h) run over 1, 2, 3. Summation convention holds only for repeated indices at different levels. Index h labels the quantities referred to the homogenized models. Comma implies partial differentiation with respect to x_i, particularly x_α. An arrow ($\to$) will denote either convergence or replacement, which should not lead to misunderstandings.

2. In-plane deformation of symmetric three-layer laminates of moderate thickness

The aim of this section is to form a new two-dimensional model of a three-layer symmetric laminate, capable of describing the independent in-plane displacements of the faces and of the internal layer. In particular, the cross-ply laminates of the $[0_m^\circ, 90_n^\circ]_s$ class are of such a type.

Consider a symmetric laminate composed of the faces of thickness d and the internal layer of thickness $2c$. The middle plane Ω of the internal layer is parametrized by Cartesian coordinates x_α; $(x_\alpha) = x \in \Omega$. The whole laminate occupies a cylindrical domain $\mathcal{B} = \Omega \times (-h, h)$; $h = c + d$. To an arbitrary point $\mathbf{x} \in \mathcal{B}$ we assign its coordinates $\mathbf{x} = (x_i) = (x_\alpha, x_3 = z)$, z-axis being perpendicular to the Ω plane.

The lower and upper faces $z = \pm h$ are assumed to be free of loads, whilst the lateral edge surface $S = \Gamma \times (-h, h)$, $\Gamma = \partial\Omega$, is subjected to the tractions $p^i(s, z)$, $s \in \Gamma$, on its part $S_\sigma = \Gamma_\sigma \times (-h, h)$. The remaining part of S, $S_w = \Gamma_w \times (-h, h)$ ($\overline{\Gamma}_w \cup \overline{\Gamma}_\sigma = \Gamma$) is clamped. For the sake of further simplifications, the loading p^i is assumed to have the following through-the-thickness distribution

$$p^\alpha(s,z) = \begin{cases} \dfrac{1}{2c}\overline{L}^\alpha(s), & |z| < c, \\ \dfrac{1}{2d}(\overline{N}^\alpha(s) - \overline{L}^\alpha(s)), & \text{otherwise;} \end{cases} \tag{2.1}$$

$$p^3(s,z) = \begin{cases} \dfrac{1}{2d}(z-h)\overline{Q}(s), & c < z < h, \\ -\dfrac{z}{2c}\overline{Q}(s), & |z| < c, \\ \dfrac{1}{2d}(z+h)\overline{Q}(s), & -h < z < -c. \end{cases} \tag{2.2}$$

The loading functions $\overline{N}^{\alpha}$, $\overline{L}^{\alpha}$, $\overline{Q}$ are defined on Γ_{σ}. The body forces are omitted.

The through-the-thickness distribution of elastic compliances has the form

$$D_{ijkl} = \begin{cases} D^m_{ijkl}, & |z| < c, \\ D^f_{ijkl}, & \text{otherwise}, \end{cases} \tag{2.3}$$

and the components of the tensors $\mathbf{D}^m$ and $\mathbf{D}^f$ may depend on $\mathbf{x} \in \mathcal{B}$. As it is usually done in most treatments, we assume that the $z = \text{const}$ planes are the planes of material symmetry, hence

$$D^n_{3\alpha\beta\gamma} = D^n_{333\alpha} = 0, \qquad n = m \quad \text{or} \quad f. \tag{2.4}$$

The tensor $\mathbf{D}$ satisfies the usual symmetry condition

$$D_{ijkl} = D_{jikl} = D_{klij}. \tag{2.5}$$

We make further the following assumption

$$(II) \quad \left| \begin{array}{l} D_{ijkl} \in L^{\infty}(\mathcal{B}); \\ \text{there exists a constant } C > 0 \text{ such that} \\ D_{ijkl}(x) T^{ij} T^{kl} \geq C |\mathbf{T}|^2, \end{array} \right.$$

for a.e. $\mathbf{x} \in \mathcal{B}$ and for each $\mathbf{T} \in \mathbb{E}^3_s$, where $\mathbb{E}^3_s$ is the space of symmetric 3×3 matrices and

$$|\mathbf{T}|^2 = \sum_{i,j=1}^{3} T^{ij} T^{ij}.$$

We recall that throughout this paper only Cartesian coordinate systems are employed, thus we may identify (T^{ij}) with (T_{ij}), etc. Moreover, C possibly with a subscript will denote a positive constant.

Within the three-dimensional framework, the problem of equilibrium of the laminate considered amounts to finding the stress fields $\tilde{\sigma}^{ij}$ as well as the displacement fields $\tilde{w}^i$ for which the two-field REISSNER [43] functional

$$I(\mathbf{w}, \boldsymbol{\sigma}) = \int_{\mathcal{B}} \left[\frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha})\sigma^{\alpha\beta} + (w_{\alpha,3} + w_{3,\alpha})\sigma^{\alpha 3} + w_{3,3}\sigma^{33} - \frac{1}{2} D_{ijkl}\sigma^{ij}\sigma^{kl} \right] d\mathbf{x} - \int_{S_{\sigma}} p^i(s,z) w_i(s,z)\, ds\, dz, \tag{2.6}$$

assumes its stationary value at a saddle point, cf. NEČAS and HLAVÁČEK [41].

To prove that the functional I possesses a unique saddle point one can use the ARNOLD and FALK [5] version of BREZZI's theorem [6].

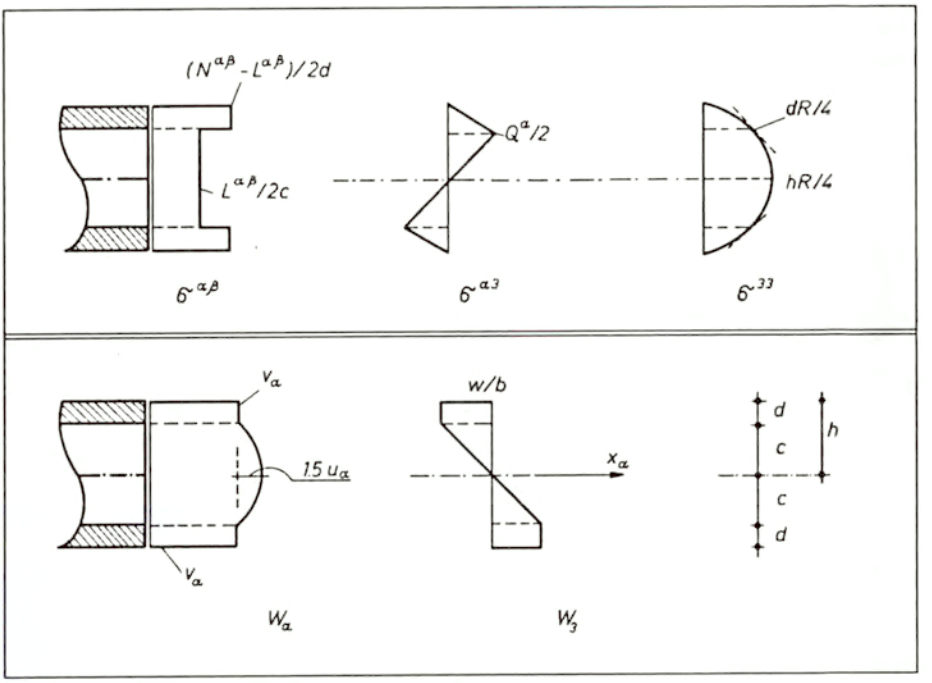

FIG. 1. Stress and displacement assumptions.

To facilitate the envisaged treatment of cracks in the interior layer $|z| < c$, it is thought to be helpful to develop a new two-dimensional laminate model. We base this modelling upon the following stress assumptions (Fig. 1)

$$\sigma^{\alpha\beta}(\mathbf{x}) = \begin{cases} \dfrac{1}{2c} L^{\alpha\beta}(x), & |z| < c, \\ \dfrac{1}{2d}\left[N^{\alpha\beta}(x) - L^{\alpha\beta}(x)\right], & \text{otherwise;} \end{cases} \tag{2.7}$$

$$\sigma^{\alpha 3}(\mathbf{x}) = \begin{cases} \dfrac{1}{2d}(z-h)Q^{\alpha}(x), & c < z < h, \\ -\dfrac{z}{2c}Q^{\alpha}(x), & |z| < c, \\ \dfrac{1}{2d}(z+h)Q^{\alpha}(x), & -h < z < -c; \end{cases} \tag{2.8}$$

$$\sigma^{33}(\mathbf{x}) = \begin{cases} \dfrac{1}{4d}(z-h)^2 R(x), & c < z < h, \\ \dfrac{1}{4c}(-z^2 + ch)R(x), & |z| < c, \\ \dfrac{1}{4d}(z+h)^2 R(x), & -h < z < -c. \end{cases} \tag{2.9}$$

Here $L^{\alpha\beta}$, $N^{\alpha\beta}$, Q^{α}, R are unknown fields defined on Ω. The stress assumptions (2.7)–(2.9) can be viewed as a modification of HASHIN's [17, 18] formulae for stresses in which counterparts of the stress resultants N^{11}, N^{12} were considered as determined by the boundary loads applied, Q^{α} and R being directly expressed in terms of $L^{\alpha\beta}$ by: $Q^{\alpha} = L^{\alpha\beta}_{,\beta}$, $R = -L^{\alpha\beta}_{,\alpha\beta}$. In our approach the stress resultants $N^{\alpha\beta}$ are unknowns of the model. Moreover, we introduce Q^{α} and R as independent stress resultants, similarly as it is usually done in the theory of plates, cf. REISSNER [42, 43].

The stress assumptions (2.7)–(2.9) determine the two-dimensional model to be derived. However, to have all information about the through-the-thickness distribution of the displacements, it is helpful to add the kinematic assumptions:

$$w_{\alpha}(\mathbf{x}) = \begin{cases} v_{\alpha}(x) + \dfrac{3}{2c^2}(c^2 - z^2)u_{\alpha}(x), & |z| < c, \\ v_{\alpha}(x), & \text{otherwise;} \end{cases} \tag{2.10}$$

$$w_3(\mathbf{x}) = \begin{cases} \dfrac{1}{b}w(x), & c < z < h, \\ \dfrac{z}{c}\dfrac{w(x)}{b}, & |z| < c, \\ -\dfrac{1}{b}w(x), & -h < z < -c; \end{cases} \tag{2.11}$$

where $b = \dfrac{d}{2} + \dfrac{c}{3}$, cf. Fig. 1. Note that

$$\int_{-h}^{h} \sigma^{\alpha\beta}\, dz = N^{\alpha\beta}, \qquad \frac{1}{2c}\int_{-c}^{c} w_{\alpha}\, dz = v_{\alpha} + u_{\alpha}. \tag{2.12}$$

The specific form of the hypotheses (2.7)–(2.11) cannot be cleared up at this stage of the analysis. We mention only that stresses (2.8) and (2.9) satisfy the boundary conditions on the faces

$$\sigma^{k3}(x, \pm h) = 0, \tag{2.13}$$

as well as the continuity conditions on the interfaces $z = \pm c$.

Deformations associated with displacements (2.10) and (2.11) are not correlated with stresses (2.7)–(2.9) by constitutive relationships. Modelling based upon the $I(\mathbf{w}, \boldsymbol{\sigma})$ functional admits such a mismatch. Nevertheless, the form of kinematic assumptions (2.10) and (2.11) has no influence on the errors of stress evaluation. The model construction could be based only upon the stress constraints (2.7)–(2.9) and then $(\mathbf{v}, \mathbf{u}, w)$ would occur formally as Lagrangian multipliers. The hypotheses (2.10) and (2.11) endow the multipliers with a physical meaning.

A more accurate evaluation of the displacement fields can be done after solving the problem by integrating the three-dimensional constitutive relationships with stresses given by (2.7)–(2.9). In this manner, displacement representations compatible with stress fields of HASHIN [18] have recently been found by MCCARTNEY [37]; similar ones can be constructed for the (2.7)–(2.9) representations. They could be helpful in forming the displacement-based models, cf. comments in Final Remarks.

Let us now proceed to form the two-dimensional model of the laminate assuming, for the sake of simplicity, that the compliances D_{ijkl} depend on $x \in \Omega$ only. We substitute the expressions (2.7)–(2.11) into the Reissner functional (2.6) and perform z-integration to obtain a new functional J:

$$J(\mathbf{v}, \mathbf{u}, w; \mathbf{N}, \mathbf{L}, \mathbf{Q}, R) = I(\mathbf{w}, \boldsymbol{\sigma}), \tag{2.14}$$

where the fields $\mathbf{w}$, $\boldsymbol{\sigma}$ have the form (2.7)–(2.11). The functional J has the form

$$J = \int_{\Omega} \Big[v_{\alpha,\beta} N^{\alpha\beta} + u_{\alpha,\beta} L^{\alpha\beta} + (u_\alpha - w_{,\alpha}) Q^\alpha + w\, R - W_c(x, \mathbf{N}, \mathbf{L}, \mathbf{Q}, R) \Big] dx - \int_{\Gamma_\sigma} \left(\overline{N}^\alpha v_\alpha + \overline{L}^\alpha u_\alpha - \overline{Q} w \right) ds, \tag{2.15}$$

where the complementary energy reads

$$2W_c = D^N_{\alpha\beta\lambda\mu} N^{\alpha\beta} N^{\lambda\mu} + D^L_{\alpha\beta\lambda\mu} L^{\alpha\beta} L^{\lambda\mu} + 2D^{NL}_{\alpha\beta\lambda\mu} N^{\alpha\beta} L^{\lambda\mu} + D^Q_{\alpha\beta} Q^\alpha Q^\beta + 2D^{RL}_{\alpha\beta} R L^{\alpha\beta} + 2D^{RN}_{\alpha\beta} R N^{\alpha\beta} + D^R R^2. \tag{2.16}$$

The generalized compliances $\mathbf{D}^N$, $\mathbf{D}^L$, $\mathbf{D}^{NL}$, $\mathbf{D}^Q$, $\mathbf{D}^{RL}$, $\mathbf{D}^{RN}$, D^R depend on $x \in \Omega$ if D_{ijkl} do, and they are given by

$$\begin{aligned}
D^N_{\alpha\beta\lambda\mu} &= \frac{1}{2d} D^f_{\alpha\beta\lambda\mu}, & D^{LN}_{\alpha\beta\lambda\mu} &= -D^N_{\alpha\beta\lambda\mu}, \\
D^L_{\alpha\beta\lambda\mu} &= D^N_{\alpha\beta\lambda\mu} + \frac{1}{2c} D^m_{\alpha\beta\lambda\mu}, & D^Q_{\alpha\beta} &= \frac{2}{3} \left(d\, D^f_{\alpha 3 \beta 3} + c\, D^m_{\alpha 3 \beta 3} \right), \\
D^{RL}_{\alpha\beta} &= -\frac{d}{12} D^f_{\alpha\beta 33} + \frac{1}{4} \left(h - \frac{c}{3} \right) D^m_{\alpha\beta 33}, & D^{RN}_{\alpha\beta} &= \frac{d}{12} D^f_{\alpha\beta 33}, \\
D^R &= \frac{d^3}{40} D^f_{3333} + \frac{c}{8} \left(h^2 - \frac{2}{3} ch + \frac{c^2}{5} \right) D^m_{3333}.
\end{aligned} \tag{2.17}$$

Their symmetry properties follow directly from those of D^f_{ijkl} and D^m_{ijkl}. The functional J attains its stationary value if the following relations are satisfied:

i) the equilibrium equations

$$-N^{\alpha\beta}{}_{,\beta} = 0, \qquad -L^{\alpha\beta}{}_{,\beta} + Q^\alpha = 0, \qquad Q^\alpha{}_{,\alpha} + R = 0, \tag{2.18}$$

ii) the constitutive relationships

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= D^{N}_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^{NL}_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RN}_{\alpha\beta} R, \\
\gamma_{\alpha\beta} &= D^{NL}_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^{L}_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RL}_{\alpha\beta} R, \\
w &= D^{RN}_{\alpha\beta} N^{\alpha\beta} + D^{RL}_{\alpha\beta} L^{\alpha\beta} + D^{R} R,
\end{aligned} \tag{2.19}
$$

$$
\kappa_{\alpha} = D^{Q}_{\alpha\beta} Q^{\beta}, \tag{2.20}
$$

where the deformation measures introduced above are defined as follows:

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \\
\gamma_{\alpha\beta} &= \gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \\
\kappa_{\alpha} &= \kappa_{\alpha}(\mathbf{u}, w) = u_{\alpha} - w_{,\alpha},
\end{aligned} \tag{2.21}
$$

iii) the stress-type boundary conditions along the line Γ_{σ}

$$
\begin{aligned}
N_n &= \overline{N}_n, \qquad N_\tau = \overline{N}_\tau, \\
L_n &= \overline{L}_n, \qquad L_\tau = \overline{L}_\tau, \qquad Q = \overline{Q}.
\end{aligned} \tag{2.22}
$$

Here

$$
\begin{aligned}
N_n &= N^{\alpha\beta} n_\alpha n_\beta, \qquad N_\tau = N^{\alpha\beta} n_\beta \tau_\alpha, \\
L_n &= L^{\alpha\beta} n_\alpha n_\beta, \qquad L_\tau = L^{\alpha\beta} n_\beta \tau_\alpha, \qquad Q = Q^{\alpha} n_\alpha, \\
\overline{N}_n &= \overline{N}^{\alpha} n_\alpha, \qquad \overline{N}_\tau = \overline{N}^{\alpha} \tau_\alpha, \quad \text{etc.}
\end{aligned} \tag{2.23}
$$

Here $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit vectors: outward normal and tangent to the Γ_{σ} line, respectively.

The constitutive relationships (2.19) and (2.20) can be inverted to the form

$$
\begin{aligned}
N^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_{v} \varepsilon_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{vu} \gamma_{\alpha\beta} + A^{\lambda\mu}_{vw} w, \\
L^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_{vu} \varepsilon_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{u} \gamma_{\alpha\beta} + A^{\lambda\mu}_{uw} w, \\
R &= A^{\alpha\beta}_{vw} \varepsilon_{\alpha\beta} + A^{\alpha\beta}_{uw} \gamma_{\alpha\beta} + A_{w} w;
\end{aligned} \tag{2.24}
$$

$$
Q^{\alpha} = H^{\alpha\beta} \kappa_{\beta}, \tag{2.25}
$$

that can be viewed as the primal one.

Note that if the equations of equilibrium (2.18) are fulfilled for every $x \in \Omega$, then the stresses σ^{ij} determined by formulae (2.7)–(2.9) satisfy the three- dimensional equilibrium equations $\sigma^{ij}{}_{,j} = 0$ identically for every $(x, z) \in \mathcal{B}$. Bearing in mind that these stresses satisfy the boundary conditions (2.13) on the faces we note that, up to the boundary zone along S, the stresses assumed are statically

admissible. This property clears up the form of the stress hypotheses (2.7)–(2.9). Therefore the model construction presented here is similar to REISSNER's [43] construction concerning transversely homogeneous plates.

The strong formulation of the equilibrium problem reads:

Find the kinematic and stress fields $\mathbf{v} = (v_\alpha)$, $\mathbf{u} = (u_\alpha)$, w; $\mathbf{N} = (N^{\alpha\beta})$, $\mathbf{L} = (L^{\alpha\beta})$, $\mathbf{Q} = (Q^\alpha)$ and R such that there are satisfied:

- the equilibrium equations (2.18),
- the constitutive relationships (2.24), (2.25),
- the strain – displacements relations (2.21),
- the boundary conditions (2.22) on Γ_σ,
- and the boundary conditions:

$$\mathbf{v} = \mathbf{0}, \qquad \mathbf{u} = \mathbf{0}, \qquad w = 0 \qquad \text{on} \quad \Gamma_w. \tag{2.26}$$

As a prerequisite for the primal variational formulation, we define the space

$$V = \Big\{(\mathbf{v}, \mathbf{u}, w) \mid \mathbf{v} \in H^1(\Omega)^2, \ \mathbf{u} \in H^1(\Omega)^2, \ w \in H^1(\Omega); \\ \mathbf{v} = \mathbf{0}, \ \mathbf{u} = \mathbf{0}, \ w = 0 \ \text{on} \ \Gamma_w\Big\}$$

representing the space of kinematically admissible fields. Let us define the bilinear form

$$a_\Omega(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}', w') = \int_\Omega \big[N^{\alpha\beta}(\mathbf{v}, \mathbf{u}, w)\varepsilon_{\alpha\beta}(\mathbf{v}') + L^{\alpha\beta}(\mathbf{v}, \mathbf{u}, w)\gamma_{\alpha\beta}(\mathbf{u}') \\ + Q^\alpha(\mathbf{u}, w)\kappa_\alpha(\mathbf{u}', w') + R(\mathbf{v}, \mathbf{u}, w)w'\big]\, dx, \tag{2.27}$$

(where $\mathbf{N}$, $\mathbf{L}$, $\mathbf{Q}$, R depend on $(\mathbf{v}, \mathbf{u}, w)$ according to the relationships (2.21), (2.24) and (2.25)) and the linear form

$$f(\mathbf{v}', \mathbf{u}', w') = \int_{\Gamma_\sigma} \left(\overline{N}^\alpha v'_\alpha + \overline{L}^\alpha u'_\alpha - \overline{Q} w'\right) ds. \tag{2.28}$$

The variational problem in its primal formulation reads:

$$(P_\Omega) \quad \left| \begin{array}{l} \text{Find} \ \ (\mathbf{v}, \mathbf{u}, w) \in V \quad \text{such that} \\ a_\Omega(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}', w') = f(\mathbf{v}', \mathbf{u}', w') \quad \forall (\mathbf{v}', \mathbf{u}', w') \in V. \end{array} \right. \tag{2.29}$$

Instead of studying problem P_Ω one can solve a more general one, that of the existence of a saddle point of the Reissner-type functional (2.15). We set

$$\mathfrak{D} = \begin{bmatrix} \mathbf{D}^N & \mathbf{D}^{NL} & \mathbf{D}^{RN} \\ \mathbf{D}^{NL} & \mathbf{D}^L & \mathbf{D}^{RL} \\ \mathbf{D}^{RN} & \mathbf{D}^{RL} & D^R \end{bmatrix}, \qquad \mathfrak{R} = (\mathbf{N}, \mathbf{L}, R), \qquad \mathfrak{R}^T = \begin{bmatrix} \mathbf{N} \\ \mathbf{L} \\ R \end{bmatrix}. \tag{2.30}$$

Thus we may write

$$W_c(x, \mathbf{N}, \mathbf{L}, \mathbf{Q}, R) = \frac{1}{2}(\mathfrak{N}\,\mathfrak{D}\,\mathfrak{N}^T + D^Q_{\alpha\beta}Q^\alpha Q^\beta). \tag{2.31}$$

The properties of the compliance tensor (D_{ijkl}) imply that each element of the matrix $\mathfrak{D}$ belongs to $L^\infty(\Omega)$; similarly $D^Q_{\alpha\beta} \in L^\infty(\Omega)$. Further, the assumption (H) implies that there exists a constant $C > 0$ such that

$$(H_1) \quad \begin{vmatrix} \mathbf{K}\mathfrak{D}(x)\mathbf{K}^T \geq C\sum_{\alpha,\beta=1}^2 (A_{\alpha\beta}A_{\alpha\beta} + B_{\alpha\beta}B_{\alpha\beta} + a^2), \\ D^Q_{\alpha\beta}(x)a^\alpha a^\beta \geq \sum_{\alpha=1}^2 a^\alpha a^\alpha, \end{vmatrix}$$

for a.e. $x \in \Omega$ and for $\mathbf{K} = (\mathbf{A}, \mathbf{B}, a) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^2$.

To corroborate this statement we write

$$\frac{1}{2}D_{ijkl}(x)\sigma^{ij}\sigma^{kl} = \frac{1}{2}D_{\alpha\beta\lambda\mu}(x)\sigma^{\alpha\beta}\sigma^{\lambda\mu} + D_{\alpha\beta33}(x)\sigma^{\alpha\beta}\sigma^{33} + 2D_{\alpha3\beta3}(x)\sigma^{\alpha3}\sigma^{\beta3} + \frac{1}{2}D_{3333}(x)(\sigma^{33})^2.$$

Because $[D_{ijkl}(x)]$ is positive definite for a.e. $x \in \Omega$, therefore $D_{3333}(x) > 0$ and consequently $D^R(x) > 0$ for a.e. $x \in \Omega$.

Similarly, we have

$$D_{\alpha3\beta3}(x)t^\alpha t^\beta > 0, \qquad \text{for a.e. } x \in \Omega \qquad \text{and for all } \mathbf{t} \in \mathbb{R}^2, \qquad \mathbf{t} \neq \mathbf{0}.$$

Treating $(\sigma^{\alpha3})$ and $\mathbf{Q} \neq \mathbf{0}$ in (2.8) as arbitrary, though sufficiently regular, we write

$$\int_{-h}^{h} D_{\alpha3\beta3}(x)\sigma^{\alpha3}\sigma^{\beta3}\,dz = \left(\int_{-h}^{h} f(z)D_{\alpha3\beta3}(x)\,dz\right) Q^\alpha(x)Q^\beta(x) = D^Q_{\alpha\beta}Q^\alpha(x)Q^\beta(x) > 0, \qquad \text{a.e. } x \in \Omega,$$

where $f(z)$ is inferred from (2.8). Hence the matrix $\mathbf{D}^Q$ is positive definite and the inequality $(H_1)_2$ follows, since for a fixed $x \in \Omega$ at which $\mathbf{Q}(x)$ makes sense one may set $\mathbf{a} = \mathbf{Q}(x) \in \mathbb{R}^2$ and treat $\mathbf{a}$ as an arbitrary element of $\mathbb{R}^2$.

For any sufficiently regular $\boldsymbol{\sigma} = (\sigma^{ij}) \neq \mathbf{0}$ such that $\sigma^{\alpha3} = 0$ $(\alpha = 1, 2)$, one has

$$j_0(x, \sigma^{\alpha\beta}, \sigma^{33}) := \frac{1}{2}D_{\alpha\beta\lambda\mu}(x)\sigma^{\alpha\beta}\sigma^{\lambda\mu} + D_{\alpha\beta33}(x)\sigma^{\alpha\beta}\sigma^{33} + \frac{1}{2}D_{3333}(x)(\sigma^{33})^2 > 0, \qquad \text{a.e. } x \in \Omega. \tag{2.32}$$

For such a σ we write Eqs. (2.7) and (2.9) concisely in the form

$$(\sigma^{\alpha\beta}, \sigma^{33}) = \mathbf{A}(z)\mathfrak{N}^T. \tag{2.33}$$

Hence, on account of (2.32) we obtain

$$j_c(x, \mathbf{N}(x), \mathbf{L}(x), R(x)) := \int_{-h}^{h} j_0(x, \sigma^{\alpha\beta}(x,z), \sigma^{33}(x,z))\, dz \tag{2.34}$$

$$= \int_{-h}^{h} j_0(x, \mathbf{A}(z)\mathfrak{N}^T(x))\, dz = \frac{1}{2}\mathfrak{N}(x)\mathfrak{D}(x)\mathfrak{N}^T(x) > 0, \qquad \text{a.e. } x \in \Omega,$$

provided that $\mathfrak{N}(x) \neq \mathbf{0}$.

Consequently, the matrix $\mathfrak{D}$ of the generalized elastic compliances is positive definite and the partial complementary potential j_c is strictly convex, cf. also ROCKAFELLAR's Th. 5.7 [46]. The condition $(H_1)_1$ is thus satisfied.

The regularity of the boundary loading is specified by

$$(H_2) \qquad \overline{N}^{\alpha} \in L^2(\Gamma_\sigma), \qquad \overline{L}^{\alpha} \in L^2(\Gamma_\sigma), \qquad \overline{Q} \in L^2(\Gamma_\sigma).$$

By applying ARNOLD and FALK [5] version of the BREZZI's theorem [6], one can now prove that the functional J possesses a unique saddle point.

3. Modelling transverse cracking in the internal layer

When stretched, the composite laminates undergo interlaminar delaminations, fiber breakage and intralaminar cracking. Only the last mode of damage mentioned is discussed in the present paper. The intralaminar cracks observed are not necessarily straight; they can also be curved or partially angled, cf. GROVES *et al.* [11]. The model proposed in Sec. 2 makes it possible to figure out only cracks, which are straight in the transverse direction.

In the present section we will show how to express the presence of such cracks in terms of the two-dimensional fields of our model. The cracks are allowed to open or close.

Consider the laminate weakened by a crack in the internal layer. The crack surface $S_F = F \times (-c, c)$ is perpendicular to the domain Ω, F being its projection on Ω, cf. Fig. 2. The crack surfaces observed in experiments are parallel to the fibers, cf. GARRETT and BAILEY [10]. In the present paper, however, we shall not associate the shape of the crack directions with anisotropy of the laminae. We assume simply, that the crack geometry is *a priori* given.

Let us extend the arc F so that it divides the domain Ω into two parts Ω_1, Ω_2, cf. Fig. 2. Consequently, the domain $\mathcal{B}$ is divided into two subdomains $\mathcal{B}_\alpha$ =

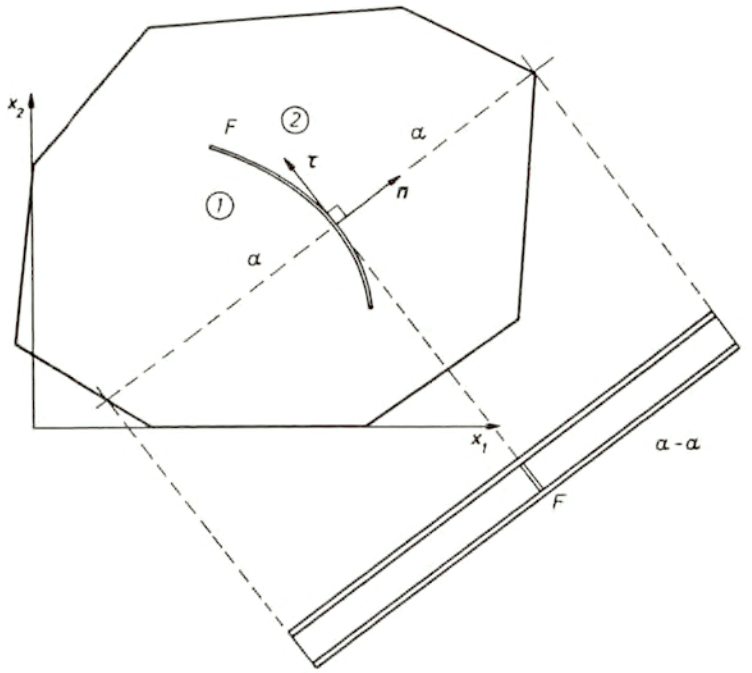

FIG. 2. Laminate with a crack in the internal layer.

$\Omega_\alpha \times (-h, h)$. Let F_{z_0} be the set of points (x, z) such that $x \in F$ and $z = z_0$. Let $\mathbf{n}^z$ be a unit vector at a point (x, z) normal to F_z and such that $\mathbf{n}^z = (n_\alpha, 0)$, where $(n_\alpha) = \mathbf{n}$ is a unit vector at $(x, 0)$ normal to F and directed outward the domain Ω_1. Similarly we define the tangent vector $\boldsymbol{\tau}^z = (\tau_\alpha, 0)$. A field g defined on $\mathcal{B}$ for $|z| < c$ can suffer a jump across the surface S_F:

$[\![g]\!]_{S_F} = g|_2 - g|_1$; here $g|_\alpha$ represents the value of g at the α-th side of S_F, viz. from the side of the domain $\mathcal{B}_\alpha$. Now we can express the contact conditions on the surface S_F of the crack as follows:

$$
\begin{aligned}
&\sigma_n = \overset{1}{\sigma}_n = \overset{2}{\sigma}_n \leq 0, \qquad [\![w_n]\!]_{S_F} \geq 0, \qquad \sigma_n [\![w_n]\!]_{S_F} = 0,\\
&\overset{1}{\sigma}_{n\tau} = \overset{2}{\sigma}_{n\tau} = 0, \qquad \overset{1}{\sigma}_{nz} = \overset{2}{\sigma}_{nz} = 0,
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
&\overset{\delta}{\sigma}_n = \sigma^{\alpha\beta}{}_{|\delta} n_\alpha n_\beta, \qquad \overset{\delta}{\sigma}_{n\tau} = \sigma^{\alpha\beta}{}_{|\delta} n_\alpha \tau_\beta,\\
&\overset{\delta}{\sigma}_{nz} = \sigma^{\alpha 3}{}_{|\delta} n_\alpha, \qquad w_n = \sum_\alpha w_\alpha n_\alpha,
\end{aligned}
\tag{3.2}
$$

and

$$
\sigma^{\alpha k} = \sigma^{\alpha k}(x, z), \qquad w_\alpha = w_\alpha(x, z).
$$

In the Signorini-type conditions (3.1) the friction between crack lips is obviously neglected. Thus the jump $[\![w_\tau]\!]_{S_F}$ assumes arbitrary values.

According to the stress hypotheses (2.7)–(2.9) one readily finds

$$\sigma_n = \begin{cases} \frac{1}{2c} L_n, & |z| < c, \\ \frac{1}{2d}(N_n - L_n), & \text{otherwise;} \end{cases}$$

$$\sigma_{n\tau} = \begin{cases} \frac{1}{2c} L_\tau, & |z| < c, \\ \frac{1}{2d}(N_\tau - L_\tau), & \text{otherwise;} \end{cases} \tag{3.3}$$

$$\sigma_{nz} = \begin{cases} \frac{z-h}{2d} Q, & c < z < h, \\ -\frac{z}{2c} Q, & |z| < c, \\ \frac{z+h}{2d} Q, & -h < z < -c, \end{cases} \tag{3.4}$$

where

$$L_n = L^{\alpha\beta} n_\alpha n_\beta, \quad N_n = N^{\alpha\beta} n_\alpha n_\beta, \quad L_\tau = L^{\alpha\beta} n_\alpha \tau_\beta,$$
$$N_\tau = N^{\alpha\beta} n_\alpha \tau_\beta, \quad Q = Q^\alpha n_\alpha. \tag{3.5}$$

In view of (2.10) and (3.3), the conditions $(3.1)_{1,2}$ can be rewritten in the form:

$$L_n = \overset{1}{L}_n = \overset{2}{L}_n \leq 0, \quad [\![u_n]\!] \geq 0, \quad L_n [\![u_n]\!] = 0, \quad \overset{1}{L}_\tau = \overset{2}{L}_\tau = 0 \quad \text{on } F. \tag{3.6}$$

Now $[\![\cdot]\!]$ denotes a jump on F and

$$\overset{\delta}{L}_n = L^{\alpha\beta}|_\delta n_\alpha n_\beta, \quad \overset{\delta}{L}_\tau = L^{\alpha\beta}|_\delta n_\alpha \tau_\beta \tag{3.7}$$

are values associated with δ-th side of F; $\delta = 1, 2$. The fields $\mathbf{v}$, w are assumed not to suffer jumps on F, the jump $[\![u_\tau]\!]$ being unconstrained.

The stress assumptions (2.7)–(2.9) do not allow for constructing a two-dimensional approximation of condition $(3.1)_2$.

In the domains of the exterior layers ($c < z < h$, $-h < z < -c$) all the fields are assumed as continuous. Hence the fields N_n, N_τ, Q do not suffer jumps on F.

The strong formulation of the equilibrium problem of the plate with a crack amounts to finding the fields $(\mathbf{v}, \mathbf{u}, w)$ satisfying:

- the equilibrium equations (2.18) for $x \in \Omega \backslash F$,
- the constitutive relations (2.24) and (2.25) for $x \in \Omega \backslash F$,
- the boundary conditions (2.26) and (2.22),
- Signorini-type conditions (3.6) on F.

Let us pass now to the variational formulation of the problem governed by (2.18), (2.24) and (2.25), (2.26) and (2.22) along with (3.6). The set of kinematically admissible fields has the form

$$\mathbb{K} = H_{\Gamma_w}(\Omega)^2 \times K(\Omega\backslash F) \times H_{\Gamma_w}(\Omega), \tag{3.8}$$

where

$$H_{\Gamma_w}(\Omega) = \left\{\mathbf{v} \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_w\right\}, \tag{3.9}$$

$$K(\Omega\backslash F) = \left\{\mathbf{u} \in H^1(\Omega\backslash F)^2 \mid \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_w \quad \text{and} \quad [\![u_n]\!] \geq 0 \quad \text{on} \quad F\right\}. \tag{3.10}$$

The variational problem reads:

$$(P_{\Omega\backslash F}) \left| \begin{array}{l} \text{Find} \quad (\mathbf{v}, \mathbf{u}, w) \in \mathbb{K} \quad \text{such that} \\ a_{\Omega\backslash F}(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}' - \mathbf{u}, w') \geq f(\mathbf{v}', \mathbf{u}' - \mathbf{u}, w') \quad \forall (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}. \end{array} \right. \tag{3.11}$$

The bilinear form $a_{\Omega\backslash F}(.,.)$ is defined by Eq. (2.27) with integration now over $\Omega\backslash F$.

By applying Th. 2.1 of KINDERLEHRER and STAMPACCHIA [21, Chap. II] we will prove that the variational inequality (3.11) admits a unique solution. In fact, in our case the set $\mathbb{K}$ is convex and closed and the linear form f, given by (2.28), is continuous in the space

$$V(\Omega\backslash F) = H_{\Gamma_w}(\Omega)^2 \times H_{\Gamma_w}(\Omega\backslash F)^2 \times H_{\Gamma_w}(\Omega),$$

and also on $L^2(\Omega)^2 \times L^2(\Omega\backslash F)^2 \times L^2(\Omega)$. It remains to verify that the bilinear form $a_{\Omega\backslash F}$ is coercive. Toward this end we set

$$\mathfrak{A} = \mathfrak{D}^{-1} = \begin{bmatrix} \mathbf{A}_v & \mathbf{A}_{vu} & \mathbf{A}_{vw} \\ \mathbf{A}_{vu} & \mathbf{A}_u & \mathbf{A}_{uw} \\ \mathbf{A}_{vw} & \mathbf{A}_{uw} & A_w \end{bmatrix}, \qquad \mathbf{H} = [H^{\alpha\beta}] = (\mathbf{D}^Q)^{-1}. \tag{3.12}$$

The explicit form of the stiffness matrix $\mathfrak{A}$ can be found by using the Fenchel conjugate of $j_c(x, ., ., .)$, i.e.

$$\begin{aligned} j_1(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) &:= j_c^*(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \\ &= \sup\left\{N^{\alpha\beta}\varepsilon_{\alpha\beta} + L^{\alpha\beta}\gamma_{\alpha\beta} + Rr - j_c(x, \mathbf{N}, \mathbf{L}, R) \mid (\mathbf{N}, \mathbf{L}, R) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}\right\} \\ &= \frac{1}{2}\mathbf{E}\mathfrak{A}(x)\mathbf{E}^T, \end{aligned} \tag{3.13}$$

where $\mathbf{E} = (\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$. The properties of the compliance matrices $\mathfrak{D}$ and $\mathbf{D}^Q$ imply that a positive constant $C_1 > 0$ exists such that for a.e. $x \in \Omega$

$$\mathbf{E}\mathfrak{A}(x)\mathbf{E}^T \geq C_1\left(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{\gamma}|^2 + r^2\right), \qquad H^{\alpha\beta}(x)a_\alpha a_\beta \geq C_1|\mathbf{a}|^2, \tag{3.14}$$

for all $\mathbf{E} = (\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^2$. By taking now into account (3.14), we obtain, for any $(\mathbf{v}', \mathbf{u}', w') \in V(\Omega \backslash F)$

$$a_{\Omega\backslash F}(\mathbf{v}', \mathbf{u}', w'; \mathbf{v}', \mathbf{u}', w') \geq \tilde{C}_1 \Big(\|\boldsymbol{\varepsilon}(\mathbf{v}')\|_{0,\Omega}^2 + \|\boldsymbol{\gamma}(\mathbf{u}')\|_{0,\Omega\backslash F}^2 + \|w'\|_{0,\Omega}^2 + \|\nabla w' - \mathbf{u}'\|_{0,\Omega}^2 \Big) \geq C_1 \left(\|\mathbf{v}'\|_{1,\Omega}^2 + \|\mathbf{u}'\|_{1,\Omega\backslash F}^2 + \|w'\|_{1,\Omega}^2 \right),$$

because Korn's inequality still holds for the domain $\Omega \backslash F$, cf. SANCHEZ–PALENCIA [47]. Consequently, the problem $P_{\Omega\backslash F}$ admits a unique solution $(\mathbf{v}, \mathbf{u}, w) \in \mathbb{K}$. It is worth noting that unique solvability is preserved in the practically important case, when F intersects the boundary Γ of Ω, cf. also CHACHA and SANCHEZ–PALENCIA [8]. The subdomains Ω_α ($\alpha = 1, 2$), however, have to be sufficiently regular.

4. Final remarks

The two-dimensional model of an elastic three-layer laminate, proposed in Sec. 2, has been shown to readily include transverse intralaminar cracks in the internal layer. In the last case, the equilibrium problem, written in the variational (weak) form, is described by variational inequality (3.11), provided that friction is precluded. In Part II of the paper [35], just this variational inequality will be of fundamental importance for modelling the macroscopic behaviour of laminates with periodically distributed transverse cracks in the internal layer.

The starting point of the modelling performed in Part I are stress-displacement assumptions (2.7)–(2.9). The modelling can be repeated without imposing the displacement fields if one appropriately interprets the Lagrangian multipliers, similarly as it was done by REISSNER [42, 43] in the case of bending problems. Then, however, the distribution of displacements $w_k(x, z)$ across the thickness cannot be uniquely recovered, which would result in ambiguities in expressing conditions on the crack lips in the internal layer. The model construction of Sec. 2 can be based upon displacement asssumptions only, provided that they are sufficiently "rich" to figure out the influence of the stresses σ^{k3}. Such displacement fields can be found in MCCARTNEY [37]. On the other hand, they cannot be too complicated, since the variational method based on displacement approach would only increase the grade of complexity of the model – a known dilemma in plate bending modelling – and the energy-inconsistent approaches, like that of LEVINSON [26] would here be impractical. One of the simplest possible forms of such assumptions has been adopted by HAN *et al.* [16], HAN and HAHN [15] and YOUNG and BOEHLER [58, Eq. (31)]. Subsequent steps of the modelling would be similar. Nevertheless, the conclusions drawn from the bending theory of plates are such that the theories based upon stress assumptions lead directly to well-assessed stiffnesses, while the same accuracy is difficult to achieve via the displacement-based models unless artificial "correction factors" are adopted.

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Stiffness loss in laminates with intralaminar cracks Part II. Periodic distribution of cracks and homogenization

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IN THE SECOND PART of the paper, under the assumption that periodically distributed cracks in the internal layer can open or close according to Signorini's conditions, the homogenization procedure is carried out. With the help of two types of scaling, three effective models of the laminate with smeared-out cracks are proposed: a model of moderately thick laminates with cracks of high density, a model for thin laminates with cracks of arbitrary density, and for thin laminates with densely distributed cracks. The models derived show nonlinear and hyperelastic features, the relevant hyperelastic potentials being convex or strictly convex. The homogenized constitutive relations assume Kachanov's form in which the damage moduli tensor is uniquely determined.

1. Introduction

BY APPLYING the new two-dimensional laminate model developed in Part I [8], two methods of homogenization will be applied in the second part of the paper. The first of them is based on the in-plane scaling of the longitudinal dimensions of periodicity cells, whilst the second method uses simultaneous scaling of all dimensions of cells.

The formulae interrelating the loss of effective characteristics with components of macrodeformations will be rigorously derived. It will be proved that the effective laminate has nonlinear hyperelastic properties, its hyperelastic potential being convex or strictly convex. The homogenization approach enables us to determine all components of the tensor of effective stiffnesses within the framework of one scheme. The reason for that is that the homogenization scheme satisfies Hill's lemma of equivalence of mutual works of stresses stored in the effective and periodic (here: periodically cracked) composites, being sufficiently flexible to produce all effective stiffnesses, see SUQUET [I.48]; cf. also TELEGA [I.49], where a generalization of this lemma to the case of discontinuous fields has been put forward. HILL'S [4] lemma can be symbolically written as

$$\langle \sigma' \varepsilon'' \rangle = \langle \sigma' \rangle \langle \varepsilon'' \rangle,$$

where $\langle \cdot \rangle$ means averaging (in the periodic case – over the basic cell of periodicity) and σ', ε'' are admissible stress and strain fields, respectively.

The papers which are based neither on the homogenization method nor on Hill's lemma, usually make use of a more restricted lemma of equivalency of energy (σ' and ε'' are then interrelated by a constitutive relation), capable of evaluating only diagonal components of the stiffness matrices, viz. the effective

Young and Kirchhoff moduli, cf. HASHIN [I.17], ABOUDI [I.1]. Then the Poisson ratios have to be assessed independently, cf. HASHIN [I.18].

Another feature of the homogenization method is that it rigorously distinguishes between micro- and macrofields, defining them precisely. Correctness of the homogenization results follows from the proof of convergence of the solution of the ε-problem to the solution to the homogenized problem as $\varepsilon \to 0$. In the problem studied here the following two reasons: presence of cracks and non-conventional space scaling make the convergence proof difficult. This proof is included in the paper by TELEGA and LEWIŃSKI [14].

The micromechanics model put forward in the present paper can include neither the effects due to delamination nor those due to fibre breakage. An extended micromechanics model has been recently proposed by YANG and BOEHLER [I.58]. Their model is capable of describing the onset of interlamination and its interaction with transverse cracks.

Supported by the own experimental results, the papers of ALLEN *at al.* [1], GROVES *et al.* [I.11], HARRIS *et al.* [3], LEE *et al.* [I.25], MOTOGI and FUKUDA [10], MOTOGI *et al.* [11] show that also a phenomenological continuum damage mechanics approach can be helpful in the description of stiffness degradation of composite laminates.

Throughout the second part of the paper Roman numeral I refers always to the first part of our contribution [8].

2. Regular crack system

In this section we assume that the internal layer incurs transverse cracks which form a fixed layout. No attempt will be made to interrelate the crack pattern geometry neither with the directions of principal stresses nor with the directions of anisotropy of the plies. For fiber-reinforced polymeric composites this would also be unrealistic since the onset of matrix cracking is caused mainly by the mismatch between the thermal expansion of the fiber and the resin, cf. YALVAÇ *et al.* [I.57].

Our aim is to find a relation between the layout of cracks and the stiffness loss of the laminate. To arrive at transparent and useful formulae we confine our attention to the case when the layout of cracks is periodic. Otherwise we would have to resort to stochastic methods or bounding techniques that require different mathematical tools and result in less viable final formulae, cf. TELEGA and LEWIŃSKI [I.53].

Consider the laminate of Sec. I.2 weakened by a family $\mathcal{F}$ of fissures F_j described in Sec. I.3. One can divide the domain Ω (except for a boundary zone) into homothetic rectangular cells Z_j, each of which is weakened by a crack F_j. Z_j may contain a finite number of cracks. We observe only that F_j should not intersect the boundary of the rectangle Z_j (cf. Fig. 1) as well as the boundary

Γ of the domain Ω. This assumption can be weakened so as to admit cracks intersecting the boundaries, cf. CHACHA and SANCHEZ–PALENCIA [I.8]. For the effective models which will be derived in the succeeding sections, the properties of the homogenized elastic potentials will also be discussed when cracks intersect the boundaries.

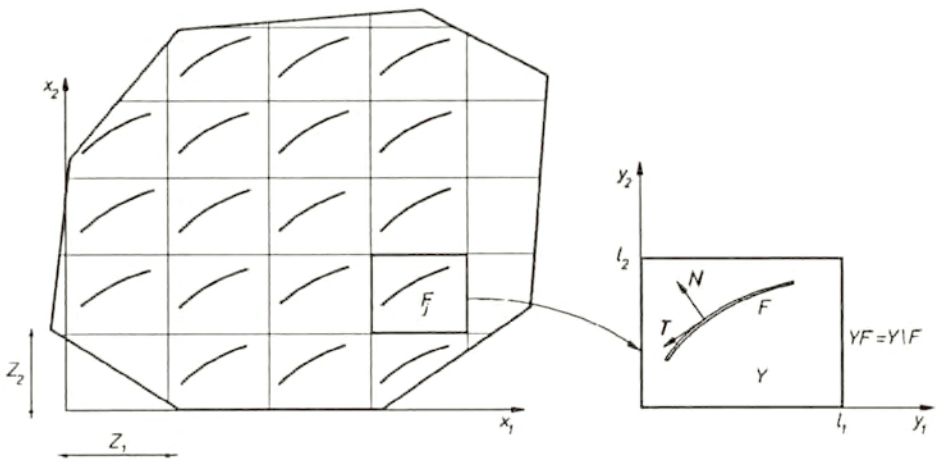


FIG. 1. Z-periodic layout of cracks. Geometry of the basic cell $YF = Y \setminus F$.

For future convenience we define a cell Z with a crack F such that all cells Z_j with cracks F_j are homothetic to it; $Z = (0, Z_1) \times (0, Z_2)$.

Mathematical description of Sec. I.3 applies here. A single crack F should be replaced by $\mathcal{F}$ which is a sum of a finite number of cracks. Both strong and variational formulations of Sec. I.3 remain formally unchanged.

If the number of cracks is very large, the solution to the relevant problem becomes unattainable even by employing numerical methods. On the other hand, a natural question arises about the overall properties of such laminates. It is the homogenization method that provides an answer to this question. As a method of averaging, this method provides a unique and perfect algorithm. As a method belonging to the family of small parameter methods, it depends upon the manner in which a small parameter ε is introduced. Two versions of the homogenization method corresponding to two methods of introducing ε into the original problem will be discussed in the sequel.

3. Moderately thick laminate weakened by transverse cracks of high density Model (h, l_0)

This section is aimed at deriving formulae for the static analysis and assessing effective stiffnesses of a moderately thick three-layer laminate densely cracked in the internal layer. The derivation will be based upon the conventional homogenization approach applied to the problem posed in the previous section. The

model thus derived will be referred to as a (h, l_0) one, which means that it applies to the case when the distances between cracks are much smaller than h, and that the thickness $2h$ can be considered as moderately thick as compared with the longitudinal dimensions of Ω.

3.1. Family of ε-problems

As a method that belongs to the family of small parameter methods, the homogenization approach requires an introduction of a small parameter. The physical nature of the problem should provide us with the hints of how to introduce this parameter. In our problem, two small parameters are present: $\max(Z_1, Z_2)$ and h (or c and d). In this section we shall assume that only the former dimension is small and we replace Z_α by εl_α and F by εF. The rectangles of periodicity are homothetic to the basic cell $Y = (0, l_1) \times (0, l_2)$. The domain $\Omega \backslash F$ is now replaced by $\Omega^\varepsilon = \Omega \backslash F^\varepsilon$, where F^ε is the sum of all cracks εF_j. The set of the kinematically admissible fields assumes the form

$$\mathbb{K}_\varepsilon := \mathbb{K}(\Omega^\varepsilon) = H_{\Gamma_w}(\Omega)^2 \times K(\Omega^\varepsilon) \times H_{\Gamma_w}(\Omega), \tag{3.1}$$

where

$$K(\Omega^\varepsilon) = \left\{ \mathbf{u} \in H^1(\Omega^\varepsilon)^2 \mid \mathbf{u} = \mathbf{0} \text{ on } \Gamma_w \text{ and } [\![u_n]\!] \geq 0 \text{ on } F^\varepsilon \right\}. \tag{3.2}$$

For a fixed $\varepsilon > 0$ the equilibrium problem assumes the form

$$(P^1_{\Omega^\varepsilon}) \quad \left| \begin{array}{l} \text{Find } (\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}_\varepsilon \text{ such that} \\ a_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \geq f(\mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \ \forall (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}_\varepsilon. \end{array} \right. \tag{3.3}$$

The bilinear form $a_{\Omega^\varepsilon}(\cdot, \cdot)$ is defined by Eq. (I.2.27), the integration over Ω being replaced here by integration over Ω^ε.

The scaling: $Z_\alpha \to \varepsilon l_\alpha$, $h \to h$ ($c \to c$, $d \to d$) used here will be referred to as an in-plane scaling; the sign ($\to$) means replacement.

REMARK 3. 1. The problem $(P^1_{\Omega^\varepsilon})$ is posed on a highly irregular domain Ω^ε, for which Korn's inequality does not apply in its standard form. However, in the paper by TELEGA and LEWIŃSKI [I.52], Korn's inequality has been derived in a form directly applicable to domains like Ω^ε presented in Fig. 1. Essential for such a domain is the assumption: $F \subset Y$, where F is closed as a set. In this case F does not intersect ∂Y. Consequently, the bilinear form a_{Ω^ε} is coercive on

$$V(\Omega^\varepsilon) = H_{\Gamma_w}(\Omega)^2 \times H_{\Gamma_w}(\Omega^\varepsilon)^2 \times H_{\Gamma_w}(\Omega),$$

and on $\mathbb{K}_\varepsilon$; the linear form f is continuous in this space. Applying now Th. 2.1 of KINDERLEHRER and STAMPACCHIA [I.21] we conclude that there exists a unique solution $(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon)$ to $P^1_{\Omega^\varepsilon}$.

The case when F intersects the boundary ∂Y of Y is obviously more complicated, cf. Fig. 2 and the paper by CHACHA and SANCHEZ–PALENCIA [I.8]. We still assume that $Y_\eta = Y \setminus F^\eta$ $(\eta > 0)$ is a connected set with a Lipschitzian boundary. Denoting by F_η^ε the sum of all holes εF_j^η $(\varepsilon > 0)$ we deduce that $\Omega_\eta^\varepsilon = \Omega \setminus F_\eta^\varepsilon$ is a domain of type I in the sense of OLEINIK *et al.* [12]. Now the boundary of the domain Ω^ε may be intersected by cracks from F^ε. By combining the results concerning extension theorems, presented in Chapter I of the book by OLEINIK *et al.* [12], with the approach used by TELEGA and LEWIŃSKI [I.52], we arrive at Korn's inequality for $V(\Omega^\varepsilon)$. Consequently, the problem $P_{\Omega^\varepsilon}^1$ still admits a unique solution for this particularly important cracking mode.

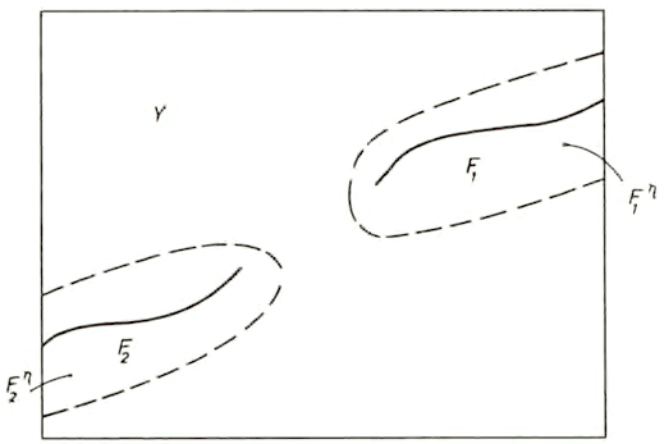


FIG. 2. Fissure $F = F_1 \cup F_2$ intersecting ∂Y.

3.2. Asymptotic solution

The asymptotic solution to the problem P_Ω^ε can be found by the method similar to that proposed by SANCHEZ–PALENCIA [I.47, Ch. 6, Sec. 6 & 7) and extended in LEWIŃSKI and TELEGA [I.29,I.30], TELEGA and LEWIŃSKI [I.52]. Therefore we shall expound here and in the succeeding sections only the ideas and results. The convergence theorem is provided in the paper by TELEGA and LEWIŃSKI [14].

Let us define the space

$$H^1_{\mathrm{per}}(Y) = \Big\{ v \in H^1(Y) \mid v \text{ assumes equal values on opposite sides of } Y \Big\}.$$

For $YF = Y \setminus F$ the space $H^1_{\mathrm{per}}(YF)$ is defined similarly. Let $\mathbf{N}$, $\mathbf{T}$ be the unit vectors: outward normal to F and tangent to F, respectively, cf. Fig. 1. Let the brackets $[\![\cdot]\!]$ denote jump on F. Let us define the sets of kinematical fields:

$$\begin{aligned} K_{YF} &= \Big\{ \mathbf{u} \in H^1_{\mathrm{per}}(YF)^2 \mid [\![u_N]\!] \geq 0 \text{ on } F \Big\}, \\ \mathbb{K}_{YF} &= H^1_{\mathrm{per}}(Y)^2 \times K_{YF} \times H^1_{\mathrm{per}}(Y). \end{aligned} \tag{3.4}$$

The solution to the problem $(P^1_{\Omega^\varepsilon})$ is predicted in the following form

$$
\begin{aligned}
v^\varepsilon_\alpha &= v^0_\alpha(x) + \varepsilon v^1_\alpha(x,y) + \varepsilon^2 v^2_\alpha(x,y) + \dots,\\
u^\varepsilon_\alpha &= u^0_\alpha(x) + \varepsilon u^1_\alpha(x,y) + \varepsilon^2 u^2_\alpha(x,y) + \dots,\\
w^\varepsilon &= w^0(x) + \varepsilon w^1(x,y) + \varepsilon^2 w^2(x,y) + \dots,
\end{aligned}
\tag{3.5}
$$

where $y = x/\varepsilon$ and v^0_α, u^0_α, $w^0 \in H_{\Gamma_w}(\Omega)$, $v^1_\alpha(x,.)$, $w^1(x,.) \in H^1_{\mathrm{per}}(Y)$, $v^k_\alpha(x,.)$, $w^k(x,.)$, $k \geq 2$ are Y-periodic and sufficiently regular; $\mathbf{u}^1(x,.) \in K_{YF}$, $v^k_\alpha(.,y)$, $w^k(.,y)$, $u^k_\alpha(.,y)$ are defined on Ω and are sufficiently regular; $y \in YF$.

The trial fields involved in the variational inequality (3.3) are expanded similarly

$$
\begin{aligned}
v'_\alpha &= v'^0_\alpha(x) + \varepsilon v'^1_\alpha(x,y) + \varepsilon^2 v'^2_\alpha(x,y) + \dots,\\
u'_\alpha &= u'^0_\alpha(x) + \varepsilon u'^1_\alpha(x,y) + \varepsilon^2 u'^2_\alpha(x,y) + \dots,\\
w' &= w'^0(x) + \varepsilon w'^1(x,y) + \varepsilon^2 w'^2(x,y) + \dots,
\end{aligned}
\tag{3.6}
$$

where

$$
y = x/\varepsilon \qquad \text{and} \qquad v'^0_\alpha,\ u'^0_\alpha,\ w'^0 \in H_{\Gamma_w}(\Omega);
$$

$$
v'^1_\alpha(x,.),\ w'^1(x,.) \in H^1_{\mathrm{per}}(Y), \qquad \mathbf{u}'^1(x,.) \in K_{YF}.
$$

The stress resultants associated with the kinematic fields (3.5) assume the form

$$
\begin{aligned}
N^{\alpha\beta}_\varepsilon &= N^{\alpha\beta}_0 + O(\varepsilon), \qquad R_\varepsilon = R_0 + O(\varepsilon),\\
L^{\alpha\beta}_\varepsilon &= L^{\alpha\beta}_0 + O(\varepsilon), \qquad Q^\alpha_\varepsilon = Q^\alpha_0 + O(\varepsilon),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
N^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_v \varepsilon^0_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{vu} \gamma^0_{\alpha\beta} + A^{\lambda\mu}_{vw} w^0,\\
L^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_{vu} \varepsilon^0_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{u} \gamma^0_{\alpha\beta} + A^{\lambda\mu}_{uw} w^0,\\
R_0 &= A^{\alpha\beta}_{vw} \varepsilon^0_{\alpha\beta} + A^{\alpha\beta}_{uw} \gamma^0_{\alpha\beta} + A_w w^0;
\end{aligned}
\tag{3.8}
$$

$$
Q^\alpha_0 = H^{\alpha\beta} \left(\kappa^h_\beta - \frac{\partial w^1}{\partial y_\beta} \right) \Big|_{y=x/\varepsilon}. \tag{3.9}
$$

The deformations are defined by

$$
\varepsilon^0_{\alpha\beta} = \varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)|_{y=x/\varepsilon}, \qquad \gamma^0_{\alpha\beta} = \gamma^h_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{u}^1)|_{y=x/\varepsilon},
$$

where

$$
\varepsilon^h_{\alpha\beta} = \varepsilon_{\alpha\beta}(\mathbf{v}^0), \qquad \gamma^h_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}^0), \qquad \kappa^h_\beta = u^0_\beta - \frac{\partial w^0}{\partial x_\beta}; \tag{3.10}
$$

$$(3.11)\qquad \varepsilon^y_{\alpha\beta}(\mathbf{v}^1) = \frac{1}{2}\left(\frac{\partial v^1_\alpha}{\partial y_\beta} + \frac{\partial v^1_\beta}{\partial y_\alpha}\right), \qquad \gamma^y_{\alpha\beta}(\mathbf{u}^1) = \frac{1}{2}\left(\frac{\partial u^1_\alpha}{\partial y_\beta} + \frac{\partial u^1_\beta}{\partial y_\alpha}\right).$$

Quantities (3.10) play the role of homogenized (or averaged by the procedure of smearing-out the cracks) deformation measures. The following simple averages are the stress resultants of the model:

$$(3.12)\qquad N^{\alpha\beta}_h = \langle N^{\alpha\beta}_0\rangle, \qquad L^{\alpha\beta}_h = \langle L^{\alpha\beta}_0\rangle, \qquad R_h = \langle R_0\rangle, \qquad Q^\alpha_h = \langle Q^\alpha_0\rangle,$$

where

$$(3.13)\qquad \langle\cdot\rangle = \frac{1}{|Y|}\int\limits_{YF}(\cdot)\,dy, \qquad |Y| = l_1 l_2\,.$$

To find the effective model of the laminate one should substitute expansions (3.5) and (3.6) into variational inequality (3.3) and let ε tend to zero. Following the lines of the derivation elaborated upon by SANCHEZ–PALENCIA [I.47] and LEWIŃSKI and TELEGA [I.29, I.30], one eventually arrives at the homogenized problem and at the local problems the solutions of which intervene in the homogenized stress-resultants (3.12) and the effective elastic potential, see Subsection 3.3.

The homogenized problem assumes the form:

$$(3.14)\qquad (P^1_h)\ \left|\begin{array}{l}\text{Find}\ \ (\mathbf{v}^0,\mathbf{u}^0,w^0)\in V\ \ \text{such that}\\ a_h(\mathbf{v}^0,\mathbf{u}^0,w^0;\ \mathbf{v}',\mathbf{u}',w') = f(\mathbf{v}',\mathbf{u}',w'), \quad \forall(\mathbf{v}',\mathbf{u}',w')\in V,\end{array}\right.$$

where

$$(3.15)\qquad a_h(\mathbf{v}^0,\mathbf{u}^0,w^0;\ \mathbf{v}',\mathbf{u}',w') = \int\limits_\Omega\Big[N^{\alpha\beta}_h(\mathbf{v}^0,\mathbf{u}^0,w^0)\varepsilon_{\alpha\beta}(\mathbf{v}')$$
$$+L^{\alpha\beta}_h(\mathbf{v}^0,\mathbf{u}^0,w^0)\gamma_{\alpha\beta}(\mathbf{u}') + R_h(\mathbf{v}^0,\mathbf{u}^0,w^0)w' + Q^\alpha_h(\mathbf{u}^0,w^0)\kappa_\alpha(\mathbf{u}',w')\Big]\,dx.$$

The homogenized stress resultants can be expressed by the formulae

$$(3.16)\qquad \begin{aligned} N^{\lambda\mu}_h &= A^{\lambda\mu\alpha\beta}_v\varepsilon^h_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{vu}\left[\gamma^h_{\alpha\beta} + \langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right] + A^{\lambda\mu}_{vw}w^h,\\ L^{\lambda\mu}_h &= A^{\lambda\mu\alpha\beta}_{vu}\varepsilon^h_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{u}\left[\gamma^h_{\alpha\beta} + \langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right] + A^{\lambda\mu}_{uw}w^h,\\ R_h &= A^{\alpha\beta}_{vw}\varepsilon^h_{\alpha\beta} + A^{\alpha\beta}_{uw}\left[\gamma^h_{\alpha\beta} + \langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right] + A_w w^h, \quad Q^\alpha_h = H^{\alpha\beta}\kappa^h_\beta\,, \end{aligned}$$

since

$$(3.17)\qquad \langle\varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\rangle = 0, \qquad \left\langle\frac{\partial w^1}{\partial y_\beta}\right\rangle = 0.$$

We recall that v^1_α and w^1 are continuous on F. The notation $w^h \equiv w^0$ indicates that this field plays the role of a deformation field.

The fields $\mathbf{v}^1$, $\mathbf{u}^1$, w^1 depend upon the homogenized deformations (3.10). This interrelation is provided for by the local or basic cell problem:

$$(3.18)\qquad (P^1_{\mathrm{loc}})\quad \left|\begin{array}{l} \text{Find } (\mathbf{v}^1, \mathbf{u}^1, w^1) \in \mathbb{K}_{YF} \text{ such that} \\ \langle N_0^{\alpha\beta} \varepsilon^y_{\alpha\beta}(\mathbf{v}') \rangle = 0, \qquad \langle L_0^{\alpha\beta} \gamma^y_{\alpha\beta}(\mathbf{u}' - \mathbf{u}^1) \rangle \geq 0, \\ \left\langle Q_0^{\alpha} \frac{\partial w'}{\partial y_\alpha} \right\rangle = 0, \quad \forall (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}_{YF}\,. \end{array}\right.$$

Here $N_0^{\alpha\beta}$, $L_0^{\alpha\beta}$, Q_0^{α} depend on $\mathbf{v}^1$, $\mathbf{u}^1$, w^1 according to the relations (3.8) and (3.9), the x-dependent fields $\varepsilon^h_{\alpha\beta}$, $\gamma^h_{\alpha\beta}$, w^h being viewed as given and x being treated as a parameter.

The problem P^1_{loc} is *equivalent* to the following minimization problem:

$$(\tilde{P}^1_{\mathrm{loc}})\quad \left|\begin{array}{l} \text{Find} \\ \inf \left\{ \dfrac{1}{|Y|} \displaystyle\int_{Y\setminus F} j_1\Big(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^h + \boldsymbol{\gamma}^y(\mathbf{u}), w^h\Big) dy \mid (\mathbf{v}, \mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF} \right\} \\ \quad + \inf \left\{ \dfrac{1}{2|Y|} \displaystyle\int_Y H^{\alpha\beta}(x) \left(\kappa^h_\alpha - \frac{\partial w}{\partial y_\alpha}\right) \left(\kappa^h_\beta - \frac{\partial w}{\partial y_\beta}\right) dy \mid w \in H^1_{\mathrm{per}}(Y) \right\}, \end{array}\right.$$

where j_1 is defined by (I.3.13). Consequently $(\mathbf{v}^1, \mathbf{u}^1, w^1) \in \mathbb{K}_{YF}$ solves the problem $(\tilde{P}^1_{\mathrm{loc}})$ and *vice versa*. On account of the properties of j_1 and $\mathbf{H}$, the local fields $\mathbf{v}^1$ and $\mathbf{u}^1$ are determined uniquely up to constant vectors while w^1 is unique up to a constant.

3.3. Hyperelastic potential

The elastic potential of the homogenized laminate is given by

$$(3.19)\qquad U_h = \frac{1}{2}\Big\langle N_0^{\alpha\beta}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + L_0^{\alpha\beta}\left[\gamma^h_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{u}^1)\right] + R_0 w^h + Q_0^{\alpha}\left(\kappa^h_\alpha - \frac{\partial w^1}{\partial y_\alpha}\right)\Big\rangle = U_1 + U_2\,,$$

where

$$(3.20)\qquad U_1(x, \boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, w^h)$$

$$= \inf \left\{ \frac{1}{|Y|} \int_{Y\setminus F} j_1\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^h + \boldsymbol{\gamma}^y(\mathbf{u}), w^h\right) dy \mid (\mathbf{v}, \mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF} \right\}$$

$$= \frac{1}{|Y|} \int_{Y\setminus F} j_1\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}^1), \boldsymbol{\gamma}^h + \boldsymbol{\gamma}^y(\mathbf{u}^1), w^h\right) dy,$$

$$U_2(x,\boldsymbol{\kappa}^h) = \frac{1}{2}H^{\alpha\beta}(x)\kappa^h_\alpha\kappa^h_\beta\,, \tag{3.21}$$

since the simple relation

$$\int_Y \frac{\partial w}{\partial y_\alpha}dy = \int_{\partial Y} w n_\alpha\, ds = 0 \qquad \alpha = 1,\ 2)$$

reduces the second infimum in the problem $(\widetilde{P}^1_{\mathrm{loc}})$ to $U_2(x,\boldsymbol{\kappa}^h)$.

Below it will be proved that

$$N^{\alpha\beta}_h = \frac{\partial U_h}{\partial \varepsilon^h_{\alpha\beta}}\,, \qquad L^{\alpha\beta}_h = \frac{\partial U_h}{\partial \gamma^h_{\alpha\beta}}\,, \qquad R_h = \frac{\partial U_h}{\partial w^h}\,, \qquad Q^\alpha_h = \frac{\partial U_h}{\partial \kappa^h_\alpha}\,. \tag{3.22}$$

It is worth noting that the expression (3.19) can be reduced to

$$2U_h = N^{\alpha\beta}_h\varepsilon^h_{\alpha\beta} + L^{\alpha\beta}_h\gamma^h_{\alpha\beta} + R_h w^h + Q^\alpha_h\kappa^h_\alpha\,. \tag{3.23}$$

To prove (3.23) let us insert $\mathbf{v}' = \mathbf{v}^1$ and $w' = w^1$ into $(3.18)_{1,3}$ and then $\mathbf{u}' = 2\mathbf{u}^1$ and once more $\mathbf{u}' = \mathbf{0}$ into $(3.18)_2$. On combining the identities obtained with (3.19), one easily finds (3.23).

Let us pass now to the study of the properties of the effective elastic potential $U_h = U_1 + U_2$.

(i) $U_h(x,.,.,.,.)$ is a strictly convex function provided that F does not separate Y into two (say) disjoint subdomains.

Firstly we demonstrate that $U_h(x,.,.,.,.)$ is *always* convex. Due to $(\mathrm{I}.3.14)_2$, the partial elastic potential $U_2(x,.)$ is *always* strictly convex. Therefore we must investigate the function U_1. Let $\mathbf{E}^{(\alpha)} = (\boldsymbol{\varepsilon}^{(\alpha)}, \boldsymbol{\gamma}^{(\alpha)}, w^{(\alpha)}) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$, $\alpha = 1,\ 2$ and assume that $(\mathbf{v}^{(\alpha)}, \mathbf{u}^{(\alpha)}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF}$ solves the following minimization problem:

$$U_1(x,\boldsymbol{\varepsilon}^{(\alpha)},\boldsymbol{\gamma}^{(\alpha)},w^{(\alpha)}) = \inf\Bigg\{\frac{1}{|Y|}$$

$$\int_{Y\setminus F} j_1\left(x,\boldsymbol{\varepsilon}^{(\alpha)} + \boldsymbol{\varepsilon}^y(\mathbf{v}),\boldsymbol{\gamma}^{(\alpha)} + \boldsymbol{\gamma}^y(\mathbf{u}),w^{(\alpha)}\right)dy \mid (\mathbf{v},\mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF}\Bigg\}.$$

To prove that $U_1(x,.,.,.)$ is a convex function we calculate

$$U_1\left[x,\frac{1}{2}(\mathbf{E}^{(1)} + \mathbf{E}^{(2)})\right] \tag{3.24}$$

$$= U_1\left[x,\frac{1}{2}(\boldsymbol{\varepsilon}^{(1)} + \boldsymbol{\varepsilon}^{(2)}),\frac{1}{2}(\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}),\frac{1}{2}(w^{(1)} + w^{(2)})\right]$$

(3.24) [cont.]

$$\leq \frac{1}{|Y|}\int\limits_{Y\setminus F}\left\{\frac{1}{2}j_1\left(x,\boldsymbol{\varepsilon}^{(1)}+\boldsymbol{\varepsilon}^y(\mathbf{v}^{(1)}),\boldsymbol{\gamma}^{(1)}+\boldsymbol{\gamma}^y(\mathbf{u}^{(1)}),w^{(1)}\right)\right.$$

$$\left.+\frac{1}{2}j_1\left(x,\boldsymbol{\varepsilon}^{(2)}+\boldsymbol{\varepsilon}^y(\mathbf{v}^{(2)}),\boldsymbol{\gamma}^{(2)}+\boldsymbol{\gamma}^y(\mathbf{u}^{(2)}),w^{(2)}\right)\right\}dy$$

$$=\frac{1}{2}U_1(x,\boldsymbol{\varepsilon}^{(1)},\boldsymbol{\gamma}^{(1)},w^{(1)})+\frac{1}{2}U_1(x,\boldsymbol{\varepsilon}^{(2)},\boldsymbol{\gamma}^{(2)},w^{(2)})$$

$$=\frac{1}{2}U_1(x,\mathbf{E}^{(1)})+\frac{1}{2}U_1(x,\mathbf{E}^{(2)}).$$

If F does not intersect Y, like for instance in Fig. 1, then $U_1(x,.,.,.)$ is strictly convex even in the situations such as in Fig. 2.

(ii) $U_h(x,.,.,.,.)$ is of class C^1.

To corroborate this statement it is sufficient to consider the partial effective potential U_1 once again. As we know, U_1 is convex and finite, thus subdifferentiable, cf. ROCKAFELLAR [I.46, Corollary 10.1.1 and Th. 23.4]. The straightforward proof, however, is more instructive also for our further developments. Let the microscopic generalized stresses $\mathbf{N}_0$, $\mathbf{L}_0$ and R_0 be specified by Eq. (3.8) for a prescribed $(\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h)\in\mathbb{E}^2_s\times\mathbb{E}^2_s\times\mathbb{R}$. For each $(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\gamma}},\overline{w})\in\mathbb{E}^2_s\times\mathbb{E}^2_s\times\mathbb{R}$ we have

(3.25)

$$U_1(x,\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\gamma}},\overline{w})-U_1(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h)$$

$$=\frac{1}{|Y|}\int\limits_{Y\setminus F}j_1\left(x,\overline{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^y(\mathbf{v}),\overline{\boldsymbol{\gamma}}+\boldsymbol{\gamma}^y(\mathbf{u}),\overline{w}\right)dy$$

$$-\frac{1}{|Y|}\int\limits_{Y\setminus F}j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}^1),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}^1),w^h\right)dy$$

$$\geq\frac{1}{|Y|}\int\limits_{Y\setminus F}\left\{N_0^{\alpha\beta}(y)\left[(\overline{\varepsilon}_{\alpha\beta}+\varepsilon^y_{\alpha\beta}(\overline{\mathbf{v}}(y)))-(\varepsilon^h_{\alpha\beta}+\varepsilon^y_{\alpha\beta}(\mathbf{v}^1(y)))\right]\right.$$

$$\left.+L_0^{\alpha\beta}(y)\left[(\overline{\gamma}_{\alpha\beta}+\gamma^y_{\alpha\beta}(\overline{\mathbf{u}}(y)))-(\gamma^h_{\alpha\beta}+\gamma^y_{\alpha\beta}(\mathbf{u}^1(y)))\right]+R_0(y)\left(\overline{w}-w^h\right)\right\}dy$$

$$\geq N_h^{\alpha\beta}\left(\overline{\varepsilon}_{\alpha\beta}-\varepsilon^h_{\alpha\beta}\right)+L_h^{\alpha\beta}\left(\overline{\gamma}_{\alpha\beta}-\gamma^h_{\alpha\beta}\right)+R_h\left(\overline{w}-w^h\right).$$

Here (3.12) and (3.18) as well as the subdifferentiability of the function $j_1(x,.,.,.)$ have been taken into account. The subgradient $(\mathbf{N}_h,\mathbf{L}_h,R_h)\in\mathbb{E}^2_s\times\mathbb{E}^2_s\times\mathbb{R}$ is unique, hence $U_1(x,.,.,.)$ and consequently $U_h(x,.,.,.,.)$ are of class C^1. Consequently, relations (3.22) follow.

(iii) There exists a constant $C_1>0$ such that

$$U_h(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h)\leq C_1(|\boldsymbol{\varepsilon}^h|^2+|\boldsymbol{\gamma}^h|^2+|\boldsymbol{\kappa}^h|^2+|w^h|^2), \tag{3.26}$$

for a.e. $x\in\Omega$ and all $(\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h)\in\mathbb{E}^2_s\times\mathbb{E}^2_s\times\mathbb{R}^2\times\mathbb{R}$.

The proof is straightforward. □

We notice that the property (3.26) holds irrespective of the form of F, the case presented in Fig. 3 being obviously included.

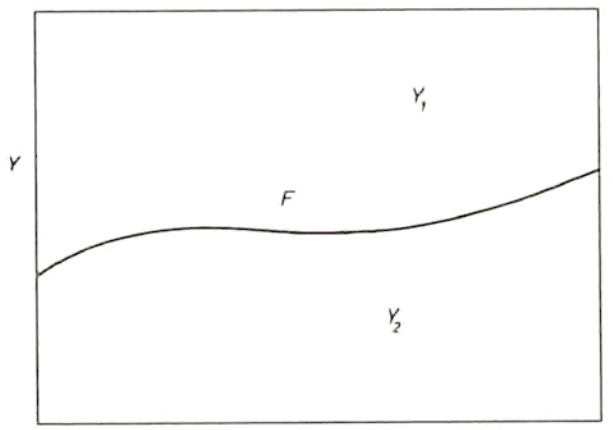


FIG. 3. Fissure dividing Y into two disjoint subdomains Y_1 and Y_2.

(iv) There exists a constant $C_0 > 0$ such that

$$U_h(x, \boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \geq C_0(|\boldsymbol{\varepsilon}^h|^2 + |\boldsymbol{\gamma}^h|^2 + |\boldsymbol{\kappa}^h|^2 + |w^h|^2), \tag{3.27}$$

for a.e. $x \in \Omega$ and all $(\boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}^2 \times \mathbb{R}$.

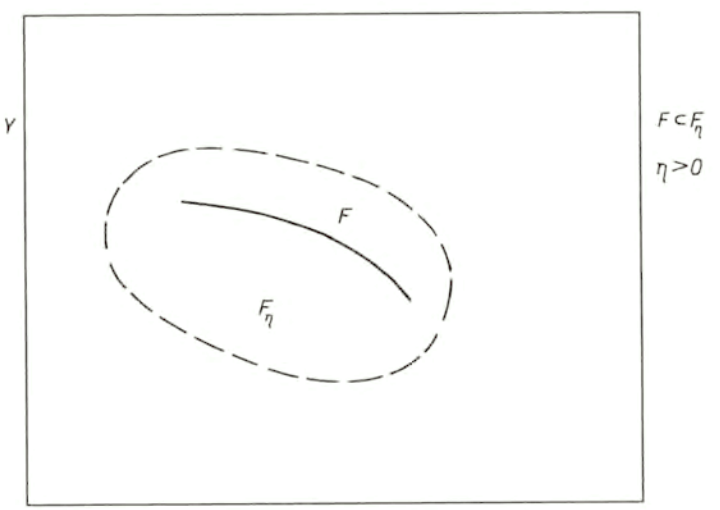


FIG. 4.

The coercivity condition (3.27) is valid only for $F \subset Y$ (Fig. 1, Fig. 4) and for situations like in Fig. 2. We assume that: a) $Y \setminus F_\eta$ has a Lipschitzian boundary, $F \subset F_\eta$, b) $|F_\eta| \to 0$ as $\eta \to 0$.

Now we have

$$
\begin{aligned}
(3.28)\qquad U_h(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h) &\geq \inf\Bigg\{\frac{1}{|Y|}\int\limits_{Y\setminus F} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}),w^h\right)dy \\
&\qquad \mid (\mathbf{v},\mathbf{u})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(YF)^2\Bigg\}+C|\boldsymbol{\kappa}^h|^2 \\
&\geq \inf\Bigg\{\frac{1}{|Y|}\int\limits_{Y\setminus F_\eta} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\gamma^h+\gamma^y(\mathbf{u}),w^h\right)dy \\
&\qquad \mid (\mathbf{v},\mathbf{u})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(Y)^2\Bigg\}+C|\boldsymbol{\kappa}^h|^2 \\
&\geq \frac{C_2}{|Y|}\int\limits_{Y\setminus F_\eta}\left(|\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\tilde{\mathbf{v}})|^2+|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\tilde{\mathbf{u}})|^2\right)dy+C_3(|\boldsymbol{\kappa}^h|^2+|w^h|^2).
\end{aligned}
$$

Here $(\tilde{\mathbf{v}},\tilde{\mathbf{u}})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(Y)^2$ is a minimizer.

We have

$$
\begin{aligned}
(3.29)\qquad \int\limits_{Y\setminus F_\eta}(|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\tilde{\mathbf{u}})|^2dy &\geq |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2+2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{Y\setminus F_\eta}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy \\
&\geq |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2+2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{Y}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy \\
&= |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy \\
&\geq |Y\setminus F_{\eta_0}||\boldsymbol{\gamma}^h|^2-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy, \qquad 0<\eta<\eta_0,\quad \eta_0 \text{ – fixed.}
\end{aligned}
$$

Let us examine the last term. One has

$$
(3.30)\qquad -\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\tilde{\mathbf{u}})dy\geq -|F_\eta||\boldsymbol{\gamma}^h|\|\boldsymbol{\gamma}^y(\tilde{\mathbf{u}})\|_{0,Y}\to 0 \qquad \text{as} \quad \eta\to 0.
$$

In the last inequality $\tilde{\mathbf{u}}\in H^1_{\mathrm{per}}(Y)^2$ and $|F_\eta|\to 0$ as $\eta\to 0$. Thus we arrive at the condition (3.27).

When F intersects Y (cf. Fig. 3), the following estimate can only be obtained

$$
\begin{aligned}
(3.31)\qquad U_h(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h) &\geq C\left(|\boldsymbol{\varepsilon}^h|^2+|\boldsymbol{\kappa}^h|^2+|w^h|^2\right) \\
&\quad+\frac{C}{|Y|}\int\limits_{Y_1\cup Y_2}|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}^1)|^2\,dy,
\end{aligned}
$$

where C is a positive constant and $\mathbf{u}^1$ enters the solution of the local problem. The last term in the inequality (3.31) can be estimated from below by zero. Then the effective potential U_h is not coercive on the space $\mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}^2 \times \mathbb{R}$. However, the potential U_h may be coercive in a restricted sense, namely on $\mathbb{E}_s^2 \times S \times \mathbb{R}^2 \times \mathbb{R}$, where S is a subset of $\mathbb{E}_s^2$. S depends on the solution of the local problem and particularly on F.

REMARK 3.2. Under the condition (3.27), the homogenized problem (P_h^1) admits a unique solution provided that the length of Γ_w is positive.

3.4. Strong formulation of the local problem

Prior to finding a strong form of (P_{loc}^1) we rearrange its variational formulation. As we already know, w^1 does not depend on y, viz. $w^1 = w^1(x)$. Let us decompose the stress resultants (3.8) and (3.9) as follows:

$$
\begin{aligned}
N_0^{\alpha\beta} &= n^{\alpha\beta} + n_0^{\alpha\beta} \qquad & L_0^{\alpha\beta} &= l^{\alpha\beta} + l_0^{\alpha\beta}, \\
R_0 &= r + r_0, & Q_0^{\alpha} &= q^{\alpha} + q_0^{\alpha},
\end{aligned} \tag{3.32}
$$

where

$$
\begin{aligned}
n^{\lambda\mu} &= A_v^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^y(\mathbf{v}^1) + A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1), \\
l^{\lambda\mu} &= A_{vu}^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^y(\mathbf{v}^1) + A_u^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1),
\end{aligned} \tag{3.33}
$$

$$
q^{\lambda} = 0 \tag{3.34}
$$

and

$$
\begin{aligned}
n_0^{\lambda\mu} &= A_v^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^h + A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^h + A_{vw}^{\lambda\mu}w^h, \\
l_0^{\lambda\mu} &= A_{vu}^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^h + A_u^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^h + A_{uw}^{\lambda\mu}w^h, \\
q^{\lambda} &= H^{\lambda\mu}\kappa_{\mu}^h .
\end{aligned} \tag{3.35}
$$

The local $(\overline{P}_{\text{loc}}^1)$ problem reads:

Find $\mathbf{v}^1 \in H_{\text{per}}^1(Y)^2$ and $\mathbf{u}^1 \in K_{YF}$ such that

$$
\left\langle n^{\alpha\beta}\frac{\partial v'_{\alpha}}{\partial y_{\beta}} \right\rangle = -\left\langle n_0^{\alpha\beta}\frac{\partial v'_{\alpha}}{\partial y_{\beta}} \right\rangle, \qquad \forall\, \mathbf{v}' \in H_{\text{per}}^1(Y)^2, \tag{3.36}
$$

$$
\left\langle l^{\alpha\beta}\frac{\partial (u'_{\alpha} - u_{\alpha}^1)}{\partial y_{\beta}} \right\rangle \geq -\left\langle l_0^{\alpha\beta}\frac{\partial (u'_{\alpha} - u_{\alpha}^1)}{\partial y_{\beta}} \right\rangle, \qquad \forall\, \mathbf{u}' \in K_{YF}. \tag{3.37}
$$

Localization of $(\overline{P}_{\text{loc}}^1)$ leads to the strong formulation: Find $\mathbf{v}^1$ and $\mathbf{u}^1$ such that:

(i) the following local equilibrium equations hold in YF:

$$A_v^{\alpha\beta\lambda\mu}\frac{\partial^2 v_\lambda^1}{\partial y_\beta \partial y_\mu} + A_{vu}^{\alpha\beta\lambda\mu}\frac{\partial^2 u_\lambda^1}{\partial y_\beta \partial y_\mu} = 0, \tag{3.38}$$

$$A_{vu}^{\alpha\beta\lambda\mu}\frac{\partial^2 v_\lambda^1}{\partial y_\beta \partial y_\mu} + A_u^{\alpha\beta\lambda\mu}\frac{\partial^2 u_\lambda^1}{\partial y_\beta \partial y_\mu} = 0. \tag{3.39}$$

(ii) $\mathbf{u}^1$ and $\mathbf{v}^1$ assume equal values at the opposite sides of YF.

(iii) $N_0^{\alpha\beta}n_\beta$, $L_0^{\alpha\beta}n_\beta$ assume opposite values at the opposite sides of Y; here $\mathbf{n} = (n_\alpha)$ represents a unit vector, outward normal to ∂Y.

(3.40) (iv) $$\overset{1}{L}{}^0_N = \overset{2}{L}{}^0_N = L_N^0 \leq 0, \qquad \overset{\sigma}{L}{}^0_N = L_{0|\sigma}^{\alpha\beta}N_\alpha N_\beta,$$
$$L_N^0[\![u_N^1]\!] = 0, \qquad [\![u_N^1]\!] \geq 0.$$

(3.41) (v) $$\overset{1}{L}{}^0_T = \overset{2}{L}{}^0_T = 0, \qquad \overset{\sigma}{L}{}^0_T = L_{0|\sigma}^{\alpha\beta}N_\alpha T_\beta.$$

(vi) $\mathbf{v}^1$ is continuous on Y.

(3.42) (vii) $$N_{0|1}^{\alpha\beta}N_\alpha N_\beta = N_{0|2}^{\alpha\beta}N_\alpha N_\beta, \qquad N_{0|1}^{\alpha\beta}N_\alpha T_\beta = N_{0|2}^{\alpha\beta}N_\alpha T_\beta.$$

3.5. Insensitivity to the l_α/h ratios

One can show that the solution to the problem (P_{loc}^1) is not sensitive to ratios $\varrho_\alpha = l_\alpha/2h$ (crack spacing/laminate thickness), hence is not sensitive to the transverse shape of the periodicity cell. This property follows from the fact that all equations and inequalities of (P_{loc}^1) involve derivatives of the same order, hence this system is free from length scales. Consequently, the effective potential U_h and formulae for the effective stiffnesses do not depend upon ϱ_α. They apply indeed to the case when ϱ_α are very small, which is a consequence of the in-plane scaling $Z_\alpha \to \varepsilon l_\alpha$, $h \to h$ (an arrow means replacement).

4. Thin laminate with transverse cracks of high density Model (h_0, l_0)

On the basis of the results of the previous section, a model will be derived suitable for describing densely distributed transverse cracks in the internal layer of very thin laminates. Notation (h_0, l_0) means that $2h \ll \operatorname{diam}\Omega$ and $l_\alpha \ll 2h$.

In the conventional engineering analysis of statics of thin laminates, the longitudinal displacements w_α are usually treated as uniform through the thickness and the influence of the stress σ^{33} is neglected. Such an approximation, justifiable for thin laminates, corresponds to the following assumptions

$$u_\alpha^0 = 0, \qquad R_0 = 0. \tag{4.1}$$

Note that only at the macrolevel the graph $w_\alpha(z)$ is assumed to be uniform; it is not stipulated that $u^1_\alpha = 0$.

Let us substitute $\mathbf{u}' = \mathbf{0}$ and $w' = 0$ into Eq. (3.14). Then we arrive at the variational equation

$$\int_\Omega N_h^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}')dx = \int_{\Gamma_\sigma} \overline{N}^\alpha v'_\alpha \, ds, \qquad \forall\, \mathbf{v}' \in H_{\Gamma_w}(\Omega)^2. \tag{4.2}$$

The field w^h involved in the constitutive relation

$$N_h^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_{vu}^{\lambda\mu\alpha\beta}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle + A_{vw}^{\lambda\mu}w^h, \tag{4.3}$$

can be eliminated by means of the equation

$$R_h = \langle R_0\rangle = 0, \qquad A_{vw}^{\alpha\beta}\varepsilon^h_{\alpha\beta} + A_{uw}^{\alpha\beta}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle + A_w w^h = 0, \tag{4.4}$$

to obtain the homogenized constitutive relations in the form of KACHANOV [6], cf. also SAYERS and KACHANOV [13], KACHANOV [7] and HORII and SAHASAKMONTRI [5].

$$N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} - A_2^{\lambda\mu\alpha\beta}\varepsilon^F_{\alpha\beta}. \tag{4.5}$$

"Crack deformation measures" $\varepsilon^F_{\alpha\beta}$ are defined by

$$\varepsilon^F_{\alpha\beta} = -\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle, \tag{4.6}$$

or, equivalently

$$\varepsilon^F_{\alpha\beta} = \frac{1}{2|Y|}\int_F \left(\llbracket u^1_\alpha\rrbracket N_\beta + \llbracket u^1_\beta\rrbracket N_\alpha\right) ds. \tag{4.7}$$

We can also write

$$\varepsilon^F_{\alpha\beta} = \frac{1}{|Y|}\int_F \left[\llbracket u^1_N\rrbracket N_\alpha N_\beta + \frac{1}{2}\llbracket u^1_T\rrbracket(T_\alpha N_\beta + T_\beta N_\alpha)\right] ds, \tag{4.8}$$

since

$$u^1_\alpha = u^1_N N_\alpha + u^1_T T_\alpha, \qquad u^1_N = u^1_1 N_1 + u^1_2 N_2 \qquad \text{and} \qquad u^1_T = u^1_1 T_1 + u^1_2 T_2.$$

The stiffnesses involved in Eq. (4.5) are given by

$$A_1^{\alpha\beta\lambda\mu} = A_v^{\alpha\beta\lambda\mu} - A_{vw}^{\alpha\beta}A_{vw}^{\lambda\mu}(A_w)^{-1}, \qquad A_2^{\alpha\beta\lambda\mu} = A_{vu}^{\alpha\beta\lambda\mu} - A_{vw}^{\alpha\beta}A_{uw}^{\lambda\mu}(A_w)^{-1}. \tag{4.9}$$

The stiffnesses $A_1^{\alpha\beta\lambda\mu}$ characterize effective properties of the uncracked laminate, the moduli $A_2^{\alpha\beta\lambda\mu}$ represent damage moduli of the cracked laminate.

To be consistent, we should appropriately reformulate the local problem (P^1_{loc}). We put $R_0 = 0$, where R_0 is given by Eq. $(3.8)_3$, and $\gamma^h_{\alpha\beta} = 0$ according to $(4.1)_1$. Hence

$$A^{\alpha\beta}_{vw}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A^{\alpha\beta}_{uw}\gamma^y_{\alpha\beta}(\mathbf{u}^1) + A_w w^h = 0. \tag{4.10}$$

Therefore, according to (3.8), one finds

$$\begin{aligned} N^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_1\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A^{\lambda\mu\alpha\beta}_2\gamma^y_{\alpha\beta}(\mathbf{u}^1), \\ L^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_3\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A^{\lambda\mu\alpha\beta}_4\gamma^y_{\alpha\beta}(\mathbf{u}^1), \end{aligned} \tag{4.11}$$

where

$$A^{\lambda\mu\alpha\beta}_3 = A^{\alpha\beta\lambda\mu}_2, \qquad A^{\alpha\beta\lambda\mu}_4 = A^{\alpha\beta\lambda\mu}_u - A^{\alpha\beta}_{uw}A^{\lambda\mu}_{uw}(A_w)^{-1}. \tag{4.12}$$

Thus the local problem assumes the form of the problem $(\overline{P}^1_{\mathrm{loc}})$ in which $n^{\lambda\mu}$ and $l^{\lambda\mu}$ are interpreted in the following way:

$$\begin{aligned} n^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_1\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A^{\lambda\mu\alpha\beta}_2\gamma^y_{\alpha\beta}(\mathbf{u}^1), \\ l^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_3\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A^{\lambda\mu\alpha\beta}_4\gamma^y_{\alpha\beta}(\mathbf{u}^1), \end{aligned} \tag{4.13}$$

while $n^{\lambda\mu}_0$ and $l^{\lambda\mu}_0$ assume the form

$$n^{\lambda\mu}_0 = A^{\lambda\mu\alpha\beta}_1\varepsilon^h_{\alpha\beta}, \qquad l^{\lambda\mu}_0 = A^{\lambda\mu\alpha\beta}_3\varepsilon^h_{\alpha\beta}. \tag{4.14}$$

The new local problem thus obtained will be referred to as (P^0_{loc}). It is *equivalent* to the following minimization problem:

PROBLEM $\tilde{P}^0_{\mathrm{loc}}$

$$\left|\begin{array}{l} \text{For } \boldsymbol{\varepsilon}^h \in \mathbb{E}_s \text{ find} \\ \mathcal{V}_h(x,\boldsymbol{\varepsilon}^h) := \inf\left\{\dfrac{1}{2|Y|}\displaystyle\int_{Y\setminus F}\left[\boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^y(\mathbf{u})\right]\mathfrak{A}_1\left[\boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^y(\mathbf{u})\right]^T dy \right. \\ \qquad\qquad \left. \Big| \ (\mathbf{v},\mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF}\right\}, \end{array}\right. \tag{4.15}$$

provided that the matrix

$$\mathfrak{A}_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \mathfrak{A}_1^T \tag{4.16}$$

is positive. Then the problem $(\widetilde{P}^0_{\mathrm{loc}})$ is a convex one, since $\mathbf{A}_3 = \mathbf{A}_2^T$. To assess the solvability of $(\widetilde{P}^0_{\mathrm{loc}})$, we assume that $\mathfrak{A}_1$ is positive definite, i.e.

$$\text{(4.17)} \qquad \left| \begin{array}{l} \exists \ C > 0 \ \text{ such that for a.e. } \ x \in \Omega \ \text{ and for all } \ \mathbf{e}_1, \ \mathbf{e}_2 \in \mathbb{E}^2_s \\ [\mathbf{e}_1, \mathbf{e}_2]\mathfrak{A}_1(x)[\mathbf{e}_1, \mathbf{e}_2]^T \geq C(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2). \end{array} \right.$$

For

$$(\mathbf{v}, \mathbf{u}) \in \widetilde{H}^1_{\mathrm{per}}(Y)^2 \times \widetilde{K}_{YF} = \left\{ (\mathbf{v}, \mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF} \mid \int_Y \mathbf{v}(y)\, dy = 0, \quad \int_{Y \setminus F} \mathbf{u}(y)\, dy = 0 \right\},$$

the minimization problem in (4.15) is coercive, provided that F is regular and does not separate Y into two disjoint subdomains like in Fig. 3. As previously, the unique solution is denoted by $(\mathbf{v}^1, \mathbf{u}^1)$. It is determined up to a constant vector as an element of $H^1_{\mathrm{per}}(Y)^2 \times K_{YF}$.

The homogenized constitutive relationship remains hyperelastic:

$$\text{(4.18)} \qquad N_h^{\alpha\beta} = \frac{\partial \mathcal{V}_h}{\partial \varepsilon^h_{\alpha\beta}},$$

where the elastic effective potential has the form

$$\text{(4.19)} \qquad \mathcal{V}_h = \langle N_0^{\alpha\beta}[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)] + L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle/2 .$$

Now let us examine the effective potential $\mathcal{V}_h$. Its properties can be verified similarly as those of U_h, cf. Subsec. 3.3 of the present paper. Therefore we will only summarize the main properties, assuming that condition (4.17) is satisfied:

(i) $\mathcal{V}_h(x, .)$ is a strictly convex function of class C^1 in the space $\mathbb{E}^2_s$.

(ii) There exist constants $C_1 > C_0 > 0$ such that for a.e. $x \in \Omega$

$$\text{(4.20)} \qquad C_0|\mathbf{e}|^2 \leq \mathcal{V}_h(x, \mathbf{e}) \leq C_1|\mathbf{e}|^2 \qquad \forall \ \mathbf{e} \in \mathbb{E}^2_s .$$

The right-hand side inequality in (4.20) is obvious since each element of the matrix $\mathfrak{A}_1$ belongs to $L^\infty(\Omega)$. The left-hand one also readily follows

$$\text{(4.21)} \qquad \mathcal{V}_h(x, \mathbf{e}) \geq \frac{C}{2|Y|}\int_Y |\mathbf{e} + \boldsymbol{\varepsilon}^y(\mathbf{v}^1)|^2 dy + \frac{C}{2|Y|}\int_{Y \setminus F} |\boldsymbol{\gamma}^y(\mathbf{u}^1)|^2 dy$$

$$\geq \frac{C}{2|Y|}\int_Y |\mathbf{e} + \boldsymbol{\varepsilon}^y(\mathbf{v}^1)|^2 dy \geq \frac{C}{2|Y|}|\mathbf{e}|^2 = C_0|\mathbf{e}|^2.$$

It is worth noting that on account of the assumption $\gamma^h = 0$, these two properties of the potential $\mathcal{V}_h$ are satisfied also when F divides Y into two disjoint subdomains. Macroscopic constitutive relation (4.18) is a by-product of (i) while (4.19) is implied by (4.15) and (4.11). Variational equation (4.2) along with (4.15) constitutes a new homogenized problem (P_h^0). Due to the property (ii), the last problem admits a unique solution.

Let us substitute $v'_\alpha = v^1_\alpha$ into Eq. (3.36), $u'_\alpha = 2u^1_\alpha$ and then $u'_\alpha = 0$ into inequality (3.37). On combining these relations with (4.19) one can reduce the expression for $\mathcal{V}_h$ to the form

$$\mathcal{V}_h = N_h^{\alpha\beta}(\boldsymbol{\varepsilon}^h)\varepsilon^h_{\alpha\beta}/2, \tag{4.22}$$

consistent with (3.23) and the simplifications assumed. Let us observe that the assumptions (4.1) are, in general, contradictory. They imply equality $w^0 = 0$ which cannot be satisfied simultaneously with equation $R_h = 0$. Such internal contradictions are inevitable in constructing engineering models of thin plates.

5. Thin laminate weakened by transverse cracks of arbitrary density Model (h_0, l)

In this section we derive formulae for assessing stiffness loss of a thin three-layer laminate with transverse cracks in the internal layer. No limitations concerning crack spacing will be imposed. The abbreviation (h_0, l) means that $h \ll \operatorname{diam} \Omega$ and l_α are arbitrary.

The homogenization process will be based upon a scaling according to which all characteristic length scales of the model of the laminate are viewed as small parameters. These length scales represent both the transverse and longitudinal dimensions of the periodicity cell of the original laminate considered as a three-dimensional structure.

5.1. Family of ε-problems

The following quantities

$$c,\ d,\ h,\ b,\ Z_1,\ Z_2 \tag{5.1}$$

are internal length scales of the model of Sec. 2. We shall assume that all these parameters depend upon a small parameter ε. In this context the following replacement is natural

$$c \to \varepsilon c, \quad d \to \varepsilon d, \quad h \to \varepsilon h, \quad b \to \varepsilon b, \quad Z_\alpha \to \varepsilon l_\alpha, \quad F \to \varepsilon F. \tag{5.2}$$

If ε tends to zero, the thickness of the laminate also diminishes to zero. To compensate for this degeneracy we scale the loading

$$\overline{N}^\beta \to \varepsilon \overline{N}^\beta, \qquad \overline{L}^\beta \to \varepsilon L^\beta, \qquad \overline{Q} \to \overline{Q}. \tag{5.3}$$

Instead of scaling the loading we could scale the elastic moduli (cf. CIARLET [2], CAILLERIE [I.7]), which seems to be even more esoteric.

The length scales scaling (5.2) implies the following scaling of the stiffnesses involved in the constitutive relationships (I.2.24) and (I.2.25)

$$
\begin{aligned}
&\left(A_v^{\alpha\beta\lambda\mu},\ A_{vu}^{\alpha\beta\lambda\mu},\ A_u^{\alpha\beta\lambda\mu}\right) \to \left(\varepsilon A_v^{\alpha\beta\lambda\mu},\ \varepsilon A_{vu}^{\alpha\beta\lambda\mu},\ \varepsilon A_u^{\alpha\beta\lambda\mu}\right),\\
&\left(A_{vw}^{\alpha\beta},\ A_{uw}^{\alpha\beta}\right) \to \left(\frac{1}{\varepsilon}A_{vw}^{\alpha\beta},\ \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\right), \qquad H^{\alpha\beta} \to \frac{1}{\varepsilon}H^{\alpha\beta}, \qquad A_w \to \frac{1}{\varepsilon^3}A_w .
\end{aligned} \tag{5.4}
$$

Scaling (5.2) concerns all dimensions of the three-dimensional periodicity cell. That is why this scaling will be referred to as the space scaling, although the problem itself is posed as a two-dimensional one.

Let us re-define the bilinear form of the problem consistently with the scaling (5.4) of the stiffnesses:

$$
\begin{aligned}
b_{\Omega^\varepsilon}&(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}', \mathbf{u}', w') \\
&= \int_{\Omega^\varepsilon} \Big[N_\varepsilon^{\alpha\beta}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon)\,\varepsilon_{\alpha\beta}(\mathbf{v}') + L_\varepsilon^{\alpha\beta}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon)\,\gamma_{\alpha\beta}(\mathbf{u}') \\
&\qquad + Q_\varepsilon^{\alpha}(\mathbf{u}^\varepsilon, w^\varepsilon)\,\kappa_\alpha(\mathbf{u}', w') + R_\varepsilon(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon)\,w' \Big] dx.
\end{aligned} \tag{5.5}
$$

The constitutive relations become

$$
\begin{aligned}
N_\varepsilon^{\lambda\mu} &= \varepsilon A_v^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \varepsilon A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}A_{vw}^{\lambda\mu}w^\varepsilon, \\
L_\varepsilon^{\lambda\mu} &= \varepsilon A_{vu}^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \varepsilon A_u^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\lambda\mu}w^\varepsilon, \\
R_\varepsilon &= \frac{1}{\varepsilon}A_{vw}^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^3}A_w w^\varepsilon,
\end{aligned} \tag{5.6}
$$

$$
Q_\varepsilon^{\alpha} = \frac{1}{\varepsilon}H^{\alpha\beta}\kappa_\beta(\mathbf{u}^\varepsilon, w^\varepsilon). \tag{5.7}
$$

The equilibrium problem reads:

$$
(P^2)_{\omega^\varepsilon} \quad \left| \begin{array}{l} \text{Find } (\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}(\Omega^\varepsilon) \text{ such that} \\ b_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon;\ \mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \geq g^\varepsilon(\mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \\ \qquad\qquad \forall\ (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}(\Omega^\varepsilon), \end{array} \right. \tag{5.8}
$$

where

$$
g^\varepsilon(\mathbf{v}', \mathbf{u}', w') = \int_{\Gamma_\sigma} \left(\varepsilon \overline{N}^\alpha v'_\alpha + \varepsilon \overline{L}^\alpha u'_\alpha - \overline{Q} w' \right) ds. \tag{5.9}
$$

The solution to problem $(P^2_{\Omega^\varepsilon})$ has been denoted similarly to the solution of the problem $(P^1_{\Omega^\varepsilon})$. This ambiguity should not lead to any misunderstanding. For a fixed $\varepsilon > 0$ the problem $(P^2_{\Omega^\varepsilon})$ admits a unique solution $(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}(\Omega^\varepsilon)$. Due to the presence of singular terms this result is not straightforward. Details are given in our separate paper [14].

5.2. Asymptotic solution

In this section a formal asymptotic procedure will be applied to find the main terms of the solution to the problem $(P^2_{\Omega^\varepsilon})$. A rigorous justification of this method via the theory of epi-convergence and dual homogenization is addressed to in the paper by TELEGA and LEWIŃSKI [14].

The solution to the problem $(P^2_{\Omega^\varepsilon})$ will be looked for in the following form:

$$v^\varepsilon_\alpha = v^0_\alpha(x) + \varepsilon v^1_\alpha(x,y) + \varepsilon^2 v^2_\alpha(x,y) + \dots, \tag{5.10}$$

$$u^\varepsilon_\alpha = \varepsilon u^1_\alpha(x,y) + \varepsilon^2 u^2_\alpha(x,y) + \dots, \tag{5.11}$$

$$w^\varepsilon = \varepsilon^2 w^2(x,y) + \varepsilon^3 w^3(x,y) + \dots, \qquad y = x/\varepsilon. \tag{5.12}$$

The trial fields are expanded similarly

$$v'_\alpha = v'^0_\alpha(x) + \varepsilon v'^1_\alpha(x,y) + \varepsilon^2 v'^2_\alpha(x,y) + \dots, \tag{5.13}$$

$$u'_\alpha = \varepsilon u'^1_\alpha(x,y) + \varepsilon^2 u'^2_\alpha(x,y) + \dots, \tag{5.14}$$

$$w' = \varepsilon^2 w'^2(x,y) + \varepsilon^3 w'^3(x,y) + \dots, \qquad y = x/\varepsilon. \tag{5.15}$$

It is assumed that

$$\begin{aligned} &v^0_\alpha,\ v'^0_\alpha \in H_{\Gamma_w}(\Omega), \\ &v^1_\alpha(x,.),\ v'^1_\alpha(x,.),\ w^2(x,.),\ w'^2x,.) \in H^1_{\mathrm{per}}(Y), \\ &\mathbf{u}^1(x,.),\ \mathbf{u}'^1(x,.) \in K_{YF}. \end{aligned} \tag{5.16}$$

The deformation measures associated with the kinematic fields (5.10)–(5.12) are

$$\begin{aligned} \varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) &= \varepsilon_{\alpha\beta}(\mathbf{v}^0) + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)|_{y=x/\varepsilon} + 0(\varepsilon), \\ \gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) &= \gamma^y_{\alpha\beta}(\mathbf{u}^1)|_{y=x/\varepsilon} + 0(\varepsilon), \\ \kappa_\alpha(\mathbf{u}^\varepsilon, w^\varepsilon) &= \varepsilon\kappa^y_\alpha(\mathbf{u}^1, w^2)|_{y=x/\varepsilon} + 0(\varepsilon^2), \end{aligned} \tag{5.17}$$

where

$$\kappa^y(\mathbf{u}^1, w^2) = u^1_\beta - \frac{\partial w^2}{\partial y_\beta}. \tag{5.18}$$

The remaining symbols have been already introduced in Sec. 3.2.

The stress resultants associated with the kinematics (5.10)–(5.12) are

$$
\begin{aligned}
N_\varepsilon^{\alpha\beta} &= \varepsilon N_0^{\alpha\beta} + 0(\varepsilon^2), \qquad & L_\varepsilon^{\alpha\beta} &= \varepsilon L_0^{\alpha\beta} + 0(\varepsilon^2),\\
Q_\varepsilon^{\alpha} &= Q_0^{\alpha} + 0(\varepsilon), & R_\varepsilon &= \frac{1}{\varepsilon} R_0 + 0(1),
\end{aligned} \tag{5.19}
$$

where the rescaled stress resultants assume the form

$$
\begin{aligned}
N_0^{\lambda\mu} &= A_v^{\lambda\mu\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_{vw}^{\lambda\mu} w^2,\\
L_0^{\lambda\mu} &= A_{vu}^{\lambda\mu\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{u}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_{uw}^{\lambda\mu} w^2,\\
R_0 &= A_{vw}^{\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{uw}^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_{w} w^2,
\end{aligned} \tag{5.20}
$$

$$
Q_0^{\alpha} = H^{\alpha\beta}\kappa_\beta^y(\mathbf{u}^1, w^2), \tag{5.21}
$$

and $\varepsilon_{\alpha\beta}^h = \varepsilon_{\alpha\beta}(\mathbf{v}^0)$. Once more we note that quantities introduced in this section are frequently denoted by the same letters as their counterparts of Sec. 3.2, but they do not coincide with them.

By using the relations (5.17)–(5.21) one can express the bilinear form (5.5) as follows:

$$
\begin{aligned}
b_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}', \mathbf{u}', w') = \varepsilon \int_{\Omega^\varepsilon} \Big\{ & N_0^{\alpha\beta}\left[\varepsilon_{\alpha\beta}(\mathbf{v}'^0) + \varepsilon_{\alpha\beta}^y(\mathbf{v}'^1)\right]\\
& + L_0^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}'^1) + Q_0^{\alpha}\kappa_\alpha^y(\mathbf{u}'^1, w'^2) + R_0 w'^2 \Big\} dx + 0(\varepsilon^2).
\end{aligned} \tag{5.22}
$$

On the other hand, the linear form (5.9) assumes the form

$$
g^\varepsilon(\mathbf{v}', \mathbf{u}', w') = \varepsilon \int_{\Gamma_\sigma} \overline{N}^{\alpha} v_\alpha'^0 \, ds + 0(\varepsilon^2). \tag{5.23}
$$

Our aim is to determine the main part of the solution to the problem $(P_{\Omega^\varepsilon}^2)$, see (5.8), in which the bilinear and linear forms are given by (5.22) and (5.23), respectively. At the first stage we put

$$
\mathbf{v}' = \pm \mathbf{v}'^0(x), \qquad \mathbf{u}' = \mathbf{0}, \qquad w' = 0, \tag{5.24}
$$

and next divide both sides of (5.8) by ε and pass with ε to zero. We arrive at the variational equation

$$
\int_\Omega N_h^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}'^0)dx = \int_{\Gamma_\sigma} \overline{N}^{\alpha} v_\alpha'^0 \, ds, \qquad \text{where} \quad N_h^{\alpha\beta} = \langle N_0^{\alpha\beta} \rangle. \tag{5.25}
$$

Let us return now to inequality (5.8), in which the left and right-hand sides are given by (5.22), (5.23). On dividing its both sides by ε, passing to zero with ε and taking account of (5.25) one obtains the following variational inequality:

$$
\begin{aligned}
(5.26)\qquad & \int_{\Omega} \langle N_0^{\alpha\beta} \varepsilon_{\alpha\beta}^{y}(\mathbf{v}'^1) + L_0^{\alpha\beta} \gamma_{\alpha\beta}^{y}(\mathbf{u}'^1 - \mathbf{u}^1) \\
& + Q_0^{\alpha} \left(u_{\alpha}'^1 - u_{\alpha}^1 - \frac{\partial w'^2}{\partial y_{\alpha}} \right) + R_0 w'^2 \rangle dx \geq 0, \qquad \forall\, (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}(\Omega^{\varepsilon}).
\end{aligned}
$$

Now we put

$$
\begin{aligned}
& \mathbf{v}'^1(x,y) = \pm \mathbf{v}'(y)\varphi(x), \qquad w'^2(x,y) = \pm w'(y)\psi(x), \\
(5.27)\qquad & \mathbf{u}'^1(x,y) = \mathbf{u}^1(x,y) + \chi(x) \left[\mathbf{u}'(y) - \mathbf{u}^1(x,y) \right], \qquad 0 \leq \chi \leq 1, \\
& \varphi, \psi, \chi \in \mathcal{D}(\Omega), \qquad (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}_{YF}.
\end{aligned}
$$

In the standard manner we find the set consisting of two variational equations and a variational inequality that constitutes the basic cell problem:

$$
(5.28)\qquad (P^2_{\mathrm{loc}}) \quad \left| \begin{array}{l}
\text{Find } (\mathbf{v}^1, \mathbf{u}^1, w^2) \in \mathbb{K}_{YF} \text{ such that} \\
\langle N_0^{\alpha\beta} \varepsilon_{\alpha\beta}^{y}(\mathbf{v}') \rangle = 0, \\
\langle L_0^{\alpha\beta} \gamma_{\alpha\beta}^{y}(\mathbf{u}' - \mathbf{u}^1) + Q_0^{\alpha}(u_{\alpha}' - u_{\alpha}^1) \rangle \geq 0, \\
\left\langle R_0 w' - Q_0^{\alpha} \dfrac{\partial w'}{\partial y_{\alpha}} \right\rangle = 0, \qquad \forall\, (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}_{YF}.
\end{array} \right.
$$

The rescaled stress resultants depend on $(\mathbf{v}^1, \mathbf{u}^1, w^2)$ according to the relationships (5.20) and (5.22).

The local problem (P^2_{loc}) is equivalent to the following minimization problem:

$$
(\widetilde{P}^2_{\mathrm{loc}}) \quad \left| \begin{array}{l}
\text{Find} \\
\inf \left\{ \dfrac{1}{|Y|} \displaystyle\int_{Y \setminus F} j\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^y(\mathbf{u}), \boldsymbol{\kappa}^y(\mathbf{u}, w), w \right) dy \mid (\mathbf{v}, \mathbf{u}, w) \in \mathbb{K}_{YF} \right\},
\end{array} \right.
$$

where $\boldsymbol{\varepsilon}^h \in \mathbb{E}^2_s$ and

$$
j(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, r) = j_1(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, r) + \frac{1}{2} H^{\alpha\beta}(x) \kappa_{\alpha} \kappa_{\beta},
$$

$$
(\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, r) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}^2 \times \mathbb{R}.
$$

The properties of the microscopic elastic stored energy function j, readily inferred from (I.3.14), imply that a solution $(\mathbf{v}^1, \mathbf{u}^1, w^2) \in \mathbb{K}_{YF}$ to $(\widetilde{P}^2_{\mathrm{loc}})$ and hence to (P^2_{loc}), exists and is such that $\mathbf{v}^1$ and $\mathbf{u}^1$ are unique up to constant vectors while w^2 is unique up to a constant.

Now we are ready to formulate the homogenized problem:

(P_h^2) Find $\mathbf{v}^0 \in H_{\Gamma_w}(\Omega)$ such that the variational equation $(5.25)_1$ holds, where $N_h^{\alpha\beta}$ are given by $(5.25)_2$ and$(5.20)_1$, and the fields $(\mathbf{v}^1, \mathbf{u}^1, w^2)$ appearing in the constitutive relation $(5.25)_2$ depend on the tensor $\varepsilon_{\alpha\beta}^h = \varepsilon_{\alpha\beta}(\mathbf{v}^0)$ according to the implicit relation determined by the problem (P_{loc}^2).

REMARK 5.1. Unlike the results based upon the in-plane scaling, the problem (P_{loc}^2) is sensitive to the change of the coefficients $\varrho_\alpha = l_\alpha/2h$. The space scaling (5.2) preserves the relations: l_α/d, l_α/c for each ε. Consequently, the homogenized constitutive relation $N_h^{\alpha\beta}(\boldsymbol{\varepsilon}^h)$ will depend on the crack spacing measured with respect to the laminate thickness.

5.3. Hyperelastic potential. Well-posedness of the problem (P_h^2)

The constitutive relationship $(5.25)_2$ can be rewritten by introducing the homogenized potential

$$W_h(x, \boldsymbol{\varepsilon}^h) = \frac{1}{|Y|} \int_{Y \setminus F} j\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}^1),\ \boldsymbol{\gamma}^y(\mathbf{u}^1),\ \boldsymbol{\kappa}^y(\mathbf{u}^1, w^2),\ w^2\right) dy \tag{5.29}$$

$$= \frac{1}{2}\left\langle N_0^{\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + L_0^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + Q_0^\alpha \kappa_\alpha^y(\mathbf{u}^1, w^2) + R_0 w^2\right\rangle.$$

One can prove that

$$N_h^{\alpha\beta} = \frac{\partial W_h}{\partial \varepsilon_{\alpha\beta}^h}. \tag{5.30}$$

The proof follows the lines of the demonstration of the property (ii) characterizing the effective potential U_h, see Sec. 3.3 and, in particular, formula (3.25). More precisely, the basic properties of W_h are specified by:

(a) $W_h(x, .)$ $(\mathbf{e}_1 \in \mathbb{E}_s^2)$ is strictly convex and of class C^1.

(b) There exist constants $C_1 > C_0 > 0$ such that for a.e. $x \in \Omega$

$$C_0|\mathbf{e}|^2 \leq W_h(x, \mathbf{e}) \leq C_1|\mathbf{e}|^2 \qquad \text{for all} \quad \mathbf{e} \in \mathbb{E}_s^2.$$

We observe that formulae (4.18), (5.30) and the properties (a) and (b) just stated are preserved when F divides the basic cell Y into two disjoint subdomains. The property (b) of the effective potential W_h ensures unique solvability of the homogenized problem (P_h^2), i.e. its well-posedness, provided that the length of

Γ_w is greater than zero. Note that the homogenized potential (5.29) assumes a simple form

$$W_h(x, \varepsilon^h) = \frac{1}{2} N_h^{\alpha\beta}(\varepsilon^h)\varepsilon_{\alpha\beta}^h . \tag{5.31}$$

To prove it, let us substitute $\mathbf{v}' = \mathbf{v}^1$, $w' = w^2$, $\mathbf{u}'^1 = 2\mathbf{u}^1$ and then $\mathbf{u}'^1 = \mathbf{0}$ into (5.28). Upon adding the equalities obtained in this manner one finds

$$\left\langle N_0^{\alpha\beta}\varepsilon_{\alpha\beta}^y(\mathbf{v}^1) + L_0^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + Q_0^{\alpha}\kappa_{\alpha}^y(\mathbf{u}^1, w^2) + R_0 w^2 \right\rangle = 0, \tag{5.32}$$

and (5.31) follows.

5.4. Kachanov's form of the homogenized constitutive relations

Let us focus our attention on the homogenized constitutive relation $(5.25)_2$. Considering $(5.20)_1$ and recalling that $\mathbf{v}^1 \in H^1_{\mathrm{per}}(Y)^2$ one obtains

$$N_h^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^h + A_{vu}^{\lambda\mu\alpha\beta}\langle\gamma_{\alpha\beta}^y(\mathbf{u}^1)\rangle + A_{vw}^{\lambda\mu}\langle w^2\rangle. \tag{5.33}$$

We show below that the formula above can be rearranged to a new one depending only on $\varepsilon_{\alpha\beta}^h$ and $\langle\gamma_{\alpha\beta}^y(\mathbf{u}^1)\rangle$. Indeed, let us substitute $w' = \mathrm{const}$ into variational equation $(5.28)_3$. One finds

$$\langle R_0\rangle = 0. \tag{5.34}$$

Taking into account relation $(5.20)_3$, one can reformulate Eq. (5.33) to the form

$$N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^h + A_2^{\lambda\mu\alpha\beta}\langle\gamma_{\alpha\beta}^y(\mathbf{u}^1)\rangle, \tag{5.35}$$

similar to the previous formula (4.5): the tensors $A_\sigma^{\alpha\beta\lambda\mu}$ are determined by Eqs. (4.9). Formula (5.35) can be written as follows

$$N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^h - A_2^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}^F , \tag{5.36}$$

where

$$\varepsilon_{\alpha\beta}^F = -\langle\gamma_{\alpha\beta}^y(\mathbf{u}^1)\rangle. \tag{5.37}$$

These quantities will be referred to as crack deformation measures, similarly to the quantities (4.6) of the model (h_0, l_0). One can represent them in the form (4.7), (4.8). Thus we conclude that the knowledge of the relations $[\![u_N^1(\varepsilon^h)]\!]$, $[\![u_T^1(\varepsilon^h)]\!]$ suffices for the determination of the homogenized constitutive relationship (5.36).

The constitutive relationship (5.36), along with the representation (4.7) of (5.37), have assumed the form of KACHANOV [6]. The crack deformation measures $\varepsilon^F_{\alpha\beta}$ turn out to play the role of internal state variables. In contrast to Kachanov's phenomenological approach, formulae (5.36) and (5.37) have not been proposed but rigorously derived. The internal state variables are directly connected with macrodeformation fields $(\varepsilon^h_{\alpha\beta})$ through the local problem, cf. also TELEGA [I.49]. The tensor $\mathbf{A}_2$ may be called a "damage moduli tensor". For the investigation of damage in laminates, ALLEN *at al.* [1] use Kachanov's concept. According to their analysis $\mathbf{A}_1 = -\mathbf{A}_2$. The formulae (4.9) do not warrant such an identification.

5.5. Characteristics of the model (h_0, l)

Let us list the main characteristics of the model (h_0, l). Within its framework, the analysis is decomposed into macro and micro- (or local) levels.

At the local level:

- the axial stresses in the external and internal layers are assumed to depend only upon the in-plane coordinates,
- the equilibrium equations are satisfied exactly,
- the interface conditions for both stresses and displacements are satisfied exactly,
- the stress-strain relations are satisfied in an average sense by requiring that the Reissner functional expressed in terms of stress resultants and generalized displacemets attain a saddle point at the solution,
- the local elasticity problem is reduced to solving the set (5.28) of two variational equalities and one inequality. In the case of cracks going parallely through the whole laminate, this problem is reduced to solving a set of three ordinary differential equations (see [9] for details).

At the macro-level:

- equilibrium equations involve the in-plane stress resultants as in the conventional plane-stress description; hence the equilibrium equations are satisfied in an average sense,
- boundary conditions are formulated as in the plane-stress model, in an average sense,
- stress-strain relations are nonlinear and their form expresses the unilateral cracking effects at the local (micro) level.

5.6. The (h_0, l_0) laminate model as a limiting case of the model (h_0, l) when $\varrho_\alpha \to 0$

We shall now prove that the thin laminate model (h_0, l_0) of Sec. 4 can be derived from the (h_0, l) model by passing to zero with ϱ_α; $\varrho_\alpha = l_\alpha/2h$. Let us introduce non-dimensional coordinates

$$\xi_\alpha = y_\alpha/l_1\,, \qquad \xi = (\xi_1, \xi_2) \in \Sigma, \qquad \Sigma = (0,1) \times (0, \xi_0), \qquad \xi_0 = l_2/l_1\,.$$

In the sequel, a re-defined cell of periodicity Σ and the transverse dimensions h, c, d will be held fixed; averaging over Σ being denoted by $\langle\cdot\rangle_0$.

Consider the consequences of passing to zero with ϱ_1 (then also ϱ_2 tends to zero). On introducing the variables ξ_α into (P^2_{loc}), muliplying both sides of $(5.28)_3$ by $(l_1)^2$, taking into account the relations (5.21) and passing to zero with ϱ_1 (h is held fixed), one obtains the following variational equation

$$\left\langle H^{\alpha\beta}\frac{\partial w^2}{\partial \xi_\alpha}\frac{\partial w'}{\partial \xi_\beta}\right\rangle_0 = 0, \tag{5.38}$$

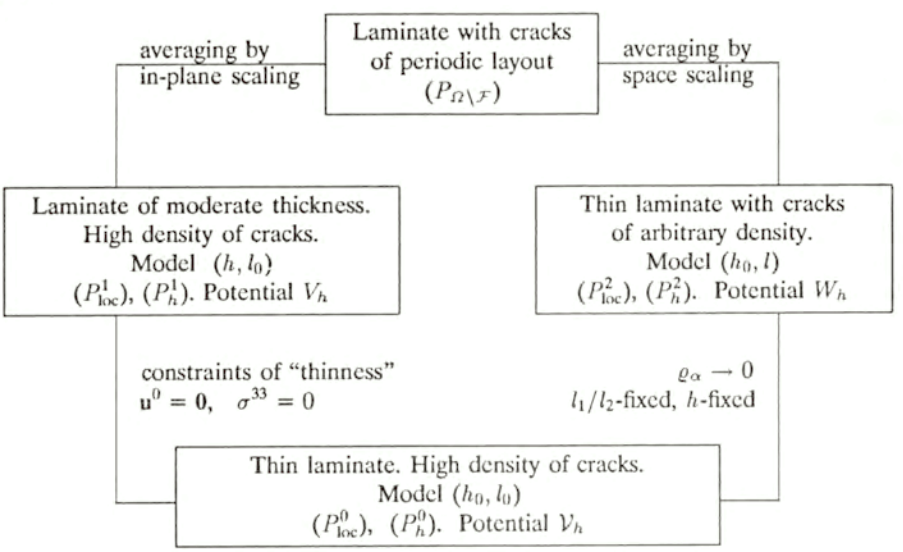
satisfied for all $w' \in H^1_{\mathrm{per}}(\Sigma)$. Hence $w^2 = \mathrm{const}$ and consequently $w^2 = \langle w^2\rangle_0$.

According to (5.34) $\langle R_0\rangle_0 = 0$ and we can eliminate w^2 from the relations (5.20) to arrive at relations (4.11). On multiplying Eqs. $(5.28)_{1,2}$ by l_1 and passing to zero with ϱ_1, one arrives at an equality and inequality of the form $(3.18)_1$ and $(3.18)_2$, respectively. Thus we obtain the problem (P^0_{loc}) formulated in Sec. 4. Averaging equation $(4.11)_1$ results in the homogenized constitutive relation (5.35). Thus we obtain the problem (P^0_h) in an alternative manner. In particular, the ϱ_α-independent potential $\mathcal{V}_h$ defined by Eq. (4.22) turns out to be a limit of the ϱ_α-dependent potential W_h (cf. (5.31)) as $\varrho_\alpha \to 0$, viz.:

$$\mathcal{V}_h = \lim_{\varrho_\alpha \to 0} W_h(\varrho_1, \varrho_2), \qquad l_1/l_2 - \text{ fixed}. \tag{5.39}$$

Two ways of deriving the model (h_0, l_0) are outlined in Diagram 1.

Diagram 1.



6. Final remarks

In our accompanying paper [9], the case of cracks going along straight lines is examined in detail. The analytical results are compared with the available experimental data and with theoretical predictions of HASHIN [I.17], ABOUDI [I.1], ALLEN *et al.* [1] and MCCARTNEY [I.37, I.38].

Partially angled and curved cracks observed in experiments, as well as delamination (cf. GROVES *et al.* [I.11]), could not easily be accounted for by two-dimensional laminate models. Curved cracks result, in general, in transverse asymmetry, thus coupling the membrane and bending effects. Such effects could be considered within the framework of the three-dimensional local problems, similar to that derived by CHACHA and SANCHEZ–PALENCIA [I.8].

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Invariance of sliding motions with respect to control matrix

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THE OBJECT of this paper is the proof of the theorem that the sliding solutions of systems of differential equations with discontinuous functions, with a linear control input, which describe mechanical systems, are independent of parameters of the control input matrix.

1. Introduction

DISTURBANCES are an element which usually accompanies physical phenomena. They can be produced either by changes of environment, in which the given phenomenon is observed, or by changes of the object itself. Hence, determination of characteristic features of the objects insensitive to disturbances is of great importance, similarly to the problem of formulation of conditions making possible a synthesis of systems which would be invariant with respect to disturbances. Also the number of the developed systems insensitive to disturbances is still growing.

Systems with variable structure belong to the systems which are insensitive to disturbances. For such systems, an invariance principle has been formulated which gives the conditions of invariance of a variable structure system, which performs sliding motions, resistant to possible disturbances [3]. The aforementioned invariance principle specifies the conditions of insensibility to a variable structure system to disturbances without separation of a special class of such systems. The object of consideration in this article is the invariance of mechanical variable structure systems. It appears that such systems are independent of the disturbances of control input parameters. This peculiar feature of mechanical variable structure systems is the object of analysis in this article.

Mechanical systems are described by differential equations of the second order. These equations can have various forms, depending upon the form of description (Newton, Lagrange). Moreover, the form of the equations also depends on the fact whether we have to do with holonomic, or non-holonomic systems. But nevertheless, irrespective of the kind of the description, we obtain, as a result, a system of differential equations of the second order. Transformation of such equations into a system of differential equations of the first order can lead to a situation, where the obtained matrices of coefficients of those equations contain many zero elements. When the equations which describe the sliding motion of a system with discontinuous functions are derived, these matrices must be appropriately transformed. A considerable number of zero elements makes it possible to put forward a hypothesis that elements of some matrices can be eliminated during

the process of construction of the equations which describe the sliding motion. Full elimination of matrices would mean independence of the sliding solutions of the elements of that matrix. This article will be devoted to determination of the conditions of such independence.

2. Systems with discontinuous functions and sliding solutions

Variable structure systems subject to sliding motion are most often described by differential equations with a linear control input. They have the following form:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ s &= Kx, \\ u &= \begin{cases} u^+ & \text{for } s > 0, \\ u^- & \text{for } s < 0. \end{cases} \end{aligned} \tag{2.1}$$

The system so described has a variable structure, determined by the control function u, varying over the surface s (in our case s is a hyperplane). The author will consider the properties of a motion, which takes place on a switching surface, which is called a sliding motion. When the sliding conditions [2, 4] are satisfied, then a sliding motion described by the differential equation (2.2) takes place on the discontinuity surface:

$$\dot{x} = Ax - B(KB)^{-1}KAx\,. \tag{2.2}$$

Let us note that the sliding solutions, that is the solutions of Eq. (2.2), depend upon the matrix B. Invariance of those equations with respect to B means a reduction of matrix B in the quotient $B(KB)^{-1}$. This means, that there exist conditions, whose satisfaction enables the elimination of matrix B in Eq. (2.2). However, in a general case, such a reduction is impossible. Another important and interesting problem is the determination of special cases, when all elements of matrix B are reduced after all the operations are performed in the case of matrices $B(KB)^{-1}$, even if such a reduction was not possible *a priori*. In such a case the sliding motion would be independent of the elements of the matrix B, in spite of the fact, that the structure of the equation (2.2) depends upon the structure of matrix B. The same can also be said about the sliding solutions. In the paper the author has verified the hypothesis that mechanical variable structure systems perform sliding motions independently of the elements of the matrix B, in spite of the fact that they depend upon its structure (even the dimensions and positions of nonzero elements).

A natural system is a system of n differential equations with one-dimensional control of the following form [1]:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n + bu, \\
s &= k_1x_1 + k_2x_2 + \ldots + k_nx_n, \\
u &= \begin{cases} u^+ & \text{for } s > 0, \\ u^- & \text{for } s < 0, \end{cases}
\end{aligned} \tag{2.3}
$$

and hence the system in a matrix form can be written as follows:

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \\
s &= k_1x_1 + k_2x_2 + \ldots + k_nx_n \\
u &= \begin{cases} u^+ & \text{for } s > 0, \\ u^- & \text{for } s < 0. \end{cases}
\end{aligned} \tag{2.4}
$$

Matrices A and B will have the following form:

$$
A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ a_1 & a_2 & a_3 & \ldots & a_n \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}.
$$

Matrix $B(KB)^{-1}$ will have the following form:

$$
B(KB)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} \left([k_1, k_2, \ldots, k_n] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} (k_nb)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{k_n} \end{bmatrix}. \tag{2.5}
$$

As a result, we will obtain an expression independent of matrix B, and hence also the sliding solution will be independent of matrix B.

In this way the following lemma has been proved.

LEMMA 1. For the system (2.3), or for an equivalent system (2.4), both the sliding solution, as well as the equation which describes it, are independent of elements of matrix B.

The hypothesis stating that the sliding solution for the mechanical systems is independent of matrix B, will be verified on an example of the system with two degrees of freedom with two intersecting discontinuity surfaces, which have the following structure:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + b_2u, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + b_4u, \\
s_1 &= k_{11}x_1 + k_{12}x_2 + k_{13}x_3 + k_{14}x_4, \\
s_2 &= k_{21}x_1 + k_{22}x_2 + k_{23}x_3 + k_{24}x_4, \\
u &= \begin{cases} u^+ & s_i > 0, \\ u^- & s_i < 0. \end{cases}
\end{aligned} \tag{2.6}
$$

Matrices A and B from (2.6) will have the following form:

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ b_2 & 0 \\ 0 & 0 \\ 0 & b_4 \end{bmatrix}, \qquad K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}.
$$

We will verify the result of the operation $B(KB)^{-1}$ being an element of description of the sliding motions for the matrices described above:

$$
B(KB)^{-1} = \begin{bmatrix} 0 & 0 \\ b_2 & 0 \\ 0 & 0 \\ 0 & b_4 \end{bmatrix} \left(\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_2 & 0 \\ 0 & 0 \\ 0 & b_4 \end{bmatrix} \right)^{-1}
$$

$$
= \frac{1}{k_{12}k_{24} - k_{14}k_{22}} \begin{bmatrix} 0 & 0 \\ k_{24} & -k_{42} \\ 0 & 0 \\ -k_{22} & k_{12} \end{bmatrix}.
$$

As a result of the operations performed, we have obtained for the example considered a matrix with elements independent of matrix B. However, the structure of the matrix thus obtained is dependent upon the structure of matrix B. In consequence, the sliding motion is dependent structurally on matrix B, but it is

independent of the values of elements of matrix B. As a result, we have obtained a partial invariance of sliding solutions with respect to matrix B.

In a general case, matrix $B(KB)^{-1}$ does not satisfy the conditions of the hypothesis of independence of matrix B. This can be illustrated by the following example:

Let $B(KB)^{-1} = C$, where C is some unknown matrix. Multiplying the left-hand side of that expression by KB we obtain

$$B = CKB.$$

If our hypothesis is true (without any exceptions), then the following condition should be satisfied:

$$CK = I \qquad (I - \text{identity matrix, that is } B = IB).$$

On the contrary, for the matrix

$$\overline{K} = \begin{bmatrix} \dfrac{b_1 - b_2 a_{12}}{b_1} & a_{12} \\ a_{21} & \dfrac{b_2 - b_1 a_{21}}{b_2} \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

the relationship $B = \overline{K}B$ is satisfied, where matrix $\overline{K}$ is not an identity matrix.

3. Mechanical systems with discontinuous functions

The preceding section included an example of a mechanical system with discontinuous functions, the sliding solutions of which were independent of matrix B. Under these circumstances a question arises, what was the real cause of elimination of elements of matrix K. We may presume that the real cause of elimination of those elements was a specific nature of mechanical systems, described by the equations, their matrices having many zero elements and a specific structure (half of elements of matrix B are zero); this was due to transformation of the second order differential equations into a system of differential equations of the first order.

For mechanical system the following theorem can be formulated.

THEOREM 1. *Consider a mechanical system of the form*

$$\dot{x} = Ax + Bu$$

with discontinuity surfaces

$$s = Dx \tag{3.1}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2,2k} \\ 0 & 1 & 0 & 1\ldots\ldots & 0 \\ a_{41} & a_{42} & a_{43} & \ldots\ldots & a_{4,2k} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & 1 \\ a_{2k,1} & a_{2k,2} & a_{2k,3} & \ldots\ldots & a_{2k,2k} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ b_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & b_4 & 0 & \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots & b_{2k} \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1,2k} \\ d_{21} & d_{22} & \ldots & d_{2,2k} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ d_{k,1} & d_{k,2} & \ldots & d_{k,2k} \end{bmatrix},$$

and matrix DB is nonsingular,

$$u_i = \begin{cases} u_i^+ & s_i > 0, \\ u_i^- & s_i < 0. \end{cases}$$

Sliding solution obtained for a mechanical variable structure system of the form (3.1) is independent of elements of the control matrix B.

P r o o f.

$$(3.2) \qquad (DB)^{-1} = \left(\begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1,2k} \\ d_{21} & d_{22} & \ldots & d_{2,2k} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ d_{k,1} & d_{k,2} & \ldots & d_{k,2k} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ b_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & b_4 & 0 & \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots & b_{2k} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \dfrac{|D_{11}|}{b_2 b_4 \ldots b_{2k} |D|} & \dfrac{|D_{21}|}{b_2 b_4 \ldots b_{2k} |D|} & \ldots & \dfrac{|D_{k1}|}{b_2 b_4 \ldots b_{2k} |D|} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{|D_{1k}|}{b_2 b_4 \ldots b_{2k} |D|} & \dfrac{|D_{2k}|}{b_2 b_4 \ldots b_{2k} |D|} & \ldots & \dfrac{|D_{kk}|}{b_2 b_4 \ldots b_{2k} |D|} \end{bmatrix},$$

D_{ji} is a matrix without the i-th line and j-th column,

$$(DB)^{-1} = \left[\frac{(-1)^{j+i}|D_{ji}|}{b_2 b_4 \dots b_{2k}|D|}\right],$$

where

$$|D_{ji} = b_2 b_4 \dots b_{2(j-1)} b_{2(j+1)} \dots b_{2k}|D^*_{ji}| = \frac{\prod b_i}{b_{2j}}(-1)^{j+i}|D^*_{ji}|,$$

where D^*_{ji} is

$$D^*_{ji} = \begin{bmatrix} d_{12} & d_{14} & \dots & d_{1,2(j-1)} & d_{1,2(j+1)} & \dots & d_{1,2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{i-1,2} & d_{i-1,4} & \dots & d_{i-1,2(j-1)} & d_{i-1,2(j+1)} & \dots & d_{i-1,2k} \\ d_{i+1,2} & d_{i+1,4} & \dots & d_{i+1,2(j-1)} & d_{i+1,2(j+1)} & \dots & d_{i-1,2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{k,2} & d_{k,4} & \dots & d_{k,2(j-1)} & d_{k,2(j+1)} & \dots & d_{k,2k} \end{bmatrix},$$

$$(DB)^{-1} = \left[(-1)^{j+i}\frac{1}{b_{2i}}\frac{|D^*_{ji}|}{|D|}\right].$$

From the results presented above we can obtain the form of the expression $B(DB)^{-1}$ by multiplying of the matrix $(DB)^{-1}$ in the form (3.2) by the matrix B.

$$(3.3) \qquad B(DB)^{-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ b_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & b_4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_{2k} \end{bmatrix} \times \begin{bmatrix} \frac{|D^*_{11}|}{b_2|D|} & -\frac{|D^*_{21}|}{b_2|D|} & \dots & \frac{(-1)^{k+1}|D^*_{k1}|}{b_2|D|} \\ -\frac{|D^*_{12}|}{b_4|D|} & \frac{|D^*_{22}|}{b_4|D|} & \dots & \frac{(-1)^{k+2}|D^*_{k2}|}{b_4|D|} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{(-1)^{k+1}|D^*_{1k}|}{b_{2k}|D|} & \frac{(-1)^{k+2}|D^*_{2k}|}{b_{2k}|D|} & \dots & \frac{(-1)^{k+k}|D^*_{kk}|}{b_{2k}|D|} \end{bmatrix}$$

(3.3) [cont.]

$$= \begin{bmatrix} 0 & 0 & \dots & 0 \\ (-1)^2 \frac{|D_{11}^*|}{|D|} & -\frac{|D_{21}^*|}{|D|} & \dots & (-1)^{k+1} \frac{|D_{k1}^*|}{|D|} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ (-1)^{k+1} \frac{|D_{1k}^*|}{|D|} & (-1)^{k+2} \frac{|D_{2k}^*|}{|D|} & \dots & (-1)^{k+k} \frac{|D_{kk}^*|}{|D|} \end{bmatrix}.$$

Hence, matrix $B(DB)^{-1}$ does not contain the elements of matrix B, which concludes the proof.

In this way we have proved the theorem that the sliding solutions of the system (3.1) are independent of the elements of control input matrices B, being dependent on its structure. The structure of matrix B exerts influence on the sliding motion of a mechanical system. The theorem being proved in that form is restricted to holonomic systems. The questions whether non-holonomic systems have the same property deserves a separate consideration.

Invariance of the sliding solutions with respect to internal disturbances was analysed simultaneously with the first investigations of the sliding motions [3]. This property was one of the main causes of taking applications of sliding in the construction of the existing technical objects. The present paper extended the scope of investigations of the problems of parametric invariance, it has shown that the mechanical systems, as well as the systems of a different physical nature, wich can be described by the same mathematical model, possess sliding solutions, which are invariant with respect to the parameters of discontinuous control input matrix.

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Hall effect on thermosolutal instability in a Maxwellian viscoelastic fluid in porous medium

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THERMOSOLUTAL instability in a Maxwellian viscoelastic fluid in porous medium is studied to include the effect of Hall current. For stationary convection, the Maxwellian viscoelastic fluid behaves like an ordinary Newtonian fluid and stable solute gradient is found to have stabilizing effect, whereas Hall currents and medium permeability are found to have destabilizing effects on the system. The sufficient conditions for the non-existence of overstability are also obtained.

1. Introduction

THE ONSET OF CONVECTION in Newtonian fluids heated from below, under varying assumptions of hydrodynamics and hydromagnetics, has been treated by CHANDRASEKHAR [1]. The effect of Hall currents on the thermal instability of a horizontal layer of conducting fluid has been studied by GUPTA [2]. VERONIS [3] has investigated the thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient. The heat and solute being two diffusing components, thermosolutal convection is the general term dealing with such phenomena.

A macroscopic equation which describes incompressible flow of a Newtonian fluid of viscosity μ through a macroscopically homogeneous and isotropic porous medium of permeability k_1 is the well known Darcy's equation. The usual viscous term in the equations of fluid motion is replaced by the resistance term $-(\mu/k_1)\mathbf{v}$, where $\mathbf{v}$ is the filter velocity of the fluid.

BHATIA and STEINER [4] have studied the problem of thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destabilizing effect, in contrast to the stabilizing effect on Newtonian fluid. BHATIA and STEINER [5] have also considered the thermal instability of a Maxwell fluid in hydromagnetics and have found that the magnetic field has stabilizing effect on viscoelastic fluid, just as in case of Newtonian fluid.

The Hall effect is likely to be important in many geophysical situations like Earth's molten core as well as in flows of laboratory plasma. SHERMAN and SUTTON [6] have considered the effect of Hall current on the efficiency of a magneto-fluid-dynamic generator. UBEROI and DEVANATHAN [7] have investigated the effects of Hall phenomenon on the propagation of small amplitude waves taking compressibility into account. As the Hall current, solute gradient and viscoelastic effects are likely to be important in geophysical situations, a reconsideration of the thermal

convection effects occurring in porous medium including these effects is certainly called for and is the object of the present paper.

2. Perturbation equations

Here we consider an infinite horizontal layer of a Maxwellian viscoelastic fluid of depth d in a porous medium, heated and soluted from below and acted on by gravity force $\mathbf{g}(0, 0, -g)$ and magnetic field $\mathbf{H}(0, 0, H)$. The Maxwell's viscoelastic fluid is described by the constitutive relations

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \tau_{ij}\,, \\ \left(1 + \lambda\frac{d}{dt}\right)\tau_{ij} &= z\mu e_{ij}\,, \\ e_{ij} &= \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right), \end{aligned} \tag{2.1}$$

where T_{ij}, $\tau_{ij}(= \boldsymbol{\tau})$, $e_{ij}(= \mathbf{e})$, δ_{ij}, p, q_i, x_i, μ and λ denote respectively the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, scalar pressure, velocity, position vector, viscosity and stress relaxation time. d/dt is the convective derivative.

When the fluid slowly percolates through the pores of the rock, the gross effect is represented by the usual Darcy's law. As a consequence, the resistance term $-(\mu/k_1)\mathbf{v}$ will replace the usual viscous term in the equation of motion. Here k_1 is the permeability of the medium and $\mathbf{v}$ is the filter velocity of the fluid.

The equations of motion, continuity and heat conduction for a viscous, incompresible fluid heated from below (CHANDRASEKHAR [1], pp. 11–16) are

$$\varrho\frac{d\mathbf{q}}{dt} = \varrho\mathbf{X} - \nabla p + \operatorname{div}\boldsymbol{\tau}\,, \tag{2.2}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{2.3}$$

$$\varrho\frac{d}{dt}(c_v T) = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right), \tag{2.4}$$

where p, ϱ, T, $\mathbf{q}$ and $\mathbf{X}$ denote respectively the fluid pressure, density, temperature, velocity and the external force acting on the fluid. k and c_v stand for the thermal conductivity and the specific heat at constant volume. The viscous dissipation term, being very small in magnitude, has not been included in (2.4). Since external forces are of non-electromagnetic origin (gravity) and of electromagnetic origin (Lorentz force per unit volume), equation of motion (2.2) may be rewritten as

$$\varrho\frac{d\mathbf{q}}{dt} = -\nabla p + \varrho\mathbf{g} + \frac{\mu_e}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H} + \operatorname{div}\boldsymbol{\tau}. \tag{2.5}$$

Using the constitutive relations (2.1) for the Maxwellian viscoelastic fluid and also using the fact that when fluid flows through a porous medium, the gross effect is represented by Darcy's law, the equations of motion and continuity for a Maxwellian viscoelastic fluid through porous medium become

$$\frac{\varrho}{\varepsilon}\left(1+\lambda\frac{d}{dt}\right)\frac{d\mathbf{v}}{dt}=\left(1+\lambda\frac{d}{dt}\right)\left[-\nabla p+\varrho\mathbf{g}+\frac{\mu_e}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H}\right]-\frac{\varrho\nu}{k_1}\mathbf{v}, \tag{2.6}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2.7}$$

where $\mathbf{v}$ is the filter velocity, ε is medium porosity and k_1 is the medium permeability. $\nu(=\mu/\varrho)$ and μ_e stand for kinematic viscosity and magnetic permeability. The fluid velocity $\mathbf{q}$ and the Darcian (filter) velocity $\mathbf{v}$ are connected by the relation $\mathbf{q}=\mathbf{v}/\varepsilon$.

When the fluid flows through a porous medium, the equation of heat conduction (JOSEPH [8],pp. 53–55) is

$$[\varrho c_f\phi+\varrho_s c_s(1-\phi)]\frac{\partial T}{\partial t}+\varrho c_f(\mathbf{v}\cdot\nabla)=k\nabla^2 T. \tag{2.8}$$

An analogous solute concentration equation is

$$\left[\varrho c_f'\phi+\varrho_s c_s'(1-\phi)\right]\frac{\partial C}{\partial t}+\varrho c_f'(\mathbf{v}\cdot\nabla)C=k'\nabla^2 C. \tag{2.9}$$

Using generalized Ohm's law to take account of the Hall current

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\frac{c}{Ne}\mathbf{j}\times\mathbf{H}, \tag{2.10}$$

and eliminating $\mathbf{E}$, $\mathbf{j}$ etc., the Maxwell's equations in terms of magnetic field become

$$\frac{\partial\mathbf{H}}{\partial t}=\frac{1}{\varepsilon}\nabla\times(\mathbf{v}\times\mathbf{H})+\eta\nabla^2\mathbf{H}-\frac{c}{4\pi Ne}\nabla\times[(\nabla\times\mathbf{H})\times\mathbf{H}], \tag{2.11}$$

$$\nabla\cdot\mathbf{H}=0. \tag{2.12}$$

Initially

$$\mathbf{v}=(0,0,0),\qquad \varrho=\varrho(z),\qquad p=p(z),$$
$$T=T(z),\qquad C=C(z)\qquad\text{and}\qquad \mathbf{H}=(0,0,H).$$

Let $\delta\varrho$, δp, θ, γ, $\mathbf{h}(h_x,h_y,h_z)$ and $\mathbf{v}(u,v,w)$ denote respectively the perturbations in density ϱ, pressure p, temperature T, solute concentration C, magnetic field $\mathbf{H}(0,0,H)$ and filter velocity (zero initially). Let κ, κ', α, α', $\beta(=|dt/dz|)$ and $\beta'(=|dC/dz|)$ stand for thermal diffusivity, solute diffusivity, thermal coefficient of expansion, an analogous solvent expansion, uniform temperature gradient and

uniform solute gradient, respectively. Then the linearized thermosolutal hydromagnetic perturbed equations of flow through porous medium (2.6)–(2.9), (2.11) and (2.12), following the Boussinesq approximation, become

$$\frac{\varrho_0}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \mathbf{v}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\delta p+\mathbf{g}\delta\varrho+\frac{\mu_e}{4\pi}(\nabla\times\mathbf{h})\times\mathbf{H}\right]-\frac{\varrho_0\nu}{k_1}\mathbf{v}, \tag{2.13}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2.14}$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta, \tag{2.15}$$

$$E'\frac{\partial\gamma}{\partial t}=\beta' w+\kappa'\nabla^2\gamma, \tag{2.16}$$

$$\varepsilon\frac{\partial\mathbf{h}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{H})+\varepsilon\eta\nabla^2\mathbf{h}-\frac{c\varepsilon}{4\pi Ne}\nabla\times[(\nabla\times\mathbf{h})\times\mathbf{H}], \tag{2.17}$$

$$\nabla\cdot\mathbf{h}=0, \tag{2.18}$$

where $E=\varepsilon+(1-\varepsilon)[(\varrho_s c_s)/(\varrho_0 c_f)]$ and ϱ_0, c_f; ϱ_s, c_s stand for density and heat capacity of fluid and solid matrix, respectively. E' is an analogous solute parameter. c, η, N and e stand for speed of light, electrical resistivity, electron number density and charge of an electron, respectively. The equation of state is

$$\varrho=\varrho_0\left[1-\alpha(T-T_0)+\alpha'(C-C_0)\right], \tag{2.19}$$

where the suffix zero refers to values at the reference level $z=0$, e.g. ϱ_0, T_0 and C_0 stand for density, temperature and solute concentration at the lower boundary $z=0$.

The change in density $\delta\varrho$, caused by the perturbations θ, γ in temperature and solute concentration, is given by

$$\delta\varrho=-\varrho_0(\alpha\theta-\alpha'\gamma). \tag{2.20}$$

Equations (2.13)–(2.18), using (2.20), give

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}\nabla^2 w-g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\alpha\theta-\alpha'\gamma)-\frac{\mu_e H}{4\pi\varrho_0}\nabla^2\frac{\partial h_z}{\partial z}\right]=-\frac{\nu}{k_1}\nabla^2 w, \tag{2.21}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial\zeta}{\partial t}-\frac{\mu_e H}{4\pi\varrho_0}\frac{\partial\xi}{\partial z}\right]=-\frac{\nu}{k_1}\zeta, \tag{2.22}$$

$$\varepsilon\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)h_z=H\frac{\partial w}{\partial z}-\frac{cH\varepsilon}{4\pi Ne}\frac{\partial\xi}{\partial z}, \tag{2.23}$$

$$\varepsilon\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\xi=H\frac{\partial\zeta}{\partial z}+\frac{cH\varepsilon}{4\pi Ne}\nabla^2\frac{\partial h_z}{\partial z}, \tag{2.24}$$

$$\left(E\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = \beta w, \tag{2.25}$$

$$\left(E'\frac{\partial}{\partial t} - \kappa'\nabla^2\right)\gamma = \beta' w, \tag{2.26}$$

where $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ and $\xi = \dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}$ denote the z-components of vorticity and current density, respectively.

The fluid is confined between the planes $z = 0$ and $z = d$ maintained at constant temperatures and solute concentrations. Since no perturbations in temperature and concentration are allowed and since normal component of the velocity must vanish on these surfaces, we have

$$w = 0, \quad \theta = 0 \quad \text{and} \quad \gamma = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{2.27}$$

Here we consider the case of two free boundaries, and the medium adjoining the fluid is electrically non-conducting. The case of two free boundaries is slightly artificial, except in stellar atmospheres (SPIEGEL [9]) and in certain geophysical situations where it is most appropriate, but it allows for an analytical solution. The condition of vanishing of tangential stresses at free surfaces implies

$$\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{and} \quad \frac{\partial \zeta}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{2.28}$$

Moreover,

$$\xi = (\nabla \times \mathbf{h})_z = 0 \quad \text{and} \quad \mathbf{h} \text{ is continuous at } z = 0 \text{ and } z = d. \tag{2.29}$$

3. The dispersion relation

Here we assume the perturbations to be of the form

$$[w, \theta, \gamma, h_z, \zeta, \xi] = [W(z), \Theta(z), \Gamma(z), K(z), Z(z), X(z)] \cdot \exp(ik_x x + ik_y y + nt), \tag{3.1}$$

where k_x, k_y are horizontal wave numbers, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number and n is a complex constant.

Using the dimensionless variables

$$a = kd, \qquad \sigma = \frac{nd^2}{\nu}, \qquad p_1 = \frac{\nu}{\kappa}, \qquad p_2 = \frac{\nu}{\eta},$$

$$q = \frac{\nu}{\kappa'}, \qquad P_l = \frac{k_1}{d^2}, \qquad x' = \frac{x}{d}, \qquad y' = \frac{y}{d}, \qquad z' = \frac{z}{d}, \qquad \text{and} \qquad D = d/dz,$$

and removing the dashes for convenience, Eqs. (2.21)–(2.26), with the help of (3.1), become

$$\left[\frac{\sigma}{\varepsilon}(1+F\sigma)+\frac{1}{P_l}\right](D^2-a^2)W+(1+F\sigma)\frac{gd^2a^2}{\nu}(\alpha\Theta-\alpha'\Gamma)$$
$$-(1+F\sigma)\frac{\mu_e Hd}{4\pi\varrho_0\nu}(D^2-a^2)DK=0, \tag{3.2}$$

$$\left[\frac{\sigma}{\varepsilon}(1+F\sigma)+\frac{1}{P_l}\right]Z=(1+F\sigma)\frac{\mu_e Hd}{4\pi\varrho_0\nu}DX, \tag{3.3}$$

$$\left[D^2-a^2-p_2\sigma\right]K=-\left(\frac{Hd}{\eta\varepsilon}\right)DW+\frac{cHd}{4\pi Ne\eta}DX, \tag{3.4}$$

$$\left[D^2-a^2-p_2\sigma\right]X=-\left(\frac{Hd}{\eta\varepsilon}\right)DZ-\frac{cH}{4\pi Ne\eta d}(D^2-a^2)DK, \tag{3.5}$$

$$\left[D^2-a^2-Ep_1\sigma\right]\Theta=-\left(\frac{\beta d^2}{\kappa}\right)W, \tag{3.6}$$

$$\left[D^2-a^2-E'q\sigma\right]\Gamma=-\left(\frac{\beta' d^2}{\kappa'}\right)W. \tag{3.7}$$

The boundary conditions (2.27)–(2.29), using expression (3.1), become

$$W=D^2W=0,\quad \Theta=0,\quad \Gamma=0,\quad DZ=0,\quad X=0 \tag{3.8}$$

and h_x, h_y, h_z are continuous at $z=0,1$.

Eliminating Θ, Z, Γ, X and K between Eqs. (3.2)–(3.7), we obtain

$$\frac{\left\{\frac{\sigma}{\varepsilon}(1+F\sigma)+\frac{1}{P_l}\right\}^2}{(1+F\sigma)}\Big[(D^2-a^2)(D^2-a^2-Ep_1\sigma)$$
$$\cdot(D^2-a^2-E'q\sigma)(D^2-a^2-p_2\sigma)^2\Big]W$$
$$+\frac{Q}{\varepsilon}\Bigg[\left\{(D^2-a^2-Ep_1\sigma)(D^2-a^2-E'q\sigma)(D^2-a^2)\right\}$$
$$\cdot\left\{2\left(\frac{\sigma}{\varepsilon}\overline{1+F\sigma}+\frac{1}{P_l}\right)(D^2-a^2-p_2\sigma)+\frac{Q}{\varepsilon}(1+F\sigma)D^2\right\}\Bigg]D^2W$$
$$+\frac{M\left(\frac{\sigma}{\varepsilon}\overline{1+F\sigma}+\frac{1}{P_l}\right)^2}{(1+F\sigma)}\Big[(D^2-a^2)^2(D^2-a^2-Ep_1\sigma)$$
$$\cdot(D^2-a^2-E'q\sigma)\Big]D^2W-\Bigg[\left\{Ra^2(D^2-a^2-E'q\sigma)\right. \tag{3.9}$$

(3.9) [cont.]
$$- Sa^2(D^2 - a^2 - Ep_1\sigma)\Big\}\Big\{\Big(\frac{\sigma}{\varepsilon}\sqrt{1 + F\sigma} + \frac{1}{P_l}\Big)$$
$$\cdot(D^2 - a^2 - p_2\sigma)^2 + \frac{Q}{\varepsilon}(1 + F\sigma)(D^2 - a^2 - p_2\sigma)D^2$$
$$+ M\Big(\frac{\sigma}{\varepsilon}\sqrt{1 + F\sigma} + \frac{1}{P_l}\Big)(D^2 - a^2)D^2\Big\}\Big]W = 0.$$

Here
$$R = \frac{g\alpha\beta d^4}{\nu\kappa}$$
is the thermal Rayleigh number,
$$S = \frac{g\alpha'\beta' d^4}{\nu\kappa}$$
is the analogous solute Rayleigh number,
$$Q = \frac{\mu_e H^2 d^2}{4\pi\varrho_0\nu\eta}$$
is the Chandrasekhar number, and
$$M = \Big(\frac{cH}{4\pi Ne\eta}\Big)^2$$
is a non-dimensional number according to the Hall currents.

Using the boundary conditions (3.8), it can be shown with the help of Eqs. (3.2)–(3.7) that all the even-order derivatives of W vanish at the boundaries, and hence the proper solution of Eq. (3.9) characterizing the lowest mode is

(3.10)
$$W = W_0 \sin \pi z,$$

where W_0 is a constant. Substituting (3.10) in Eq. (3.9) and letting
$$a^2 = \pi^2 x, \qquad R_1 = \frac{R}{\pi^4}, \qquad S_1 = \frac{S}{\pi^4}, \qquad Q_1 = \frac{Q}{\pi^2},$$
$$i\sigma_1 = \frac{\sigma}{\pi^2} \qquad \text{and} \qquad P = \pi^2 P_l,$$
we obtain the dispersion relation

(3.11)
$$R_1 x = \Big[\frac{(1 + x)(1 + x + iEp_1\sigma_1)}{(1 + i\pi^2 F\sigma_1)}\Big(\frac{i\sigma_1}{\varepsilon}\sqrt{1 + i\pi^2 F\sigma_1} + \frac{1}{P}\Big)^2$$
$$\Big\{(1 + x + ip_2\sigma_1)^2 + M(1 + x)\Big\} + \frac{Q_1}{\varepsilon}(1 + x)(1 + x + iEp_1\sigma_1)$$

(3.11) [cont.]
$$\left\{2(1+x+ip_2\sigma_1)\left(\frac{i\sigma_1}{\varepsilon}\overline{1+i\pi^2F\sigma_1}+\frac{1}{P}\right)+\frac{Q_1}{\varepsilon}(1+i\pi^2F\sigma_1)\right\}\Bigg]$$
$$\Big/\left[\left(\frac{i\sigma_1}{\varepsilon}\overline{1+i\pi^2F\sigma_1}+\frac{1}{P}\right)\left\{(1+x+ip_2\sigma_1)^2+M(1+x)\right\}\right.$$
$$+\frac{Q_1}{\varepsilon}(1+i\pi^2F\sigma_1)(1+x+ip_2\sigma_1)$$
$$\left.+S_1x\frac{(1+x+iEp_1\sigma_1)}{(1+x+iE'q\sigma_1)}\right].$$

4. The stationary convection

For stationary convection, $\sigma = 0$ and Eq. (3.11) reduces to

(4.1)
$$R_1=\left(\frac{1+x}{x'}\right)\frac{\left(\frac{1+x}{P}+\frac{Q_1}{\varepsilon}\right)^2+\frac{M(1+x)}{P^2}}{\frac{1+x}{P}+\frac{Q_1}{\varepsilon}+\frac{M}{P}}+S_1,$$

and the Maxwellian viscoelastic fluid behaves like an ordinary Newtonian fluid. In order to investigate the effects of Hall current, stable solute gradient and medium permeability, we examine the behaviour of dR_1/dM, dR_1/dS_1 and dR_1/dP analytically.

Equation (4.1) yields

(4.2)
$$\frac{dR_1}{dM}=-\left(\frac{1+x}{\varepsilon x}\right)Q_1\frac{\left(\frac{1+x}{P}+\frac{Q_1}{\varepsilon}\right)}{\left(\frac{1+x+M}{P}+\frac{Q_1}{\varepsilon}\right)^2},$$

which is negative. The Hall current, therefore, has a destabilizing effect on the thermosolutal convection in porous medium. It is evident from Eq. (4.1) that

(4.3)
$$\frac{dR_1}{S_1}=+1,$$

implying threby that stable solute gradient has a stabilizing effect on the thermosolutal convection in porous medium.

Equation (4.1) also yields

(4.4)
$$\frac{dR_1}{dP}=-\frac{(1+x)}{xP^2}\frac{\frac{(1+x)}{P^2}(1+x+M)^2+\frac{2Q_1(1+x)}{\varepsilon P}+\frac{Q_1^2}{\varepsilon^2}(1+x-M)}{\left(\frac{1+x+M}{P}+\frac{Q_1}{\varepsilon}\right)^2},$$

which is negative if $i + x > M$. The condition $i + x > M$ is met for all wave numbers as the Hall current parameter $M \ll 1$. The medium permeability, therefore, has a destabilizing effect on thermosolutal convection in porous medium in a Maxwellian viscoelastic fluid for the stationary convection.

5. The overstable case

Here we discuss the possibility as to whether instability may occur as overstability. Equating real and imaginary parts of Eq. (3.11) and eliminating R_1 between them, we obtain

$$A_6 c_1^6 + A_5 c_1^5 + A_4 c_1^4 + A_3 c_1^3 + A_2 c_1^2 + A_1 c_1 + A_0 = 0, \tag{5.1}$$

where we have written $c_1 = \sigma_1^2$, $b = 1 + x$ and

$$A_6 = -\frac{\pi^6 F^3 p_2^4 E' q b}{\varepsilon^3}\left[2EE' p_1 q + \pi^2 F b(Ep_1 - E'q)\right], \tag{5.2}$$

$$\begin{aligned} A_0 = {} & \frac{1}{P^2}\left(\frac{1}{\varepsilon} - \frac{\pi^2 F}{P}\right) b^8 + \left[\frac{Ep_1}{P^3} + \frac{2M}{\varepsilon P^2} + \frac{2\pi^2 F Q_1}{\varepsilon P^2}(b-1)\right. \\ & \left. + \frac{2Q_1}{\varepsilon p}\left(\frac{1}{\varepsilon} - \frac{\pi^2 F}{P}\right)\right] b^7 + \left[\frac{2Q_1^2 \pi^2 F(b-1)}{\varepsilon^2 P} + \frac{2MEp_1}{P^3}\right. \\ & + \frac{Q_1}{\varepsilon P^2}(Ep_1 - p_2) + \frac{2Q_1}{\varepsilon P^2}\left(Ep_1 - \pi^2 FM\right) + \left(\frac{M}{P} + \frac{Q_1}{\varepsilon}\right)^2 \\ & \left.\left(\frac{1}{\varepsilon} - \frac{\pi^2 F}{P}\right)\right] b^6 + \left[\frac{MQ_1}{\varepsilon P^2}(p_2 + 3Ep_1) + \frac{M^2 Ep_1}{P^3}\right. \\ & \left. + \frac{Q_1^2}{\varepsilon^2}\left\{\frac{2}{P}(Ep_1 - p_2) + \frac{\pi^2 FM}{P} + \left(\frac{Ep_1}{P} - \frac{M}{\varepsilon}\right)\right\}\right] b^5 \\ & + \frac{Q_1^2}{\varepsilon^2}\left[\frac{Q_1}{\varepsilon}(Ep_1 - p_2) + \frac{MEp_1}{P}\right] b^4 + S_1(b-1)\left(\frac{M+b}{P}\right) \\ & (Ep_1 - E'q)\left\{\frac{b}{P} + \frac{Q_1}{\varepsilon} + \frac{M}{P}\right\} b^3. \end{aligned} \tag{5.3}$$

The six values of c_1, σ_1 being real, are positive. The product of the roots $(= A_0/A_6)$ is positive.

A_6 is negative if

$$Ep_1 > E'q, \tag{5.4}$$

and A_0 is positive if

$$Ep_1 > p_2, \quad Ep_1 > E'q, \quad \frac{1}{\varepsilon} > \frac{\pi^2 F}{P}, \quad Ep_1 > M\pi^2 F \quad \text{and} \quad \frac{Ep_1}{P} > \frac{M}{\varepsilon}. \tag{5.5}$$

The inequalities (5.4) and (5.5) imply that the sufficient conditions for non-existence of overstability are

$$Ep_1 > E'q, \qquad \frac{1}{\varepsilon} > \frac{\pi^2 F}{P}$$

and

$$Ep_1 > \text{maximum of } \left(p_2, \frac{MP}{\varepsilon}\right),$$

i.e

$$E'\kappa < E\kappa', \qquad \lambda < \frac{k_1}{\nu\varepsilon}$$

and

$$\frac{E\nu}{\kappa} > \text{maximum of } \left[\frac{\nu}{\eta}, \left(\frac{\pi cH}{4\pi Ne}\right)^2 \frac{k_1}{\varepsilon d^2}\right],$$

$$E'\kappa < E\kappa', \qquad \lambda < \frac{k_1}{\nu\varepsilon}$$

and

$$\frac{E\nu}{\kappa} > \text{maximum of } \left[\frac{\nu}{\eta}, \left(\frac{\pi cH}{4\pi Ne}\right)^2 \frac{k_1}{\varepsilon d^2}\right].$$

These are, therefore, the sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

6. Conclusions

A Maxwellian viscoelastic fluid layer heated and soluted from below in a porous medium is considered to include the effect of the Hall currents. For stationary convection, the Maxwellian viscelastic fluid behaves like an ordinary Newtonian fluid and stable solute gradient is found to postpone the onset of instability, whereas medium permeability and Hall currents speed up the onset of instability. The sufficient conditions for the non-existence of overstability are obtained, the violation of which does no necessarily imply the occurrence of overstability. The problem and the results have relevance and importance for geophysics.

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BRIEF NOTES

Some explicit formulae for heteroclinic solutions to scalar second order ordinary differential equations

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WE DERIVE some explicit formulae for heteroclinic pairs for scalar ordinary differential equations of the form $u'' + v\lambda(u)u' - su(u-a)(u-1)(1+\phi(u)) = 0$, where λ is an arbitrary polynomial in ϕ and is properly chosen. The results are used in a simple model of plasma sustained by a laser beam.

Problem

LET US CONSIDER the equation:

$$u'' + vu' - su(u-a)(u-1) = 0, \tag{1}$$

where $'$ denotes differentiation with respect to $\xi \in (-\infty, \infty)$. If $a \in (0, 1)$, then it is known (see e.g. [1]) that there exists a unique heteroclinic pair $(v, u(\xi))$ satisfying the conditions $u(0) = 1/2$, $u(-\infty) = 0$ and $u(\infty) = 1$. It has a very simple form:

$$v = -\left(\frac{1}{2} - a\right)\sqrt{2s}, \qquad u(\xi) = \left(1 + \exp\left[-\sqrt{\frac{s}{2}}\,\xi\right]\right)^{-1}.$$

In many applications it seems that the coefficient multiplying u' should not be treated as a constant. On the other hand, one would like to have still at least one solution in an explicit form. It is rather difficult to satisfy these two demands. However, in this work we find a partial solution to these problems. First, we have the following lemma:

LEMMA 1. The function

$$u(\xi) = \left(1 + \exp\left[-\sqrt{s}\sqrt{\frac{A}{B}}\,\xi\right]\right)^{-1},$$

where $A := ca + b$, $B := 2b + c$, satisfies the equation

$$u'' + v(b + cu)u' - su(u-a)(u-1) = 0$$

for $v = -\sqrt{s}(1-2a)\left(\sqrt{AB}\right)^{-1}$. □

P r o o f. The claim of the lemma follows by straightforward calculation. □

REMARK 1. It is easy to note that in the case of equation of the form $u'' + vu' + Luu' - su(u-a)(u-1) = 0$, a heteroclinic pair is the following one:

$$\left(v = -(2M)^{-1}[LM + s(1-2a)], \quad u(\xi) = \left(1 + \exp\left[-\frac{1}{4}\xi M\right]\right)^{-1}\right),$$

where $M = L + \sqrt{L^2 + 8s}$. □

REMARK 2. If a function $f(u)$ behaves qualitatively as $-u(u-a)(u-1)$, i.e. so that $f_{,u}(0) < 0$, $f_{,u}(1) < 0$ and $\lambda(u)$ is of one sign, the heteroclinic pair (v, u) for the equation $u'' + v\lambda(u)u' + f(u) = 0$ is uniquely determined. The proof follows, for example, from the usual phase plane analysis (see e.g. [2]). □

In general, determination of an explicit heteroclinic solution for the equation $u'' + v\lambda(u)u' - su(u-a)(u-1) = 0$ is rather difficult, even if λ is in a polynomial form. However, one can analyze an equation of the kind

$$u'' + v\lambda(u)u' - su(u-a)(u-1)(1+\phi(u)) = 0 \tag{2}$$

and try to find a term $\phi(u)$ so that such an equation still has a solution of the form $(1 + \exp[-\sigma\xi])^{-1}$ for a unique value of the constant v. Below, we will consider two simple and important cases.

CASE A. $\lambda(u)$ – *arbitrary polynomial*

Let

$$\lambda(u) := b + \sum_{i=1}^{I} c_i u^i.$$

Let $I \geq 1$ be natural,

$$A_I := b + \sum_{i=1}^{I} c_i a^i, \qquad B_I = 2b + \sum_{i=1}^{I} c_i a^{i-1}.$$

Let b and c_i be such that A_I and B_I are positive. Then, the following lemma is true:

LEMMA 2. Let

$$v = -\sqrt{s}(1-2a)\left(\sqrt{A_I B_I}\right)^{-1} \qquad \text{and} \qquad \phi(u) = (1-2a)B_I^{-1}\sum_{j=1}^{I-1}\chi_j u^j,$$

where

$$\chi_j = \left(\sum_{i=j+1}^{I} c_i a^{i-j-1}\right).$$

Then, for any finite I, the function

$$u(\xi) = \left(1 + \exp\left[-\xi\sqrt{s}\sqrt{A_I B_I^{-1}}\right]\right)^{-1}$$

satisfies Eq. (2).

P r o o f. The claim of the lemma follows by straightforward calculation. □

By means of the above lemma one can check the following

COROLLARY. Suppose that $b = 0$ and $c_i = 0$ for $i \neq \zeta$, $\zeta \geq 2$.
Then, for

$$v = -\sqrt{\frac{s}{a}}(1 - 2a)c_\zeta a^\zeta \qquad \text{and} \qquad \phi(u) = (1 - 2a)a^{-\zeta+1}uS(a, u, \zeta - 2),$$

the function

$$u(\xi) = (1 + \exp[-\xi\sqrt{as}])^{-1}$$

satisfies Eq. (2). Here $S(a, u, \zeta)$ denotes the symmetric polynomial of the variables a and u of the order $(\zeta-2)$ for $\zeta > 2$, i.e.

$$S(a, u, \zeta) = \sum_{i=0}^{\zeta-2} a^i u^{\zeta-i-2}.$$

Thus, this time, ϕ is independent of λ. □

CASE B. $\lambda(u) = b + \gamma(\chi + u)^{-1}$

Now, we assume that

$$\gamma + b(a + \chi) \neq 0 \qquad \text{and} \qquad (\chi + u) \neq 0 \quad \text{for} \quad u \in (0, 1).$$

LEMMA 3. If

$$v = -\sqrt{\frac{s}{2}}(1 - 2a)\frac{a + \chi}{\gamma + b(a + \chi)} \qquad \text{and} \qquad \phi(u) = \frac{1}{2}\frac{\gamma(1 - 2a)}{\gamma + b(a + \chi)}(\chi + u)^{-1},$$

then the function

$$u(\xi) = \left(1 + \exp\left[-\sqrt{\frac{s}{2}}\xi\right]\right)^{-1}$$

satisfies Eq. (2). □

P r o o f. The claim of the lemma follows by straightforward calculation. □

By means of the above lemma one can check the following

COROLLARY. Suppose that $b = 0$. Then, for

$$v = -\sqrt{\frac{s}{2}}(1-2a)(a+\chi)\gamma^{-1} \qquad \text{and} \qquad \phi(u) = (1-2a)\frac{1}{2}(u+\chi)^{-1},$$

the function

$$u(\xi) = \left(1 + \exp\left[-\xi\sqrt{\frac{s}{2}}\right]\right)^{-1}$$

satisfies Eq. (2). If additionally $\chi = 0$, then ϕ is independent of λ. However, this time the source function

$$-su(u-a)(u-1)\left[1 + (1-2a)\frac{1}{2u}\right]$$

does not tend to 0 for $u \to 0$ (for $a \neq 2^{-1}$). Likewise, if $\chi = -1$, then $\phi(u) = \frac{1}{2}(1-2a)(u-1)^{-1}$ and the source function does not tend to 0 for $u \to 1$. □

A simple application

Let us consider the plasma sustained by a laser beam. It can be described by the equation (see [3, 4]):

$$\varrho c_p \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{grad}\right) T = \operatorname{div}(k\,\mathbf{grad}\,T) + F, \tag{3}$$

where T the temperature of the gas, ϱ is the mass density, c_p the heat capacity per unit volume (under a constant pressure), k is the effective heat conductivity coefficient and $\mathbf{v}$ is a velocity of the flowing gas. F is a nonlinear source term and (after rescaling T) it can be qualitatively modeled by a cubic polynomial $-ST(T_0 - T)(1-T)$, $T_0 \in (0, 1)$. We are interested in temperature front solutions connecting the two asymptotic states of gas: 1. Cold gas corresponding to $T = 0$ (about 300 K before scaling), and 2. Hot gas (partially ionized plasma) corresponding to $T = 1$ (15000 – 20000 K before scaling). When the velocity of the cold incoming gas is parallel to the direction of the laser beam, the problem has a cylindrical symmetry with respect to this direction. Let us analyze Eq. (3) at the points lying on the axis of symmetry. It is convenient to introduce a curvilinear system of coordinates with base vectors parallel and perpendicular to the lines of constant temperature. When we assume that $(\partial T/\partial t) = 0$, then at the axis of symmetry Eq. (3) becomes

$$\varrho c_p \|\mathbf{v}\| T_{,\xi} = k_{,T}(T_{,\xi})^2 + k(T_{,\xi\xi} + \kappa(x)T_{,\xi}) + F, \tag{4}$$

where $\kappa(x)$ is the curvature of an isotherm passing through the point x. This curvature is nonzero due to the finite width of the laser beam. If we replace

$\varrho c_p \|\mathbf{v}\|$ and k (which, in general, depend on T) by suitable constants, then Eq. (4) can be rewritten as:

$$T_{,\xi\xi} - \left(q - \kappa(x)k^{-1}\right) T_{,\xi} + f = 0, \tag{5}$$

where $f(T) = F(T)k^{-1} = -sT(T_0 - T)(1 - T)$, $s = Sk^{-1}$, and q corresponds to the mass speed of the gas. According to the experimental, numerical and theoretical results [4, 5], the curvature can be qualitatively modeled as a function of T in the following way: $\kappa(T)k^{-1} = [L_0 + TL]$, with appropriate positive L and L_0. Now, using Remark 1 after Lemma 1 we can find $q(L_0, L)$ for which there exists a heteroclinic solution to Eq. (5) joining the states 0 and 1. Namely, we have

$$q = (2M)^{-1}[LM + s(1 - 2a)] + L_0\,,$$

where

$$M = L + \sqrt{L^2 + 8s} \qquad \text{and} \qquad u(\xi) = \left(1 + \exp\left[-\frac{1}{4}\xi M\right]\right)^{-1}.$$

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