Polish Academy of Sciences

Institute of Fundamental Technological Research

Archives of Mechanics

Archiwum Mechaniki Stosowanej

volume 48 issue 3

Polish Scientific Publishers PWN

Warszawa 1996

ARCHIVES OF MECHANICS IS DEVOTED TO
Theory of elasticity and plasticity • Theory of nonclassical continua • Physics of continuous media • Mechanics of discrete media • Nonlinear mechanics • Rheology • Fluid gas-mechanics • Rarefied gas • Thermodynamics

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Printed in Poland, Editorial Office: Świętokrzyska 21,
00-049 Warszawa (Poland)

Arkuszy wydawniczych 10,25. Arkuszy drukarskich 9
Papier offset. kl. III 70 g. B1. Oddano do składania w marcu 1996 r.
Druk ukończono w maju 1996 r.
Skład i łamanie: "MAT–TEX"
Druk i oprawa: Drukarnia Braci Grodzickich, Żabieniec ul. Przelotowa 7

Arch. Mech., 48, 3, pp. 447–473, Warszawa 1996

Gas filtration through porous coal medium Effect of the gas constrained in micropores

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GAS FILTRATION through the macropores in porous coal media, with diffusion of a gas constrained in micropores, is investigated by using the homogenization process for periodic structures. This technique leads to the macroscopic model of the considered phenomenon by starting from the description at the pore level. No prerequisite is imposed at the macroscopic scale. Three different macroscopic models are obtained. Their ranges of validity are defined by appropriate dimensionless numbers that describe the geometrical structure and the physico-chemical properties of the coal. In two of these models, the micropore diffusion is coupled to the filtration process by a source term in the macroscopic mass balance. Finally, we investigate a one-dimensional flow through a semi-infinite coal seam, when the coal is assumed to be composed of grains. This simple example demonstrates the strong influence of the characteristic sizes of the grains and of the macroscopic sample on the filtration process.

1. Introduction

ONE OF THE GREATEST DANGERS occurring in some underground coal mines are gas-coal outbursts. During this violent process, gas moving with a high velocity and crushed coal mixture endangers the health and lives of the miners. To reduce the hazard connected with such disastrous explosions, it is necessary to know their causes.

The mechanism of a coal outburst is investigated in several papers [e.g. 1–4]. Many factors are shown to be responsible for its occurrence. Large pressures, the kind of gas, the exploitation stresses, the physico-chemical and physico-mechanical properties of coal and internal structure of the coal porous medium play here the most important role. Many factors lead to the numerous formulae for an outburst danger. For instance, the influence of the geometrical structure on the outburst peril is represented by the following empirical relation [5]:

$$G = \frac{P_1 + 10P_2 + 100P_3}{P_1 + P_2 + P_3}, \tag{1.1}$$

where P_1 is the mass of the grain fraction of a diameter greater than 4 mm, P_2 is the mass of the grains of a diameter within the range 0.5–4 mm, and P_3 is the mass of grains of a diameter smaller than 0.5 mm. All these values are obtained from the grain size distribution of a coal specimen that was primarily crushed according to definite prescription. When $G > 13$, the presence of an outburst danger is assumed.

However, a quantitative model describing such an instantaneous phenomenon is not available. We limit ourselves to the investigation of the early stage, before the explosion.

One of the most important factors is the gas seepage through the porous coal structure, representing a triple porosity system, with three different pore scales [6]:

• The scale of network sorption is characterized by capillaries with the pore radii up to 0.3 – 0.5 nm, in which the absorption process resembles the phenomenon of dissolution.

• The scale of micropores comprises capillaries with the radii up to 1.2 – 1.5 nm.

• The scale of macropores comprises pores with greater radii, where single- and multilayer adsorption takes place and where free gas is present.

Only a small part of the gas is in a free state. The main part of the gas is constrained at the two smaller scales, i.e., the scale of micropores and the scale of network sorption. Depending on the magnitude of its pressure, the free gas in the macropores may be or may not be in a thermodynamic equilibrium with the constrained gas. When the equilibrium is disturbed, the constrained gas acts on the gas filtration in the macropores by its emission through the internal surface of the coal. The intensity of gas emission through the internal surface directly depends on the geometrical structure and the physico-chemical properties of the skeleton [7]. Therefore it often results in a strong coupling between the gas filtration intensity and the parameters mentioned above.

The aim of this paper is to show the influence of the geometrical structure and the physico-chemical properties of the skeleton on the gas filtration process. The description of such complicated systems as porous media, with strong heterogeneities of high density, is practically possible at the macroscopic level only, where an equivalent continuous medium is defined. This can be obtained in the following two ways. The first way is the phenomenological approach. It was used in [3] to investigate the behaviour of the gas-coal system. The second way includes all the different averaging (homogenization) processes for investigating the passage from the local to the macroscopic level. The main characteristics of these processes can be found in [8].

Here we use the multiple scale asymptotic method. This technique has been already used in several papers to model porous materials. Some of them concern multiple porosity media. Deformable double porosity media saturated by an incompressible fluid are investigated in [9], by starting from the Navier – Stokes equations in the micropores and in the macropores. The analysis is extended to compressible fluids in [10]. In [11], the authors assume a rigid skeleton and a compressible fluid, with Darcy's law satisfied in the micropores and in the macropores. The analysis presented here is an extension of these works to the study of a porous coal medium.

In the Sec. 2, after introducing the local description of the gas-coal system, we briefly present the homogenization process. The flow in the macropores is described by the Navier–Stokes equations for compressible fluids. Because of the small radii of micropore capillaries, we assume that the mass transport of the gas constrained in the micropores is a molecular diffusion process. For simplicity, the porous matrix is considered to be rigid. Since random and periodic microstructures lead to the same macroscopic description, [14], we assume a periodic porous matrix. Then, the homogenization process is applied to our problem and different macroscopic equivalent descriptions are obtained. The main result consists in the fact that the macroscopic gas filtration can be modelled by three different kinds of macroscopic descriptions. Their respective ranges of validity are defined by the values of appropriate dimensionless numbers. The reader who is not familiar with the mathematical approach used in the Sec. 2, can directly go over to the Sec. 3, where the results are summarized.

The quantitative influence of the gas constrained in the microporous part is illustrated in Sec. 4 of the paper. For this purpose, a one-dimensional flow through a semi-infinite coal seam is investigated, when the geometry of the internal structure of coal is assumed to be composed of spherical grains. In particular, we investigate the distribution of the gas pressure and its gradient near the long-wall head, depending on the grain radius. Determination of the small parameter of scale separation in each point of the seam enables us to show the domains of validity of the three descriptions.

2. The homogenization process

Let us introduce the physics at the different capillary and pore scales. We assume that these scales are well separated from the macroscopic scale. The local physics and the separation of scales represent the basic assumptions that lead to the macroscopic descriptions. The method of multiple scale developments does not introduce any prerequisite concerning the macroscopic scale.

2.1. Local description

Let us simplify the coal system to a single porosity medium composed of a solid part V_s and pores V_p. The solid part V_s comprises the porous matrix of coal and the capillaries of the two smaller scales. Pores V_p are the macropores introduced in Sec. 1. We assume that:

a. Flow of the gas in the macropores (in V_p) is described by the Navier–Stokes equations of a barotropic liquid.

b. Motion of the constrained gas (in V_s) obeys the Fick molecular diffusion law.

c. The solid is undeformable.

With these assumptions, the local description (at the pore level) is given by:

• the Navier-Stokes equation:

$$\Delta \mathbf{v} + (\lambda + \mu)\,\mathbf{grad}\,(\operatorname{div} \mathbf{v}) - \mathbf{grad}\, p = \varrho \frac{\partial \mathbf{v}}{\partial t} + \varrho(\mathbf{v}\,\mathbf{grad})\mathbf{v} \qquad \text{in } V_p, \tag{2.1}$$

• the equation of mass conservation for free gas:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \varrho v = 0 \qquad \text{in } V_p, \tag{2.2}$$

• the ideal gas law for isothermic processes:

$$\varrho = \frac{\varrho_a}{p_a} p \qquad \text{in } V_p, \tag{2.3}$$

• the equation of mass conservation for molecular diffusion:

$$\frac{\partial C}{\partial t} - \operatorname{div}(\mathbf{D}\,\mathbf{grad}\, C) = 0 \qquad \text{in } V_s. \tag{2.4}$$

Here $\mathbf{v}$ is the velocity vector of the free gas in the macropores, p is the gas pressure, ϱ is the gas density, C is the overall concentration of constrained gas in the solid, $\mathbf{D}$ is the effective micropore diffusion coefficient, p_a is the atmospheric pressure, ϱ_a is the gas density at atmospheric pressure, and μ and λ are the gas viscosities.

The set (2.1)–(2.4) is completed with the boundary conditions on the interface Γ between the solid and the macropores, i.e. continuity of the mass flux:

$$(\varrho \mathbf{v} + \mathbf{D}\,\mathbf{grad}\, C)\mathbf{n} = 0 \tag{2.5}$$

and continuity of the gas pressure. Due to relation (2.3), it is reduced to the condition of continuity of the density. The overall gas concentration C in the solid part can be equated to the overall gas density $\phi_s \varrho$. Therefore, the condition of continuity of the gas pressure on Γ is written in the form

$$C = \phi_s \varrho. \tag{2.6}$$

The adhesion condition:

$$\mathbf{v}\eta = 0. \tag{2.7}$$

Here $\mathbf{n}$ and η are unit vectors, normal and tangent to the common surface Γ, respectively. ϕ_s is the volume occupied by the gas constrained in the unit volume of the solid. In addition, we assume the thermodynamic equilibrium between the phases at the initial instant.

In many practical cases the bulk volume of the considered porous medium is very large compared to the size of the heterogeneities. Therefore a very good separation of scales exists which enables us to determine the equivalent continuous macroscopic description.

2.2. Homogenization principle

The separation of scales implies the existence of an elementary representative volume (ERV). In the very particular case of a periodic medium, the spatial period represents the ERV. If l is a characteristic length of the ERV and if L is a characteristic length of the sample of coal or of the phenomenon under consideration, we have

$$\varepsilon = \frac{l}{L} \ll 1.$$

If the order of magnitude of l is known for a given material, L is determined by the solution of the macroscopic boundary value problem (see Sec. 4). Therefore the value of ε is known *a posteriori* only. It is generally assumed that $\varepsilon = 0.1$ is the limit for the separation of the scales to exist.

When the medium is random, the separation of scales implies a local asymptotic invariance. The volume averages of physical quantities in the ERV remain constant under a translation $O(l)$. When the medium is periodic, it results in the local periodicity of the physical quantities. However, independently of whether the medium is random or periodic, the structure of the macroscopic equivalent description remains unchanged [14]. Therefore it will be assumed that the medium is periodic, since in this case the process is much more powerful. Nevertheless, it must be mentioned that the determination of effective coefficients needs *a priori* different approaches for the two kinds of media considered. A periodic medium is shown in Fig. 1. Ω is the unit cell, Ω_s is the solid part of Ω, Ω_p is the porous part of Ω and Γ is the interface. The geometry of the pores inside the unit cell can be chosen arbitrarily. Variation of the geometry does not modify the structure of the macroscopic description, but only the effective coefficients appearing in it.

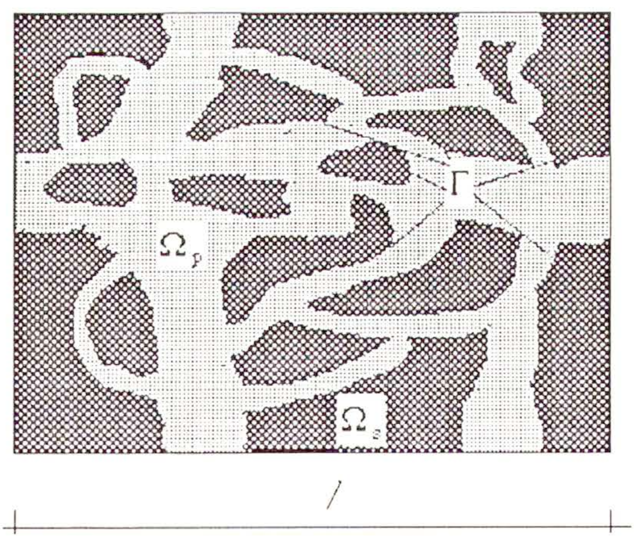

FIG. 1. Schematic view of the medium at the microscopic level: unit cell (2D case).

Two characteristic lengths l and L introduce two dimensionless space variables $\mathbf{x}$, $\mathbf{y}$ and each physical quantity F is a function of these two variables and time t.

$$\mathbf{x} = \frac{\mathbf{X}}{L}, \qquad \mathbf{y} = \frac{\mathbf{X}}{l}, \qquad F = F(\mathbf{x}, \mathbf{y}, t).$$

Variable $\mathbf{x}$ is the macroscopic space variable well suited to describe the macroscopic variations, while $\mathbf{y}$ is the macroscopic space variable well suited for the local description.

The existence of two dimensionless space variables has to be taken into account in the expressions of the differential operators. Two equivalent descriptions are then possible. The first description corresponds to the microscopic point of view. We get:

$$\begin{aligned} \mathbf{grad} &= \frac{1}{l}(\varepsilon\, \mathbf{grad_x} + \mathbf{grad_y}), \\ \Delta &= \frac{1}{l^2}\left(\varepsilon^2 \Delta_x + 2\varepsilon \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial y_j}\right) + \Delta_y\right), \\ \mathrm{div} &= \frac{1}{l}(\varepsilon\, \mathrm{div}_x + \mathrm{div}_y). \end{aligned} \tag{2.8}$$

The second description corresponds to the macroscopic point of view:

$$\begin{aligned} \mathbf{grad} &= \frac{1}{L}(\mathbf{grad_x} + \varepsilon^{-1}\, \mathbf{grad_y}), \\ \Delta &= \frac{1}{L^2}\left(\Delta_x + 2\varepsilon^{-1} \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial y_j}\right) + \varepsilon^{-2}\Delta_y\right), \\ \mathrm{div} &= \frac{1}{L}(\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_y). \end{aligned} \tag{2.9}$$

Subscripts x and y denote partial derivatives with respect to x and y, respectively. By taking advantage of the small parameter ε, all the physical quantities are sought for in the form of asymptotic expansions

$$F(\mathbf{x}, \mathbf{y}t) = F^{(0)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon F^{(1)}(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 F^{(2)}(\mathbf{x}, \mathbf{y}, t) + \ldots, \tag{2.10}$$

where $F^{(i)}$ is Ω-periodic in y.

The method consists in incorporating such expansions into the set of equations that describes the phenomenon at the local scale, and in identifying terms with the same powers of ε. Before that, it is necessary to normalize all equations of the local descriptions. This means that the local description is made dimensionless and the dimensionless numbers are evaluated according to the powers of ε. A quantity q is said to be $O(\varepsilon^p)$ if $\varepsilon^{p+1} \ll q \ll \varepsilon^{p-1}$.

The result of the homogenization process is a set of equations satisfied by the first terms of the asymptotic expansions, that represents the macroscopic description, within an approximation of the order of ε.

2.3. Estimations

Equations (2.1), (2.2), (2.4) and (2.5) introduce the following dimensionless numbers:

$$
\begin{aligned}
Q &= \frac{|\mathbf{grad}\, p|}{|\mu \Delta \mathbf{v}|}, & H &= \frac{|(\lambda+\mu)\mathbf{grad}(\mathrm{div}\,\mathbf{v})|}{|\mu \Delta \mathbf{v}|},\\
R_t &= \frac{\left|\varrho \dfrac{\partial \mathbf{v}}{\partial t}\right|}{|\mu \Delta \mathbf{v}|}, & R_e &= \frac{|\varrho(\mathbf{v}\,\mathbf{grad})\mathbf{v}|}{|\mu \Delta \mathbf{v}|},\\
S_t &= \frac{\left|\dfrac{\partial \varrho}{\partial t}\right|}{\mathrm{div}\, \varrho\mathbf{v}|}, & M_t &= \frac{\left|\dfrac{\partial C}{\partial t}\right|}{|\mathrm{div}(\mathbf{D}\,\mathbf{grad}\, C)|},\\
P_e &= \frac{|\varrho \mathbf{v}|}{|\mathbf{D}\,\mathbf{grad}\, C|}.
\end{aligned}
\tag{2.11}
$$

Let us use the microscopic point of view. Therefore l is the characteristic length for estimating the dimensionless numbers (2.11). Using the characteristic values v_c, p_c, ϱ_c, C_c, t_c of the velocity, pressure, density, concentration and time, respectively, the dimensionless numbers (2.11) can be expressed by

$$
\begin{aligned}
Q_l &= \frac{p_c l}{\mu v_c}, & H_l &= \frac{\lambda+\mu}{\mu},\\
R_{tl} &= \frac{\varrho_c l^2}{\mu t_c}, & R_{el} &= \frac{\varrho_c v_c l}{\mu},\\
S_{tl} &= \frac{l}{t_c v_c}, & M_{tl} &= \frac{l^2}{D t_c},\\
P_{el} &= \frac{\varrho_c v_c l}{D C_c}.
\end{aligned}
\tag{2.12}
$$

We limit our study to the case when the gas flow in macropores is slow and quasi-permanent. It means that the Reynolds numbers R_{el} and R_{tl} are assumed to be small, i.e.,

$$R_{el} \ll O(\varepsilon) \qquad \text{and} \qquad R_{tl} \ll O(\varepsilon).$$

We assume that the gas viscosities λ and μ are of the same order of magnitude (with respect to ε). The dimensionless number H_l becomes

$$H_l = O(1).$$

The number Q_l can be estimated by physical considerations [15]. The gas flow is forced by a macroscopic gradient of pressure. Therefore,

$$|\mathbf{grad}\, p| = O\left(\frac{p_c}{L}\right).$$

Since the gas is flowing through pores of size l, the characteristic length in evaluating the viscous term is l:

$$|\mu\Delta\mathbf{v}| = O\left(\frac{\mu v_c}{l^2}\right).$$

For slow and permanent flows, the pressure term in Eq. (2.1) is equilibrated by the viscous term. It follows that

$$\frac{\mu v_c}{l^2} = O\left(\frac{p_c}{L}\right),$$

and the dimensionless number Q_l becomes

$$Q_l = O\left(\frac{p_c l}{\mu v_c}\right) = O(\varepsilon^{-1}).$$

Estimates of the dimensionless numbers S_{tl} and M_{tl} are obtained from the conditions for the homogenization to be possible. As it was shown in [16], number S_{tl} should fulfill the following inequality:

$$S_{tl} \leq O(\varepsilon). \tag{2.13}$$

In the same way it is easy to obtain a similar restriction on M_{tl}:

$$M_{tl} \leq O(1). \tag{2.14}$$

Now, by taking into account the definitions (2.12) of P_{el}, S_{tl} and M_{tl}, the following relation can be written:

$$P_{el} = \frac{M_{tl}}{S_{tl}}\frac{\varrho_c}{C_c}.$$

Assuming that ϱ_c and C_c are of the same order of magnitude, and assuming for the moment that

$$M_{tl} = O(\varepsilon^m) \qquad \text{and} \qquad S_{tl} = O(\varepsilon^s),$$

the following estimation of P_{el} is obtained:

$$P_{el} = O(\varepsilon^{m-s}),$$

where m and s are non-negative integers.

It is well known that the filtration coefficient is very much larger than the coefficient of the micropore diffusion, and that the main flux of the gas flow through the porous medium is due to the filtration process. Therefore we confine our study to the case

$$P_{el} \geq O(1).$$

This restriction, together with the above estimation of P_{el}, leads to the inequality for m and s:

$$m - s \leq 0.$$

In the following, numbers m and s will be used to distinguish between different types of the considered phenomenon.

On the other hand, it is interesting to introduce two characteristic times T_D and T_S of the fluid diffusion and fluid seepage, respectively:

$$T_D = \frac{l^2}{D}, \qquad T_S = \frac{L}{v_c}.$$

Their ratio A can be put in the form

$$A = \frac{T_S}{T_D} = \varepsilon^{-1}\frac{D}{lv_c} = \varepsilon^{-1}\frac{S_{tl}}{M_{tl}} = \varepsilon^{s-m-1}. \tag{2.16}$$

Finally, by defining the dimensionless variables

$$v^* = \frac{v}{v_c}, \qquad p^* = \frac{p}{p_c}, \qquad \varrho^* = \frac{\varrho}{\varrho_c}, \qquad C^* = \frac{C}{C_c}, \tag{2.17}$$

and by taking into account the above estimates of the dimensionless numbers and the relations (2.8), we obtain the following dimensionless form of the local description:

$$\left(\varepsilon^2 \Delta_x + 2\varepsilon \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y_i}\right) + \Delta_y\right)\mathbf{v}^* + (\varepsilon\, \mathbf{grad}_x + \mathbf{grad}_y)(\varepsilon\, \mathrm{div}_x + \mathrm{div}_y)\mathbf{v}^*$$
$$-(\mathbf{grad}_x + \varepsilon^{-1}\mathbf{grad}_y)p^* = \varepsilon\varrho^*\frac{\partial \mathbf{v}^*}{\partial t^*} + \varepsilon\varrho^*\mathbf{v}^*((\varepsilon\, \mathbf{grad}_x + \mathbf{grad}_y)\mathbf{v}^*), \tag{2.18}$$

$$\varepsilon^s \frac{\partial \varrho^*}{\partial t^*} + (\varepsilon\, \mathrm{div}_x + \mathrm{div}_y)(\varrho^*\mathbf{v}^*) = 0, \tag{2.19}$$

$$\varrho^* = \frac{p_c}{\varrho_c}\frac{\varrho_a}{p_a}p^* \qquad \text{in } \Omega_p, \tag{2.20}$$

$$\varepsilon^m \frac{\partial C^*}{\partial t^*} - (\varepsilon\, \mathrm{div}_x + \mathrm{div}_y)\mathbf{D}(\varepsilon\, \mathbf{grad}_x + \mathbf{grad}_y)C^* = 0 \qquad \text{in } \Omega_s, \tag{2.21}$$

$$(\varepsilon^{m-s}\varrho^*\mathbf{v}^* + \mathbf{D}(\varepsilon\, \mathbf{grad}_x + \mathbf{grad}_y)C^*)\mathbf{n} = 0, \tag{2.22}$$

$$C^* = \frac{\varrho_c}{C_c}\phi_s\varrho^*, \tag{2.23}$$

$$\mathbf{v}^*\eta = 0 \qquad \text{on } \Gamma. \tag{2.24}$$

At the initial instant of time, the thermodynamical equilibrium requires that

$$C^* = \frac{\varrho_c}{C_c}\phi_s\varrho^* \qquad \text{everywhere.}$$

2.4. Macroscopic description

We introduce into the normalized set (2.18)–(2.24) asymptotic expansions (2.10) for v^*, p^*, ϱ^* and C^*. Grouping the terms with the same powers of ε, we get sets of equations to be satisfied by the consecutive terms of the asymptotic expansions. For the sake of simplicity, the asterisk marking the dimensionless variables is omitted in the following considerations.

From Eqs. (2.18), (2.20), (2.23) and (2.24) we obtain:

$$\begin{aligned} \mathbf{grad}_y p^{(0)} &= 0, \\ \Delta_y \mathbf{v}^{(0)} + \mathbf{grad}_y(\mathbf{div}_y)\mathbf{v}^{(0)} - \mathbf{grad}_x p^{(0} - \mathbf{grad}_y p^{(1)} &= 0, \end{aligned} \tag{2.25}$$

$$\varrho^{(0)} = \frac{p_c}{\varrho_c}\frac{\varrho_a}{p_a}p^{(0)} \qquad \text{in } \Omega_p, \tag{2.26}$$

$$C^{(0)} = \frac{\varrho_c}{C_c}\phi_s\varrho^{(0)}, \tag{2.27}$$

$$\mathbf{v}^{(0)}\eta = 0, \qquad \mathbf{v}^{(1)}\eta = 0 \qquad \text{on } \Gamma. \tag{2.28}$$

Equations (2.19), (2.21) and (2.22) directly depend on the values of the parameters m and s. Therefore, to obtain the sequence of equations for the consecutive powers of ε, it is needed to assume the accurate values of m and s. Different values of m and s lead to different sets of equations and, as a consequence, to different equivalent macroscopic descriptions. Four cases of interest can be distinguished:

CASE I. **Model I.** Diffusion-filtration coupling with memory effects, $s = 1$ and $m = 0$, $A = O(1)$, $T_D = O(T_S)$.

CASE II. **Model II.** Classical diffusion-filtration coupling, $s = 1$ and $m = 1$, $A = O(\varepsilon^{-1})$, $T_D \gg O(T_S)$.

CASE III. **Model III.** Classical seepage law, $s \geq 2$ and $m \geq 0$, $A = O(\varepsilon)$, $T_D \ll O(T_S)$.

CASE IV. Non-homogenizable situation, $s = 0$ and $m = 0$, $A = O(\varepsilon^{-1})$. Clearly in this case the condition (2.13) of homogenizability is not fulfilled. Case IV leads to a non-homogenizable situation, i.e. a situation where an equivalent macroscopic description is not possible. A direct proof of that is presented in the Appendix.

Model I. Diffusion-filtration coupling with memory effects, $s = 1$, $m = 0$, $S_{tl} = O(\varepsilon)$, $M_{tl} = O(1)$, $P_{el} = O(\varepsilon^{-1})$, $T_D = O(T_S)$.

With this estimation we get from (2.19), (2.21) and (2.22) the following equations:

$$
\begin{aligned}
& \operatorname{div}_y(\varrho^{(0)}\mathbf{v}^{(0)}) = 0, \\
& \frac{\partial \varrho^{(0)}}{\partial t} + \operatorname{div}_x(\varrho^{(0)}\mathbf{v}^{(0)}) + \operatorname{div}_y(\varrho^{(1)}\mathbf{v}^{(0)} + \varrho^{(0)}\mathbf{v}^{(1)}) = 0 \qquad \text{in } \Omega_p ,
\end{aligned} \tag{2.29}
$$

$$
\frac{\partial C^{(0)}}{\partial t} - \operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_y C^{(0)}) = 0 \qquad \text{in } \Omega_s , \tag{2.30}
$$

$$
\begin{aligned}
& \varrho^{(0)}\mathbf{v}^{(0)}\mathbf{n} = 0, \\
& (\varrho^{(0)}\mathbf{v}^{(1)} + \varrho r^{(1)}\mathbf{v}^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(0)})\mathbf{n} = 0 \qquad \text{on } \Gamma.
\end{aligned} \tag{2.31}
$$

Equations (2.25)–(2.31) give a sequence of boundary value problems for the first terms of the asymptotic expansions.

The first problem following from $(2.25)_1$, (2.26) leads to:

$$
\begin{aligned}
p^{(0)} &= p^{(0)}(\mathbf{x}, t), \\
\varrho^{(0)}(\mathbf{x}, t) &= \frac{p_c}{\varrho_c}\frac{\varrho_a}{p_a} p^{(0)}(\mathbf{x}, t).
\end{aligned} \tag{2.32}
$$

The first terms of the gas pressure and of the gas density are locally constant over the macropores Ω_p.

The second problem is given by (2.27) and (2.30). It is similar to that discussed in [17]. To solve it, the following substitution is applied:

$$
U(\mathbf{x}, \mathbf{y}, t) = C^{(0)} - \frac{\varrho_c}{C_c}\phi_s \varrho^{(0)}(\mathbf{x}, t).
$$

This leads to the set of equations

$$
\begin{aligned}
& \frac{\partial U}{\partial t} - \operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_y U) = -\frac{\varrho_c}{C_c}\phi_s \frac{\partial \varrho^{(0)}}{\partial t} \qquad \text{in } \Omega_s , \\
& U(\mathbf{x}, \mathbf{y}, t) = 0 \qquad \text{on } \Gamma.
\end{aligned} \tag{2.33}
$$

The thermodynamic equilibrium at the initial time gives

$$
U(\mathbf{x}, \mathbf{y}, 0) = 0.
$$

By using the Laplace transform, we obtain

$$
\begin{aligned}
& a\mathcal{L}(U) - \operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_y \mathcal{L}(U)) = -\frac{\varrho_c}{C_c}\phi_s \mathcal{L}\left(\frac{\partial \varrho^{(0)}}{\partial t}\right), \\
& \mathcal{L}(U) = 0 \qquad \text{on } \Gamma,
\end{aligned} \tag{2.34}
$$

where a is the complex Laplace variable and

$$\mathcal{L}(U) = \int_0^\infty U e^{-at}\, dt.$$

The right-hand side of $(2.34)_1$ does not depend on the microscopic space variable y.

Therefore the solution of (2.34) is a linear function of this forcing term:

$$\mathcal{L}(U) = -\frac{\varrho_c}{C_c}\phi_s \mathcal{L}(G(\mathbf{y},t))\mathcal{L}\left(\frac{\partial \varrho^{(0)}}{\partial t}\right), \tag{2.35}$$

where $\mathcal{L}(G(y,t))$ is the solution of (2.34), when the right-hand side of $(2.34)_1$ is equated to unity. We use now the volume average defined by the formula

$$\langle {}^*\rangle = \frac{1}{\Omega}\int {}^* d\Omega,$$

and we apply the inverse Laplace transform to (2.35). We obtain from the convolution theorem

$$\langle U\rangle = -\frac{\varrho_c}{C_c}\phi_s \int_0^t \frac{\partial \varrho^{(0)}}{\partial t}\langle G(t-\tau)\rangle d\tau. \tag{2.36}$$

Finally, introduction of the concentration gives the solution of the considered second boundary value problem in the form:

$$\langle C^{(0)}\rangle = \frac{\varrho_c}{C_c}\phi_s\left((1-\phi)\varrho^{(0)} - \int_0^t \frac{\partial \varrho^{(0)}}{\partial t}\langle G(t-\tau)\rangle d\tau\right), \tag{2.37}$$

where ϕ is the porosity, $\phi = \Omega_p/\Omega$. The average is evaluated by assuming the concentration $C^{(0)}$ to be zero in Ω_p.

Relation (2.37) shows that the gas concentration depends on the history of the first time-derivative of the gas density. Function $G(t)$ represents a memory function.

The third problem to be solved is given by the equations $(2.25)_2$, $(2.28)_1$, $(2.29)_1$, $(2.31)_1$ and the condition of Ω-periodicity of $p^{(1)}$ and $\mathbf{v}^{(0)}$. By taking into account the relations (2.32), this set becomes

$$\begin{gathered}\Delta_y \mathbf{v}^{(0)} - \mathbf{grad}_x p^{(0)} - \mathbf{grad}_y p^{(1)} = 0,\\ \mathrm{div}_y(\mathbf{v}^{(0)}) = 0, \qquad \mathbf{v}^{(0)}|_\Gamma = 0.\end{gathered} \tag{2.38}$$

The system (2.38) represents the classical problem of flow of an incompressible fluid through a rigid porous medium. At this stage, $p^{(0)}$ is considered as a known function of $\mathbf{x}$. The unknowns $\mathbf{v}^{(0)}$ and $p^{(1)}$ are linear functions of the macroscopic gradient $\mathbf{grad}_x p^{(0)}$ (see for example [18, 19]). In what follows, only $\mathbf{v}^{(0)}$ is needed:

$$v_i^{(0)} = -k_{ij}(\mathbf{y})\frac{\partial p^{(0)}}{\partial x_j} \qquad \text{in } \ \Omega_p\,.$$

By taking the volume average of $\mathbf{v}^{(0)}$, we obtain the well-known Darcy law:

$$\left\langle v_i^{(0)}\right\rangle = -\langle k_{ij}\rangle\frac{\partial p^{(0)}}{\partial x_j}\,. \tag{2.39}$$

The fourth problem leads to the macroscopic mass conservation law and is given by $(2.29)_2$, (2.30) and $(2.31)_2$. By integrating $(2.29)_2$ with respect to $\mathbf{y}$ on Ω_p and by using the divergence theorem, we obtain

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} + \operatorname{div}_x\left(\varrho^{(0)}\left\langle v^{(0)}\right\rangle\right) + \frac{1}{|\Omega|}\int\limits_{d\Omega_p}\left(\varrho^{(1)}\mathbf{v}^{(0)} + \varrho^{(0)}\mathbf{v}^{(1)}\right)\mathbf{n}\,dS = 0.$$

By taking now into account (2.30) and $(2.31)_2$, the above equation leads to the following form of the macroscopic mass conservation law:

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} + \operatorname{div}_x\left(\varrho^{(0)}\left\langle \mathbf{v}^{(0)}\right\rangle\right) + \frac{\partial\langle C^{(0)}\rangle}{\partial t} = 0. \tag{2.40}$$

The last term in the mass balance equation (2.40) represents a source term due to the diffusion process in the micropores.

Equations (2.32), (2.37), (2.39) and (2.40) represent the macroscopic description. Returning to the physical variables, they assume the form

$$\begin{aligned}
&p^{(0)} = p^{(0)}(\mathbf{X}, t),\\
&\varrho^{(0)}(\mathbf{X}, t) = \frac{\varrho_a}{p_a}p^{(0)}(\mathbf{X}, t),\\
&\langle C^{(0)}\rangle = \phi_s\left((1-\phi)\varrho^{(0)} - \int\limits_0^t \frac{\partial \varrho^{(0)}}{\partial t}\langle G(t-\tau)\rangle d\tau\right),\\
&\left\langle v_i^{(0)}\right\rangle = -\frac{\langle k_{ij}\rangle}{\mu}l^2\frac{\partial p^{(0)}}{\partial X_j},\\
&\phi\frac{\partial \varrho^{(0)}}{\partial t} + \operatorname{div}_X\left(\varrho^{(0)}\left\langle \mathbf{v}^{(0)}\right\rangle\right) + \frac{\partial\langle C^{(0)}\rangle}{\partial t} = 0.
\end{aligned} \tag{2.41}$$

The set (2.41) exhibits the memory effects, similarly to [9, 10] or [11].

Model II. Clasical diffusion-filtration couplings, $s = 1$, $m = 1$, $S_{tl} = O(\varepsilon)$, $M_{tl} = O(\varepsilon)$, $P_{el} = O(1)$, $T_D = O(\varepsilon^{-1}T_S)$.

In this case we get from (2.20), (2.22) and (2.23) the following sequence of equations.

$$
\begin{aligned}
&\operatorname{div}_y(\varrho^{(0)}\mathbf{v}^{(0)}) = 0,\\
&\frac{\partial\varrho^{(0)}}{\partial t} + \operatorname{div}_x(\varrho^{(0)}\mathbf{v}^{(0)}) + \operatorname{div}_y(\varrho^{(1)}\mathbf{v}^{(0)} + \varrho^{(0)}\mathbf{v}^{(1)}) = 0 \quad \text{in } \Omega_p,
\end{aligned} \tag{2.42}
$$

$$
\begin{aligned}
&\operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_y C^{(0)}) = 0,\\
&\frac{\partial C^{(0)}}{\partial t} - \operatorname{div}_x(\mathbf{D}\,\mathbf{grad}_y C^{(0)}) - \operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_x C^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(1)}) = 0 \quad \text{in } \Omega_s,
\end{aligned} \tag{2.43}
$$

$$
\begin{aligned}
&(\varrho^{(0)}\mathbf{v}^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(0)})\mathbf{n} = 0,\\
&(\varrho^{(0)}\mathbf{v}^{(1)} + \varrho^{(1)}\mathbf{v}^{(0)} + \mathbf{D}\,\mathbf{grad}_x C^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(1)})\mathbf{n} = 0 \quad \text{on } \Gamma.
\end{aligned} \tag{2.44}
$$

Case II is described by the above system, together with Eqs. (2.25) – (2.28).

As before in the Case I, the first boundary value problem to be investigated is given by $(2.25)_1$ and (2.26), and it leads to the relations (2.32).

Equations $(2.43)_1$ and (2.27) constitute the second boundary value problem. By using an equivalent variational formulation, [17], and by taking into account the equation $(2.31)_2$, we obtain

$$
C^{(0)} = \frac{\varrho_c}{C_c}\phi_s\varrho^{(0)}(\mathbf{x},t) \qquad \text{in } \Omega_s. \tag{2.45}
$$

The third problem is described by $(2.25)_2$, $(2.28)_1$, $(2.42)_1$ and $(2.44)_1$. The above result (2.45) transforms Eq. $(2.44)_1$ into the relation $(2.31)_1$, and the set under consideration becomes equivalent to the corresponding one investigated in the Case I. Therefore the Darcy law (2.39) is valid in this case too.

The macroscopic mass conservation law follows from the fourth boundary value problem. It is given by the set $(2.42)_2$, $(2.43)_2$ and $(2.44)_2$. Using the above results, the considered system can be rewritten in a simpler form:

$$
\begin{aligned}
\frac{\partial\varrho^{(0)}}{\partial t} + \operatorname{div}_x(\varrho^{(0)}\mathbf{v}^{(0)}) + \operatorname{div}_y(\varrho^{(1)}\mathbf{v}^{(0)} + \varrho^{(0)}\mathbf{v}^{(1)}) &= 0,\\
\frac{\partial C^{(0)}}{\partial t} - \operatorname{div}_y(\mathbf{D}\,\mathbf{grad}_x C^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(1)}) &= 0,\\
(\varrho^{(0)}\mathbf{v}^{(1)} + \varrho^{(1)}\mathbf{v}^{(0)} + \mathbf{D}\,\mathbf{grad}_x C^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(1)})\mathbf{n} &= 0.
\end{aligned} \tag{2.46}
$$

By applying the same method as in the Case I, the set (2.42), (2.46) yields the macroscopic mass conservation law:

$$
\phi\frac{\partial\varrho^{(0)}}{\partial t} + \operatorname{div}_x\left(\varrho^{(0)}\left\langle\mathbf{v}^{(0)}\right\rangle\right) + (1-\phi)\frac{\partial C^{(0)}}{\partial t} = 0. \tag{2.47}
$$

As in the Case I, the last term occurring in the above equation is a source term due to the diffusion process. Therefore, as in Case I, the gas constrained in the micropores interacts with the filtrating gas. However, the coupling is now clasical, and it does not introduce the memory effects.

The macroscopic equivalent description is given by Eqs. (2.32), (2.39), (2.45) and (2.47). When they are expressed in terms of physical variables, they have the following form:

$$
\begin{aligned}
&p^{(0)} = p^{(0)}(\mathbf{X}, t),\\
&\varrho^{(0)}(\mathbf{X}, t) = \frac{\varrho_a}{p_a} p^{(0)}(\mathbf{X}, t),\\
&C^{(0)} = \phi_s \varrho^{(0)}(\mathbf{X}, t),\\
&\left\langle v_i^{(0)} \right\rangle = -\frac{\langle k_{ij} \rangle}{\mu} l^2 \frac{\partial p^{(0)}}{\partial X_j},\\
&\phi \frac{\partial \varrho^{(0)}}{\partial t} + \operatorname{div}_X \left(\varrho^{(0)} \left\langle \mathbf{v}^{(0)} \right\rangle \right) + (1-\phi) \frac{\partial C^{(0)}}{\partial t} = 0.
\end{aligned}
\tag{2.48}
$$

Model III. Classical seepage law, $s \geq 2$ and $m \geq 0$, $S_{tl} \leq O(\varepsilon^2)$, $M_{tl} \leq O(1)$, $P_{el} \geq O(1)$, $T_D = O(\varepsilon T_S)$.

For simplicity, we do not present here the homogenization process. The procedure is very similar to that of the Cases I and II. It results in a macroscopic description similar to (2.48), without the time derivatives.

The Case III describes, at the macroscopic level, the stationary gas filtration in the micropores, without any influence of the diffusion. The macroscopic equivalent description is given by the following set:

$$
\begin{aligned}
&\left\langle v_i^{(0)} \right\rangle = -\frac{\langle k_{ij} \rangle}{\mu} l^2 \frac{\partial p^{(0)}}{\partial X_j},\\
&\operatorname{div}_X \left(\varrho^{(0)} \left\langle \mathbf{v}^{(0)} \right\rangle \right) = 0.
\end{aligned}
\tag{2.49}
$$

Moreover, the gas concentration in the solid is given at the first order of magnitude by

for $M_{tl} \leq O(\varepsilon)$

$$C^{(0)} = \phi_s \varrho^{(0)}(\mathbf{X}, t);$$

for $M_{tl} = O(1)$

$$\langle C^{(0)} \rangle = \phi_s \left((1-\phi) \varrho^{(0)} - \int_0^t \frac{\partial \varrho^{(0)}}{\partial t} \langle G(t-\tau) \rangle \, d\tau \right),$$

where

$$p^{(0)} = p^{(0)}(\mathbf{X}, t), \qquad \varrho^{(0)}(\mathbf{X}, t) = \frac{\varrho_a}{p_a} p^{(0)}(\mathbf{X}, t).$$

3. Remarks on the macroscopic behaviour

The passage from the pore scale to the macroscopic scale shows three different equivalent macroscopic descriptions, depending on the value of the dimensionless numbers:

Case I. Diffusion-filtration coupling with memory effects

$$
\begin{aligned}
&\phi\frac{\partial p^{(0)}}{\partial t} - \operatorname{div}\left(\frac{\langle k_{ij}\rangle l^2}{2\mu}\,\mathbf{grad}\,(p^{(0)})^2\right)\\
(3.1)\qquad &\qquad + \phi_s(1-\phi)\frac{\partial p^{(0)}}{\partial t} - \phi_s\frac{\partial}{\partial t}\left(\int\limits_0^t \frac{\partial p^{(0)}}{\partial t}\langle G(t-\tau)\rangle\,d\tau\right) = 0,\\
&\langle C^{(0)}\rangle = \phi_s\left((1-\phi)\varrho^{(0)} - \int\limits_0^t \frac{\partial \varrho^{(0)}}{\partial t}\langle G(t-\tau)\rangle\,d\tau\right).
\end{aligned}
$$

Case II. Classical diffusion-filtration coupling

$$
(3.2)\qquad
\begin{aligned}
&\phi\frac{\partial p^{(0)}}{\partial t} - \operatorname{div}\left(\frac{\langle k_{ij}\rangle l^2}{2\mu}\mathbf{grad}\,(p^{(0)})^2\right) + \phi_s(1-\phi)\frac{\partial p^{(0)}}{\partial t} = 0,\\
&C^{(0)} = \phi_s\varrho^{(0)}(\mathbf{x},t).
\end{aligned}
$$

The coupling is represented here by the term $\phi_s(1-\phi)\dfrac{\partial p^{(0)}}{\partial t}$. As in the Case I, the coupling term disappears when $\phi_s = 0$.

Case III. Classical seepage law

$$
(3.3)\qquad \operatorname{div}\left(\frac{\langle k_{ij}\rangle l^2}{\mu}\mathbf{grad}\,(p^{(0)})^2\right) = 0,
$$

and, additionally,

for $M_{tl} \le O(\varepsilon)$

$$
C^{(0)} = \phi_s\varrho^{(0)}(\mathbf{x},t);
$$

for $M_{tl} = O(1)$

$$
\langle C^{(0)}\rangle = \phi_s\left((1-\phi)\varrho^{(0)} - \int\limits_0^t \frac{\partial \varrho^{(0)}}{\partial t}\langle G(t-\tau)\rangle\,d\tau\right).
$$

The above equations have to be supplemented by suitable initial and boundary conditions for $p^{(0)}$.

We remark here that the physical meanings of the macroscopic quantities $p^{(0)}$, $\varrho^{(0)}$ and $\langle C^{(0)}\rangle$ do not pose any problems since they are equal or proportional to the corresponding local quantities. Relations $(3.1)_2$ and $(3.2)_2$ represent the constitutive equations of the gas. They give the concentration in the micropores as a function of the gas density in the macropores. The gas filtration is governed by the classical Darcy law. The macroscopic mass balance is represented by $(3.1)_1$ or $(3.2)_1$ or $(3.3)_1$. Their respective ranges of validity are obtained from the values of the dimensionless numbers. However, the description I is the most powerful since it comprises the descriptions II and III as particular behaviours. The descriptions II and III are obtained in the limit from description I for slow and rapid transient excitations, respectively.

Let us now study the total mass flux of the gas. It is the sum of the filtration flux and the diffusion flux,

$$\mathbf{F} = \varrho\langle \mathbf{v}\rangle - \langle \mathbf{D}\,\mathbf{grad}\, C\rangle.$$

To determine the contribution of filtration and diffusion in the total flux, we use again dimensionless variables. For the sake of clarity of the description, we do not omit now the asterisk which denotes the dimensionless variables. Within the approximation of $O(\varepsilon)$, the above relation becomes:

$$\mathbf{F}^{(0)} = \varrho_c v_c \left[\varrho^{*(0)}\left\langle \mathbf{v}^{*(0)}\right\rangle - \frac{\mathbf{D}\, C_c}{\varrho_c v_c l}\left(\varepsilon\,\mathbf{grad}_x\langle C^{*(0)}\rangle + \langle \mathbf{grad}_y C^{*(0)}\rangle\right)\right].$$

Now, from the definition (2.12) of the surface Peclet number

$$\frac{DC_c}{\varrho_c v_c l} = P_{el}^{-1},$$

we have

$$\mathbf{F}^{(0)} = \varrho_c v_c \left[\varrho^{*(0)}\langle \mathbf{v}^{*(0)}\rangle - P_{el}^{-1}\left(\varepsilon\,\mathbf{grad}_x\langle C^{*(0)}\rangle + \langle \mathbf{grad}_y C^{*(0)}\rangle\right)\right].$$

By using the estimations presented in the Sec. 2, it becomes in all cases

$$\left|\mathbf{F}^{(0)} - \varrho_c v_c \varrho^{*(0)}\langle \mathbf{v}^{*(0)}\rangle\right| \leq O(\varepsilon).$$

The total mass flux is equal to the pore filtrating flux within the approximation $O(\varepsilon)$.

4. One-dimensional problem

To emphasize the influence of the gas diffusion, let us consider the one-dimensional macroscopic boundary value problem. Consider the gas filtration through a horizontal and semi-infinite coal seam. In addition, we assume that:

- the coal stratum is an isotropic and homogeneous porous medium of constant thickness,
- the roof and the floor are impermeable to the gas,

- the mine opening is maintained at the atmosphere pressure p_a,
- the initial pressure p_i in the coal seam is constant,
- the long-wall head moves with a constant velocity ω.

With the above assumptions it is possible to change the problem to a steady state problem. We introduce the moving system of coordinates (ξ_1, ξ_2, ξ_3), Fig. 2, with $\xi_1 = x_1 - \omega t$ and $\xi_1 = 0$ on the long-wall head. The derivatives are transformed into the form:

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi_1}, \qquad \frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \xi_1}. \tag{4.1}$$

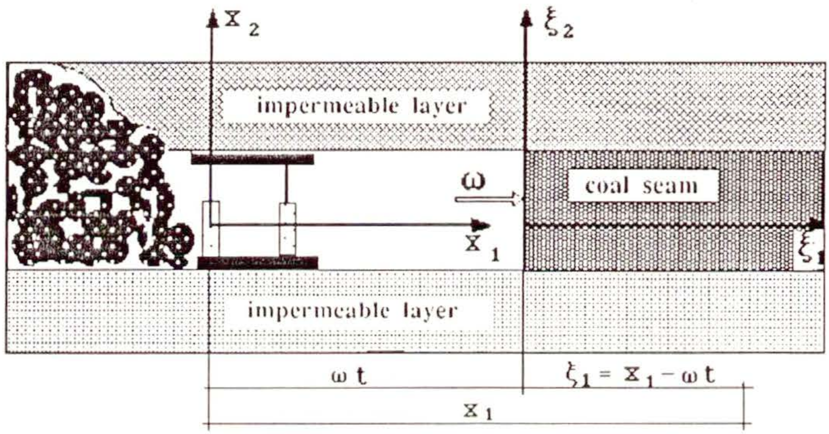

FIG. 2. Geometrical scheme of the one-dimensional problem.

We investigate three boundary value problems where one of the three descriptions is assumed to be valid everywhere throughout the seam:

I. Gas filtration with diffusion in the solid part and with memory effects (the model (3.1), Case I).

II. Gas filtration with gas diffusion in the solid part and without memory effect (3.2), Case II.

III. Gas filtration without any gas diffusion in the solid part (the classical model described by (3.3), Case III).

The solution of the Problem III can be obtained by direct integration of the differential equation describing this case. Taking into account the boundary conditions

$$\frac{\partial p}{\partial \xi_1} = 0, \qquad p = p_i \qquad \text{at } \xi_1 \to \infty,$$
$$p = p_a \qquad \text{at } \xi_1 = 0,$$

gives the gas pressure distribution and its gradient in the form [4]:

$$\xi_1 = \frac{\langle k \rangle l^2}{\phi \mu \omega} \left[p_a - p + p_i \ln \left(\frac{p_i - p_a}{p_i - p} \right) \right], \tag{4.2}$$

$$\frac{\partial p}{\partial \xi_1} = \frac{\phi\omega\mu}{\langle k\rangle l^2}\left[\frac{p_i}{p} - 1\right]. \tag{4.3}$$

Consider now the Problem II. It is easy to conclude that its solution can be obtained by changing ϕ into $\phi + \phi_s(1-\phi)$:

$$\xi_1 = \frac{\langle k\rangle l^2}{(\phi + \phi_s[1-\phi])\mu\omega}\left[p_a - p + p_i \ln\left(\frac{p_i - p_a}{p_i - p}\right)\right], \tag{4.4}$$

$$\frac{\partial p}{\partial \xi_1} = \frac{[\phi + \phi_s(1-\phi)]\omega\mu}{\langle k\rangle l^2}\left[\frac{p_i}{p} - 1\right]. \tag{4.5}$$

Solution of the Problem I necessitates the memory function $\langle G(t)\rangle$. It is defined from the set (2.34), where the right-hand side of $(2.34)_1$ is equal to unity. In order to present a closed analytical (not numerical) form of the memory function, we confine our study to a very simple model of the periodic cell. We assume spherical grains with radii R. The spatial structure of the grain packing is shown in Fig. 3. The grains are assumed to constitute of a homogeneous and isotropic microporous medium.

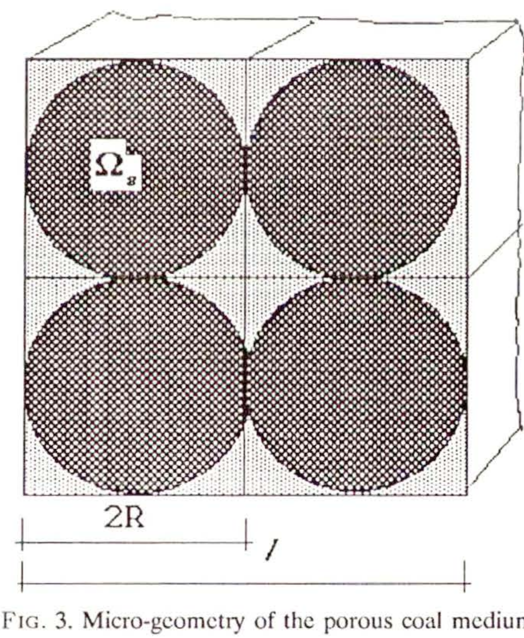

FIG. 3. Micro-geometry of the porous coal medium.

By using spherical coordinates and by putting $H(r,t) = G(r,t)\cdot r$, the set (2.33) can be written in the form:

$$\begin{aligned} &a\mathcal{L}(H(r,t)) - D\frac{d^2\mathcal{L}(H(r,t))}{dr^2} = r, \\ &\mathcal{L}(H(r,t)) = 0 \qquad \text{for} \qquad r = 0 \qquad \text{and} \qquad r = R, \end{aligned} \tag{4.6}$$

where r represents the radial coordinate.

The eigenvalues and eigenfunctions associated with the set (4.6) are:

$$\lambda_m = D\left(\frac{m\pi}{R}\right)^2, \qquad \varphi_m = \sqrt{\frac{2}{R}}\sin\left(\frac{m\pi}{R}r\right).$$

By looking for $\mathcal{L}(H(r,t))$ in the form:

$$\mathcal{L}(H(r,t)) = \sum_{m=1}^{\infty} d_m\varphi_m,$$

we obtain

$$d_m = -\frac{1}{a+\lambda_m}\frac{\sqrt{2R^3}}{m\pi}\cos(m\pi),$$

and

$$\mathcal{L}(H(r,t)) = -2R\sum_{m=1}^{\infty}\frac{1}{a+\lambda_m}\frac{\cos(m\pi)}{m\pi}\sin\left(\frac{m\pi}{R}r\right).$$

The Laplace transform of the function $G(r,t)$ is

$$\mathcal{L}(G(r,t)) = -\frac{2R}{r}\sum_{m=1}^{\infty}\frac{1}{a+\lambda_m}\frac{\cos(m\pi)}{m\pi}\sin\left(\frac{m\pi}{R}r\right).$$

Finally, by taking the volume average of the above equation and applying the inverse Laplace transform, we obtain $\langle G(t)\rangle$ in the form

$$\langle G(t)\rangle = \sum_{m=1}^{\infty}\frac{1}{m^2\pi}e^{-D(m\pi/R)^2t}. \tag{4.7}$$

Let us return to the Problem I. The memory effect in Eq. (3.1) is given by the convolution product of the memory function by the time derivative of the pressure. By integration by parts, this product can be presented in the following equivalent form:

$$\begin{aligned} &\int_0^t \frac{\partial p}{\partial\tau}\langle G(t-\tau)\rangle\,d\tau \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+1}\frac{1}{m^2\pi}\left[\frac{1}{D}\left(\frac{R}{m\pi}\right)^2\right]^n\left\{\left[\frac{\partial^n p}{\partial\tau^n}\right]_{\tau=t}-\left[\frac{\partial^n p}{\partial\tau^n}\right]_{\tau=0}e^{-D(k\pi/R)^2t}\right\}. \end{aligned} \tag{4.8}$$

By using the transformation rules (4.1) and by taking into account the above relation, we reduce the mathematical model of the Problem I to

$$\begin{aligned} &-\phi\omega\frac{\partial p}{\partial\xi_1}-\frac{\langle k_{ij}\rangle l^2}{2\mu}\frac{\partial^2p^2}{\partial\xi_1^2}-\phi_s(1-\phi)\omega\frac{\partial p}{\partial\xi_1} \\ &\qquad -\phi_s\sum_{n=1}^{\infty}\omega^{n+1}\frac{R^{2n}}{D^n\pi^{2n+1}}d_n\frac{\partial^{n+1}p}{\partial\xi_1^{n+1}} = 0, \end{aligned} \tag{4.9}$$

where

$$d_n = \sum_{m=1}^{\infty} \left(\frac{1}{m^2}\right)^{n+1} .$$

Clearly, Eq. (4.9) is too complicated for analytical solution. Therefore, a numerical iteration procedure is introduced to obtain an approximate solution. It gives the distributions of the gas pressure and its gradient. The results are plotted in Fig. 4 and 5, together with the results of III and II. The gas is carbon dioxide. The following typical values have been used in the calculations:

- macropore porosity: $\phi = 0.05$,
- micropore porosity: $\phi_S = 0.11$,
- coefficient of filtration: $\dfrac{\langle k\rangle l^2}{\mu} = 10^{-4}\dfrac{\mathrm{m}^4}{\mathrm{MNs}}$,
- diffusion coefficient in the micropores: $D = 10^{-11}\dfrac{\mathrm{m}^2}{\mathrm{s}}$,
- radius of grain (three cases): $R_1 = 10^{-3}$ m, $R_2 = 2\times10^{-3}$ m, $R_3 = 4\times10^{-3}$ m,
- initial gas pressure in the coal seam: $p_i = 4$ MPa,
- velocity of the long-wall head: $\omega = 8 \times 10^{-5}$ m/s.

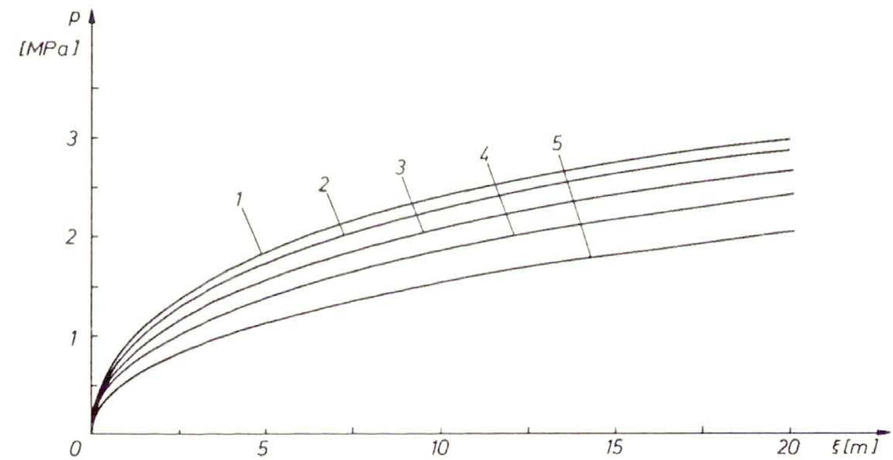

FIG. 4. Distribution of the gas pressure in the coal seam: *1* – Solution II, *2* – Solution I for $R = 1$ mm, *3* – Solution I for $R = 2$ mm, *4* – Solution I for $R = 4$ mm, *5* – Solution III.

Figure 4 and Fig. 5 show that II yields larger values of the gas pressure and of its gradient, whereas III gives lower values. The solutions III and II can be considered as bounds for the solution I. When there is no available information about the geometrical structure of the coal, they can be used as rough approximations of the pressure and its gradient. Note, hovever, the large difference between the two solutions III and II, in particular between the pressure gradients at the long-wall head.

The most important factor responsible for the occurrence of a gas-coal outburst is the gradient of the gas pressure at the long-wall head [4]. It is shown in

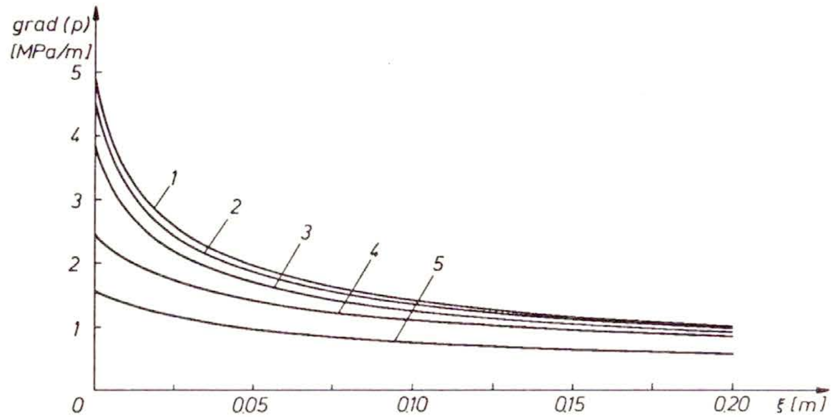

FIG. 5. Distribution of the gradient of the gas pressure in the vicinity of the long-wall head: *1* – Solution II, *2* – Solution I for $R = 1\,\mathrm{mm}$, *3* – Solution I for $R = 2\,\mathrm{mm}$, *4* – Solution I for $R = 4\,\mathrm{mm}$, *5* – Solution III.

Fig. 6 as a function of the grain radius. We conclude that the solution I converges to the solution II when the radius of the grain becomes smaller and smaller, and converges to the solution III when the radius becomes larger and larger. The curve in Fig. 6 shows also that a smaller radius yields a larger value of the gas pressure gradient at the long-wall head. We can immediately see the important role played by the grain radius or, more generally, the geometrical structure of coal. Our results agree with the empirical relation (1.1).

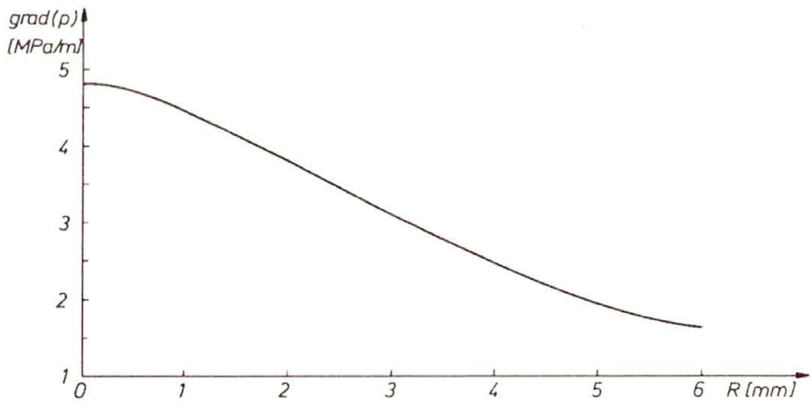

FIG. 6. The gradient of the gas pressure at the long-wall head versus the grain radius.

It is interesting to investigate the domain of validity of each description in the seam. It is now possible to estimate the macroscopic characteristic length $L(\xi)$ in each point of the seam, by using

$$L = \frac{\partial p}{\partial \xi_1} \Big/ \frac{\partial^2 p}{\partial \xi_1^2}.$$

The solutions I, II and III give approximately the same result. The resulting parameter ε is shown in the Fig. 7. It is seen that ε is small everywhere, except in a thin layer at the long-wall head where it goes to infinity. In this region there is no separation of scale and, consequently, there is no macroscopic description. The solutions I, II and III remain valid outside this boundary layer, i.e., approximately where $\varepsilon \geq 0.1$. The results in Fig. 6 are nevertheless valid because of the momentum balance applied to the boundary layer.

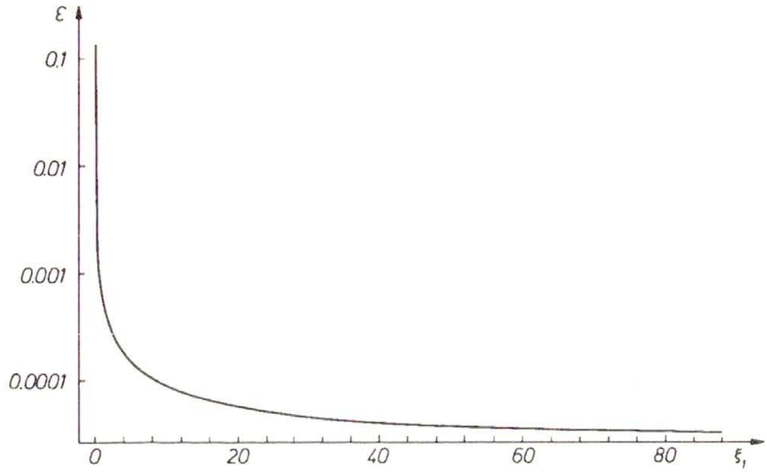

FIG. 7. Distribution of the parameter of scale separation ε in the coal seam.

The domain of validity of each description can be investigated by using the dimensionless number:

$$A = \frac{T_D}{T_S} = \frac{\varepsilon^{-1} S_{tl}}{M_{tl}} = \frac{DL}{l^2 v_c},$$

where v_c is given by

$$v_c = \frac{k}{\mu} l^2 \frac{\partial p}{\partial \xi_1}.$$

We have $A = O(1)$, $O(\varepsilon^{-1})$ and $O(\varepsilon)$ in the Case I, II and III, respectively. A, ε and ε^{-1} are plotted for comparison in the Fig.8. The figure shows four regions:

$$\xi_1 < 0.01 \text{ m}, \qquad \text{i.e.} \qquad \varepsilon > 0.1,$$

corresponds to the boundary layer where no macroscopic description is possible.

$$0.01\,\text{m} < \xi_1 < 0.3\,\text{m},$$

near the boundary layer, $A = O(\varepsilon)$, $T_D = O(\varepsilon T_S)$, and the classical description III can be applied.

$$0.3\,\text{m} < \xi_1 < 20\,\text{m}, \qquad A = O(1), \qquad T_D = O(T_S),$$

and the description I, with memory effects has to be considered.

$$\xi_1 > 20\,\text{m}, \qquad A = O(\varepsilon^{-1}), \qquad T_D = O(\varepsilon^{-1}T_S),$$

and the description II, classical coupling, is valid.

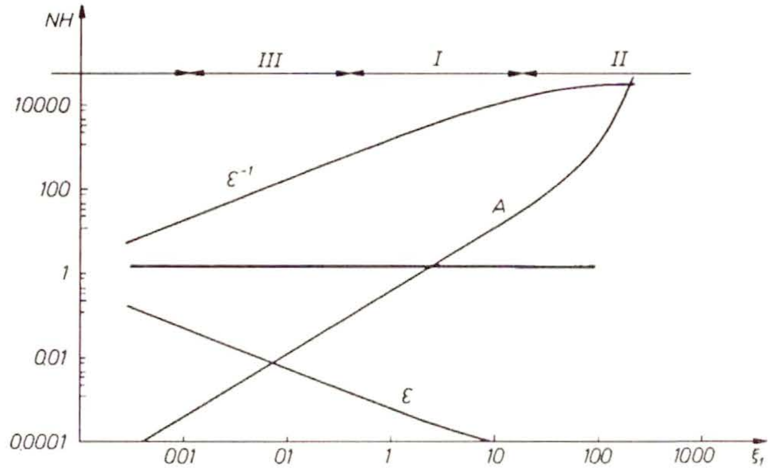

FIG. 8. Domains of validity of the three models. I: $A = O(1)$, Model I. II $A = O(\varepsilon^{-1})$ Model II. III $A = O(\varepsilon)$, Model III. NH: non-homogenizable.

5. Conclusions

The above study shows that the influence of the diffusion process in the micropores on the gas filtration in the macropores depends on a source term in the macroscopic equation of mass conservation. The filtration and the simultaneous diffusion of the gas are modelled by three different macroscopic descriptions. Appropriate dimensionless numbers, related to the physico-chemical properties and the geometrical structure of the coal, determine the model to be used. In particular, it is shown that the gas concentration exhibits memory effects if A, the ratio of the diffusion to the convection characteristic times, is of $O(1)$. When A decreases to $A = O(\varepsilon)$, the memory effects disappear and the model converges to the classical filtration model. The diffusion in the solid part is ignored. When A increases to $A = O(\varepsilon^{-1})$, the memory effects disappear too, and the model converges to a filtration-like model. The behaviour is described by an equation similar to the classical filtration process, but where the porosity of the macropores is replaced by the total porosity of the micropores and the macropores. The two last behaviours, i.e., the filtration without any diffusion and the filtration with the classical diffusion process, give bounds for the solution of the filtration with memory effects.

Appendix

Non-homogenization situation: $s = 0$, $m = 0$ (Case IV)

From (2.19), (2.21) and (2.22) we get

$$\frac{\partial \varrho^{(0)}}{\partial t} + \mathrm{div}_y(\varrho^{(0)}\mathbf{v}^{(0)}) = 0 \qquad \text{in } \Omega_p, \tag{A.1}$$

$$\frac{\partial C^{(0)}}{\partial t} - \mathrm{div}_y(\mathbf{D}\,\mathbf{grad}_y C^{(0)}) = 0 \qquad \text{in } \Omega_s, \tag{A.2}$$

$$(\varrho^{(0)}\mathbf{v}^{(0)} + \mathbf{D}\,\mathbf{grad}_y C^{(0)})\mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{A.3}$$

The above set, together with Eqs. (2.25)–(2.28), yields the sequence of the boundary value problems to be solved.

The first one is described by $(2.25)_1$, (2.26) and leads again to the relation (2.32).

Equations (A.2) and (2.27) determine the second problem. They are equivalent to the corresponding ones in the Case I. Therefore the first term of the gas concentration satisfies the relation (2.37).

Now we solve the fourth boundary value problem described by (A.1), (A.2) and (A.3). Taking the volume average and using the divergence theorem, Eq. (A.1) takes the form:

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} + \frac{1}{|\Omega|}\int\limits_{d\Omega_p} (\varrho^{(0)}\mathbf{v}^{(0)})\mathbf{n}\, dS = 0.$$

The condition (A.3) transforms the above equation into:

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} - \frac{1}{|\Omega|}\int\limits_{d\Omega_s} (\mathbf{D}\,\mathbf{grad}_y C^{(0)})\mathbf{n}\, dS = 0.$$

Now, by using (A.2) and again the divergence theorem, we obtain the following relation:

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} - \frac{\partial \langle C^{(0)}\rangle}{\partial t} = 0.$$

Substitution of (2.37) leads to

$$\phi\frac{\partial \varrho^{(0)}}{\partial t} - \frac{\varrho_c}{C_c}\phi_s\left((1-\phi)\frac{\partial \varrho^{(0)}}{\partial t} - \frac{\partial}{\partial t}\left[\int\limits_0^t \frac{\partial \varrho^{(0)}}{\partial \tau}\langle G(t-\tau)\rangle\, d\tau\right]\right) = 0.$$

Application of the Laplace transform and the convolution theorem leads to the equation

$$\mathcal{L}\left(\frac{\partial \varrho^{(0)}}{\partial t}\right)\left[\phi - \frac{\varrho_c}{C_c}\phi_s(1-\phi) + a\frac{\varrho_c}{C_c}\phi_s\mathcal{L}(\langle G(t)\rangle)\right] = 0,$$

where a is the complex Laplace variable.

The above relation must be valid for any values of a and for any geometry of the period Ω. Therefore, it is clear that

$$\mathcal{L}\left(\frac{\partial \varrho^{(0)}}{\partial t}\right) = 0,$$

and then

$$\frac{\partial \varrho^{(0)}}{\partial t} = 0.$$

This condition leads to the rescaling of the dimensionless number S_{tl}. This one becomes of the order of magnitude $O(\varepsilon)$, that is in a contradiction with our initial assumption $S_{tl} = O(1)$. Remark that $S_{tl} = O(1)$ does not satisfy the condition (2.13). We conclude that the case under consideration is not homogenizable.

Acknowledgement

D. Lydzba wishes to thank Ministère Francais de la Recherche et de l'Espace for the support by the grant No: 080350B 1 2215E.

References

1. W. BURGERT and H. LIPPMANN, *Models of translatory rock bursting in coal*, Int. J. Rock Mech. Min. Sci. and Geomech. Abstr., **18**, 285–294, 1981.
2. J. LITWINISZYN, *A model for the initiation of coal gas outbursts*, Int. J. Rock Mech. Min. Sci. and Geomech. Abstr., **22**, 39–46, 1985.
3. D. LYDZBA, *State of strength and stress in a gas-coal system near a wall head* [in Polish], Ph.D. Thesis, Inst. of Geotechn. of Technological University of Wrocław, 1990.
4. L. PATERSON, *A model for outbursts in coal*, Int. J. Rock Mech. Min. Sci. and Geomech. Abstr., **23**, 327–332, 1986.
5. M. LASON, *Specific surface of porous materials*, Arch. Min. Sci., **33**, 475-497, 1988.
6. H. GIL and A. SWIDZINSKI, *Rock and gas outbursts in underground mines* [in Polish], Silesian University of Technology, 1982.
7. J. LITWINISZYN, *Gas emission from a crushed rock medium during its sudden outburst*, Arch. Min. Sci., **32**, 155–168, 1987.
8. J.-L. AURIAULT and D. CAILLERIE, *Quelques remarques sur les méthodes d'homogénéisation*, Rev. Fr. Géotech., **49**, 43–50, 1989.
9. J.-L. AURIAULT and C. BOUTIN, *Deformable porous media with double porosity. Quasi-statics. I. Coupling effects*, TiPM, **7**, 63–82, 1992; *Quasi-statics. II. Memory effects*, TiPM, **10**, 153–169, 1993; *III. Acoustics*, TiPM, **14**, 143–162, 1994.
10. J.-L. AURIAUILT and P. ROYER, *Ecoulement d'un fluide très compressible dans un milieu poreux à double porosité*, CRAS, **317**, II, 431–436, 1993.
11. T. ARBOGAST, J. DOUGLAS and U. HORNUNG, *Derivation of the double porosity model of single phase flow via homogenization theory*, SIAM J. Math. Anal., **21**, 4, 823–836, 1990.
12. A. BENSOUSSAN, J.-L. LIONS and PAPANICOLAOU, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam 1978.
13. E. SANCHEZ-PALENCIA, *Non-homogeneous media and vibration theory*, Springer Verlag, New York 1980.
14. J.-L. AURIAULT, *Heterogeneous media. Is an equivalent macroscopic description possible?* Int. J. Engng. Sci., **29**, 785–795, 1991.

15. C.C. MEI and J.-L. AURIAULT, *The effect of weak inertia on the flow through a porous medium*, J. Fluid Mech., **222**, 647–663, 1991.

16. J.-L. AURIAULT, T. STRZELECKI, J. BAUER and S. HE, *Porous deformable media saturated by a very compressible fluid: quasi-statics*, Eur. J. Mech., A/Solids, **9**, 373–392, 1990.

17. J.-L. AURIAULT, *Effective macroscopic description for heat conduction in periodic composites*, Int. J. Heat Mass Transfer., **26**, 861–869, 1983.

18. J.-L. AURIAULT and T. STRZELECKI, *On electro-osmotic flow in a saturated porous medium*, Int. J. Engng. Sci., **19**, 915–928, 1981.

19. J.-L. AURIAULT, *Non-saturated deformable porous media: quasi-statics*, TiPM, **2**, 45–64, 1987.

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Received March 30, 1995.

Arch. Mech., 48, 3, pp. 475–492, Warszawa 1996

Influence of the Schulgasser inequality on effective moduli of two-phase isotropic composites

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THE AIM of this paper is to study the effective transport coefficients λ_e of macroscopically isotropic two-phase composites for the case, where dielectric coefficients λ_1 and λ_2 of components are real. As an input we take: (i) N coefficients of the power expansion of $\lambda_e(x)$ at $x = 0$, where $x = (\lambda_2/\lambda_1) - 1$; (ii) the analytical property of $\lambda_e(x)$, namely $\lambda_e(-1) \geq 0$; (iii) the Schulgasser inequality $\lambda_e(x)\lambda_e(y) = (\lambda_1)^2$, $y = -x/(x+1)$. By starting from (i), (ii) and (iii), an infinite set of bounds on $\lambda_e(x)$ has been established and compared with the corresponding ones reported in literature. As an example of illustration of the obtained results, the regular arrays of spheres has been investigated numerically.

1. Introduction

THE EFFECTIVE TRANSPORT coefficients λ_e of composite materials may be evaluated by the method of bounds [5, 6, 7, 8, 12, 19, 20]. The bounds become increasingly narrow, when more information concerning the geometrical properties of the medium is available.

Milton has derived in the complex λ_e-plane an infinite set of narrowing bounds on λ_e. The calculation of his bounds requires the knowledge of successive terms of the power expansion of λ_e in $\lambda_2 - \lambda_1$. The coefficients of the expansion are geometrical in nature and their values are determined by the correlation functions of disordered geometry. Milton's approach is based on an analytic representation of the effective dielectric constant due to BERGMAN [4]. The problem of complex bounds was also discussed by FELDERHOF [12], who obtained the estimation of λ_e with the help of four characteristic geometrical functions introduced by BERGMAN [5]. Recently, interesting continued fraction representations for the set of complex bounds on λ_e were presented by BERGMAN [6] for three-, and by CLARK and MILTON [8] for two-dimensional systems.

The fundamental estimations of $\lambda_e(x)$ reported in literature [20] do not exploit the well known Schulgasser inequality $\lambda_e(x)\lambda_e(y) \geq (\lambda_1)^2$, $y = -x/(x+1)$ [22]. Direct links of this inequality with bounds for isotropic, inhomogeneous materials has been advocated by MILTON [20, p. 5297], see also [7, p. 927]. He suggested that some of the existing bounds on $\lambda_e(x)$ are not the best, cf. [20, p. 5297]. A simple case of incorporation of $\lambda_e(x)\lambda_e(y) \geq (\lambda_1)^2$ into the second order bounds on $\lambda_e(x)$ only, was studied in [6].

The main aim of this paper is to include the Schulgasser inequality $\lambda_e(x)\lambda_e(y) \geq (\lambda_1)^2$, $y = -x/(x+1)$ into an infinite set of fundamental real-valued bounds on

$\lambda_e(x)$ reported by MILTON [20]. This aim is achieved by applying Padé approximants and continued fractions to the formulation of a method of incorporation of Schulgasser inequality into lower and upper bounds on scalar, bulk transport coefficients of two-phase media, see Theorem 2.

2. Basic definitions and assumptions

This study is concerned with the effective dielectric constant Λ_e of a composite consisting of two isotropic components of dielectric moduli λ_1, λ_2 and volume fractions φ_1 and $\varphi_2 = 1 - \varphi_1$, respectively. The overall dielectric coefficient Λ_e is defined by the linear relationship between the volume-averaged electric field $\langle \mathbf{U} \rangle$ and volume-averaged displacement $\langle \mathbf{D} \rangle$:

$$\langle \mathbf{D} \rangle = \Lambda_e \langle \mathbf{U} \rangle. \tag{2.1}$$

The value $\langle \cdot \rangle$ is averaged over a representative volume or a basic cell. In general, Λ_e will be a second-order symmetric tensor, even when λ_1 and λ_2 are both scalars, and will depend on the microstructure of composite. Our consideration will be limited to one of the diagonal element of Λ_e, say λ_e, which has a well known Stieltjes integral representation [4, 9, 10]

$$G(x) = \frac{\lambda_e(x)}{\lambda_1} - 1 = x \int_0^1 \frac{d\gamma(u)}{1 + xu}, \tag{2.2}$$

where

$$x = h - 1, \qquad h = \frac{\lambda_1}{\lambda_2}. \tag{2.3}$$

Here $G(x)$ is defined for $x \in (-1, \infty)$, cf. [6, 12]. The spectrum $\gamma(u)$ appearing in (2.2) is a real, bounded and non-decreasing function determined for $0 \le u < \infty$. The representation (2.2) was introduced by BERGMAN [6] and referred to as characteristic, geometrical function.

Let us consider the power expansion of (2.2)

$$G(x) = \sum_{n=1}^{\infty} G_n x^n, \tag{2.4}$$

where

$$G_n = (-1)^{n-1} \int_0^{\infty} u^{n-1}\, d\gamma(u). \tag{2.5}$$

For composite materials the coefficients G_n ($n = 1, 2, \ldots,$) are finite and series (2.4) is convergent for $|x| < 1$.

Now we are in a position to introduce the Padé approximants to Stieltjes series (2.4). To this end we consider the following rational functions

$$(2.6)\qquad [M+J/M] = \frac{L_{M+J}(x)}{P_M(x)} = \frac{a_1^{(J)}x + \cdots + a_{M+J}^{(J)}x^{M+J}}{1 + b_1^{(J)}x + \cdots +_M^{(J)} x^M}, \qquad J = 0,1,$$

with the power expansion of $[M + J/M]$ at $x = 0$

$$(2.7)\qquad [M+J/M](x) = \sum_{n=1}^{\infty} G_{n,J}x^n, \qquad J = 0,1.$$

The functions (2.6) are the subdiagonal ($J = 0$) and diagonal ($J = 1$) Padé approximants $[M + J/M]$ to the Stieltjes function (2.2), provided that

$$(2.8)\qquad G_{n,J} = G_n \qquad \text{for} \qquad n = 1,2,\ldots,2M+J, \qquad J = 0,1.$$

Padé approximants (2.6) can also be expressed in the form of S-continued fractions

$$(2.9)\qquad [M+J/M](x) = \frac{g_1x}{1} \, \frac{g_2x}{+\;1} \, +\cdots+ \frac{g_{2M+J}x}{1}, \qquad J = 0,1,$$

equivalent to the following explicit expression, see [1, 26]

$$[M+J/M](x) = \cfrac{g_1x}{1 + \cfrac{g_2x}{\cdots\; 1 + \cfrac{g_{2M+J-1}x}{1 + g_{2M+1}x}}}.$$

The coefficients $g_1, \ldots, g_{2M+J}$ appearing in (2.9) are positive and uniquely determined by the $2M + J$ coefficients G_n ($n = 1,2,...,2M + J$; $J = 0,1$) of a Stieltjes series (2.4).

After this preparation, we can recall the infinite set of fundamental bounds on $\lambda_e(x)$ derived by MILTON in [20]. By expanding his estimations $U_{N,0}(\varrho)$ and $V_{N,0}(\varrho)$ ($\varrho = x/(x + 2)$; $x = \lambda_2/\lambda_1 - 1$) [20, p. 5296] into S-continued fractions dependent on x, we obtain:

THEOREM 1. *For two-phase inhomogeneous media, the S-continued fractions* (2.9) *generated by power expansion* (2.4) *obey the following inequalities:*

(i) *If $x \geq 0$ then*

$$(2.10)\qquad V_{N,0}(x) \geq (-1)^N \frac{\lambda_e}{\lambda_1} \geq (-1)^N N_{N,0}(x).$$

(ii) *If $-1 \leq x \leq 0$, then*

$$(2.11)\qquad V_{N,0}(x) \leq \frac{\lambda_e}{\lambda_1} \leq U_{N,0}(x),$$

where C_{N+1} *is given by the following recurrence formula*

$$C_1 = 1, \qquad C_p = \frac{g_p}{1 - C_{p+1}}, \qquad p = 1, 2, \ldots N, \tag{2.12}$$

while $U_{N,0}(x)$ *and* $V_{N,0}(x)$ *take the following* S*-continued fraction forms*

$$\begin{aligned} U_{N,0}(x) &= 1 + \frac{g_1 x}{1} \underset{+}{} \frac{g_2 x}{1} \underset{+}{} \cdots \underset{+}{} \frac{g_N x}{1}, \\ V_{N,0}(x) &= 1 + \frac{g_1 x}{1} \underset{+}{} \frac{g_N x}{1} \underset{+}{} \cdots \underset{+}{} \frac{C_{N+1} x}{1}. \end{aligned} \tag{2.13}$$

Here $U_{N,0}(x)$ *is a Padé approximant given by* (2.9) *to power series* (2.4), N *denotes the number of known coefficients of a power series expansion* (2.4), *while* $x = (\lambda_2/\lambda_1) - 1$.

For macroscopically isotropic composites the well known Schulgasser inequality holds [22]:

$$\frac{\lambda_e(x)}{\lambda_1}\frac{\lambda_e(y)}{\lambda_1} \geq 1, \qquad \text{if} \qquad y = -\frac{x}{x+1} \qquad \text{and} \qquad x > -1. \tag{2.14}$$

The main purpose of this paper is to incorporate the relation (2.14) into S-fraction bounds (2.10)–(2.11).

3. Schulgasser inequality $\lambda_e(x)\lambda_e(y) \geq (\lambda_1)^2$

Let us consider the following class of S-continued fractions

$$\psi_{N+1}(x, q_{N+1}) = 1 + \frac{g_1 x}{1} \underset{+}{} \frac{g_2 x}{1} \underset{+}{} \cdots \underset{+}{} \frac{g_N x}{1} \underset{+}{} \frac{q_{N+1} x}{1}. \tag{3.1}$$

Here $g_j > 0$ $(j = 1, 2, ..., N)$ are uniquely determined by N terms of a power expansion of λ_e/λ_1, while q_{N+1} is a free parameter belonging to the interval

$$R_{N+1,0} = \Big\{ q_{N+1} \mid q_{N+1} \geq 0 \Big\}. \tag{3.2}$$

Now we will seek the interval $R_{N+1,1}(x)$ of admissible values of q_{N+1} defined by

$$R_{N+1,1}(x) = \Big\{ q_{N+1} \mid \psi_{N+1}(x, q_{N+1})\psi_{N+1}(y, q_{N+1}) \geq 1 \Big\}, \tag{3.3}$$

where $y = -x/(x+1)$. It is obvious that q_{N+1} determined by (3.3) satisfy the Schulgasser relation (2.14)

$$\psi_{N+1}(x, q_{N+1})\psi_{N+1}(y, q_{N+1}) \geq 1. \tag{3.4}$$

Of interest is the equality

$$\psi_{N+1}(x, q_{N+1})\psi_{N+1}(y, q_{N+1}) = 1, \qquad y = -x/(x+1), \tag{3.5}$$

i.e.:

$$\left(1 + \frac{g_1 x}{1} {}_{+} \cdots {}_{+} \frac{g_N x}{1} {}_{+} \frac{q_{N+1} x}{1}\right)\left(1 + \frac{g_1 y}{1} {}_{+} \cdots {}_{+} \frac{g_{N+1} y}{1} {}_{+} \frac{q_{N+1} y}{1}\right) = 1. \tag{3.6}$$

The recurrence formula for S-continued fractions reported in [2, Chap. 4.2] yields

$$\left(1 + \frac{g_1 z}{1} {}_{+} \cdots {}_{+} \frac{g_N z}{1} {}_{+} \frac{q_{N+1} z}{1}\right) = \frac{A_N(z) + A_{N-1}(z)q_{N+1}}{B_N(z) + B_{N-1}(z)q_{N+1}}, \tag{3.7}$$

where $A_N(z)$ and $B_N(z)$ are polynomials determined by

$$A_{-1} = 1, \quad A_0 = 1, \quad A_j(z) = A_{j-1}(z) + zg_jA_{j-2}(z), \quad j = 1, 2, ..., N, \tag{3.8}$$

$$B_{-1} = 0, \quad B_0 = 1, \quad B_j(z) = B_{j-1}(z) + zg_jB_{j-2}(z), \quad j = 1, 2, ..., N. \tag{3.9}$$

On the basis of (3.7), relation (3.6) takes the form

$$\frac{A_N(x) + xA_{N-1}(x)q_{N+1}}{B_N(x) + xB_{N-1}(x)q_{N+1}} \frac{A_N(y) + yA_{N-1}(y)q_{N+1}}{B_N(y) + yB_{N-1}(y)q_{N+1}} = 1. \tag{3.10}$$

Here $y = -x/(x+1)$. Simple rearrangements of (3.10) yield

$$\alpha_{N+1}(x)q_{N+1}^2 + \beta_{N+1}(x)q_{N+1} + \delta_{N+1}(x) = 0, \tag{3.11}$$

where

$$\alpha_{N+1}(x) = xy\left[A_{N-1}(x)A_{N-1}(y) - B_{N-1}(x)B_{N-1}(y)\right], \tag{3.12}$$

$$\begin{aligned}\beta_{N+1}(x) = {} & x\left[A_{N-1}(x)A_N(y) - B_{N-1}(x)B_N(y)\right] \\ & + y\left[A_N(x)A_{N-1}(y) - B_N(x)B_{N-1}(y)\right],\end{aligned} \tag{3.13}$$

$$\delta_{N+1}(x) = A_N(x)A_N(y) - B_N(x)B_N(y). \tag{3.14}$$

The solutions of (3.11) are given by

$$\begin{aligned} q'_{N+1}(x) &= -\frac{\beta_{N+1}(x)}{2\alpha_{N+1}(x)}\left[1 + \sqrt{1 - \frac{4\alpha_{N+1}(x)\delta_{N+1}(x)}{\beta_{N+1}^2(x)}}\right], \\ q''_{N+1}(x) &= -\frac{\beta_{N+1}(x)}{2\alpha_{N+1}(x)}\left[1 - \sqrt{1 - \frac{4\alpha_{N+1}(x)\delta_{N+1}(x)}{\beta_{N+1}^2(x)}}\right]. \end{aligned} \tag{3.15}$$

On account of (3.3) and (3.15) we have

(i) if $\alpha_{N+1}(x) \leq 0$, then

$$R_{N+1,1}(x) = \left\{q_{N+1} \mid q''_{N+1}(x) \leq q_{N+1} \leq q'_{N+1}(x)\right\}, \tag{3.16}$$

(ii) if $\alpha_{N+1}(x) \geq 0$, then

$$R_{N+1,1}(x) = \left\{q_{N+1} \mid q_{N+1}\tau \leq q''_{N+1} \vee q_{N+1}\tau \geq q'_{N+1}\right\}. \tag{3.17}$$

According to definition (3.3), a class of bounds given by

$$\psi_{N+1}(x, q_{N+1}), \qquad q_{N+1} \in R_{N+1,1}(x) \tag{3.18}$$

satisfies the Schulgasser inequality (2.14).

4. Inequality $\lambda_e(x)/\lambda_1 \geq \Lambda(x)$

Let us assume now that for fixed $x = (\lambda_2/\lambda_1) - 1$, the lower bound $\Lambda(x)$ on the effective modulus $\lambda_e(x)/\lambda_1$ is known,

$$\lambda_e(x)/\lambda_1 \geq \Lambda(x). \tag{4.1}$$

By using (3.1) we can write

$$\psi_{N+1}(x, q_{N+1}) \geq \Lambda(x). \tag{4.2}$$

Of interest is the equality, cf. $(2.10)_2$ and (2.14),

$$\psi_{N+1}(x, C_{N+1}) = \left(1 + \frac{g_1 x}{1} {}_{+} \cdots {}_{+} \frac{g_N x}{1} {}_{+} \frac{C_{N+1} x}{1}\right) = \Lambda(x). \tag{4.3}$$

By applying recurrence formulae (3.8)–(3.9) to continued fraction (4.3), we obtain

$$\Lambda(x) = \frac{A_N(x) + x A_{N-1}(x) C_{N+1}}{B_N(x) + x B_{N-1}(x) C_{N+1}}. \tag{4.4}$$

Hence

$$C_{N+1}(x) = \frac{\Lambda(x) B_N(x) - A_N(x)}{x[A_{N-1}(x) - \Lambda(x) B_{N-1}(x)]}. \tag{4.5}$$

Now we are in a position to introduce the interval $R_{N+1,2}(x)$ of admissible values of q_{N+1} given by

$$R_{N+1,2}(x) = \left\{q_{N+1} \mid \psi_{N+1}(x, q_{N+1}) \geq \Lambda(x)\right\}. \tag{4.6}$$

On account of (4.5) and (4.6), $R_{N+1,2}(x)$ takes a form

$$R_{N+1,2}(x) = \left\{q_{N+1} \mid q_{N+1} \leq C_{N+1}(x)\right\}. \tag{4.7}$$

Note that, according to (4.6) and (4.7), a class of bounds determined by

$$\psi_{N+1}(x, q_{N+1}), \qquad q_{N+1} \in R_{N+1,2}(x) \tag{4.8}$$

satisfies the inequality (4.2).

5. Bounds exploiting Schulgasser inequality

Let us introduce an interval $R_{N+1}(x)$

$$R_{N+1}(x) = R_{N+1,0} \cap R_{N+1,1}(x) \cap R_{N+1,2}(x), \tag{5.1}$$

where $R_{N+1,0}$, $R_{N+1,1}(x)$ and $R_{N+2,2}(x)$ are defined by (3.2), (3.16)–(3.17) and (4.7), respectively. Note that the class of functions

$$\psi_{N+1}(x, q_{N+1}), \qquad q_{N+1} \in R_{N+1}(x) \tag{5.2}$$

satisfy the inequalities (2.14) and (4.1). For $x \to -1^+$ the lower estimation of $\lambda_e(x)$ is well known, cf. [4, 5, 6, 23]

$$\Lambda(-1^+) = 0. \tag{5.3}$$

For such a case it is convenient to introduce the notation

$$\lim_{x \to -1^+} Q(x) \equiv Q(-1^+) \equiv Q(-1), \tag{5.4}$$

consequently used in the sequel. Now we are ready to formulate the theorem solving the problem of incorporation of the Schulgasser inequality (2.14) into bounds (2.10)–(2.12).

THEOREM 2. *For macroscopically isotropic two-phase inhomogeneous media, the S-continued fractions* (2.9) *generated by power expansion* (2.4) *obey the following inequalities:*

(i) *If* $x \geq 0$ ($x = (\lambda_2/\lambda_1) - 1$), *then*

$$\begin{gathered} (-1)^N \psi_{N+1}(x, E_{N+1}) \geq (-1)^N \frac{\lambda_e(x)}{\lambda_1} \geq (-1)^N \psi_N(x), \\ \psi_N(x) = 1 + \frac{g_1 x}{1} \, \frac{}{+} \, \frac{g_2 x}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{g_N x}{1}, \\ \psi_{N+1}(x, E_{N+1}) = 1 + \frac{g_1 x}{1} \, \frac{}{+} \, \frac{g_2 x}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{g_N x}{1} \, \frac{}{+} \, \frac{E_{N+1} x}{1}. \end{gathered} \tag{5.5}$$

(ii) *If* $-1 \leq x \leq 0$ $(x = (\lambda_2/\lambda_1) - 1)$, *then*

$$\begin{aligned} \psi_N(x) \geq \frac{\lambda_e(x)}{\lambda_1} \geq \psi_{N+2}(x, E_{N+1}, H_{N+2}), \\ \psi_N(x) = 1 + \frac{g_1 x}{1} {}_{+\cdots+} \frac{g_N x}{1}, \\ \psi_{N+2}(x, E_{N+1}, H_{N+2}) = 1 + \frac{g_1 x}{1} {}_{+\cdots+} \frac{g_N x}{1} {}_{+} \frac{E_{N+1} x}{1} {}_{+} \frac{H_{N+2} x}{1}. \end{aligned} \tag{5.6}$$

Here the coefficients H_{N+2} *and* E_{N+1} *are given by*

$$H_{N+2} = \frac{A_N(-1) - E_{N+1} A_{N-1}(-1)}{A_N(-1)}, \quad E_{N+1} = \min\{D_{N+1}, C_{N+1}\}, \tag{5.7}$$

$$D_{N+1} = \max\left\{q'_{N+1}(-1), q''_{N+1}(-1)\right\}, \quad C_{N+1} = \frac{A_N(-1)}{A_{N-1}(-1)}, \tag{5.8}$$

where $q'_{N+1}(-1)$, $q''_{N+1}(-1)$ *are determined by* (3.15). *Relation* $(5.8)_2$ *is a consequence of* (4.5) *and* (5.3), *while* N *appearing in* (5.5)–(5.8) *denotes the number of known coefficients of power series* (2.4).

P r o o f. It follows from Appendix A that $\alpha_{N+1}(-1) \leq 0$ and $\delta_{N+1}(-1) \geq 0$. Thus the roots of (3.11) q'_{N+1} and q''_{N+1} have opposite signs, cf. (3.15). On account of (5.1), (5.7) and (5.8), we get

$$R_{N+1}(-1) = \left\{\tau \mid 0 \leq \tau \leq E_{N+1}\right\}. \tag{5.9}$$

Hence the class of bounds (5.2) takes a form

$$\psi_{N+1}(x, \tau) = 1 + \frac{g_1 x}{1} {}_{+\cdots+} \frac{g_N x}{1} {}_{+} \frac{\tau x}{1}, \qquad 0 \leq \tau \leq E_{N+1}. \tag{5.10}$$

The first derivative of $\psi_{N+1}(x, \tau)$ with respect to τ satisfies

$$\begin{aligned} \frac{\partial \psi_{N+1}(x,\tau)}{\partial \tau} > 0, \quad \text{for} \quad x \in (0, \infty), \quad 0 \leq \tau \leq E_{N+1} \quad \text{and} \quad N = 0, 2, \ldots, \\ \frac{\partial \psi_{N+1}(x,\tau)}{\partial \tau} < 0, \quad \text{for} \quad x \in (0, \infty), \quad 0 \leq \tau \leq E_{N+1} \quad \text{and} \quad N = 1, 3, \ldots. \end{aligned} \tag{5.11}$$

Hence the continued fraction $\psi_{N+1}(x, \tau)$ $(x \in (0, \infty))$ defined by (5.10) assumes its extremal values for

$$\tau = 0 \qquad \text{and} \qquad \tau = E_{N+1}. \tag{5.12}$$

By substituting (5.12) into (5.10) we obtain the formula (5.5).

If $-1 \leq x \leq 0$, the inequalities (5.6) result from the relations:

$$0 < g_{N+1} \leq E_{N+1}, \quad \lambda_e/\lambda_1 \geq \psi_{N+2}(x, C_{N+2}) \geq \psi_{N+2}(x, E_{N+1}, H_{N+2}), \tag{5.13}$$

where

$$\psi_{N+2}(x, C_{N+2}) = 1 + \frac{g_1 x}{1} \, _{+\cdots+} \, \frac{g_N x}{1} \, _+ \, \frac{g_{N+1} x}{1} \, _+ \, \frac{C_{N+2} x}{1}, \tag{5.14}$$

$$\psi_{N+2}(x, E_{N+1}, H_{N+2}) = 1 + \frac{g_1 x}{1} \, _{+\cdots+} \, \frac{g_N x}{1} \, _+ \, \frac{E_{N+1} x}{1} \, _+ \, \frac{H_{N+2} x}{1}. \quad \square \tag{5.15}$$

Note that for $D_{N+1} \geq C_{N+1}$, the bounds determined by Th. 2 reduce to the existing ones defined by Th. 1, since the parameters C_{N+1} given by (2.12) and $(5.8)_2$ coincide, while $H_{N+2} = 0$. Hence the estimations (5.5)–(5.6) obtained in the present paper can not be worse than the previous bounds (2.10)–(2.11) reported in literature [20]. Moreover, for some cases they have to be better. In the next section we demonstrate the analytical form of a low order bounds on $\lambda_e(x)/\lambda_1$ given by (5.5) and (5.6).

6. Low order bounds on λ_e

To illustrate Th. 2 we will evaluate bounds on an effective dielectric constant $\lambda_e(x)$ for the cases, where (i) no coefficients ($N = 0$), (ii) one coefficient ($N = 1$) and (iii) two coefficients ($N = 2$) of the power expansion of $\lambda_e(x)/\lambda_1$ are available.

(i) The recurrence formulae (3.8) and (3.9) give:

$$A_{-2} = 0, \qquad A_{-1} = 1, \qquad A_0 = 1, \qquad B_{-2} = 0, \qquad B_{-1} = 0, \qquad B_0 = 1. \tag{6.1}$$

Then relations (3.12)–(3.14) yield

$$\alpha_1(x) = xy, \qquad \beta_1(x) = x + y, \qquad \delta_1(x) = 0. \tag{6.2}$$

Hence from (3.15), (4.5) we get

$$q_1' = -\frac{x + y}{xy}, \qquad q_1'' = 0, \qquad C_1 = -\frac{1}{x}, y = -x/(x + 1). \tag{6.3}$$

For $x = -1^+$ the equations (6.3) reduce to

$$q_1' = 1, \qquad q_1'' = 0, \qquad C_1 = 1. \tag{6.4}$$

From (5.7) and (5.8), it follows that

$$D_1 = 1, \qquad E_1 = 1. \tag{6.5}$$

Hence, on the basis of Th. 2 the bounds on λ_e are given by

$$1 \geq \frac{\lambda_e}{\lambda_1} \geq 1 + x, \quad \text{if} \quad -1 \leq x \leq 0; \qquad 1 + x \geq \frac{\lambda_e}{\lambda_1} \geq 1, \quad \text{if} \quad x \geq 0. \tag{6.6}$$

(ii) $N = 1$. Then

$$A_{-2} = 0, \qquad A_{-1} = 1, \qquad A_0 = 1, \qquad B_{-1} = 0, \qquad B_0 = 1, \tag{6.7}$$

$$\begin{gathered}\alpha_2(x) = 0, \qquad \beta_2(x) = 2g_1xy,\\ \delta_2(x) = g_1x + g_1y + g_1^2xy, \qquad y = -x/(x+1),\end{gathered} \tag{6.8}$$

$$q_2' = -\infty, \qquad q_2'' = -\frac{g_1x + g_1y + g_1^2xy}{2g_1xy}, \qquad C_2 = \frac{-(1+g_1x)}{x}. \tag{6.9}$$

For $x = -1^+$ we have

$$\begin{gathered} q_2' = -\infty, \qquad q_2'' = \frac{1-g_1}{2}, \qquad C_2 = 1 - q_1, \\ D_2 = \frac{1-g_1}{2}, \qquad E_2 = \frac{1-g_1}{2}, \qquad H_3 = \frac{1}{2}. \end{gathered} \tag{6.10}$$

From (5.5), (5.6) and (6.10) we readily obtain

$$1 + \frac{g_1x}{1}{}_{+}\frac{(1-g_1)x/2}{1} \le \frac{\lambda_e}{\lambda_1} \le 1 + g_1x. \tag{6.11}$$

(iii) $N = 2$. Now we have

$$\alpha_3(x) = xy[(1+g_1x)(1+g_1y) - 1], \qquad y = -x/(x+1), \tag{6.12}$$

$$\beta_3(x) = xg_1[x + y + (g_1+g_2)xy] + yg_2[x + y + (g_1+g_2)xy], \qquad y = -x/(x+1), \tag{6.13}$$

$$\delta_3(x) = g_1x(1+g_2y) + g_1y(1+g_2x) + g_2g_2xy, \qquad y = -x/(x+1). \tag{6.14}$$

Thus for $x = -1^+$

$$q_3' = \frac{1-g_1-g_2}{1-g_1}, \qquad q_3'' = 0, \qquad C_3 = \frac{1-g_1-g_2}{1-g_1}. \tag{6.15}$$

Hence

$$\begin{gathered} 1 + \frac{g_1x}{1}{}_{+}\frac{g_2x}{1}{}_{+}\frac{\dfrac{(1-g_1-g_2)x}{1-g_1}}{1} \ge \frac{\lambda_e}{\lambda_1} \ge 1 + \frac{g_1x}{1}{}_{+}\frac{g_2x}{1}, \quad \text{if} \quad -1 \le x \le 0, \\ 1 + \frac{g_1x}{1}{}_{+}\frac{g_2x}{1}{}_{+}\frac{\dfrac{(1-g_1-g_2)x}{1-g_1}}{1} \le \frac{\lambda_e}{\lambda_1} \le 1 + \frac{g_1x}{1}{}_{+}\frac{g_2x}{1}, \quad \text{if} \quad x \ge 0. \end{gathered} \tag{6.16}$$

It is interesting to compare the low order bounds existing in literature (Th. 1) with the bounds incorporating the Schulgasser inequality (Th. 2). The basic bounds (6.6) are the same, the estimations (6.11) are more restrictive than the well known Wiener bounds [27] (Fig. 1), while the inequalities (6.16) coincide with Hashin–Shtrikman bounds reported in [14].

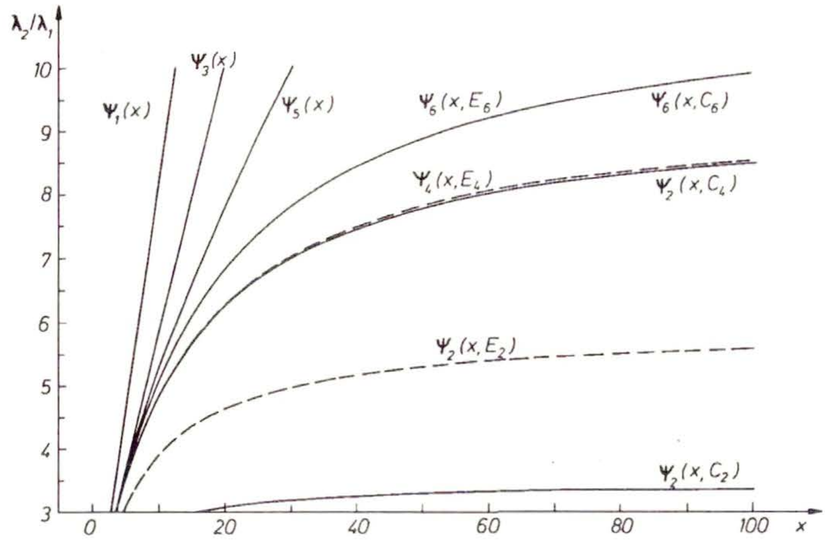

FIG. 1. Existing (—) and improved (- - -) bounds on the effective dielectric constant of a face-centered lattice of spheres for volume fraction $\varphi_2 = 0.71$. Upper bounds $\Psi_N(x)$ ($N = 1, 3, 5$) coincide, while lower ones $\Psi_{N+1}(x, C_{N+1})$ and $\Psi_{N+1}(x, E_{N+1})$ differ significantly for $N = 1$ and slightly for $N = 3, 5$.

7. Even number of terms of a power expansion of λ_e

In this section we will compare the known (2.10)–(2.11) and obtained (5.5)–(5.6) bounds calculated from an even number ($N = 0, 2, 4, ...$) of coefficients of power series (2.4). To this end we prove that for $x \to -1^+$, thus $y = -x/(x+1) \to \infty$ ($N = 0, 2, ...$), the expressions (3.15) reduce via (3.12)–(3.14) to

$$\lim_{x \to -1} 2\alpha_{N+1}(x) \neq 0, \tag{7.1}$$

$$\begin{aligned} q'_{N+1} &= \lim_{x\to -1^+} -\frac{\beta_{N+1}(x)}{2\alpha_{N+1}(x)}\left(1+\sqrt{1-\frac{4\alpha_{N+1}(x)\delta_{N+1}(x)}{\beta^2_{N+1}(x)}}\right) = \frac{A_N(-1)}{A_{N-1}(-1)},\\ q''_{N+1} &= \lim_{x\to -1^+} -\frac{\beta_{N+1}(x)}{2\alpha_{N+1}(x)}\left(1-\sqrt{1-\frac{4\alpha_{N+1}(x)\delta_{N+1}(x)}{\beta^2_{N+1}(x)}}\right) = 0. \end{aligned} \tag{7.2}$$

P r o o f. The recurrence formulae (3.8) and (3.9) for S-continued fractions [2] and the Schulgasser inequality (3.4) yields

$$\begin{gathered} \frac{A_N(x)A_N(y)}{B_N(x)B_N(y)} \geq 1,\\ A_N(x)A_N(y) > 0, \qquad B_N(x)B_N(y) > 0,\\ \lim_{x\to -1^+} \frac{A_N(y)}{A_{N-1}(y)} \leq \infty, \qquad \lim_{x\to -1^+} \frac{B_N(y)}{A_N(y)} \leq \infty. \end{gathered} \tag{7.3}$$

For even N, on the basis of (3.12), (3.14) and (7.3), we have

$$\alpha_{N+1}(x) = xy\delta_N(x) = xyB_{N-1}(x)B_{N-1}(y)\left(\frac{A_{N-1}(x)A_{N-1}(y)}{B_{N-1}(x)B_{N-1}(y)} - 1\right) \neq 0, \tag{7.4}$$

$$\lim_{x\to-1^+} \frac{\beta_{N+1}(x)}{2\alpha_{N+1}(x)} = \frac{1}{y}\,\frac{A_N(y))}{A_{N-1}(y)}\,\frac{\left(A_{N-1}(x) - B_{N-1}(x)\dfrac{B_N(y)}{A_N(y)}\right)}{\left(A_{N-1}(x) - B_{N-1}(x)\dfrac{B_{N-1}(y)}{A_{N-1}(y)}\right)}$$
$$+\frac{1}{x}\,\frac{\left(A_N(x) - B_N(x)\dfrac{B_{N-1}(y)}{A_{N-1}(y)}\right)}{\left(A_{N-1}(x) - B_{N-1}(x)\dfrac{B_{N-1}(y)}{A_{N-1}(y)}\right)} = -\frac{A_N(-1)}{A_{N-1}(-1)}, \tag{7.5}$$

$$\lim_{x\to-1^+} \frac{\alpha_{N+1}(x)}{\beta_{N+1}(x)}\,\frac{\delta_{N+1}(x)}{\beta_{N+1}(x)} = \lim_{x\to-1^+} \frac{\alpha_{N+1}(x)}{\beta_{N+1}(x)}\lim_{x\to-1^+}\frac{\delta_{N+1}(x)}{\beta_{N+1}(x)}$$
$$= \frac{A_N(-1)}{A_{N-1}(-1)}\lim_{x\to-1^+}\frac{\delta_{N+1}(x)}{\beta_{N+1}(x)},$$

$$\lim_{x\to-1^+}\frac{\delta_{N+1}(x)}{\beta_{N+1}(x)} = \lim_{x\to-1^+}$$
$$\frac{\left(A_N(x) - B_N(x)\dfrac{B_N(y)}{A_N(y)}\right)}{x\left(A_{N-1}(x) - B_{N-1}(x)\dfrac{B_N(y)}{A_N(y)}\right) + \dfrac{yA_{N-1}(y)}{A_N(y)}\left(A_N(x) - B_N(x)\dfrac{B_{N-1}(y)}{A_{N-1}(y)}\right)} = 0. \tag{7.6}$$

From (7.4)–(7.6), follow the relations (7.1) and (7.2). $\square$

For $A(-1) = 0$ and even N ($N = 0, 2, \ldots$), the relation (4.5) coincides with $(7.2)_1$. Hence inequalities (5.5) and (5.6) agree with (2.10) and (2.11). Consequently for even N, the S-continued fraction method based on the Schulgasser inequality (2.14) does not provide better bounds than the approaches neglecting this inequality. Therefore an improvement of the existing bounds on $\lambda_e(x)$ can be expected for odd N ($N = 1, 3, ..$) of coefficients of power expansion of $\lambda_e(x)$ only.

8. Regular arrays of spheres

Now we are prepared to apply Th. 2 to regular lattices of spheres embedded in an infinite matrix. By λ_e, λ_2 and λ_1 we denote the dielectric constants of the composite, spheres and matrix, respectively. The first three coefficients of the power expansion of $(\lambda_e/\lambda_1) - 1$ are as follows [4], cf. (2.2), (2.4):

$$\frac{\lambda_e}{\lambda_1} - 1 = \varphi_2 x - \frac{1}{3}\varphi_1\varphi_2 x^2 + O(x^3), \tag{8.1}$$

where, as previously, $x = (\lambda_2/\lambda_1) - 1$. Here φ_2, φ_1 denote volume fractions of the spheres and matrix. On the basis of (2.6), S-continued fractions (2.9) associated with (8.1) are expressed by

$$[0/0] = 0, \qquad [1/0] = \frac{\varphi_2 x}{1}, \qquad [1/1] = \frac{\varphi_2 x}{1\;+}\,\frac{(\varphi_1/3)x}{1}, \tag{8.2}$$

where

$$g_1 = \varphi_2, \qquad g_2 = \varphi_1. \tag{8.3}$$

Hence from (6.6), (6.11) and (6.16) we have:

(i) for $N = 0$

$$\begin{aligned} &1 \geq \lambda_e/\lambda_1 \geq 1 + x, \qquad \text{if} \quad \lambda_2 \leq \lambda_1, \\ &1 \leq \lambda_e/\lambda_1 \leq 1 + x, \qquad \text{if} \quad \lambda_2 \geq \lambda_1; \end{aligned} \tag{8.4}$$

(ii) for $N = 1$

$$1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x/2}{1} \leq \frac{\lambda_e}{\lambda_1} \leq 1 + \frac{\varphi_2 x}{1}; \tag{8.5}$$

(iii) for $N = 2$

$$\begin{aligned} &1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x/3}{1} \geq \frac{\lambda_e}{\lambda_1} \geq 1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x/3}{1\;+}\,\frac{2x/3}{1}, \qquad \text{if} \quad \lambda_2 \leq \lambda_1, \\ &1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x/3}{1} \leq \frac{\lambda_e}{\lambda_1} \geq 1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x/3}{1\;+}\,\frac{2x/3}{1}, \qquad \text{if} \quad \lambda_2 \geq \lambda_1. \end{aligned} \tag{8.6}$$

According to the results of Sec. 7 valid for even N, the bounds (8.4) and (8.6) agree with the existing bounds following from Th. 1, where (8.6) are Hashin–Shtrikman bounds. Of interest is the case (8.5). For $N = 1$, from Th. 1 follow the well known Wiener bounds [27]

$$1 + \frac{\varphi_2 x}{1\;+}\,\frac{\varphi_1 x}{1} \leq \frac{\lambda_e}{\lambda_1} \leq 1 + \frac{\varphi_2 x}{1}. \tag{8.7}$$

By comparing (8.5) with (8.7) we conclude that incorporation of the Schulgasser inequality (Th. 2) improves lower bound of WIENER [27], while the upper one remains the same (Fig. 1). To determine bounds more exactly, further terms of the power expansion of $\lambda_e(x)/\lambda_1$ are required. For simple, body-centered and face-centered, cubic lattices of spheres, MCPHEDRAN and MILTON [16] evaluated the coefficients of a power series expansion of $\lambda_e(\alpha)/\lambda_1$, $\alpha = x/(x+2)$ at $\alpha = 0$, and gathered them in tables as discrete functions of φ_2. In [25] we derive a simple formula relating the terms of a power series of $\lambda_e(x)/\lambda_1$ to the terms of

Table 1. Low order coefficients G_n, g_n, C_{N+1}, E_{N+1}, H_{N+2} for evaluation of S-continued fraction bounds for the effective conductivity of regular arrays of spheres.

Arrays of spheres		$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$\varphi_2 = 0.52$	G_n	0.52	0.0832	0.0248	0.0102	0.0050	0.0028	
Simple	g_n	0.52	0.1600	0.1380	0.2420	0.1727	0.2579	
cubic	C_n	1.00	0.4800	0.6667	0.7930	0.6949	0.7514	0.6568
	E_n	1.00	0.2400	0.6667	0.7427	0.6949	0.7473	0.6568
	H_n		0.0000	0.5000	0.0000	0.0634	0.0000	0.0055
$\varphi_2 = 0.67$	G_n	0.67	0.0737	0.0155	0.0053	0.0025	0.0015	
Body-	g_n	0.67	0.1100	0.1009	0.2761	0.2020	0.2566	
centered	C_n	1.00	0.3300	0.6667	0.8486	0.6747	0.7006	0.6337
	E_n	1.00	0.1650	0.6667	0.8082	0.6747	0.6960	0.6337
	H_n		0.0000	0.5000	0.0000	0.0476	0.0000	0.0066
$\varphi_2 = 0.71$	G_n	0.71	0.0686	0.0147	0.0058	0.0030	0.0018	
Face-	g_n	0.71	0.0967	0.1171	0.3342	0.1221	0.3168	
centered	C_n	1.00	0.2900	0.6667	0.8244	0.5947	0.7947	0.6013
	E_n	1.00	0.1450	0.6667	0.7794	0.5947	0.7889	0.6013
	H_n		0.0000	0.5000	0.0000	0.0546	0.0000	0.0074

the power expansion of $\lambda_e(\alpha)/\varepsilon_1$, $\alpha = x/(x+2)$. From the coefficients given in [16, Tabs. 6, 7, 8] we have calculated, by using the algorithm proposed by us in [25], the coefficients G_n of power series (2.4). The coefficients g_n, C_{N+1} and E_{N+1} gathered in Table 1 are evaluated by means of the numerical procedure proposed in [25]. Note that for even n (odd N), the coefficients E_{N+1} ($n = N+1$) are smaller than C_{N+1}, while for odd n (even N) they take the same values. For face-centered cubic arrays of spheres (fcc) the existing bounds and the improved ones are presented in Tables 2 and 3.

Table 2. Existing $\{\psi_N(x), \psi_{N+1}(x, C_{N+1})$, Th. 1$\}$ and improved $\{\psi_N(x), \psi_{N+2}(x, E_{N+1}, H_{N+2})$ Th. 2$\}$ low order bounds on $\lambda_e(x)/\lambda_1$ for the fcc lattice of spheres.

φ_2	N	x	$\psi_N(x)$	$\psi_{N+2}(x, E_{N+1}, H_{N+2})$	$\psi_{N+1}(x, C_{N+1})$
	1	−0.5	0.6450	0.607011	0.584795
0.71	3	−0.5	0.6258	0.624909	0.624863
	5	−0.5	0.6255	0.625497	0.625497
	1	−0.7	0.5030	0.411030	0.376512
0.71	3	−0.7	0.4634	0.457736	0.457466
	5	−0.7	0.4621	0.461837	0.461835
	1	−0.9	0.3610	0.162217	0.135318
0.71	3	−0.9	0.2921	0.252278	0.250850
	5	−0.9	0.2872	0.282345	0.282319

Table 3. Existing $\{\psi_N(x), \psi_{N+1}(x, C_{N+1}),$ Th. 1$\}$ and improved $\{\psi_N(x), \psi_{N+1}(x, E_{N+1})$ Th. 2$\}$ low order bounds on $\lambda_e(x)/\lambda_1$ for the fcc lattice of spheres.

φ_2	N	x	$\psi_N(x)$	$\psi_{N+1}(x, E_{N+1})$	$\psi_{N+1}(x, C_{N+1})$
0.71	1	50.0	36.500	5.303030	3.290323
	3	50.0	21.817	7.806020	7.768516
	5	50.0	13.861	8.872180	8.870695
0.71	1	70.0	50.700	5.457399	3.333333
	3	70.0	29.629	8.206098	8.163556
	5	70.0	17.539	9.442256	9.440478
0.71	1	90.0	64.900	5.548043	3.357934
	3	90.0	37.427	8.449407	8.403644
	5	90.0	21.133	9.796655	9.794679

9. Concluding remarks

By starting from: (i) N coefficients of the power expansion of $\lambda_e(x)$ at $x = 0$, (ii) – the analytical property $\lambda_e(-1) > 0$, and (iii) – the Schulgasser inequality (2.14), an infinite set of upper and lower bounds on the effective transport coefficient $\lambda_e(x)$ of two-phase, isotropic composites have been established (Theorem 2) and investigated in detail.

With respect to the corresponding estimations reported in literature (Th. 1), the improvement has been obtained for the case of lower bounds on $\lambda_e(x)$ constructed from an odd number N of coefficients of a power expansion of $\lambda_e(x)$, cf. Fig. 1, Tables 2 and 3. For even N the incorporation of the Schulgasser inequality (2.14) does not provide better bounds in comparison to the approaches neglecting this inequality [7, 8, 22].

As an example of illustration of Theorem 2, the existing and improved bounds on the effective dielectric constant for regular, face-centered arrays of spheres have been evaluated and depicted in Fig. 1, Tabs. 2 and 3. A significant improvement has been observed for $N = 1$. For $N = 2$ the difference between the bounds reported in the literature [20] and in the present paper is relatively small, while for $N = 3$ it is negligible (Fig. 1). Note that the above conclusion is valid for a special geometry of two-phase composite, namely a regular array of spheres. For such a composite and for $n = 4, 6$, from Table 1 we have $E_n/C_n \simeq 1$. In the case of other geometrical structures, when the ratio E_n/C_n satisfies for instance $E_n/C_n < 0.5$ (Tab. 1), it is possible to get much better improvement.

Appendix A

In this Appendix we demonstrate the lemma indispensable for incorporating the Schulgasser inequality (2.14) into the bounds on λ_e.

LEMMA A.1. If a Stieltjes function

$$\frac{\lambda_e(x)}{\lambda_1} = 1 + x\int_0^1 \frac{d\gamma(u)}{1+xu} \tag{A.1}$$

satisfies the relations

$$\frac{\lambda_e(x)}{\lambda_1}\frac{\lambda_e(y)}{\lambda_1} \geq 1, \qquad y = -x/(1+x), \qquad x \in (-1,\infty), \tag{A.2}$$

then Padé approximants $A_N(x)/B_N(x)$ to $\lambda_e(x)/\lambda_1$ obey the inequalities

$$\frac{A_N(x)}{B_N(x))}\frac{A_N(y)}{B_N(y)} \geq 1 \quad (N = 0,1,2...), \quad y = -x/(1+x), \quad x \in (-1,\infty). \tag{A.3}$$

Here $A_N(x)$ and $B_N(x)$ are polynomials determined by recurrence formulae (3.8) – (3.9).

P r o o f. The analytical properties of $A_N(x)/B_N(x)$ $(N = 0,1,2...)$ yield:

$$\text{if} \quad \lim_{x\to -1^+} \frac{A_N(x)}{B_N(x)}\frac{A_N(y)}{B_N(y)} = 1 \tag{A.4}$$

$$\text{then} \quad \frac{A_N(x)}{B_N(x)}\frac{A_N(y)}{B_N(y)} \geq 1 \qquad \text{in } x \in (-1,\infty),$$

where $y = -x/(x+1)$. Hence of interest is the inequality (A.3) taken for $x \to -1^+$. On the basis of Theorem 1 we have:

(i) if N is odd, then

$$\frac{A_N(-1^+)}{B_N(-1^+)} \geq \frac{\lambda_e(-1^+)}{\lambda_1}, \qquad \text{and} \qquad \frac{A_N(\infty)}{B_N(\infty)} \geq \frac{\lambda_e(\infty)}{\lambda_1}, \quad \text{if} \quad x \geq 0. \tag{A.5}$$

(ii) If N is even, then

$$\begin{aligned} &\frac{A_N(-1^+)}{B_N(-1^+)} \geq \frac{\lambda_e(-1^+)}{\lambda_1}, \qquad &\text{if} \quad -1 \leq x \leq 0, \\ &\frac{A_N(\infty)}{B_N(\infty)} \leq \frac{\lambda_e(\infty)}{\lambda_1}, \qquad &\text{if} \quad x \geq 0. \end{aligned} \tag{A.6}$$

According to Th. 1 and Th. 15.2 reported in [1], Padé approximants $A_N(-1^+)/B_N(-1^+)$ and $A_N(\infty)/B_N(\infty)$ $(N = 0,2,...)$ are the best bounds for Stieltjes function $\lambda_e(-1^+)/\lambda_1$ and $\lambda_e(\infty)/\lambda_1$ with respect to a given number of coefficients of a power expansion of $\lambda_e(x)/\lambda_1$ at $x = 0$. Hence the relations

$$\frac{A_N(-1^+)}{B_N(-1^+)}\frac{A_N(\infty)}{B_N(\infty)} \geq 1, \qquad N = (0,2,...) \tag{A.7}$$

have to be satisfied. From (A.4) – (A.7) one can easily derive the inequality (A.3). □

Acknowledgments

The authors were supported by the State Committee for Scientific Research under Grant No 3 P404 013 06.

References

1. G.A. BAKER, *Essentials of Padé approximants*, Acad. Press, London 1975.
2. G.A. BAKER and P. GRAVES-MORRIS, *Padé approximants. Part I. Basic theory*, Encyclopedia of Mathematics and its Applications, Vol. **13**, Addison-Wesley Publ. Co., London 1981.
3. G.A. BAKER and P. GRAVES-MORRIS, *Padé approximants, Part II. Extensions and applications*, Encyclopedia of Mathematics and its Applications, Vol. **14**, Addison-Wesley Publ. Co., London 1981.
4. D.J. BERGMAN, *The dielectric constant of a composite material-A problem in classical physics*, Phys. Report., **C 43**, 377–407, 1978.
5. D.J. BERGMAN, *Bounds for the complex dielectric constants of two-component composite material*, Phys. Rev., **B 23**, 6, 3058–3065, 1981.
6. D.J. BERGMAN, *Rigorous bounds for the complex dielectric constants of a two-component composite material*, Ann. Phys., **138**, 78–114, 1982.
7. D.J. BERGMAN, *Hierarchies of Stieltjes functions and their applications to the calculation of bounds for the dielectric constant of a two-component composite medium*, SIAM J. Appl. Math., **53**, 4, 915–930, 1993.
8. K.E. CLARK and G.W. MILTON, *Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites*, submitted to Proc. Roy. Soc. Lond. A, 448, 161–190, 1995.
9. K. GOLDEN, *Bounds on the complex permittivity of a multicomponent material*, J. Mech. Phys. Solids, **34**, 333–358, 1986.
10. K. GOLDEN and G. PAPANICOLAOU, *Bounds for effective parameters of heterogeneous media by analytic continuation*, Comm. Math. Phys., **90**, 473–491, 1983.
11. A.M. DYKHNE, *Conductivity of a two-dimensional two-phase system* [in Russian], Zh. Eksp. Teor. Fiz., **59**, 110–115, 1970.
12. B.U. FELDERHOF, *Bounds for the complex dielectric constant of a two-phase composite*, Physica A, **126**, 430–442, 1984.
13. J. GILEWICZ, *Approximants de Padé*, Springer-Verlag, Berlin 1978.
14. Z. HASHIN and S. SHTRIKMAN, *A variational approach to the theory of the effective magnetic permeability of multiphase materials*, J. Appl. Phys. Sol., **33**, 3125–3131, 1962.
15. J.B. KELLER, *A theorem on the conductivity of a composite medium*, J. Math. Phys., **33**, 548–449, 1964.
16. R.C. MCPHEDRAN and G.W. MILTON, *Bounds and exact theories for the transport properties of inhomogeneous media*, Appl. Phys., **A 26**, 207–220, 1981.
17. S. MAY, S. TOKARZEWSKI, A. ZACHARA and B. CICHOCKI, *Continued fraction representation for the effective thermal conductivity coefficient of a periodic two-component composite*, Int. J. Heat and Mass Transf., **37**, 2165–2173, 1994.
18. K.S. MENDELSON, *A theorem on the effective conductivity of two-dimensional heterogeneous medium*, J. Appl. Phys., **46**, 4740–4741, 1975.
19. G.W. MILTON, *Bounds on the complex permittivity of two-component composite material*, J. Appl. Phys., **52**, 8, 5288–5293, 1981.
20. G.W. MILTON, *Bounds on the transport and optical properties of two-components material*, J. Appl. Phys., **52**, 5294–5304, 1981.
21. G.W. MILTON, *Bounds on the complex dielectric constant of a composite material*, Appl. Phys. Lett., **37**, 300–302, 1980.
22. K. SCHULGASSER, *On a phase interchange relationship for composite materials*, J. Math. Phys., **17**, 378–381, 1976.
23. S. TOKARZEWSKI, *Inequalities for the effective transport coefficients of two-components composite materials*, Arch. Mech., **46**, 611–623, 1994.

24. S. TOKARZEWSKI, J. BŁAWZDZIEWICZ and I.V. ANDRIANOV, *Effective conductivity of densely packed highly conducting cylinders*, Appl. Phys. A, 601–604, 1994.

25. S. TOKARZEWSKI and J.J. TELEGA, *S-continued fraction representation for effective transport coefficients of two-phase media*, IFTR Reports, 16/95.

26. H.S. WALL, *Analytic theory of continued fractions*, Van Nostrand, New York 1948.

27. O. WIENER, *Abhandlungen der mathematisch-physischen Klasse der Königlichen Sächsischen Gesellschaft der Wissenschaften*, Abh. Sächs. Akad. Wiss. Leipzig Math.-Naturwiss. Kl., **32**, 509, 1912.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received August 7, 1995.

Arch. Mech., 48, 3, pp. 493–512, Warszawa 1996

Surface stress waves in a nonhomogeneous elastic half-space Part I. General results based on spectral analysis Existence and analyticity theorems

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EXISTENCE of surface waves in a nonhomogeneous elastic half-space is proved on the basis of the stress elastodynamics formulation (cf. [1]). It is demonstrated that in the case when nonhomogeneity depends on depth of the semi-space, both the velocity and amplitude of a surface wave are analytical functions of the wave number.

1. Introduction

IN 1971 (cf. [1]) J. IGNACZAK showed that the problem of surface wave propagation in nonhomogeneous isotropic elastic half-space can be reduced to the following eigenvalue problem: find a positive number λ and a real-valued symmetric tensor field

$$\alpha_{ij} = \alpha_{ij}(x_2) \qquad (\alpha_{ij} \in C^2[0,\infty), \qquad i,j = 1,2)$$

satisfying the following equation:

$$\mathbf{A}(s)\,\boldsymbol{\alpha} - \lambda \mathbf{B}\,\boldsymbol{\alpha} = \mathbf{0}, \tag{1.1}$$

together with conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0, \tag{1.2}$$

where

$$\boldsymbol{\alpha}(x_2) = [\alpha_{11}(x_2) \quad \alpha_{22}(x_2) \quad \alpha_{12}(x_2)]^T, \tag{1.3}$$

$$\mathbf{A} \equiv \mathbf{A}(s,\varrho) \equiv \begin{bmatrix} \dfrac{s^2}{\varrho} & 0 & \dfrac{s}{\varrho}D \\ 0 & -D\dfrac{1}{\varrho}D & sD\dfrac{1}{\varrho} \\ -sD\dfrac{1}{\varrho} & -\dfrac{s}{\varrho}D & \dfrac{s^2}{\varrho} - D\dfrac{1}{\varrho}D \end{bmatrix}, \tag{1.4}$$

$$
\text{(1.5)} \qquad \mathbf{B} \equiv B(\mu,\nu) \equiv \begin{bmatrix} \frac{1-\nu}{2\mu} & \frac{-\nu}{2\mu} & 0 \\ \frac{-\nu}{2\mu} & \frac{1-\nu}{2\mu} & 0 \\ 0 & 0 & \frac{1}{\mu} \end{bmatrix}.
$$

Tensor $\boldsymbol{\alpha}$ defines the stress tensor amplitude and symbol D denotes differentiation with respect to x_2 ($D = d/dx_2$). Number s is the wave-number, and $\varrho = \varrho(x_2)$, $\mu = \mu(x_2)$ and $\nu = \nu(x_2)$ are density, shear modulus, and Poisson's ratio, respectively ($0 \le x_2 < \infty$).

The formulation (1.1)–(1.5) is based on a pure stress method of classical elastodynamics.(1)

In an earlier paper [4] J. IGNACZAK showed, that the problem of surface wave propagation in a nonhomogeneous isotropic elastic half-space with shear moduls μ and Poisson's ratio ν depending on x_2, and with constant density, can be reduced to the following one: find a pair $(c_R, \beta(x))$ satisfying the ordinary differential equation of the fourth order

$$
\text{(1.6)} \qquad \left(\frac{1}{s^2} D \frac{1}{1-\Omega} D - 1\right) \frac{1}{1-\kappa} \frac{\Omega}{2-\Omega} \left[D^2 - s^2(1-\Omega\kappa)\right] \beta
$$
$$
+4\left[\frac{1}{2-\Omega} D^2 - D \frac{1}{1-\Omega} D \frac{1-\Omega}{2-\Omega}\right] \beta = 0 \qquad \text{for} \quad x_2 \in (0,\infty),
$$

and the boundary conditions

$$
\text{(1.7)} \qquad \begin{aligned} &\beta(0) = \beta(\infty) = 0, \\ &\frac{1}{s^2(2-\Omega)} D \left\{\frac{\Omega}{2-\Omega} \frac{1}{1-\kappa} \left[D^2 - s^2(1-\Omega\kappa)\right] \beta - 4s^2 \frac{1-\Omega}{2-\Omega} \beta\right\}\Big|_{\substack{x_2=0 \\ x_2=\infty}} = 0, \end{aligned}
$$

where

$$
\text{(1.8)} \qquad \kappa(x_2) = \frac{1-2\nu(x_2)}{2-2\nu(x_2)}, \qquad \Omega(x) = \frac{c_R^2}{\mu(x_2)}.
$$

(1) The problem (1.1)–(1.2) can be discussed in a class of square integrable functions, i.e.:

$$
\boldsymbol{\alpha} = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T \in L^2(0,\infty) \times L^2(0,\infty) \times L^2(0,\infty) = [L^2(0,\infty)]^3 \quad \mathbf{A}, \mathbf{B} \in [L^2(0,\infty)]^3,
$$

and it is correctly posed when the condition $R(\mathbf{A}) = R(\mathbf{B})$ is satisfied; $R(\mathbf{A})$, $R(\mathbf{B})$ denote the ranges of operators $\mathbf{A}, \mathbf{B}$ (cf. [2] p. 16). From equality $R(\mathbf{A}) = R(\mathbf{B})$ it follows that:

$$
R(\mathbf{A}) = R(\mathbf{B}) = \Big\{(\alpha_{11}, \alpha_{22}, \alpha_{12}) \in [C^2[0,\infty)]^3 :
$$
$$
-\left[\frac{\alpha_{11} - \nu\alpha_{ii}}{2\mu}\right]'' + \frac{s^2(\alpha_{22} - \nu\alpha_{ii})}{2\mu} - s\left[\frac{\alpha_{12}}{\mu}\right]' = 0; \quad i = 1,2\Big\}.
$$

The differential equation ($' = D$) in brackets corresponds to the compatibility condition (cf. [3] p. 345) for the problem.

The surface wave velocity c_R is the eigenvalue of the problem ((1.6)–(1.8)). Function $\beta(x_2)$ describing the variation of normal stress is the eigenfunction associated with eigenvalue c_R, ($\beta(x_2) = \alpha_{22}(x_2)$). In 1967 C.R.A. RAO [5] extended the formulation (1.6)–(1.7) to the case when density ϱ, shear modulus μ, and Poisson's ratio ν are arbitrary functions of x_2. In the particular case, when

$$\varrho(x_2) \equiv 1, \qquad \mu(x_2) \equiv \text{const}, \qquad \varepsilon > 0; \tag{1.9}$$

$$\begin{aligned} \nu_0 &= \nu(0), \qquad \nu_\infty = \nu(\infty), \\ \nu(x_2) &= 1 - (1-\nu_\infty)\left[1 + \frac{\nu_0 - \nu_\infty}{1-\nu_0}(1+\varepsilon x_2)^{-2}\right]^{-1}, \end{aligned} \tag{1.10}$$

J. IGNACZAK (cf. [4]) obtained an analytical closed-form solution. C.R.A. RAO (cf. [6, 7]) investigated the problem in case:

$$\varrho(x_2) \equiv 1, \qquad \nu(x_2) \equiv \nu_0, \qquad \mu(x_2) = \mu_\infty + (\mu_0 - \mu_\infty)e^{-\varepsilon x_2} \tag{1.11}$$

using the power series expansion method.

The problem (1.6)–(1.7) was also investigated by T. ROŻNOWSKI, (cf. [8, 9, 10]).

In [8] a solution was found under the assumptions that density and Poisson's ratio are constant, and shear modulus μ is a "weakly" variable exponential function such that the term

$$4\left(\frac{1}{2-\Omega(x_2)}\frac{d^2}{dx_2^2} - \frac{d}{dx_2}\frac{1}{1-\Omega(x_2)}\frac{d}{dx_2}\frac{1-\Omega(x_2)}{2-\Omega(x_2)}\right)\beta \tag{1.12}$$

can be neglected.

In [9] T. ROŻNOWSKI analysed the equations of motion for a transversely isotropic nonhomogeneous elastic semispace, using the stress motion equations, and formulated the problem of surface wave of the Rayleigh type. He showed that the problem can be also reduced to an ordinary differential equation of the fourth order with variable coefficients. T. ROŻNOWSKI in [10] analysed five particular cases of the wave phenomena:

a) transversely isotropic body with a "small nonhomogeneity",
b) "weakly anisotropic" nonhomogeneous body,
c) "weakly anisotropic" body with a "small nonhomogeneity",
d) transversely isotropic homogeneous body,
e) isotropic nonhomogeneous body.

The surface wave problem can be formulated in an alternative way starting from the displacement equations.

A.G. ALENITSYN (cf. [11, 12, 13, 14]) investigated the equations of motion in the displacement formulation for large wave numbers using asymptotic methods. As a result, he obtained an approximate dispersion relation (cf. [15]).

In this paper some new properties of the surface waves will be presented. The stress formulation will be used. This paper consists of four sections. Sec. 2 is devoted to general formulation of the problem. In Sec. 3 qualitative properties of the solution are discussed. It is demonstrated that for density, shear modulus, and Poisson's ratio being bounded and of class $C^2[0,\infty)$, the wave velocity and stress amplitude are analytical functions of the wave number. In Sec. 4 it is shown that at least one solution exists (and at most a finite number of solutions) under the assumptions, that density and shear modulus are constant and Poisson's ratio is a bounded function from $C^2[0,\infty)$. The obtained results are limited to the surface waves propagating in a nonhomogeneous half-space under isothermal conditions.

2. Stress formulation of a surface wave problem

Let us consider the two-dimensional stress equation of the linear elastodynamics (cf. [1]) for a nonhomogeneous isotropic medium (²)

$$(2.1)\qquad \mu^{-1}(x)\left[\frac{\partial^2}{\partial t^2}\tau_{\alpha\beta}(x,t)-\nu(x)\delta_{\alpha\beta}\frac{\partial^2}{\partial t^2}\tau_{\gamma\gamma}(x,t)\right]-\left[\varrho^{-1}(x)\tau_{\alpha\gamma,\gamma}(x,t)\right]_{,\beta} - \left[\varrho^{-1}(x)\tau_{\beta\gamma,\gamma}(x,t)\right]_{,\alpha}=0,$$

where

$$\tau_{\alpha\beta}=\tau_{\alpha\beta}(x,t),\qquad (\alpha,\beta)=(1,2),\qquad [x=(x_1,x_2)]$$

denotes nondimensional stress tensor, $\mu(x)$, $\varrho(x)$ are nondimensional shear modulus and density, $\nu(x)$ is Poisson's ratio. Nondimensional time is defined by the formula

$$(2.2)\qquad t=\frac{\tau\mu_0^{1/2}}{x_0\varrho_0^{1/2}},$$

where τ is real time and μ_0, ϱ_0 and x_0 are units of stress, density and length, respectively. Moreover

$$\dot{\tau}_{\alpha\beta}=\frac{\partial\tau_{\alpha\beta}}{\partial t},\qquad \tau_{\alpha\beta,\gamma}=\frac{\partial\tau_{\alpha\beta}}{\partial x_\gamma}.$$

It is assumed, that the functions $\varrho(x), \mu(x)$ and $\nu(x)$ depend on x_2 $(x_2\in[0,\infty))$ and $\varrho(x_2)$, $\mu(x_2)$, $\nu(x_2)\in C_2[0,\infty)$, and

$$(2.3)\qquad \begin{aligned} &0<\varrho_0\le\varrho(x_2)\le\varrho_1<\infty,\\ &0<\mu_0\le\mu(x_2)\le\mu_1<\infty,\\ &-1<\nu_0\le\nu(x_2)\le\nu_1<1/2 \qquad \text{for}\quad x_2\in[0,\infty). \end{aligned}$$

(²) See IGNACZAK [4], RAO [5].

The triplets $(\varrho_0, \mu_0, \nu_0)$ and $(\varrho_1, \mu_1, \nu_1)$ represent minimal and maximal values of (ϱ, μ, ν).

The solution $\tau_{\alpha\beta}$ of Eq. (2.1) will be considered in the half-space

$$U = \{(x_1, x_2): \quad x_2 \geq 0, \quad -\infty < x_1 < \infty\}, \tag{2.4}$$

for every $t \in [0, \infty)$. We shall look for a solution in the form:

$$\begin{aligned} \tau_{11}(x,t) &= \operatorname{Re} \alpha_{11}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \\ \tau_{22}(x,t) &= \operatorname{Re} \alpha_{22}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \\ \tau_{12}(x,t) &= \operatorname{Re} i\alpha_{12}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \end{aligned} \tag{2.5}$$

where $i = \sqrt{-1}$, $s > 0$, $\lambda > 0$ and Re stands for the real part of a complex-valued function. Moreover it is assumed that the solution satisfies the conditions

$$\tau_{22}(x_1, 0, t) = \tau_{12}(x_1, 0, t) = 0 \qquad \text{for} \quad x_1 \in (-\infty, \infty), \quad t \geq 0, \tag{2.6}$$

$$\tau_{22}(x_1, \infty, t) = \tau_{12}(x_1, \infty, t) = \tau_{11}(x_1, \infty, t) = 0 \qquad \text{for} \quad x_1 \in (-\infty, \infty), \quad t \geq 0. \tag{2.7}$$

The wave velocity, wave period and wave length are $c_R = \sqrt{\lambda}/s$, $T = 2\pi/\sqrt{\lambda}$, and $l = 2\pi/s$. The functions $\alpha_{11}(x,t)$, $\alpha_{22}(x,t)$, $\alpha_{12}(x,t)$, and the velocity c_R should be chosen in such a way that tensor field $\boldsymbol{\tau}(x,t)$ defined by (2.5) should satisfy the field equation (2.1) and the conditions (2.6) – (2.7).

Introducing (2.5) to (2.1), (2.6), (2.7) we obtain (cf. [1])

$$\begin{aligned} \varrho^{-1}(s\alpha_{11} + s\dot{\alpha}_{12}) - \lambda(2\mu)^{-1}(\alpha_{11} - \nu\alpha_{\gamma\gamma}) &= 0, \\ -\left[\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12})\right]^{\bullet} - \lambda(2\mu)^{-1}(\alpha_{22} - \nu\alpha_{\gamma\gamma}) &= 0, \\ -\left[\varrho^{-1}(s\dot{\alpha}_{12} + s\alpha_{11})\right]^{\bullet} - s\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12}) - \lambda(2\mu)^{-1}2\alpha_{12} &= 0 \end{aligned} \tag{2.8}$$

$$\text{for} \quad x_2 \in (0, \infty),$$

and the boundary conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0, \tag{2.9}$$

where

$$\boldsymbol{\alpha} = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T \in [C^2[0, \infty)]^3.$$

Starting from Eq. (2.8), the dot over a symbol will denote differentiation with respect to x_2. We shall also use the symbol D for the operator $D = d/dx_2$. C.R.A. RAO showed (cf. [5]) that the linear eigenvalue problem (2.8) – (2.9) can be further reduced, by elimination of α_{11} and α_{12}, to the nonlinear eigenvalue problem

$$\begin{aligned}
(2.10)\qquad & \left[\left\{\left[D-\left(H_1-\frac{2h}{2-\Omega}\right)\right]\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]-1\right\}\right.\\
&\left.\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}(D^2+hD-b^2)\right\}\right]\alpha_{22}+4\left\{\frac{1}{2-\Omega}(D^2+hD)\right.\\
&\left.-\left[D-\left(H_1-\frac{2h}{2-\Omega}\right)\right]\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{a^2}{2-\Omega}\right\}\alpha_{22}=0\\
&\text{for}\quad x_2\in(0,\infty),
\end{aligned}$$

$$(2.11)\qquad \alpha_{22}(0)=\alpha_{22}(\infty)=0,$$

$$\begin{aligned}
(2.12)\qquad & \left\{\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right.\right.\\
&\left.\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\right\}\Bigg|_{\substack{x_2=0\\ x_2=\infty}}=0,
\end{aligned}$$

$$\begin{aligned}
&\kappa(x_2)=\frac{1-2\nu(x_2)}{2-2\nu(x_2)}, && \nu(x_2)=\frac{1-2\kappa(x_2)}{2-2\kappa(x_2)},\\
(2.13)\qquad &h=\varrho D(\varrho^{-1}), && \Omega(x_2)=\frac{c_R^2\varrho(x_2)}{\mu(x_2)},\\
&a^2=s^2(1-\Omega), && b^2=s^2(1-\Omega\kappa),\\
&H_1=[\Omega/(2-\Omega)]\cdot[h/(2-2\kappa)], && e^2=DH_1-(1-2\kappa)H_1^2.
\end{aligned}$$

From a solution $(\lambda,\alpha_{22}(x_2))$ of Eqs. (2.10)–(2.12) one can obtain the functions $\alpha_{11}(x_2)$ and $\alpha_{12}(x_2)$ using the formulae

$$\begin{aligned}
(2.14)\qquad \alpha_{11}(x_2)=&-\frac{1}{s^2(2-\Omega)}\left\{\left[s^2\Omega+2(D^2+hD)\right]\alpha_{22}\right.\\
&+h\frac{1}{a^2-e^2}\,\frac{1}{1-\kappa}\left[D-(1-2\kappa)H_1\right]\frac{2}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right.\\
&\left.\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\right\},
\end{aligned}$$

$$\begin{aligned}
(2.15)\qquad -2s\alpha_{12}(x_2)=&\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right.\\
&\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\,.
\end{aligned}$$

For the special case when the density is constant, $\varrho = 1$, $h = \varrho D(\varrho^{-1}) = 0$ and Eqs. (2.10)–(2.12) reduce to (cf. [4])

$$\text{(2.16)}\qquad \left(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\right)\frac{1}{1-\kappa}\,\frac{\Omega}{2-\Omega}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22} + 4\left[\frac{1}{2-\Omega}D^2 - D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\alpha_{22} = 0 \quad \text{for} \quad x_2 \in (0,\infty),$$

$$\text{(2.17)}\qquad \alpha_{22}(0) = \alpha_{22}(\infty) = 0,$$
$$\left[D\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22} - 4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}\right]\Bigg|_{\substack{x_2=0\\ x_2=\infty}} = 0,$$

$$\text{(2.18)}\qquad \alpha_{11}(x_2) = -\frac{1}{s^2(2-\Omega)}\left(s^2\Omega + 2D^2\right)\alpha_{22},$$

$$\text{(2.19)}\qquad \alpha_{12}(x_2) = \frac{-1}{s^3(1-\Omega)}D\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22} - 4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}.$$

Clearly, in the eigenvalue problem (2.10)–(2.12) (or (2.16)–(2.17)) the eigenvalue λ enters in a nonlinear way. Also, note that the problem (2.1), (2.6), (2.7) is not a regular one ([3]). Indeed, writing (2.1) more explicitly, we have:

$$\text{(2.20)}\qquad \begin{bmatrix} \dfrac{1-\nu}{\mu} & \dfrac{-\nu}{\mu} & 0 \\ \dfrac{-\nu}{\mu} & \dfrac{1-\nu}{\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2}{\partial t^2}\tau_{11} \\ \dfrac{\partial^2}{\partial t^2}\tau_{22} \\ \dfrac{\partial^2}{\partial t^2}\tau_{12} \end{bmatrix}$$
$$= \begin{bmatrix} 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} & 0 & 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} \\ 0 & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix}.$$

The characteristic determinant associated with R.H.S of (2.20) takes the form

$$\text{(2.21)}\qquad \begin{vmatrix} -2\varrho^{-1}\xi_1^2 & 0 & -2\varrho^{-1}\xi_1\xi_2 \\ 0 & -2\varrho^{-1}\xi_1^2 & -2\varrho^{-1}\xi_2\xi_1 \\ -\varrho^{-1}\xi_2\xi_1 & -\varrho^{-1}\xi_1\xi_2 & -\varrho^{-1}(\xi_1^2+\xi_2^2) \end{vmatrix}$$

([3]) See [16, 17, 18, 19].

and it is equal to zero for any point (ξ_1, ξ_2). It can be shown that if suitable restrictions are imposed on $\boldsymbol{\tau}$ at $t = 0$, Eq. (2.20) implies the compatibility condition of the two-dimensional elasticity theory ([4])

$$\left\{\mu^{-1}[(1-\nu)\tau_{11} - \nu\tau_{22}]\right\}_{,22} + \left\{\mu^{-1}[(1-\nu)\tau_{22} - \nu\tau_{11}]\right\}_{,11} - 2\left\{\mu^{-1}\tau_{12}\right\}_{,12} = 0 \quad \text{for} \quad (x,t) \in U \times [0,\infty). \tag{2.22}$$

So, the system (2.20) subject to the condition (2.22) can be considered as a regular one.

The condition (2.22) follows from (2.20), if the stress field $\tau_{\alpha\beta}$ is sufficiently smooth on $U \times [0,\infty)$, and the L.H.S. of Eq. (2.22) together with its first time derivative vanishes for $t = 0$. The last conditions are equivalent to the assumption that deformation and its velocity satisfy the compatibility condition for $t = 0$. Vanishing of the determinant (2.21) implies that the operator

$$\sum \tau_{\alpha\beta}(x,t) \equiv \begin{bmatrix} 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} & 0 & 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} \\ 0 & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} \tag{2.23}$$

defined on the domain

$$D_1(\Sigma) = \Big\{(\tau_{11}, \tau_{22}, \tau_{12}) \in [C^2(U \times [0,\infty))]^3 :$$
$$\tau_{22}(x_1;0,t) = \tau_{12}(x_1;0,t) = \tau_{22}(x_1;\infty,t) = \tau_{12}(x_1;\infty,t) = \tau_{11}(x_1;\infty,t) = 0\Big\}$$

or

$$D_2(\Sigma) = \Big\{(\tau_{11}, \tau_{22}, \tau_{12}) \in [L^2(U \times [0,\infty))]^3 :$$
$$\tau_{22}(x_1;0,t) = \tau_{12}(x_1;0,t) = \tau_{22}(x_1;\infty,t) = \tau_{12}(x_1;\infty,t) = \tau_{11}(x_1;\infty,t) = 0\Big\}$$

is not invertible, unless the condition (2.22) is satisfied.

([4]) The compatibility condition restricted to the field $\boldsymbol{\alpha}$ takes the form:

$$\left\{\mu^{-1}[(1-\nu)\alpha_{11} - \nu\alpha_{22}]\right\}^{\cdot\cdot} - s\left\{\mu^{-1}[(1-\nu)\alpha_{22} - \nu\alpha_{11}]\right\} + 2s\left\{\mu^{-1}\alpha_{12}\right\}^{\cdot} = 0, \qquad \left(\cdot = \frac{d}{dx_2}\right).$$

3. On the analytical dependence of velocity and amplitude of the surface wave on the wave number

In this section we shall analyse the problem (2.8)–(2.9) using B-holomorphic perturbation theory for linear operators proposed by T. KATO (cf. [2]). We will demonstrate that velocity and amplitude of the wave are analytical functions of the wave number s.

In the complex Hilbert space H generated by the scalar product [5]

$$(3.1)\qquad (\boldsymbol{\alpha},\boldsymbol{\beta}) = \int_0^\infty (\overline{\alpha}_{11}\beta_{11} + \overline{\alpha}_{22}\beta_{22} + \overline{\alpha}_{12}\beta_{12})\, dx_2$$

with norm

$$(3.2)\qquad \|\boldsymbol{\alpha}\|^2 = \int_0^\infty (|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2)\, dx_2 < \infty,$$

Eq. (2.8) can be written in the operator form

$$(3.3)\qquad \mathbf{A}(s)\boldsymbol{\alpha} - \lambda\mathbf{B}\,\boldsymbol{\alpha} = \mathbf{0},$$

or in the expanded form

$$(3.4)\qquad \mathbf{A}(s,\varrho)\boldsymbol{\alpha} - \lambda\mathbf{B}(\mu,\nu)\boldsymbol{\alpha} = \mathbf{0}.$$

The domain of operators **A** and **B** may be defined as follows

$$(3.5)\qquad \mathcal{D}(\mathbf{A}) = \Big\{\boldsymbol{\alpha}\colon \alpha_{ij} \in C^2[0,\infty);\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = 0\Big\},$$

$$(3.6)\qquad \mathcal{D}(\mathbf{B}) = \Big\{\boldsymbol{\alpha}\colon \alpha_{ij} \in C^2[0,\infty)\Big\}, \qquad i,j = 1,2.$$

The sets $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\mathbf{B})$ are dense in H since the set $C_0^\infty[0,\infty) \times C_0^\infty[0,\infty) \times C_0^\infty[0,\infty)$ is dense in H and is contained in $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\mathbf{B})$. We have

PROPOSITION 1. Operators **A** and **B** are symmetric in the Hilbert space H.

The symmetry of operator **A** results from the fact that operators on both sides of the principal diagonal are formally adjoint, e.g. $\frac{s}{\varrho}D$ with $-sD\frac{1}{\varrho}$, $-\frac{s}{\varrho}D$ with $sD\frac{1}{\varrho}$.

[5] In oder to be able to apply Kato's perturbation theory, we have to extend the problem to the complex plane.

For arbitrary $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}(\mathbf{A}) \subset H$ we have

$$(\mathbf{A}\boldsymbol{\alpha},\boldsymbol{\beta}) = \int_0^\infty \{\varrho^{-1}(s^2\overline{\alpha}_{11} + s\dot{\overline{\alpha}}_{12})\beta_{11} - \left[\varrho^{-1}(\dot{\overline{\alpha}}_{22} + s\overline{\alpha}_{12})\right]^{\bullet} \beta_{12}$$

$$- \left[\varrho^{-1}(\dot{\overline{\alpha}}_{12} + s\overline{\alpha}_{11})\right]^{\bullet} \beta_{12} - s\varrho^{-1}(\dot{\overline{\alpha}}_{12} - s\overline{\alpha}_{22})\beta_{12}\} dx_2 .$$

Integration by parts with the use of boundary conditions shows that

$$(\mathbf{A}\boldsymbol{\alpha},\boldsymbol{\beta}) = (\boldsymbol{\alpha}, \mathbf{A}\boldsymbol{\beta}).$$

The symmetry of operator $\mathbf{B}$ is obvious. Matrix $\mathbf{B}$ is positive definite and for every $\boldsymbol{\alpha} \in \mathcal{D}(\mathbf{B}) \subset H$ we have(6)

$$(\mathbf{B}\boldsymbol{\alpha},\boldsymbol{\alpha}) \geq k(\boldsymbol{\alpha},\boldsymbol{\alpha}),$$

where

$$k = \min_{x_2 \in [0,\infty)} \left(\frac{1-2\nu}{2\mu}, \frac{1}{\mu}, \frac{1}{2\mu}\right).$$

Let us consider the forms $\mathcal{U}[\boldsymbol{\alpha}] = (\mathbf{A}\boldsymbol{\alpha},\boldsymbol{\alpha})$, $\mathcal{B}[\boldsymbol{\alpha}] = (\mathbf{B}\boldsymbol{\alpha},\boldsymbol{\alpha})$ described by the formulae

$$(\mathbf{A}\boldsymbol{\alpha},\boldsymbol{\alpha}) = \int_0^\infty \frac{1}{\varrho}\left[|\dot{\alpha}_{22} - s\alpha_{12}|^2 + |\dot{\alpha}_{12} + s\alpha_{11}|^2\right] dx_2 ,$$

$$(\mathbf{B}\boldsymbol{\alpha},\boldsymbol{\alpha}) = \int_0^\infty (2\mu)^{-1} \left[(1-\nu)|\alpha_{22}|^2 + (1-\nu)|\alpha_{22}|^2 + 2|\alpha_{12}|^2 - 2\nu \mathrm{Re}\,(\alpha_{11}\overline{\alpha}_{22})\right] dx_2 .$$

In view of (2.3) we have $(\mathbf{A}\boldsymbol{\alpha},\boldsymbol{\alpha}) \geq 0$. Operators $\mathbf{A}$ and $\mathbf{B}$ being symmetric, are closable in the space H. Let $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ denote the closures of operators $\mathbf{A}$ and $\mathbf{B}$. Let us set in H the form:

$$\mathcal{U}[\boldsymbol{\alpha}] = \sum_{i=0}^\infty \mathcal{U}^{(i)}(s_0)[\boldsymbol{\alpha}](z - s_0)^i \tag{3.7}$$

for z belonging to a certain neighbourhood of the real semi-axis s, $s_0 \in (0,\infty)$(7), where

$$\mathcal{U}^{(0)}[\boldsymbol{\alpha}] = (\mathbf{A}(s_0)\boldsymbol{\alpha},\boldsymbol{\alpha}) = \int_0^\infty \varrho^{-1}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right) dx_2 , \tag{3.8}$$

(6) The eigenvalues of matrix $\mathbf{B}$ are $\frac{1-2\nu}{2\mu}, \frac{1}{2\mu}, \frac{1}{\mu}$. The symmetric matrix $\mathbf{B}$ is positive definite iff all its eigenvalues λ_i are positive and $(\mathbf{B}\boldsymbol{\alpha},\boldsymbol{\alpha}) \geq \min\limits_i \lambda_i(\boldsymbol{\alpha},\boldsymbol{\alpha})$ (cf. [20]).

(7) The neighbourhood is a set: $V = \left\{z\colon |z - s_0| < \frac{1}{b+c} \text{ and } z \notin (-\infty, 0]\right\}$ where $b = \frac{1}{\varepsilon}$, $c = \frac{2}{\varepsilon}$, $\varepsilon > 0$. We can expand the region of holomorphicity by choosing a suitable ε. The meaning of b, c, ε will be made clear in the sequel.

$$(3.9)\qquad \mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}] = \int_0^\infty \frac{1}{\varrho}\Big\{2s_0|\alpha_{12}|^2 + 2s_0|\alpha_{11}|^2 - 2\mathrm{Re}\,(\alpha_{12}\dot{\bar{\alpha}}_{22}) + 2\mathrm{Re}\,(\alpha_{11}\dot{\bar{\alpha}}_{12})\Big\}dx_2,$$

$$(3.10)\qquad \mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}] = \int_0^\infty \frac{2}{\varrho}\left(|\alpha_{12}|^2 + |\alpha_{11}|^2\right)dx_2,$$

$$(3.11)\qquad \mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}] = 0, \qquad n = 3, 4, \ldots .$$

The form $\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]$ is a derivative of $(\mathbf{A}(s)\boldsymbol{\alpha}, \boldsymbol{\alpha})$ with respect to the real parameter s at $s = s_0$,

$$\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0}\frac{(\mathbf{A}(s)\boldsymbol{\alpha}, \boldsymbol{\alpha}) - (\mathbf{A}(s_0)\boldsymbol{\alpha}, \boldsymbol{\alpha})}{s - s_0}.$$

Similarly,

$$\mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0}\frac{\mathcal{U}^{(1)}(s)[\boldsymbol{\alpha}] - \mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]}{s - s_0},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0}\frac{\mathcal{U}^{(n-1)}(s)[\boldsymbol{\alpha}] - \mathcal{U}^{(n-1)}(s_0)[\boldsymbol{\alpha}]}{s - s_0}.$$

We shall prove the following lemma:

LEMMA 1. The closure $\widetilde{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generates a family of operators $\widetilde{\mathbf{A}}(z)$ which is B-holomorphic([8]).

In order to demonstrate that $\widetilde{\mathbf{A}}(z)$ is a B-holomorphic family of operators we shall use Kato's B-holomorphism criterion([9]).

Let $\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]$ be a sequence of sesquilinear form in H ($n = 0, 1, 2\ldots$), and let the form $\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]$ be sectorial([10]) and closable, and with the domain $D(\mathcal{U}^{(0)}) = D$. Assume that the forms $\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]$ for $n \geq 1$ are bounded with respect to $\mathcal{U}^{(0)}[\boldsymbol{\alpha}]$, i.e. $D \subset D(\mathcal{U}^{(n)})$, and

$$(*)\qquad \left|\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]\right| \leq c^{n-1}(a\|\boldsymbol{\alpha}\|^2 + b\mathrm{Re}\,\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]), \qquad \boldsymbol{\alpha} \in D, \quad n > 1, \quad a, b \geq 0, \quad c > 0.$$

Then operators $\widetilde{\mathbf{A}}(z)$ corresponding to the forms $\widetilde{\mathcal{U}}(z)[\boldsymbol{\alpha}]$ are a B-holomorphic family of operators for $|z - s_0| < \dfrac{1}{b + c}$.

To show that the assumptions of this criterion are satisfied, let us observe that $\mathcal{U}^{(0)} = \mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}] = (A(s_0)\boldsymbol{\alpha}, \boldsymbol{\alpha})$ is a non-negative, symmetric and hence the

([8]) (cf. [2] p. 395–397).
([9]) (cf. [2] p. 398).
([10]) (cf. [2], p. 310).

sectorial form fixed in the dense set D. The density of D results from the fact that the set $\mathcal{D}(\mathbf{A}) \subset D \subset H$ and $\mathcal{D}(\mathbf{A})$ is dense. Thus the form $\mathcal{U}^{(0)}$ is closable.

From the inequalities(11)

$$
\begin{aligned}
(3.12)\qquad \left|\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]\right| &= \left|\int\limits_0^\infty \frac{1}{\varrho}\Big[-\overline{\alpha}_{12}(\dot{\alpha}_{22} - s_0\alpha_{12}) - (\dot{\overline{\alpha}}_{22} - s_0\overline{\alpha}_{12})\alpha_{12}\right.\\
&\qquad \left. +\overline{\alpha}_{11}(\dot{\alpha}_{22} + s_0\alpha_{11}) + (\dot{\overline{\alpha}}_{12} + s_0\overline{\alpha}_{11})\alpha_{11}\Big]\,dx_2\right|\\
&\le \left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}\left|\dot{\alpha}_{22} - s_0\alpha_{12}\right|^2\,dx_2\right)^{1/2}\\
&\quad + \left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}\left|\dot{\alpha}_{22} - s_0\alpha_{12}\right|^2\,dx_2\right)^{1/2}\\
&\quad + \left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\,dx_2\right)^{1/2}\\
&\quad + \left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\,dx_2\right)^{1/2}\\
&= 2\left[\left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{22} - s_0\alpha_{12}|^2\,dx_2\right)^{1/2}\right.\\
&\quad \left.+ \left(\int\limits_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\,dx_2\right)^{1/2}\left(\int\limits_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\,dx_2\right)^{1/2}\right]\\
&\le \varepsilon \max_{x_2\in[0,\infty)} \frac{1}{\varrho}\left(\int\limits_0^\infty \left(|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2\right)dx_2\right)\\
&+\frac{1}{\varepsilon}\int\limits_0^\infty \frac{1}{\varrho}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right)dx_2 = \frac{\varepsilon}{\varrho_0}\|\boldsymbol{\alpha}\|^2 + \frac{1}{\varepsilon}\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]
\end{aligned}
$$

(11) To prove inequalities (3.12), (3.13) we use the inequalities

$$\left|\int \sum u_i v_i\,dx\right| \le \left(\int \sum |u_i|^2\,dx\right)^{1/2}\left(\int \sum |v_i|^2\,dx\right)^{1/2},$$

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2,$$

where v_i and u_i are complex function, a and b are real functions, and $\varepsilon > 0$.

and

$$(3.13) \qquad \left|\mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}]\right| = \int_0^\infty \frac{2}{\varrho}\left(|\alpha_{11}|^2 + |\alpha_{22}|^2\right) dx_2$$

$$\leq \max_{x_2\in[0,\infty)} \frac{2}{\varrho}\int_0^\infty \left(|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2\right) dx_2$$

$$+\frac{2}{\varepsilon^2}\int_0^\infty \frac{1}{\varrho}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right) dx_2$$

$$= \frac{2}{\varrho_0}\|\boldsymbol{\alpha}\|^2 + \frac{2}{\varepsilon^2}\mathrm{Re}\,\mathcal{U}^0(s_0)[\boldsymbol{\alpha}],$$

it follows that $D(\mathcal{U}^{(n)}) \supset D(\mathcal{U}^{(0)})$, $n = 1, 2, 3, \ldots$, and that there exist $a = \dfrac{\varepsilon}{\varrho_0}$, $b = \dfrac{1}{\varepsilon}$, $c = \dfrac{2}{\varepsilon}$. Thus the operator $\widetilde{A}(z)$ forms a holomorphic family of type (B). From Lemma 1 it follows that the following Proposition is valid.

PROPOSITION 2. The form $\mathcal{U}(z)$ given by (3.7) is defined for $|z - s_0| < \varepsilon/2$, and for $|z - s_0| < \varepsilon/3$ it is sectorial and closable. The closure $\widetilde{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generates a B-holomorphic family of operator $\widetilde{\mathbf{A}}(z)$ where $\widetilde{\mathbf{A}}(z)$ is the maximal and closed operator.

Now we shall consider eigenvalue problem given by

$$(3.14) \qquad \widetilde{\mathbf{A}}(z)\boldsymbol{\alpha} - \lambda\widetilde{\mathbf{B}}\boldsymbol{\alpha} = \mathbf{0},$$

where $\widetilde{\mathbf{A}}(z)$ is the operator defined in Proposition 2 and $\widetilde{\mathbf{B}}$ is the closure of $\mathbf{B}$. From KATO'S theorems (cf. [2] p. 416–423) it follows:

THEOREM 1. *If the pair $(\lambda(z), \boldsymbol{\alpha}(z))$ is a solution of the eigenvalue problem* (3.14), *then it is an analytical function with respect to z for $z \in V = \{z : |z - s_0| < \varepsilon/3$ and $z \notin (-\infty, 0]\}$.*

THEOREM 2. *If the pair $(\lambda(s), \boldsymbol{\alpha}(x_2, s))$ is a solution of the eigenvalue problem* (3.3), *then it is an analytical function of the wave-number s.*

It means that

$$(\lambda(s), \boldsymbol{\alpha}(x_2, s)) \equiv \left(\sum_{n=0}^\infty \lambda_n (s - s_0)^n, \quad \boldsymbol{\alpha} = \sum_{n=0}^\infty \boldsymbol{\alpha}_n(x_2)(s - s_0)^n\right),$$

where

$$\lambda_n = \frac{1}{n!}\left(\frac{d^n\lambda}{ds^n}\right)_{s=s_0}, \qquad \boldsymbol{\alpha}_n(x_2) = \frac{1}{n!}\left(\frac{\partial^n\boldsymbol{\alpha}}{\partial s^n}\right)_{s=s_0}, \qquad s_0 \in (0, \infty)\ x_2 \geq 0.$$

The proof of Theorem 2 follows directly from Theorem 1 and from the fact that each solution of (3.3) is also a solution of (3.14).

Natural approach to the considered eigenvalue problem

$$\mathbf{A}\boldsymbol{\alpha} - \lambda\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$$

is investigation of the generalized resolvent

$$(\mathbf{A} - \xi\mathbf{B})^{-1}.$$

Let us introduce the spaces X and Y defined by

$$X = \Big\{(\alpha_{11}, \alpha_{22}, \alpha_{12}) \in [L^2(0,\infty)]^3, \quad \left[C^2[0,\infty)\right]^3 : \qquad -\left[\frac{\alpha_{11} - \nu\alpha_{ii}}{2\mu}\right]'' + s^2\frac{\alpha_{22} - \nu\alpha_{ii}}{2\mu} - s\left[\frac{\alpha_{12}}{\mu}\right]' = 0, \qquad i = 1,2 \quad \text{for every} \quad x_2 \geq 0\Big\},$$

$$Y = \Big\{(g_{11}, g_{22}, g_{12}) \in [L^2(0,\infty)]^3, \quad [C^2[0,\infty)]^3 : \qquad -\ddot{g}_{11}(x_2) + s^2 g_{22}(x_2) - s\,\dot{g}_{12}(x_2) = 0, \qquad \text{for every} \quad x_2 \geq 0\Big\}.$$

It is easy to check that the spaces X, Y are linear subspaces of $[L^2(0,\infty)]^3$ and $[C^2[0,\infty)]^3$.

Let $\mathcal{C}(X,Y)$ be a space of closed operators from X to Y.

Let $\mathcal{B}(X,Y)$ be a space of bounded operators from X to Y.

Since $\widetilde{\mathbf{A}} \in \mathcal{C}(X,Y)$, $\widetilde{\mathbf{B}} \in \mathcal{B}(X,Y)$ and $\widetilde{\mathbf{B}^{-1}} \in \mathcal{B}(X,Y)$, thus $\widetilde{\mathbf{B}^{-1}\mathbf{A}} \in \mathcal{C}(X,X) = \mathcal{C}(X)$, $\widetilde{\mathbf{A}\,\mathbf{B}^{-1}} \in \mathcal{C}(Y,Y) = \mathcal{C}(Y)$ and the eigenvalue problems

$$\widetilde{\mathbf{A}}\boldsymbol{\alpha} - \lambda\widetilde{\mathbf{B}}\boldsymbol{\alpha} = \mathbf{0}, \qquad \widetilde{\mathbf{B}^{-1}\mathbf{A}}\boldsymbol{\alpha} - \lambda\boldsymbol{\alpha} = \mathbf{0}, \qquad \widetilde{\mathbf{A}\mathbf{B}^{-1}}\boldsymbol{\alpha} - \lambda\boldsymbol{\alpha} = \mathbf{0}$$

are equivalent (cf. [2] p. 417, 418).

To investigate the resolvent $(A - \xi B)^{-1}$, let us take the homogeneous case $\varrho = \text{const}$, $\mu = \text{const}$, $\nu = \text{const}$, as an illustration.

A solution of the equation $\mathbf{A}\boldsymbol{\alpha} - \xi\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\alpha} \in D(\mathbf{A}) \cap D(\mathbf{B}) \subset X$ is $\boldsymbol{\alpha} = [0,0,0]^T$ if $\xi \notin \{\omega_1, \omega_2, \omega_3\}$, where ω_1, ω_2, ω_3 are the roots of equation

$$(2-\omega)^2 - 4\sqrt{(1-\omega)(1-\omega\kappa)} = 0, \qquad \kappa = (1-2\nu)(2-2\nu)^{-1}.$$

To prove this, note that a solution of the equation $\mathbf{A}\boldsymbol{\alpha} - \xi\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$, takes the form:

$$\begin{aligned}
\alpha_{11} &= -\beta_0\left[e^{-x_2h_2} - \frac{2+\xi(1-2\kappa)}{2-\xi}e^{-x_2h_1}\right],\\
\alpha_{22} &= \beta_0\left[e^{-x_2h_2} - e^{-x_2h_1}\right],\\
\alpha_{12} &= -\frac{2}{s}\,\frac{\beta_0}{2-\xi}h_1\left[e^{-x_2h_2} - e^{-x_2h_1}\right],\\
h_1 &= s\sqrt{1-\xi\kappa}, \qquad h_2 = s\sqrt{1-\xi}.
\end{aligned}$$

Introducing such $\boldsymbol{\alpha}$ to the compatibility condition (cf. [6] p. 7) we get

$$\frac{\beta_0 s^2}{2\mu(2-\xi)} e^{-x_2 s\sqrt{1-\xi}} \left[(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)}\right] + \frac{\beta_0 s^2}{2\mu(2-\xi)(1-\nu)} e^{-x_2 s\sqrt{1-\xi\kappa}} \, [\, 0 \,] = 0.$$

Therefore if $\xi \notin \{\omega_1, \omega_2, \omega_3\}$ then $(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)} \neq 0$ and $\beta_0 = 0$. In this case $(\mathbf{A} - \xi\mathbf{B})^{-1}$ exists.

Let us consider the multiplicity of eigenvalue $\lambda = 0$. This problem can be written in the form

$$\mathbf{A}(s)\boldsymbol{\alpha} = \mathbf{0}.$$

As the domain of the operator A we take the set:

$$D(\mathbf{A}) = \Big\{\boldsymbol{\alpha} = [\alpha_{11}\,\alpha_{22}\,\alpha_{12}]^T \in [L^2(0,\infty)]^3, \quad [C^2[0,\infty)]^3 : \alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0\Big\}.$$

We have

$$\mathbf{A}(s)\boldsymbol{\alpha} = \mathbf{0} \;\Leftrightarrow\; \begin{cases} s\alpha_{11} + \dot{\alpha}_{12} = 0 \\ -s\alpha_{12} + \dot{\alpha}_{22} = 0 \end{cases} \;\Leftrightarrow\; \begin{cases} \alpha_{11}(x_2) = C_1\varphi''(x_2), \\ \alpha_{22}(x_2) = -s^2 C_1 \varphi(x_2), \\ \alpha_{12}(x_2) = -s C_1 \varphi'(x_2), \end{cases}$$

where $\varphi = \varphi(x_2)$ is an arbitrary differential function. Selecting $\varphi(x_2)$ in such a way as to meet the boundary conditions, we obtain

$$\ker \mathbf{A}: \qquad \begin{aligned} \alpha_{11}(x_2) &= C_1(2 - 4\alpha_k x_2 + \alpha_k^2 x_2^2)e^{-\alpha_k x_2}, \\ \alpha_{22}(x_2) &= -s^2 x_2^2 C_1 e^{-\alpha_k x_2}, \\ \alpha_{12}(x_2) &= -s C_1(2x_2 - \alpha_k x_2^2)e^{-\alpha_k x_2}, \end{aligned}$$

where

$$C_1 \in R, \qquad x_2 \in [0,\infty), \qquad \alpha_k > 0.$$

It is clear that in this case

$$\dim \ker \mathbf{A} = \infty.$$

Note that in the case, when the domain of the operator is a subspace of the functions satisfying the compatibility condition,

$$\dim \ker \mathbf{A} = 0.$$

4. Existence of surface waves in nonhomogeneous isotropic elastic half-space with arbitrary variation of Poisson's ratio

The problem of propagation of surface waves in a nonhomogeneous isotropic elastic half-space with variable Poissons's ratio can be reduced to the following eigenproblem (cf. [4]): find a nonvanishing pair $(c_R, \alpha_{22}(x_2))$ satisfying the relations:

$$(4.1)\qquad \left[\frac{1}{s^2(1-\Omega_0)}D^2 - 1\right]\frac{1}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = 0 \qquad \text{for} \quad x_2 \in (0,\infty),$$

$$(4.2)\qquad \begin{cases} \alpha_{22}(0) = \alpha_{22}(\infty) = 0 \\ D\left\{\dfrac{\Omega_0}{2-\Omega_0}\dfrac{1}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right] - 4s^2\dfrac{1-\Omega_0}{2-\Omega_0}\alpha_{22}\right\}\Big|_{\substack{x_2=0\\x_2=\infty}} = 0. \end{cases}$$

Here

$$(4.3)\qquad \kappa(x_2) = \frac{1-2\nu(x_2)}{2-2\nu(x_2)}, \qquad \Omega(x) = \frac{c_R^2}{\mu_0}, \qquad D = \frac{d}{dx_2},$$

$\nu(x_2)$ and μ_0 are the Poisson's ratio and shear modulus, respectively; symbol $c_R = p/s$, where $2\pi/p$ is the wave period and $2\pi/s$ is the wave length, denotes the velocity of surface wave. The eigenvalue c_R corresponding to the eigenfunction α_{22} is to be identified with the Rayleigh velocity.

Now we consider the case

$$(4.4)\qquad \begin{cases} \kappa = \kappa(x_2) \in C^2[0,\infty), & 0 < \kappa_0 \le \kappa(x_2) \le \kappa_1 < 3/4, \\ \mu_0 = 1, & \Omega(x_2) \equiv \Omega_0 \equiv c_R^2. \end{cases}$$

These hypotheses assure that the elastic energy of the half-space is strictly positive. We shall look for an eigenfunction $\alpha_{22} \in K$, where

$$K := \left\{\alpha_{22} = \alpha_{22}(x_2) \in C^4[0,\infty), \qquad \alpha_{22}(\infty) = 0\right\}.$$

The system (4.1)–(4.2) subject to the conditions (4.4) is equivalent to

$$(4.5)\qquad \frac{1}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = C_1\exp\left(-s\sqrt{1-\Omega_0}x_2\right) \qquad \text{for} \quad x_2 \in (0,\infty),$$

$$(4.6)\qquad \alpha_{22}(0) = 0,$$

$$(4.7)\qquad D\left\{\frac{\Omega_0}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} - 4s^2(1-\Omega_0)\alpha_{22}\right\}\Big|_{x_2=0} = 0.$$

It is shown in [1] that if there exists a solution of eigenproblem (4.1)–(4.2), the eigen-value $\Omega_0 = c_R^2$ is strictly positive. This fact with (4.5)–(4.7) implies that an admissible Ω_0 belongs to the interval $(0, 1)$. Consider now the homogeneous differential equation corresponding to (4.5):

$$\frac{1}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = 0 \tag{4.8}$$

which, by virtue of (4.4), is equivalent to

$$\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = 0. \tag{4.8$'$}$$

We have the following theorem

THEOREM 3. *Equation* (4.8) *subject to* (4.4) *has two linearly independent solutions:*

$$\alpha_{22}^{(1)}(x_2, \Omega_0, s), \quad \alpha_{22}^{(2)}(x_2, \Omega_0, s)$$

of the form:

$$\alpha_{22}^{(i)} = \alpha_{22}^{(i)}(0, \Omega_0, s) \exp \int_0^{x_2} \xi_i(\tau, s, \Omega_0)\, d\tau \qquad (i = 1, 2), \tag{4.9}$$

where $\xi_1(\tau, \Omega_0, s)$, $\xi_2(\tau, \Omega_0, s)$ *satisfy the inequalities*

$$a \le \xi_1 \le b < c \le \xi_2 \le d \qquad \textit{for every} \qquad (\tau, \Omega_0, s) \in (0, \infty) \times (0, 1) \times (0, \infty). \tag{4.10}$$

Constants a, b, c and d in (4.10) are defined by

$$\begin{aligned} a &= -s\sqrt{1-\Omega_0\kappa_0}, \qquad & b &= -s\sqrt{1-\Omega_0\kappa_1}, \\ c &= s\sqrt{1-\Omega_0\kappa_1}, \qquad & d &= s\sqrt{1-\Omega_0\kappa_0}. \end{aligned} \tag{4.11}$$

The proof of this theorem is based on a theorem due to OLECH (cf. [21], p. 323) and will not be given here.

It follows from Theorem 3 and the conditions (4.6), (4.7) that an admissible solution of Eq. (4.5) takes the form

$$\alpha_{22}(x_2, \Omega_0, s) = A_1 \exp\left(\int_0^{x_2} \xi_i(\tau, s, \Omega_0)\, d\tau\right) - \frac{1}{\Omega_0 s^2} C_1 \exp\left(-s x_2 \sqrt{1-\Omega_0}\right), \tag{4.12}$$

where $(\Omega_0, s) \in (0, 1) \times (0, \infty)$. Clearly, this solution belongs to the class $C^4[0, \infty)$.

Therefore, applying the theorem (cf. [21], p. 56) on analytical dependence on the parameters to the equation

$$\ddot{\alpha}_{22} - s^2(1 - \Omega_0 \kappa(x_2))\alpha_{22} = 0 \tag{4.13}$$

subject to the conditions

$$\alpha_{22}(0) = 1, \qquad \alpha_{22} \in K,$$

we conclude that the solution of (4.13) given by $\alpha_{22}^{(0)} = \exp\left(\int\limits_0^{x_2} \xi_1(\tau, s, \Omega_0)\, d\tau\right)$ is analytic with respect to $(\Omega_0, s) \in (0,1) \times (0,\infty)$. 01

Therefore $\xi_1(\tau, \Omega_0, s)$ is also analytic for $(\Omega_0, s) \in (0,1) \times (0, \infty)$.

It is clear that analyticity of α_{22} satisfying (4.13) subject to $\alpha_{22}(0) = 0$, $\alpha_{22} \in C^4[0,\infty)$ implies analyticity of α_{22} satisfying (4.5)–(4.7). Substituting (4.12) into (4.6) and (4.7), and using condition $C_1 \neq 0$, we arrive at the dispersion equation

$$(2 - \Omega_0)^2 + \frac{4\sqrt{1 - \Omega_0}\,\xi_1(0, \Omega_0, s)}{s} = 0. \tag{4.14}$$

Since

$$-s\sqrt{1 - \Omega_0 \kappa_0} \leq \xi_1(0, \Omega_0, s) \leq s\sqrt{1 - \Omega_0 \kappa_1}\,,$$

for every $(\Omega_0, s) \in \langle 0, 1) \times (0, \infty)$, thus

$$\begin{aligned} -4\sqrt{(1 - \Omega_0)(1 - \Omega_0\kappa_0)} + (2 - \Omega_0)^2 &\leq \frac{4\sqrt{1 - \Omega_0}\,\xi_1(0, \Omega_0, s)}{s} + (2 - \Omega_0) \\ &\leq -4\sqrt{(1 - \Omega_0)(1 - \Omega_0\kappa_1)} + (2 - \Omega_0)^2, \end{aligned} \tag{4.15}$$

for every $(\Omega_0, s) \in \langle 0, 1) \times (0, \infty)$.

Now, introducing the notations

$$\begin{aligned} f_0(\Omega_0) &= -4\sqrt{(1 - \Omega_0)(1 - \Omega_0\kappa_0)} + (2 - \Omega_0)^2, \\ f(\Omega_0, s) &= \frac{4\sqrt{1 - \Omega_0}\,\xi_1(0, \Omega_0, s)}{s} + (2 - \Omega_0), \qquad 2 \\ f_1(\Omega_0) &= -4\sqrt{(1 - \Omega_0)(1 - \Omega_0\kappa_1)} + (2 - \Omega_0)^2, \end{aligned}$$

we reduce (4.15) to the form

$$f_0(\Omega_0) \leq f(\Omega_0, s) \leq f_1(\Omega_0). \tag{4.16}$$

It follows from the definitions of f_0, f and f_1, and from the analyticity of $\xi_1(0, \Omega_0, s)$ that the functions f_0, f and f_1 are analytic for every $(\Omega_0, s) \in (0,1) \times$

$(0, \infty)$. Moreover, f_0 and f_1 vanish for $\Omega_0 = 0$ and for $\Omega_0 = c_1^2$, $\Omega_0 = c_2^2$, respectively. c_1^2 and c_2^2 are the squares of velocities of surface waves in the semi-space with $\kappa(x) \equiv \kappa_0$, $\mu \equiv 1$ and $\kappa(x) \equiv \kappa_1$, $\mu \equiv 1$, respectively. Therefore, the analyticity of $f(\Omega_0, s)$ for every $(\Omega_0, s) \in (0, 1) \times (0, \infty)$ together with the inequalities (4.16) imply that there exists at least one root (or at most, a countable number of roots) of the equation $f(\Omega_0, s) = 0$ for every $(\Omega_0, s) \in [c_1^2, c_2^2] \times (0, \infty)$. This completes the proof of existence of at least one solution to the eigenproblem discussed in the present section. The Fig. 1 shows the graphs of $f_0(\Omega)$ and $f_1(\Omega)$ corresponding to $\kappa_0 = 0.1$ and $\kappa_1 = 0.7$, respectively, as well as a hypothetical graph of f over the interval $0 < \Omega < 1$.

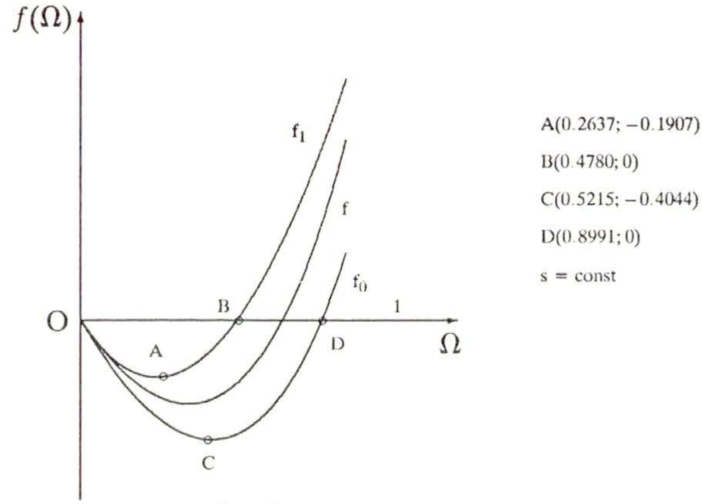

FIG. 1.

We have the following theorem:

THEOREM 4. *For every $s > 0$, the equation $f(\Omega_0, s) = 0$ has at most a finite number of solutions.*

P r o o f. If the number of the solutions of the equation $f(\Omega_0, s) = 0$ for a given $s > 0$ is infinite, then the set $S = \{f(\Omega_0, s) = 0\}$ has an accumulation point in $[c_2^2, c_1^2]$. Since the function $f(\Omega_0, s)$ is analytical in the domain $(\Omega_0, s) \in (0, 1) \times (0, \infty)$, f vanishes in the interval $[c_2^2, c_1^2]$ which contradicts the inequality (4.15).

REMARK. If the branches of the dispersion relation (4.14) intersect, then the intersection points are algebraic branch-points (cf. [23] p. 119 part II), (cf. [24] p. 174 – 181).

References

1. J. IGNACZAK, *Upper and lower bounds on the velocity of surface waves in nonhomogeneous isotropic elastic semi-space*, Arch. Mech. Stos., **23**, 789–800, 1971.
2. T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin - Heidelberg - New York 1966.

3. A.C. ERINGEN and E.S. SUHUBI, *Elastodynamics, vol. II, Linear theory*, Academic Press, New York, San Francisco, London 1975.

4. J. IGNACZAK, *Rayleigh waves in a nonhomogeneous isotropic elastic semi-space*, Arch. Mech. Stos., **15**, 341–345, 1963.

5. C.R.A. RAO, *Separation of the stress equation of motion in nonhomogeneous elastic media*, JASA, **41**, 3, 612–614, 1967.

6. C.R.A. RAO, *Separation of stress equations of motion in nonhomogeneous isotropic elastic media*, Doctoral Thesis submitted to the Monash University, Australia 1969.

7. C.R.A. RAO, *Wave propagation in elastic media with prescribed variation in the parameters. Modern problems in elastic wave propagation*, Symposium held at Northwestern University, Evanston, Illinois USA, A. Wiley, Interscience Publication, 1978.

8. T. ROŻNOWSKI, *Surface waves in an isotropic elastic semispace with small nonhomogeneity*, Bull. Pol. Sci. Tech., **25**, 67–77, 1977.

9. T. ROŻNOWSKI, *Surface stress waves in a transversely isotropic, nonhomogeneous semispace* [in Polish], Prace IPPT 31/1990, Warszawa 1990.

10. T. ROŻNOWSKI, *Surface stress waves in transversely isotropic nonhomogeneous elastic semispace, Part I. Equation of motion and equation of a Rayleigh-type surface wave*, Arch. Mech., **44**, 4, 417–435, 1992; *Part II. Surface stress wave in a weakly anisotropic semi-space with small nonhomogenity*, Arch. Mech., **44**, 4, 437–451, 1992.

11. A.G. ALENITSYN, *Rayleigh waves in an inhomogeneous elastic semi-space* [in Russian], Prik. Mat. Mech., **27**, 3, 547–550, 1963.

12. A.G. ALENITSYN, *Rayleigh waves in an inhomogeneous elastic semispace of wave guide type* [in Russian], Prik. Mat. Mech., **31**, 2, 222–230, 1967.

13. A.G. ALENITSYN, *Rayleigh waves in inhomogeneous elastic semi-space* [in Russian], Prik. Mat. Mech., **37**, 5, 895–899, 1993.

14. A.G. ALENITSYN, *Asymptotic limiting condition on the boundary of a liquid layer and an elastic inhomogeneous semi-space* [in Russian], Akust. Żurn., **34**, 9, 969–971, 1988.

15. L.M. BREKHOVSKIJ, *Waves in layered media* [in Russian], Nauka, Moskva 1973.

16. I.G. PIETROVSKIJ, *On certain problems of equations with partial derivatives* [in Russian], UMN, **1**, 3-4, 13–14, 44–70, 1956.

17. V.D. KUPRADZE, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity* [in Russian], Nauka, Moskva 1976.

18. L. BERS and J.F. SCHECHTER, *Partial differential equations*, Interscience Publ., New York - London - Sydney 1964.

19. I.M. GELFAND, I.G. PIETROVSKIĬ and G.E. SILOV, *Theory of the systems of partial differential equations* [in Russian], Trudy III Vsiesojuznogo Matiematičeskogo Sjezda, 3, Izdatielstwo A.N. SSSR, Moskva 65 - 72, 1958.

20. G. STRANG, *Linear algebra and its applications* [in Russian, Nauka, Moskva 1980], Academic Press, New York 1976.

21. C. OLECH, *Asymptotic behaviour of the solution of second order differential equations*, Bull. Acad. Polon. Sci., Série Sci. Math., **6**, 6, 319–326, 1959.

22. E. KAMKE, *Differentialgleichungen, Lösungsmethoden und Lösungen*, Leipzig 1959.

23. K. KNOPP, *Theory of functions*, Parts I, II, New York, Dover 1945, 1947.

24. F. LEJA, *Analytical functions* [in Polish], PWN, Warszawa 1969.

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Received August 18, 1995.

Arch. Mech., 48, 3, pp. 513–539, Warszawa 1996

Surface stress waves in a nonhomogeneous elastic half-space Part II. Existence of surface waves for an arbitrary variation of Poisson's ratio Approximate solution based on perturbation methods

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TWO APPROACHES to the solution of the nonlinear eigenvalue problem of propagation of surface waves in a nonhomogeneous isotropic elastic half-space are considered. In Sec. 1 the nonlinear eigenvalue problem is transformed to the equivalent integral equation, and the method of solving this equation is proposed. In Sec. 2 Friedrich's perturbation theory [6] is used to solve an eigenvalue problem describing the surface stress waves in a "weakly" nonhomogeneous isotropic elastic half-space. Two cases are discussed in detail: a) a half-space with a "weak" variation of density, b) a half-space with a "weak" variation of the shear modulus. In both cases an asymptotic solution is obtained and numerical results are given.

1. Effective form of amplitude of surface stress waves in a non-homogeneous isotropic elastic half-space

1.1. Formulation of the problem

IT IS SHOWN in [1] that the problem of propagation of surface waves in a non-homogeneous isotropic elastic half-space can be reduced to the following eigenvalue problem: to find a nonvanishing pair $(\beta(x), c_R)$ satisfying the relations

$$(1.1)\qquad \left(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\right)\frac{1}{1-\kappa}\,\frac{\Omega}{2-\Omega}\left[D^2-s^2(1-\Omega\kappa)\right]\beta \\ +4\left[\frac{1}{2-\Omega}D^2-D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\beta=0 \qquad \text{for}\quad x\in(0,\infty),$$

and

$$(1.2)\qquad \begin{cases}\beta(0)=\beta(\infty)=0,\\[4pt] \dfrac{1}{s^2(2-\Omega)}D\left\{\dfrac{\Omega}{2-\Omega}\,\dfrac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\beta-4s^2\dfrac{1-\Omega}{2-\Omega}\beta\right\}\Big|_{\substack{x_2=0\\x_2=\infty}}=0.\end{cases}$$

Here

$$(1.3)\qquad \frac{1-2\nu(x)}{2-2\nu(x)}=\kappa(x),\qquad \Omega(x)=\frac{C_R^2}{\mu(x)},\qquad D=\frac{d}{dx},$$

in which $\nu(x)$ and $\mu(x)$ are the Poisson's ratio and shear modulus, respectively, while the symbol $c_R = p/s$, where $2\pi/p$ is the wave period and $2\pi/s$ is the wave length, denotes the velocity of surface wave. The eigenvalue c_R corresponding to the eigenfunction β is to be identified with the Rayleigh velocity.

Next we consider the case

$$(1.4) \qquad \begin{cases} \beta \in C^4[0,\infty), & \\ \kappa = \kappa(x) \in C^2[0,\infty), & 0 < \kappa_0 \le \kappa(x) \le \kappa_1 < 3/4, \\ \mu(x) \equiv \mu_0 = 1, & \Omega(x) \equiv \Omega_0 = C_R^2. \end{cases}$$

These hypotheses assure that the elastic energy of the half-space is stricly positive. The system (1.1) – (1.2) subjected to conditions (1.4) is equivalent to the following equations:

$$(1.5) \qquad \frac{1}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta = C_1 e^{-s\sqrt{1-\Omega_0}x} + C_2 e^{s\sqrt{1-\Omega_0}x},$$

$$(1.6) \qquad \beta(0) = \beta(\infty) = 0,$$

$$(1.7) \qquad \begin{aligned} D\left\{\frac{\Omega_0}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta - 4s^2(1-\Omega_0)\beta\right\}\Big|_{x=0} &= 0, \\ D\left\{\frac{\Omega_0}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta - 4s^2(1-\Omega_0)\beta\right\}\Big|_{x=\infty} &= 0, \end{aligned}$$

where C_1 is an arbitrary constant, $C_2 = 0$, $\beta \in C^4[0,\infty)$, $\beta(\infty) = 0$.

The aim of this section is to transform the problem (1.5) – (1.7) to an equivalent integral equation and to construct an iteration method of solving this equation. To this end consider the differential operator L associated with (1.5):

$$(1.8) \qquad L\beta \equiv -D^2\beta + s^2(1-\Omega_0\kappa(x))\beta,$$

$$(1.9) \qquad \beta(0) = \beta(\infty) = 0.$$

Let $g = g(x,t;\Omega_0,s)$; $(x,t;\Omega_0,s) \in [0,\infty)\times[0,\infty)\times(0,1)\times(0,\infty)$ be the Green function for the operator L with a "frozen" coefficient κ. In other words, the Green function g fulfills the relations:

$$(1.10) \qquad \frac{\partial^2 g}{\partial t^2} - s^2(1-\Omega_0\kappa(x))g = 0 \qquad \text{for} \quad t \neq x,$$

$$(1.11) \qquad g = 0 \qquad \text{for} \quad t = 0,$$

$$(1.12) \qquad \frac{\partial g}{\partial t}\Big|_{t=x+0} - \frac{\partial g}{\partial t}\Big|_{t=x-0} = -1.$$

For the operator with a variable coefficient κ, the Green function $G = G(x, t; \Omega_0, s)$ satisfies the following equation (c.f. [2], 123–149):

$$G(x,t;\Omega_0,s) = g(x,t;\Omega_0,s) - s^2\Omega_0 \int\limits_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]G(\xi,t;\Omega_0,s)\,d\xi \tag{1.13}$$

for every $(x,t;\Omega_0,s) \in [0,\infty)\times[0,\infty)\times(0,1)\times(0,\infty)$.

It is easy to show, that the function $g = g(x,t;\Omega_0,s)$ fulfilling the conditions (1.10)–(1.12) has the following form

$$g(x,t;\Omega_0,s) = \begin{cases} \dfrac{1}{2s\sqrt{1-\Omega_0\kappa(x)}}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] \\ \qquad\qquad \text{for} \quad x \le t < \infty, \\ \dfrac{1}{2s\sqrt{1-\Omega_0\kappa(x)}}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] \\ \qquad\qquad \text{for} \quad 0 \le t \le x. \end{cases} \tag{1.14}$$

In the subsequent part, the properties of the Green functions $G = G(x, t; \Omega_0, s)$ will be investigated and the solution of eigenvalue problem (1.5)–(1.7) will be expressed using the function G.

1.2. Integral equation for Green function

Let us denote by X the Banach space of real functions $A(x,t)$, $(x,t) \in [0,\infty)\times[0,\infty)$ with norm $\|\cdot\|_X$ given by

$$\|A(x,t)\|_X^2 = \int\limits_0^\infty \left\{\int\limits_0^\infty |A(x,t)|^2\,dt\right\} dx < \infty. \tag{1.15}$$

Let N be the operator in X of the form:

$$NA(x,t) = s^2\Omega_0 \int\limits_0^\infty g(x,\xi;\Omega_0,s)\,[\kappa(x)-\kappa(\xi)]\,A(\xi,t)\,d\xi\,, \tag{1.16}$$

where $g(x,\xi;\Omega_0,s)$ is defined by Eq. (1.14).

One can observe, that for every $(x,\xi) \in [0,\infty)\times[0,\infty)$ there exists such m, that

$$|\kappa(x)-\kappa(\xi)| \le m|x-\xi|. \tag{1.17}$$

The existence of m follows from assumption (1.4) and from the fact, that $\kappa(x) \in C^2[0,\infty)$. It can be assumed that

$$m = \sup_{x\in[0,\infty)} \left|\frac{d\kappa}{dx}\right|.$$

The following lemma is valid:

LEMMA 1. If the inequality

$$q \equiv \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1 \tag{1.18}$$

is satisfied, then operator N is a contraction in the space X, i.e.

$$\|NA\|_X \le q\|A\|_X .$$

P r o o f. Due to (1.16) and (1.4) we obtain

$$\begin{aligned}
&NA(x,t) \equiv M(x,t,\Omega_0,s)\\
&= \frac{1}{2}s\Omega_0\int_x^\infty (1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right][\kappa(x)-\kappa(\xi)]A(\xi,t)d\xi\\
&+\frac{1}{2}s\Omega_0\int_0^x (1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(x-\xi)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right][\kappa(x)-\kappa(\xi)]A(\xi,t)d\xi\\
&\equiv a(x,t;\Omega_0,s) + b(x,t;\Omega_0,s).
\end{aligned}$$

Hence, the following estimate can be deduced

$$|M(x,t;\Omega_0,s)| \le |a(x,t;\Omega_0,s)| + |b(x,t;\Omega_0,s)|. \tag{1.19}$$

Estimating from above the function a we get

$$\begin{aligned}
a^2(x,t;\Omega_0,s) \le \frac{1}{4}s^2\Omega_0^2\cdot\Bigg\{\int_x^\infty (1-\Omega_0\kappa_1)^{-1/2}\\
\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right]\cdot[\kappa(x)-\kappa(\xi)]\cdot|A(\xi,t)|\,d\xi\Bigg\}^2.
\end{aligned} \tag{1.20}$$

From the inequalities (1.17) and (1.4) we have

$$\begin{aligned}
(1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right]\cdot[\kappa(x)-\kappa(\xi)]\cdot|A(\xi,t)|\\
\le (1-\Omega_0\kappa_1)^{-1/2}e^{-s\sqrt{1-\Omega_0\kappa_1}(\xi-x)}m(\xi-x)\cdot|A(\xi,t)|,
\end{aligned} \tag{1.21}$$

and finally

$$\begin{aligned}
a^2(x,t;\Omega_0,s) \le \frac{1}{4}s^2m^2\Omega_0^2(1-\Omega_0\kappa_1)^{-1}\\
\cdot\left\{\int_x^\infty (\xi-x)e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)}|A(\xi,t)|\,d\xi\right\}^2.
\end{aligned} \tag{1.22}$$

Integrating inequality (1.22) with respect to x on the interval $[0,\infty)$ and changing the variables we obtain

$$(1.23)\qquad \int_0^\infty a^2(x,t;\Omega_0,s)\,dx \le \frac{1}{4}m^2\Omega_0^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int_0^\infty |A(x,t)|^2\,dx.$$

Integrating the inequality (1.23) with respect to t on the interval $[0,\infty)$ we get

$$(1.24)\qquad \|a(x,t;\Omega_0,s)\|_X \le \frac{1}{2}\Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1}\|A(x,t)\|_X\,.$$

Now we shall estimate the norm $\|b(x,t;\Omega_0,s)\|_X$. From the definition of the function $b(x,t;\Omega_0,s)$ we have:

$$(1.25)\qquad b(x,t;\Omega_0,s) = \frac{1}{2}s\Omega_0\int_0^x(1-\Omega_0\kappa)^{-1/2}$$
$$\left[e^{-s\sqrt{1-\Omega_0\kappa}(x-\xi)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right]\cdot[\kappa(x)-\kappa(\xi)]\cdot|A(\xi,t)|\,d\xi.$$

The inequalities (1.26) and (1.4) lead to

$$(1.26)\qquad b^2(x,t;\Omega_0,s) \le \frac{1}{4}s^2\Omega_0 m^2(1-\Omega_0\kappa_1)^{-1}$$
$$\left\{\int_0^x(x-\xi)e^{-s\sqrt{1-\Omega_0\kappa_1}(x-\xi)}|A(\xi,t)|\,d\xi\right\}^2.$$

Similarly to the case of inequality (1.22), from (1.26) we get the following estimate

$$(1.27)\qquad \|b(x,t;\Omega_0,s)\|_X \le \frac{1}{2}\Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1}\|A(x,t)\|_X\,.$$

From the inequality (1.19), (1.24) and (1.27) it follows that the operator N is a contraction in the space X, if

$$(1.28)\qquad q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1$$

which ends the proof of Lemma 1.

In the further analysis it will be convenient to introduce two other Banach spaces $X_2^{(1)}, X_1^{(-1/2)}$ with the following norms

$$(1.29)\qquad \|A(x,y)\|^2_{X_2^{(1)}} = \sup_{y\in[0,\infty)}\int_0^\infty |A(x,y)|\,dx,$$

$$(1.30)\qquad \|A(x,y)\|^2_{X_1^{(-1/2)}} = \sup_{y\in[0,\infty)}\int_0^\infty \frac{|A(x,y)|^2}{s^2[1-\Omega_0\kappa(x)]}\,dx.$$

The following lemma are valid:

LEMMA 2. The operator N given by formula (1.16) is a contraction in $X_2^{(1)}$, i.e.

$$\|NA\|_{X_2^{(1)}} \le q_1 \|A\|_{X_2^{(1)}}, \qquad \text{if} \qquad q_1 = \sqrt{q} < 1. \tag{1.31}$$

Here q is defined by the formula (1.18).

LEMMA 3. The operator N given by formula (1.16) is a contraction in $X_1^{(-1/2)}$, i.e.

$$\|NA\|_{X_1^{(-1/2)}} \le q \|A\|_{X_1^{(-1/2)}}, \qquad \text{if} \qquad q < 1. \tag{1.32}$$

Here q is defined by the formula (1.18).

The proof of Lemma 2 and 3 is given in the Appendix I (I_e, I_a, I_d and I_f).

We need the following lemma:

LEMMA 4. For every $(\Omega_0, s) \in (0,1) \times (0, \infty)$ the functions $g(x,t;\Omega_0,s)$, $\dfrac{\partial g(x,t;\Omega_0,s)}{\partial t}$, $\dfrac{\partial^2 g(x,t;\Omega_0,s)}{\partial t^2}$ belong to $X_2^{(1)}$, $X_1^{(-1/2)}$.

P r o o f. First, we show that the function $\dfrac{\partial g(x,t;\Omega_0,s)}{\partial t}$ belongs to $X_2^{(1)}$. Indeed, differentiating the functions defined by (1.14) with respect to t we obtain

$$\frac{\partial g(x,t;\Omega_0,s)}{\partial t} = \begin{cases} \dfrac{1}{2}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\right] \\ \qquad\qquad\qquad\qquad \text{for} \quad x < t < \infty, \\ \dfrac{1}{2}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] \\ \qquad\qquad\qquad\qquad \text{for} \quad 0 < t \le x, \end{cases} \tag{1.33}$$

and we obtain the estimate:

$$\begin{aligned} \int_0^\infty \left|\frac{\partial g(x,t;\Omega_0,s)}{\partial t}\right| dt &\le \frac{1}{2}\int_x^\infty \left|e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\right| dt \\ &+ \frac{1}{2}\int_0^x \left|e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(x+t)}\right| dt \\ &\le \frac{1}{2}\left(\int_x^\infty e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\,dt + \int_0^x e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)}\,dt\right. \\ &\left. + \int_0^x e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\,dt\right) \le \frac{1}{2}\left(\int_x^\infty e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\,dt\right. \end{aligned} \tag{1.34}$$

(1.34) [cont.]

$$
\begin{aligned}
&+\int\limits_0^x e^{-s\sqrt{1-\Omega_0\kappa_1}(x-t)}\,dt+\int\limits_0^x e^{-s\sqrt{1-\Omega_0\kappa_1}(t+x)}\,dt\Bigg)\\
&=\frac{1}{2}\Bigg(\frac{-1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\Big|_{t=x}^{t=\infty}+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\Big|_{t=0}^{t=x}\\
&+\frac{-1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t+x)}\Big|_{t=0}^{t=x}\Bigg)=\frac{1}{2}\Bigg(\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}\\
&-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{s\sqrt{1-\Omega_0\kappa_1}x}-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-2s\sqrt{1-\Omega_0\kappa_1}x}\\
&+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}x}\Bigg)\le\frac{1}{2}\left(\frac{3}{s\sqrt{1-\Omega_0\kappa_1}}\right),
\end{aligned}
$$

due to

$$
-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{s\sqrt{1-\Omega_0\kappa_1}x}<0,\qquad -\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-2s\sqrt{1-\Omega_0\kappa_1}x}<0,
$$

and

$$
\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}x}\le\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}\,.
$$

And finally

$$
\sup_{x\in[0,\infty)}\int\limits_0^\infty\left|\frac{\partial g}{\partial t}\right|dt\le\frac{3}{2s\sqrt{1-\Omega_0\kappa_1}}<\infty. \tag{1.35}
$$

This implies that $\dfrac{\partial g(s,t;\Omega_0,s)}{\partial t}\in X_2^{(1)}$.

For the other functions the proof is similar.

Using the formula (1.16), Eq. (1.13) can be written in the form

$$
G(s,t;\Omega_0,s)=g(s,t;\Omega_0,s)-NG(s,t;\Omega_0,s) \tag{1.36}
$$

and a solution to this equation can be obtained by the iteration procedure ([3] pp. 30 – 31)

$$
g_{n+1}=-Ng_n+g_0\,, \tag{1.37}
$$

where

$$
g_0=g(s,t;\Omega_0,s). \tag{1.38}
$$

1.3. Time derivatives of the Green function G

From Eq. (1.13) by formal differentiation with respect to t, we obtain:

$$(1.39)\qquad \frac{\partial G(s,t;\Omega_0,s)}{\partial t} = \frac{\partial g(s,t;\Omega_0,s)}{\partial t} - s^2\Omega_0 \int_0^\infty g(s,t;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial}{\partial t}G(\xi,t;\Omega_0,s)\,d\xi.$$

From Lemma 2 and Lemma 4 it follows that the solution of Eq. (1.39) belongs to the $X_2^{(1)}$ space. It is easy to show:

THEOREM 1. *Function* $\dfrac{\partial G(s,t;\Omega_0,s)}{\partial t}$ *is continuous for every*

$$(x,t;\Omega_0,s) \in [0,\infty)\times[0,\infty)\times(0,1)\times(0,1)$$

such that $t \neq x$.

P r o o f. Equation (1.39) may be written in the form:

$$(1.40)\qquad \frac{\partial G(s,t;\Omega_0,s)}{\partial t} - \frac{\partial g(s,t;\Omega_0,s)}{\partial t} = -s^2\Omega_0 \int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial g(\xi,t;\Omega_0 s)}{\partial t}d\xi$$
$$-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\left\{\frac{\partial G(\xi,t;\Omega_0,s)}{\partial t} - \frac{\partial g(\xi,t;\Omega_0,s}{\partial t}\right\}dt.$$

Applying the estimates similar to those used in the proof of Lemma 2 one can show, that the function

$$(1.41)\qquad l(x,t;\Omega_0,s) = -s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial g(\xi,t;\Omega_0,s)}{\partial t}d$$

is continuous with respect to t, for every $x \in [0,\infty)$ and $(\Omega_0,s)\in(0,1)\times(l,\infty)$. Indeed, the integral is uniformly convergent with respect to t, due to the estimates used in the proof of Lemma 2.

Continuity of l with respect to t and Eq. (1.41) imply that $\dfrac{\partial G}{\partial t} - \dfrac{\partial g}{\partial t}$ beongs to X, $X_2^{(1)}$ or $X_1^{(-1/2)}$ if the condition (1.18) is fulfilled.

Applying the iterative procedure to Eq. (1.40) one can show, that the function

$$\frac{\partial}{\partial t}G(x,t;\Omega_0,s) - \frac{\partial}{\partial t}g(x,t;\Omega_0,s)$$

is continuous with respect to t, for $t\neq x$, which ends the proof.

One can prove:

THEOREM 2. *The function* $\frac{\partial}{\partial t}G(x,t;\Omega_0,s)$ *has the same points of discontinuity as the function* $\frac{\partial}{\partial t}g(x,t;\Omega_0,s)$.

P r o o f. From formula (1.33) it follows that

$$\frac{\partial}{\partial t}g(x,t;\Omega_0,s)\Big|_{t=x+0}-\frac{\partial}{\partial t}g(x,t;\Omega_0,s)\Big|_{t=x-0}=-1. \tag{1.42}$$

Due to Theorem 1 the function $\frac{\partial G(x,t;\Omega_0,s)}{\partial t}$ is continuous with respect to t, except $t=x$, where the discontinuity of the first kind appears, i.e.

$$\frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x+0}-\frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x-0}=-1. \tag{1.43}$$

The type of discontinuity of function G follows from the definition of the Green function for the operator L. In order to establish the properties of the second derivative with respect to t, we shall transform Eq. (1.39) to the form

$$\begin{aligned}\mathcal{L}(x,t;\Omega_0,s) &= \widehat{l}(x,t;\Omega_0,s)\\ &\quad -s^2\Omega_0\int\limits_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\mathcal{L}(\xi,t;\Omega_0,s)\,d\xi,\end{aligned} \tag{1.44}$$

where

$$\begin{aligned}\mathcal{L}(x,t;\Omega_0,s) &= \frac{\partial}{\partial t}G(x,t;\Omega_0,s)-\frac{\partial}{\partial t}g(x,t;\Omega_0,s),\\ \widehat{l}(x,t;\Omega_0,s) &= -s^2\Omega_0\int\limits_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial\mathcal{L}(\xi,t;\Omega_0,s)}{\partial t}d\xi.\end{aligned} \tag{1.45}$$

Taking the derivative with respect to t we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial t} &= \frac{\partial\widehat{l}}{\partial t}-s^2\Omega_0\int\limits_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\\ &\quad \cdot\frac{\partial\mathcal{L}(\xi,t;\Omega_0,s)}{\partial t}d\xi \qquad \text{for} \quad t\neq x.\end{aligned} \tag{1.46}$$

Denote the first term on the R.H.S. of Eq. (1.46) by $\widetilde{m}(x,t;\Omega_0,s)$ and consider the equation

$$\begin{aligned}M(x,t;\Omega_0,s) &= \widetilde{m}(x,t;\Omega_0,s)\\ &\quad -s^2\Omega_0\int\limits_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]M(\xi,t;\Omega_0,s)\,d\xi\end{aligned} \tag{1.47}$$

in which $M(x, t; \Omega_0, s)$ is an unknown function. It can be shown that the function $\tilde{m} \in X_1^{(-1/2)}$. The proof is analogous to that proposed by KOSTUČENKO [2].

From Lemma 3 it follows that if

$$q = \Omega_0 m(1 - \Omega_0 \kappa_1)^{-3/2} s^{-1} < 1,$$

then a solution of Eq. (1.47) belongs to $X_1^{(-1/2)}$. We are to show that this solution for $t \neq x$ is identical with the function

$$\frac{\partial^2}{\partial t^2}[G(x, t; \Omega_0, s) - g(x, t; \Omega_0, s)].$$

In order to do this we integrate (1.47) with respect to t over the interval $[0, t]$ and we get

$$\begin{aligned} (1.48) \qquad \int_0^t M(x, \hat{t}; \Omega_0, s)\, d\hat{t} &= \int_0^t \tilde{m}(x, \hat{t}; \Omega_0, s)\, d\hat{t} \\ &\quad - s^2 \Omega_0 \int_0^\infty g(x, \xi; \Omega_0, s)[\kappa(x) - \kappa(\xi)] \left\{ \int_0^t M(\xi, \hat{t}; \Omega_0, s)\, d\hat{t} \right\} d\xi. \end{aligned}$$

From (1.44) it follows that the equation

$$\begin{aligned} (1.49) \qquad \mathcal{L}(x, t; \Omega_0, s) - \mathcal{L}(x, 0; \Omega_0, s) &= \hat{l}(x, t; \Omega_0, s) - \hat{l}(x, 0; \Omega_0, s) \\ &\quad - s^2 \Omega_0 \int_0^\infty g(x, \xi; \Omega_0, s)[\kappa(x) - \kappa(\xi)][\mathcal{L}(\xi, t; \Omega_0, s) - \mathcal{L}(x, 0; \Omega_0, s)]\, d\xi, \end{aligned}$$

and existence as well as uniqueness of the solution of Eq. (1.48) imply that

$$(1.50) \qquad \int_0^t M(x, \hat{t}; \Omega_0, s)\, d\hat{t} = \mathcal{L}(x, t; \Omega_0, s) - \mathcal{L}(x, 0; \Omega_0, s).$$

The last relation implies

$$\begin{aligned} (1.51) \qquad M(x, t; \Omega_0, s) &= \frac{\partial}{\partial t} \mathcal{L}(x, t; \Omega_0, s) \\ &= \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} G(x, t; \Omega_0, s) - \frac{\partial}{\partial t} g(x, t; \Omega_0, s) \right]. \end{aligned}$$

Because $\dfrac{\partial^2}{\partial t^2} g(x, t; \Omega_0, s)$ for $x \neq t$ belongs to $X_1^{(-1/2)}$, therefore $\dfrac{\partial^2 G}{\partial t^2}$ belongs to $X_1^{(-1/2)}$. From $\dfrac{\partial^2 G}{\partial t^2} \in X_1^{(-1/2)}$ and Eq. (1.51) we obtain:

THEOREM 3. *If $q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1$, then $G(x,t;\Omega_0,s)$ satisfies the equation*

$$(1.52)\qquad \begin{aligned} \frac{\partial^2 G(x,t;\Omega_0,s)}{\partial t^2} &= s^2(1-\Omega_0\kappa(t))G(x,t;\Omega_0,s),\\ G(x,t;\Omega_0,s) &= G(t,x;\Omega_0,s), \end{aligned}$$

and conditions

$$(1.53)\qquad \begin{aligned} G(x,t;\Omega_0,s)\big|_{t=0} &= 0,\\ \frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x+0} - \frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x-0} &= -1. \end{aligned}$$

In other words, $G(x,t;\Omega_0,s)$ is a Green function for the boundary value problem:

$$(1.54)\qquad \begin{aligned} L\beta(x) &= 0,\\ \beta(0) &= 0. \end{aligned}$$

Clearly, a solution to (1.5)–(1.6) expressed by G takes the form

$$(1.55)\qquad \beta(x;\Omega_0,s) = C_1\int_0^\infty G(x,t;\Omega_0,s)[1-\kappa(t)]e^{-s\sqrt{1-\Omega_0}t}\,dt \qquad (C_1 = \text{const}).$$

Since condition (1.7) can be written in the form

$$(1.56)\qquad \beta(0;\Omega_0,s) = -C_1\Omega_0\sqrt{1-\Omega_0}/4s(1-\Omega_0),$$

a solution to the eigen-problem (1.5)–(1.7) is defined by the pair $(\Omega_0,\beta(x))$ in which Ω_0 is a solution to the equation

$$(1.57)\qquad \int_0^\infty\left[\frac{\partial}{\partial t}G\right]_{x=0}[1-\kappa(t)]e^{-st\sqrt{1-\Omega_0}}\,dt + \Omega_0\sqrt{1-\Omega_0}/4s(1-\Omega_0) = 0,$$

and $\beta(x)$ is given by the formula (1.55).

Using the formulae (1.37)–(1.38), (1.55) and (1.57), we get a solution of the eigen-problem if $q < 1$, e.g.

$$(1.58)\qquad \Omega_0 m < s(1-\Omega_0\kappa_1)^{3/2}.$$

In general, the Eq. (1.57) has a finite number of solutions $\Omega_0 = \Omega_0(s)$, (cf. [4]).

2. Surface waves in a weakly nonhomogeneous isotropic elastic half-space

The problem of propagation of a surface stress wave of the form

$$
\begin{aligned}
\tau_{11}(x_1,x_2,t) &= \alpha_{11}(x_2)\cos(sx_1 - t\sqrt{\lambda}),\\
\tau_{22}(x_1,x_2,t) &= \alpha_{22}(x_2)\cos(sx_1 - t\sqrt{\lambda}),\\
\tau_{12}(x_1,x_2,t) &= -\alpha_{12}(x_2)\sin(sx_1 - t\sqrt{\lambda}),
\end{aligned}
\tag{2.1}
$$

in a nonhomogeneous elastic half-space

$$
X = \{(x_1,x_2):\quad x_2 \geq 0,\ \ |x_1| < \infty\}
$$

reduces to the following eigenvalue problem [5]: find a real symmetric tensor field $\alpha_{ij} = \alpha_{ij}(x_2)$ ($\alpha_{ij} \in C^2[0,\infty)$; $i,j = 1,2$) and a real number λ ($\lambda > 0$) satisfying the system of equations

$$
\begin{aligned}
\varrho^{-1}(s^2\alpha_{11} + s\,\dot{\alpha}_{12}) - \lambda(2\mu)^{-1}(\alpha_{11} - \nu\alpha_{\gamma\gamma}) &= 0,\\
-[\varrho^{-1}(\dot{\alpha}_{22} + s\alpha_{12})]^{\cdot} - \lambda(2\mu)^{-1}(\alpha_{22} - \nu\alpha_{\gamma\gamma}) &= 0,\\
-[\varrho^{-1}(s^2\,\dot{\alpha}_{12} + s\alpha_{11})]^{\cdot} - s\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12}) - \lambda(2\mu)^{-1}2\alpha_{12} &= 0
\end{aligned}
\tag{2.2}
$$

$$
\text{for}\quad x_2 \in (0,\infty)\quad (\gamma = 1,2)
$$

and the boundary conditions

$$
\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0,
\tag{2.3}
$$

s being the wave number ($s > 0$), and $\varrho = \varrho(x_2)$, $\mu = \mu(x_2)$, $\nu = \nu(x_2)$ denoting, respectively, the density of the medium, the shear modulus, and the Poisson ratio. The functions are assumed to be of the $C^2[0,\infty)$ class, and to satisfy the following inequalities

$$
\begin{aligned}
0 < \varrho_0 \leq \varrho(x_2) \leq \varrho_1 < \infty,\\
0 < \mu_0 \leq \mu(x_2) \leq \mu_1 < \infty,\\
-1 < \nu_0 \leq \nu(x_2) \leq \nu_1 < 1/2.
\end{aligned}
\tag{2.4}
$$

A dot over a symbol denotes differentiation with respect to the variable x_2; we will also use the symbol D to denote the derivative.

The aim of this paper is to give an approximate solution of the eigenvalue problem (2.2)–(2.3) in the following two cases:

1) density $\varrho = \varrho(x_2)$ is a "weakly" variable function, and μ and ν are constant;

2) shear modulus $\mu = \mu(x_2)$ is a "weakly" variable function, and ϱ and ν are constant.

In both cases we obtain the approximate solution by using the perturbation method proposed by FRIEDRICHS in [6].

2.1. Analysis of the case $\dfrac{1}{\varrho(x_2)} = \dfrac{1}{\varrho_1} + \dfrac{\varepsilon}{\widehat{\varrho}(x_2)}$, $\mu(x_2) \equiv \mu_1$, $\nu(x_2) \equiv \nu_1$

Let us consider in the real Hilbert space $\mathcal{H}$ generated by the scalar product

$$(\alpha, \beta) = \int_0^\infty (\alpha_{11}\beta_{11} + \alpha_{22}\beta_{22} + \alpha_{12}\beta_{12})\, dx_2$$

and satisfying the condition

$$\|\alpha\|^2 = \int_0^\infty (\alpha_{11}^2 + \alpha_{22}^2 + \alpha_{12}^2)\, dx_2 < \infty.$$

Equation (2.2) written in operator form

$$A\alpha - \lambda B\alpha = 0, \tag{2.5}$$

where

$$\alpha = \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{12} \end{pmatrix},$$

$$A \equiv A(s;p) \equiv \begin{bmatrix} \dfrac{s^2}{\varrho} & 0 & \dfrac{s}{\varrho} D \\ 0 & -D\dfrac{1}{\varrho}D & sD\dfrac{1}{\varrho} \\ -sD\dfrac{1}{\varrho} & -\dfrac{s}{\varrho}D & -D\dfrac{1}{\varrho}D + \dfrac{s^2}{\varrho} \end{bmatrix},$$

$$B \equiv B(\mu;\nu) \equiv \begin{bmatrix} \dfrac{1-\nu}{2\mu} & \dfrac{-\nu}{2\mu} & 0 \\ \dfrac{-\nu}{2\mu} & \dfrac{1-\nu}{2\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix}.$$

The domains of operators A and B may be defined as

$$\begin{gathered} \mathcal{D}(A) = \left\{\alpha\colon\ \alpha_{ij} \in C^2[0,\infty),\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = 0\right\}, \\ \mathcal{D}(B) = \left\{\alpha\colon\ \alpha_{ij} \in C^2[0,\infty)\right\}, \quad i,j = 1,2. \end{gathered} \tag{2.6}$$

The sets $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are dense in $\mathcal{H}$ since the set $C_0^\infty[0,\infty) \times C_0^\infty[0,\infty) \times C_0^\infty[0,\infty)$ is dense in $\mathcal{H}$ and is contained in $\mathcal{D}(A)$ and $\mathcal{D}(B)$.

It can be demonstrated that operators A and B are symmetric. The symmetry of operator A results from the fact that operators on both sides of the principal diagonal are formally adjoint, e.g. $-sD\dfrac{1}{\varrho}$ and $\dfrac{s}{\varrho}D$, $sD\dfrac{1}{\varrho}$ and $\dfrac{-s}{\varrho}D$. For any $\alpha, \beta \in \mathcal{D}(A)$ we have

$$(A\alpha, \beta) = \int\limits_0^\infty \Big\{\varrho^{-1}(s^2\alpha_{11} + s\,\dot{\alpha}_{12})\beta_{11} - \left[\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12})\right]^{\cdot}\beta_{22}$$
$$- \left[\varrho^{-1}(\dot{\alpha}_{12} + s\alpha_{11})\right]^{\cdot}\beta_{12} - s\varrho^{-1}(s^2\,\dot{\alpha}_{12} - s\alpha_{22})\beta_{12}\Big\}dx_2.$$

Integration by parts with the use of boundary conditions shows that

$$(A\alpha, \beta) = (\alpha, A\beta).$$

The symmetry of operator B is obvious. Matrix B is positive definite and for every $\alpha \in \mathcal{D}(B) \subset \mathcal{H}$ we have ([1])

$$(B\alpha, \alpha) \geq k(\alpha, \alpha), \tag{2.7}$$

where

$$k = \min_{x_2\in[0,\infty)}\left(\frac{1-2\nu}{2\mu}, \frac{1}{\mu}, \frac{1}{2\mu}\right).$$

If in Eq. (2.5) we put $\varrho \equiv \widetilde{\varrho} = \text{const}$, $\mu \equiv \widetilde{\mu} = \text{const}$, $\nu \equiv \widetilde{\nu} = \text{const}$ (homogeneous medium), the problem has precisely one solution $(\widetilde{\alpha}, \widetilde{\lambda})$ of the form

$$\widetilde{\alpha}(\widetilde{\varrho}, \widetilde{\mu}, \widetilde{\nu}) = \begin{bmatrix} -\widetilde{\beta}_0\left[e^{-x_2\widetilde{h}_2} - \dfrac{2+\widetilde{\omega}(1-2\widetilde{\kappa})}{2-\widetilde{\omega}}e^{-x_2\widetilde{h}_1}\right] \\ \widetilde{\beta}_0\left[e^{-x_2\widetilde{h}_2} - e^{-x_2\widetilde{h}_1}\right] \\ -\dfrac{2}{s}\dfrac{\widetilde{\beta}_0}{2-\widetilde{\omega}}\widetilde{h}_1\left[e^{-x_2\widetilde{h}_2} - e^{-x_2\widetilde{h}_1}\right] \end{bmatrix}, \tag{2.8}$$

where

$$\widetilde{\kappa} = \frac{1-2\widetilde{\nu}}{2-2\widetilde{\nu}}, \qquad \widetilde{h}_1 = s\sqrt{1-\widetilde{\omega}\widetilde{\kappa}}, \qquad \widetilde{h}_2 = s\sqrt{1-\widetilde{\omega}}$$

and $\widetilde{\omega}$ is a root of the equation

$$(2-\widetilde{\omega})^2 = 4\sqrt{(1-\widetilde{\omega})(1-\widetilde{\omega}\widetilde{\kappa})} \tag{2.9}$$

([1]) The eigenvalues of matrix B are $\dfrac{1-2\nu}{2\mu}, \dfrac{1}{\mu}, \dfrac{1}{2\mu}$. From the theorem in [8] saying that a symmetric matrix B is positive definite iff all its eigenvalues are positive and $(B\alpha, \alpha) \geq \min\limits_i \lambda_i(\alpha, \alpha)$, results the Eq. (2.7).

such that $0 < \widetilde{\omega} < 1$; $\widetilde{\beta}_0$ is an arbitrary real number. The surface wave velocity in this homogeneous medium is given by

$$\widetilde{C}_R = \sqrt{\frac{\widetilde{\mu}}{\widetilde{\varrho}}} \cdot \sqrt{\widetilde{\omega}}.$$

The relation between $\widetilde{\lambda}$ and $\widetilde{C}_R$ is of the form

$$\sqrt{\widetilde{\lambda}} = s\widetilde{C}_R.$$

Let us consider the case when

$$\frac{1}{\varrho(x_2)} = \frac{1}{\varrho_1} + \frac{\varepsilon}{\widehat{\varrho}(x_2)}, \qquad \nu \equiv \nu_1, \qquad \mu \equiv \mu_1 \tag{2.10}$$

and ε is a sufficiently small positive real number. Moreover, ϱ_1 is a positive constant, and $\widehat{\varrho}(x_2)$ is a positive function (cf. (2.4)). After substituting (2.10) into (2.5) we get the equation

$$A_0\alpha + \varepsilon V\alpha = \lambda B\alpha, \tag{2.11}$$

where

$$\begin{aligned} A_0 &= A(s;\varrho_1), \\ V &= V(s;\widehat{\varrho}), \\ B &= B(\mu_1;\nu_1). \end{aligned}$$

The constraints on ϱ (cf. (2.4)) and (2.10) yield the constraints on $\widehat{\varrho}$ for $x_2 \in [0,\infty)$. Hence for every $\alpha \in \mathcal{D}(A) \equiv \mathcal{D}(V) \subset \mathcal{H}$

$$(V\alpha,\alpha) < \infty. \tag{2.12}$$

Moreover the operators A_0, V and B are symmetric in the space $\mathcal{H}$. From the fact that $\widetilde{\lambda}(\varrho_1,\mu_1,\nu_1)$ is a simple eigenvalue (the eigenspace is one-dimensional) it follows that $(A_0 - \widetilde{\lambda}B)^{-1}$ is defined in the subspace $\mathcal{H}$ orthogonal to the vector $\widetilde{\alpha}(\varrho_1,\mu_1,\nu_1)$ (2). Hence for sufficiently small ε in a neighbourhood of $(\widetilde{\lambda}(\varrho_1,\nu_1,\mu_1), \widetilde{\alpha}(\varrho_1,\nu_1,\mu_1))$ there exists a solution $(\lambda_\varepsilon,\alpha_\varepsilon)$ satisfying Eq. (2.11), analytical with respect to ε, of the form

$$\lambda_\varepsilon = \widetilde{\lambda} + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \tag{2.13}$$

$$\alpha_\varepsilon = \widetilde{\alpha} + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots, \tag{2.14}$$

(2) FRIEDRICHS [6] formulates the following assumptions of a perturbation theory; the operator must be symmetric, it must allow for a spectral decomposition, it must have a simple eigenvalue $\widetilde{\lambda}$ with the corresponding eigenvector $\widetilde{\alpha}$. Hence the equation $[A_0 - \widetilde{\lambda}B]\alpha = \Psi$ has a solution for any right-hand side of Ψ orthogonal to $\widetilde{\alpha}$ in the space $\mathcal{H}$. It is easily seen that Friedrich's assumptions are satisfied for the problem (2.5)–(2.6).

where

$$\alpha_i = \begin{pmatrix} \alpha_{11}^{(i)} \\ \alpha_{22}^{(i)} \\ \alpha_{12}^{(i)} \end{pmatrix}, \qquad i = 1, 2, 3 \ldots .$$

Substituting (2.13) and (2.14) into (2.11) and comparing the expressions appearing at suitable powers of ε we get

$$\begin{aligned}
&(A_0 - \widetilde{\lambda} B)\widetilde{\alpha} \ = 0, \\
&(A_0 - \widetilde{\lambda} B)\alpha_1 = -(V - \lambda_1 B)\widetilde{\alpha}, \\
&(A_0 - \widetilde{\lambda} B)\alpha_2 = -(V - \lambda_1 B)\alpha_1 + \lambda_2 B\widetilde{\alpha}, \\
&(A_0 - \widetilde{\lambda} B)\alpha_3 = -(V - \lambda_1 B)\alpha_2 + \lambda_2 B\alpha_1 + \lambda_3 B\widetilde{\alpha}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned} \tag{2.15}$$

Multiplying the first equation by α_1 and the second by $\widetilde{\alpha}$, and subtracting we get

$$\lambda_1 = \frac{(V\widetilde{\alpha}, \widetilde{\alpha})}{(B\widetilde{\alpha}, \widetilde{\alpha})} . \tag{2.16}$$

Analogously, multiplying the first equation by α_2, and the third one by $\widetilde{\alpha}$ and subtracting we obtain

$$\lambda_2 = \frac{(V\alpha_1, \widetilde{\alpha}) - \lambda_1(B\alpha_1, \widetilde{\alpha})}{(B\widetilde{\alpha}, \widetilde{\alpha})} . \tag{2.17}$$

In general, we get λ_i ($i \geq 3$) by multiplying the first equation by α_i, multiplying the $(i + 1)$-th equation by $\widetilde{\alpha}$ and subtracting both sides of the relations.

Equations (2.13), (2.16) and (2.17) effectively determine the approximate eigenvalue in the problem with weakly variable density in the considered half-space.

We now proceed to construct the series α_i. It is easy to demonstrate that the right-hand sides of the system (2.15) are elements of a subspace $\mathcal{H}$ orthogonal to the vector $\widetilde{\alpha}$. The construction of the series α_i is thus reduced to finding an operator $[A_0 - \widetilde{\lambda}(\varrho_1, \mu_1, \nu_1)B(\mu_1, \nu_1)]^{-1}$ on a subspace orthogonal to $\widetilde{\alpha}(\varrho_1, \mu_1, \nu_1)$. To this end, let us consider the equation

$$A_0\widehat{\alpha} - \widetilde{\lambda}(\varrho_1, \mu_1, \nu_1)B\widehat{\alpha} = g, \tag{2.18}$$

where

$$\widehat{\alpha} = \begin{pmatrix} \widehat{\alpha}_{11} \\ \widehat{\alpha}_{22} \\ \widehat{\alpha}_{12} \end{pmatrix}, \qquad g = \begin{pmatrix} g_{11} \\ g_{22} \\ g_{12} \end{pmatrix},$$

and g is a vector of the subspace $\mathcal{H}$ satisfying the condition

$$(g, \tilde{\alpha}) = 0. \tag{2.19}$$

The vector α_ε given by (2.14) should belong to $\mathcal{D}(A)$. Thus to construct α_i it is enough to find $\hat{\alpha}$ satisfying (2.18), (2.19) such that $\hat{\alpha} \in \mathcal{D}(A)$. It can be shown that vector $\hat{\alpha}$ is of the form

$$\hat{\alpha} = \begin{bmatrix} \hat{\alpha}_{11}(x_2) \\ \hat{\alpha}_{22}(x_2) \\ \hat{\alpha}_{12}(x_2) \end{bmatrix} = \begin{bmatrix} \int\limits_0^\infty K_1(x_2,t)F(t)dt \\ \int\limits_0^\infty K_2(x_2,t)F(t)dt \\ \int\limits_0^\infty K_3(x_2,t)F(t)dt \end{bmatrix} + \begin{bmatrix} G_{11}(x_2) \\ G_{22}(x_2) \\ G_{12}(x_2) \end{bmatrix}, \tag{2.20}$$

where

$$K_1(x_2,t) = \begin{cases} l_3 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(t-x_2)} + l_4 e^{-s\sqrt{1-\tilde{\omega}}(t-x_2)} \\ \qquad + \left[\dfrac{2(1-\tilde{\omega}\kappa_1)+\tilde{\omega}}{\tilde{\omega}-2}\right] a_1 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + \left[\dfrac{2(1-\tilde{\omega}\kappa_1)+\tilde{\omega}}{\tilde{\omega}-2}\right] a_2 e^{-s(\sqrt{1-\tilde{\omega}}t+\sqrt{1-\tilde{\omega}\kappa_1}x_2)} \\ -b_1 e^{-s(\sqrt{1-\tilde{\omega}\kappa_1}t+\sqrt{1-\tilde{\omega}}x_2)} - b_2 e^{-s\sqrt{1-\tilde{\omega}}(t+x_2)} \quad \text{for } t \geq x_2, \\ l_3 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(x_2-t)} + l_4 e^{-s\sqrt{1-\tilde{\omega}}(x_2-t)} \\ \qquad + \left[\dfrac{2(1-\tilde{\omega}\kappa_1)+\tilde{\omega}}{\tilde{\omega}-2}\right] a_1 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + \left[\dfrac{2(1-\tilde{\omega}\kappa_1)+\tilde{\omega}}{\tilde{\omega}-2}\right] a_2 e^{-s(\sqrt{1-\tilde{\omega}}t+\sqrt{1-\tilde{\omega}\kappa_1}x_2)} \\ -b_1 e^{-s(\sqrt{1-\tilde{\omega}\kappa_1}t+\sqrt{1-\tilde{\omega}}x_2)} - b_2 e^{-s\sqrt{1-\tilde{\omega}}(t+x_2)} \quad \text{for } x_2 \geq t, \end{cases} \tag{2.21}$$

$$K_2(x_2,t) = \begin{cases} l_1 e^{s\sqrt{1-\tilde{\omega}\kappa_1}(x_2-t)} + l_2 e^{-s\sqrt{1-\tilde{\omega}}(x_2-t)} + a_1 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + a_2 e^{-s(\sqrt{1-\tilde{\omega}}t+\sqrt{1-\tilde{\omega}\kappa_1}x_2)} + b_1 e^{-s(\sqrt{1-\tilde{\omega}\kappa_1}t+\sqrt{1-\tilde{\omega}}x_2)} \\ \qquad + b_2 e^{-s\sqrt{1-\tilde{\omega}}(t+x_2)} \quad \text{for } t \geq x_2, \\ l_1 e^{s\sqrt{1-\tilde{\omega}\kappa_1}(x_2-t)} + l_2 e^{s\sqrt{1-\tilde{\omega}}(x_2-t)} + a_1 e^{-s\sqrt{1-\tilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + a_2 e^{-s(\sqrt{1-\tilde{\omega}}t+\sqrt{1-\tilde{\omega}\kappa_1}x_2)} + b_1 e^{-s(\sqrt{1-\tilde{\omega}\kappa_1}t+\sqrt{1-\tilde{\omega}}x_2)} \\ \qquad + b_2 e^{-s\sqrt{1-\tilde{\omega}}(t+x_2)} \quad \text{for } x_2 \geq t, \end{cases} \tag{2.22}$$

$$(2.23)\qquad K_3(x_2,t)=\begin{cases} l_5e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t-x_2)}+l_6e^{-s\sqrt{1-\widetilde{\omega}}(t-x_2)}+a_1l_7e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2+t)} \\ \quad +a_2l_7e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)}+b_1l_8e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \quad +b_2l_8e^{-s\sqrt{1-\widetilde{\omega}}(x_2+t)} \qquad \text{for } t\ge x_2, \\ -l_5e^{s\sqrt{1-\widetilde{\omega}\kappa_1}(t-x_2)}-l_6e^{s\sqrt{1-\widetilde{\omega}}(t-x_2)}+a_1l_7e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2+t)} \\ \quad +a_2l_7e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)}+b_1l_8e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \quad +b_2l_8e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \qquad \text{for } x_2\ge t, \end{cases}$$

$$(2.24)\qquad F(t)=-\varrho_1\left[D^2-k_1^2\right]g_{22}(t)+\varrho_1\widetilde{\omega}^{-1}\left[2-2\widetilde{\omega}-2\kappa_1+\widetilde{\omega}\kappa_1\right]\left[D^2+k_2^2\right]g_{11}(t)$$
$$+2\varrho_1s(2-\widetilde{\omega})(1-\kappa_1)\widetilde{\omega}^{-1}Dg_{12}(t),$$

$$(2.25)\qquad G_{11}(x_2)=\frac{2\varrho_1}{s^2(\widetilde{\omega}-2)}(g_{22}-g_{11}),$$

$$(2.26)\qquad G_{22}(x_2)=0,$$

$$(2.27)\qquad G_{12}(x_2)=0.$$

The coefficients $l_1, l_2, ..., l_8$, k_1^2, k_2^2, a_1, a_2, b_1, b_2, appearing in Eqs. (2.21) – (2.27) are given in the Appendix II.

Using Eqs. (2.18) – (2.27) and the relations (2.16) and (2.17) we can find successively (λ_i, α_i).

Let us now analyse the eigenvalue λ_ε (cf. (2.13)) in the case when the function $\widehat{\varrho}=\widehat{\varrho}(x_2)$ (cf. (2.10)) is a monotonic function of the half-space depth coordinate.

Assume that

$$(2.28)\qquad \frac{1}{\varrho(x_2)}=\frac{1}{\varrho_1}+\frac{\varepsilon}{\widehat{\varrho}_\infty}(1-e^{-ax_2}),\qquad (a\ge 0),\qquad (\widehat{\varrho}_\infty>0).$$

Since $\dfrac{1}{\varrho_1}\le\dfrac{1}{\varrho(x_2)}\le\dfrac{1}{\varrho_0}$ we have on the one hand $\max\limits_{x_2\in[0,\infty)}\dfrac{1}{\varrho(x_2)}=\dfrac{1}{\varrho_0}$ and on the other hand, $\max\limits_{x_2\in[0,\infty)}\dfrac{1}{\varrho(x_2)}=\dfrac{1}{\varrho_1}+\dfrac{\varepsilon}{\widehat{\varrho}_\infty}$. Comparing these values we get

$$(2.29)\qquad \varepsilon=\widehat{\varrho}_\infty\left(\frac{1}{\varrho_0}-\frac{1}{\varrho_1}\right),$$

where $\varrho_1/\varrho_0\sim 1$.

Substituting (2.28) and (2.29) into (2.16), taking into account the relations

$$\lambda_\varepsilon = (C_R^2)_\varepsilon s^2, \qquad \widetilde{\lambda} = \widetilde{C}_R^2 s^2, \qquad \lambda_1 = (C_R^2)_1 s^2, \qquad \kappa_1 = \frac{1-2\nu_1}{2-2\nu_1}$$

and limiting ourselves to two terms in Eq. (2.13) we get for the square of surface wave velocity the relation

$$(2.30) \qquad (C_R^2)_\varepsilon = \widetilde{C}_R^2 + \left(\frac{1}{\varrho_0} - \frac{1}{\varrho_1}\right)\frac{\mu_1 a}{2(1-\widetilde{\omega})} \times \left[\frac{P_0(\widetilde{\omega})}{\sqrt{1-\widetilde{\omega}\kappa_1}(a+2s\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_1(\widetilde{\omega})}{\sqrt{1-\widetilde{\omega}}(a+2s\sqrt{1-\widetilde{\omega}})} + \frac{P_2(\widetilde{\omega})}{(\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1})(a+s\sqrt{1-\widetilde{\omega}}+s\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \times \left[\frac{P_3(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}}} + \frac{P_4(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_5(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}$$

(the polynomials $P_0, P_1, P_2, P_3, P_4, P_5$ are given in the Appendix II). Introducing the following notation

$$\theta = \frac{\varrho_1}{\varrho_0}, \quad C_2^2 = \frac{\mu_1}{\varrho_1}, \qquad a = \frac{s}{2\pi}\widehat{a} \qquad (\widehat{a} \in [0,\infty)),$$

$$\widetilde{\omega} = \frac{\widetilde{C}_R^2}{C_2^2}, \qquad \omega = \frac{(C_R^2)}{C_2^2},$$

we rewrite the formula (2.30) in the form

$$(2.31) \qquad \omega = \widetilde{\omega} + (\theta-1)\frac{\widehat{a}}{2(1-\widetilde{\omega})}\left[\frac{P_0}{\sqrt{1-\widetilde{\omega}\kappa_1}(\widehat{a}+4\pi\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_1}{\sqrt{1-\widetilde{\omega}}(\widehat{a}+4\pi\sqrt{1-\widetilde{\omega}})} + \frac{P_2}{(\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1})(\widehat{a}+2\pi\sqrt{1-\widetilde{\omega}}+2\pi\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \cdot \left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.$$

It is easy to demonstrate that the function $\omega = \omega(\widehat{a}, \theta, \kappa_1)$ described by (2.31) is for every fixed θ and κ_1 an increasing function of the variable $\widehat{a}$. Figure 1 shows the function ω for: 1) $\kappa_1 = \frac{1}{2}$, $\theta = 1.1$; 2) $\kappa_1 = \frac{1}{2}$, $\theta = 1.01$; 3) $\kappa_1 = \frac{1}{2}$, $\theta = 1$.

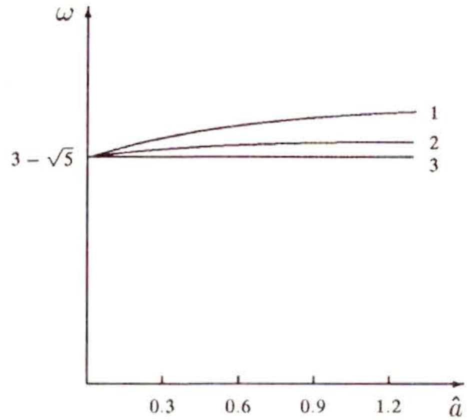

FIG. 1.

2.2. Surface wave in an elastic half-space with a weakly variable shear modulus (3)

Assume now that

$$\frac{1}{\mu(x_2)} = \frac{1}{\mu_1} + \frac{\varepsilon}{\widehat{\mu}(x_2)}, \qquad \varrho \equiv \varrho_1, \qquad \nu \equiv \nu_1 . \tag{2.32}$$

Substituting (2.32) into (2.5) we get

$$A(s, \varrho_1)\alpha - \lambda[B(\mu_1, \nu_1) + \varepsilon B(\widehat{\mu}, \nu_1)]\alpha = 0. \tag{2.33}$$

Operators $A(s, \varrho_1)$, $B(\mu_1, \nu_1)$, $B(\widehat{\mu}, \nu_1)$ are symmetric in $\mathcal{H}$. Moreover, B is a positive definite operator. According to the perturbation theory, there exists a solution of the eigenproblem (2.33) determined in some neighbourhood of $(\widetilde{\lambda}, \widetilde{\alpha})$ which is an analytical function of the parameter ε. The pair $(\lambda_\varepsilon, \alpha_\varepsilon)$ is given by (2.13) and (2.14), while $(\widetilde{\lambda}, \widetilde{\alpha})$ is given by (2.8) and (2.9), where $\widetilde{\varrho} = \varrho_1$, $\widetilde{\mu} = \mu_1$, $\nu = \nu_1$. Substituting (2.13) and (2.14) into (2.33) and comparing the values at suitable powers of ε we obtain the following system

$$\begin{aligned}
&[A(s, \varrho_1) - \widetilde{\lambda} B(\mu_1, \nu_1)]\widetilde{\alpha} \;= 0,\\
&[A(s, \varrho_1) - \widetilde{\lambda} B(\mu_1, \nu_1)]\alpha_1 = [\widetilde{\lambda} B(\widehat{\mu}, \nu_1) + \lambda_1 B(\mu_1, \nu_1)]\widetilde{\alpha},\\
&[A(s, \varrho_1) - \widetilde{\lambda} B(\mu_1, \nu_1)]\alpha_2 = \widetilde{\lambda} B(\widehat{\mu}, \nu_1)\alpha_1 + \lambda_1[B(\mu_1, \nu_1)\alpha_1 + B(\widehat{\mu}, \nu_1)\widetilde{\alpha}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \lambda_2 B(\mu_1, \nu_1)\widetilde{\alpha},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned} \tag{2.34}$$

Performing scalar multiplication of $(2.34)_1$ by α_1, of $(2.34)_2$ by $\widetilde{\alpha}$ and subtracting by sides, we get

$$\lambda_1 = \frac{-\widetilde{\lambda}[B(\widehat{\mu}, \nu_1)\widetilde{\alpha}, \widetilde{\alpha}]}{[B(\mu_1, \nu_1)\widetilde{\alpha}, \widetilde{\alpha}]} . \tag{2.35}$$

(3) This problem was also analysed in [7], using another approach.

Proceeding similarly as in the derivation of the series (2.16) and (2.17), we get

$$\lambda_2 = \frac{-\widetilde{\lambda}[B(\widehat{\mu},\nu_1)\alpha_1,\widetilde{\alpha}] - \lambda_1\{[B(\mu_1,\nu_1)\alpha_1,\widetilde{\alpha}] + [B(\widehat{\mu},\nu_1)\widetilde{\alpha},\widetilde{\alpha}]\}}{[B(\mu_1,\nu_1)\widetilde{\alpha},\widetilde{\alpha}]} \tag{2.36}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

The vectors α_i are defined by the equations

$$\begin{aligned}
\alpha_1 &= [A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]^{-1}\widetilde{\lambda}B(\widehat{\mu},\nu_1)\widetilde{\alpha} + \lambda_1 B(\mu_1,\nu_1)\widetilde{\alpha},\\
\alpha_2 &= [A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]^{-1}\{\widetilde{\lambda}B(\widehat{\mu},\nu_1)\alpha_1 + \lambda_1[B(\mu_1,\nu_1)\alpha_1\\
&\qquad + B(\widehat{\mu},\nu_1)\widetilde{\alpha}] + \lambda_2 B(\mu_1,\nu_1)\widetilde{\alpha}\},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .
\end{aligned} \tag{2.37}$$

We continue similarly to the case of the half-space with "weakly variable" density and we assume

$$\frac{1}{\widehat{\mu}(x_2)} = \frac{1}{\widehat{\mu}_\infty}(1 - e^{-ax_2}) \qquad (a \geq 0), \quad \nu \equiv \nu_1, \quad \varrho \equiv \varrho_1 ,$$

where

$$\varepsilon = \widehat{\mu}_\infty\left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right), \qquad \frac{\mu_1}{\mu_0} \sim 1.$$

From relation (2.35) for the square of wave velocity we get

$$\begin{aligned}
(C_R^2)_\varepsilon = \widetilde{C}_R^2 - \left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right)\widetilde{C}_R^2 a\mu_1 \Bigg[&\frac{P_3}{\sqrt{1-\widetilde{\omega}}(a + 2s\sqrt{1-\widetilde{\omega}})}\\
&+ \frac{P_4}{(\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1})(a + s\sqrt{1-\widetilde{\omega}} + s\sqrt{1-\widetilde{\omega}\kappa_1})}\\
&+ \frac{P_5}{(\sqrt{1-\widetilde{\omega}\kappa_1}(a + 2s\sqrt{1-\widetilde{\omega}\kappa_1})}\Bigg]\\
&\cdot\left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.
\end{aligned} \tag{2.38}$$

Introducing the following notations

$$\frac{\mu_1}{\varrho_1} = C_2^2, \qquad \theta = \frac{\mu_1}{\mu_0}, \qquad a = \frac{s}{2\pi}\widehat{a},$$

$$\widetilde{\omega} = \frac{\widetilde{C}_R^2}{C_2^2}, \qquad \omega = \frac{(C_R^2)_\varepsilon}{C_2^2},$$

we reduce (2.38) to the form

$$(2.39)\qquad \omega = \widetilde{\omega} - \widetilde{\omega}(\theta - 1)\widehat{a}\left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}(\widehat{a} + 4\pi\sqrt{1-\widetilde{\omega}})} + \frac{P_4}{(\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1})(\widehat{a} + 2\pi\sqrt{1-\widetilde{\omega}} + 2\pi\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_5}{(\sqrt{1-\widetilde{\omega}\kappa_1}(\widehat{a} + 4\pi\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \times \left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.$$

The function ω given by (2.39) for a fixed θ and κ_1 is a decreasing function of the argument $\widehat{\alpha}$. Figure 2 shows the diagrams of the function $\omega(\theta, \kappa_1, \widehat{\alpha})$ for 1) $\kappa_1 = 0.5,\ \theta = 1.1$; 2) $\kappa_1 = 0.5,\ \theta = 1.01$; 3) $\kappa_1 = 0.5,\ \theta = 1$.

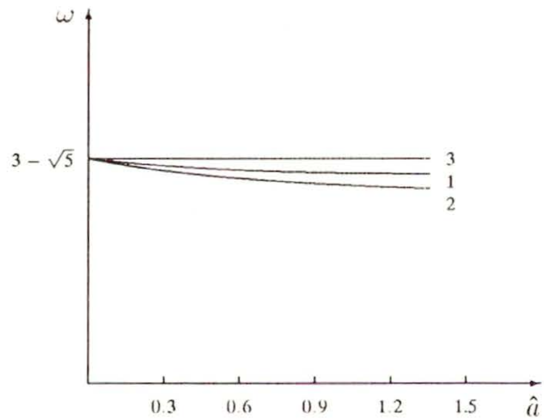

FIG. 2.

Appendix I

I_a: To obtain (1.23) from (1.22) we calculate the integral

$$(\mathrm{A.1})\qquad I_a = \int_0^\infty dx \left[\int_x^\infty a(\xi - x) b(\xi)\, d\xi\right]^2,$$

where

$$(\mathrm{A.2})\qquad \begin{aligned} a(p) &= pe^{-s\sqrt{1-\Omega_0\kappa_1 p}}, \\ b(p) &= |A(p,t)| \qquad (p > 0). \end{aligned}$$

Changing the variables in (A.1) and using the Fubini theorem, we obtain

$$\text{(A.3)}\qquad I_a = \int_0^\infty dx \left[\int_0^\infty a(p)b(p+x)dp\right]^2$$
$$= \int_0^\infty dx \left[\int_0^\infty a(p)b(p+x)dp\right]\cdot\left[\int_0^\infty a(\widehat{p})b(\widehat{p}+x)\,d\widehat{p}\right]$$
$$= \int_0^\infty a(p)\,dp\cdot\int_0^\infty a(\widehat{p})\,d\widehat{p}\cdot\int_0^\infty dx[b(p+x)b(\widehat{p}+x)].$$

From the Schwartz inequality it follows

$$\text{(A.4)}\qquad \int_0^\infty b(p+x)b(\widehat{p}+x)\,dx \le \left[\int_0^\infty b^2(p+x)dx\right]^{1/2}\cdot\left[\int_0^\infty b^2(\widehat{p}+x)\,dx\right]^{1/2}$$
$$= \left[\int_p^\infty b^2(\xi)\,d\xi\right]^{1/2}\cdot\left[\int_{\widehat{p}}^\infty b^2(\xi)\,d\xi\right]^{1/2} \le \int_0^\infty b^2(\xi)\,d\xi.$$

Finally we obtain

$$\text{(A.5)}\qquad I_a \le \left[\int_0^\infty a(p)\,dp\right]^2\cdot\int_0^\infty b^2(\xi)\,d\xi.$$

Similiarly we estimate the integrals I_b, I_c, I_d:

$$\text{(A.6)}\qquad I_b = \int_0^\infty dx\left\{\int_0^x (x-\xi)e^{-s\sqrt{1-\Omega_0\varkappa_1}(x-\xi)}\cdot|A(\xi,t)|\,d\xi\right.$$
$$\left.\cdot\int_0^x (x-\xi')e^{-s\sqrt{1-\Omega_0\varkappa_1}(x-\xi')}\cdot|A(\xi',t)|\,d\xi'\right\}$$
$$\le s^{-4}(1-\Omega_0\varkappa_1)^{-2}\int_0^\infty |A(x,t)|^2\,dx,$$

$$\text{(A.7)}\qquad I_c = \int_0^\infty |a(x,t;\Omega_0,s)|\,dx \le \frac{1}{2}\Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx,$$

$$\text{(A.8)}\qquad I_d = \int_0^\infty |b(x,t;\Omega_0,s)|\,dx \le \frac{1}{2}\Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx.$$

Integrating the inequality (cf. (1.19))

$$|NA(x,t)| \le |a(x,t;\Omega_0,s)| + |b(x,t;\Omega_0,s)| \tag{A.9}$$

with respect to x over the interval $[0,\infty)$ and using the estimate of the integral I_c and I_d, we obtain

$$\begin{aligned}\int_0^\infty |NA(x,t)|\,dx &\le \int_0^\infty |a(x,t;\Omega_0,s)|\,dx + \int_0^\infty |b(x,t;\Omega_0,s)|\,dx \\ &\le \Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx,\end{aligned} \tag{A.10}$$

and finally

$$\begin{aligned}\|NA(x,t)\|^2_{X_2^{(1)}} &= \sup_{t\in[0,\infty)}\int_0^\infty |NA(x,t)|\,dx \\ &\le \Omega_0 m(1-\Omega_0\kappa_1)^{3/2}s^{-1}\sup_{t\in[0,\infty)}\int_0^\infty |A(x,t)| \\ &= \Omega_0 m(1-\Omega_0\kappa_1)^{3/2}s^{-1}\|A(x,t)\|^2_{X_2^{(1)}}.\end{aligned} \tag{A.11}$$

From the last inequality it results that NA is a contraction operator in $X_2^{(1)}$, if

$$q_1 = \sqrt{q} < 1. \tag{A.12}$$

Let us consider the integral

$$I_e = \int_0^\infty \frac{b^2(x,t;\Omega_0,s)}{s^2(1-\Omega_0\kappa)}\,dx.$$

Due to (A.12) we get

$$\begin{aligned}I_e \le \frac{s^2\Omega_0^2 m^2}{4(1-\Omega_0\kappa)}\int_0^\infty \Bigg\{ &\frac{1}{s\sqrt{1-\Omega_0\kappa}} \\ &\cdot \int_0^x (x-\xi)\exp\left[-s\sqrt{1-\Omega_0\kappa_1}(x-\xi)\right]|A(\xi,t)|\,d\xi\Bigg\}^2 dx.\end{aligned} \tag{A.13}$$

Hence, by making estimates similar to those for the integral I_b we obtain

$$I_e \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int_0^\infty \frac{|A(x,t)^2|}{s^2(1-\Omega_0\kappa)}\,dx, \tag{A.14}$$

and

$$\text{(A.15)}\qquad \|b(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} = \sup_{t\in[0,\infty)} I_e \leq \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{-1/2}}.$$

Applying a similar procedure to that used for integral I_a, we get

$$\text{(A.16)}\qquad I_f = \int\limits_0^\infty \frac{a^2(x,t;\Omega_0,s)}{s^2(1-\Omega_0\kappa(x))}dx \leq \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int\limits_0^\infty \frac{|A(x,t)^2|}{s^2(1-\Omega_0\kappa(x))}dx.$$

Since

$$\text{(A.17)}\qquad NA(x,t) = a(x,t;\Omega_0,s) + b(x,t;\Omega_0,s)$$

and

$$\text{(A.18)}\qquad \begin{aligned} \|a(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} &= \sup_{t\in[0,\infty)} I_e \leq \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{(-1/2)}},\\ \|b(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} &= \sup_{t\in[0,\infty)} I_e \leq \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{(-1/2)}},\end{aligned}$$

the operator N is a contraction in $X_1^{(-1/2)}$ if the following condition is fulfilled:

$$\text{(A.19)}\qquad q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1.$$

Appendix II

$$\begin{aligned} P_0(\widetilde{\omega}) &= 2\widetilde{\omega}^2(1-\widetilde{\omega}) + \frac{1}{8}(2-\widetilde{\omega})^4\widetilde{\omega},\\ P_1(\widetilde{\omega}) &= \frac{1}{2}\widetilde{\omega}^2(1-\widetilde{\omega})(2-\widetilde{\omega})^2 + \frac{1}{2}(2-\widetilde{\omega})^4,\\ P_2(\widetilde{\omega}) &= -[4\widetilde{\omega}^2(1-\widetilde{\omega})(2-\widetilde{\omega}) + (2-\widetilde{\omega})^4\widetilde{\omega}],\end{aligned}$$

$$
\begin{aligned}
P_3(\tilde{\omega},\kappa_1) &= 8-4\tilde{\omega}+\tilde{\omega}^2-4\tilde{\omega}\kappa_1,\\
P_4(\tilde{\omega},\kappa_1) &= -32+8\tilde{\omega}+24\tilde{\omega}\kappa_1-4\tilde{\omega}^2\kappa_1,\\
P_5(\tilde{\omega},\kappa_1) &= 8+\kappa_1\tilde{\omega}^2-8\tilde{\omega}\kappa_1,
\end{aligned}
$$

$$
\begin{aligned}
k_1^2 &= \tilde{\omega}^{-1}[4s^2(1-\tilde{\omega})(1-\kappa_1)(\tilde{\omega}-\tilde{\omega}\kappa_1-1)],\\
k_2^2 &= s^2(1-\tilde{\omega})(1-2\kappa_1)\tilde{\omega}[2-2\tilde{\omega}-2\kappa_1+\tilde{\omega}\kappa_1]^{-1},\\
l_1 &= [2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega}\kappa_1)^{1/2}]^{-1},\\
l_2 &= -[2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega})^{1/2}]^{-1},\\
l_3 &= [2(1-\tilde{\omega}\kappa_1)+\tilde{\omega}][2s^3\tilde{\omega}(1-\kappa_1)(\tilde{\omega}-2)(1-\tilde{\omega}\kappa_1)^{1/2}]^{-1},\\
l_4 &= [2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega})^{1/2}]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
l_5 &= [-2(1-\tilde{\omega}\kappa_1)(1-\kappa_1)\tilde{\omega}+(1-2\kappa_1)(\tilde{\omega}-2)\tilde{\omega}-4(\tilde{\omega}-2)-\tilde{\omega}^2(1-\kappa_1)]\\
&\qquad \times[8s^3(1-\kappa_1)^2(\tilde{\omega}-1)(\tilde{\omega}-2)\tilde{\omega}]^{-1},\\
l_6 &= [8s^3(1-\kappa_1)^2(\tilde{\omega}-1)(\tilde{\omega}-2)\tilde{\omega}]^{-1}[-2(\tilde{\omega}-1)\tilde{\omega}-(1-2\kappa_1)(\tilde{\omega}-2)\tilde{\omega}\\
&\qquad +4(\tilde{\omega}-2)+\tilde{\omega}^2(1-\kappa_1)],\\
l_7 &= (1-\tilde{\omega}\kappa_1)^{1/2}[4(1-\kappa_1)(\tilde{\omega}-1)(\tilde{\omega}-2)]^{-1}[2\tilde{\omega}(1-\tilde{\omega}\kappa_1)-\tilde{\omega}(1-2\kappa_1)(\tilde{\omega}-2)\\
&\qquad +4(1-\kappa_1)(\tilde{\omega}-2)+\tilde{\omega}^2],\\
l_8 &= (1-\tilde{\omega}\kappa_1)^{1/2}[16(1-\kappa_1)(\tilde{\omega}-1)(\tilde{\omega}-2)(1-\tilde{\omega}\kappa_1)]^{-1}\\
&\qquad \times[8\tilde{\omega}(1-\tilde{\omega}\kappa_1)^2-\tilde{\omega}(1-2\kappa_1)(\tilde{\omega}-2)^3+4(1-\kappa_1)(\tilde{\omega}-2)^3+\tilde{\omega}^2(\tilde{\omega}-2)^2],\\
a_1 &= [8s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}]^{-1}[2(1-\tilde{\omega}\kappa_1)\tilde{\omega}-(\tilde{\omega}-2)(1-2\kappa_1)\tilde{\omega}\\
&\qquad +2(\tilde{\omega}-2)(\tilde{\omega}+6)(1-\kappa_1)],\\
a_2 &= [16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}(\tilde{\omega}-2)^2]^{-1}[-4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-2)^2\\
&\qquad +2(\tilde{\omega}-2)^3(1-2\kappa_1)\tilde{\omega}+16(1-\tilde{\omega}\kappa_1)(\tilde{\omega}-1)(1-\kappa_1)\\
&\qquad -8(\tilde{\omega}-2)^3(1-\kappa_1)-\tilde{\omega}^2(\tilde{\omega}-2)(1-\kappa_1)],\\
b_1 &= -[16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}][4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-1)\\
&\qquad -2(\tilde{\omega}-2)(1-2\kappa_1)\tilde{\omega}+8(\tilde{\omega}-2)(1-\kappa_1)+\tilde{\omega}^2)(1-\kappa_1)+4(\tilde{\omega}-1)(1-\tilde{\omega}\kappa_1)],\\
b_2 &= [16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}(\tilde{\omega}-2)^2]^{-1}\times[4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-2)^2\\
&\qquad -2(\tilde{\omega}-2)^3(1-2\kappa_1)\tilde{\omega}+8(\tilde{\omega}-2)^3(1-\kappa_1)\tilde{\omega}+\tilde{\omega}^2(\tilde{\omega}-2)^2(1-\kappa_1)\\
&\qquad +16(1-\tilde{\omega}\kappa_1)(\tilde{\omega}-1)(1-\kappa_1)].
\end{aligned}
$$

Acknowledgement

I gratefully thank Professor J. IGNACZAK of the Polish Academy of Sciences for his suggestions during the preparation of this paper.

References

1. J. IGNACZAK, *Rayleigh waves in a nonhomogeneous isotropic elastic semi-space*, Arch. Mech. Stos., **15**, 341–345, 1963.
2. A.G. KOSTUČENKO and I.S. SARGSJAN, *Distribution of eigenvalues* [in Russian], Nauka, Moskva, 123–149, 1979.
3. T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin - Heidelberg - New York 1966.
4. T. KLECHA, *Existence of surface waves in nonhomogeneous isotropic elastic semi-space with arbitrary variation of Poisson's ratio*, Bull. Acad. Polon. Sci., Série Sci. Tech., **25**, 347–351, 1977.
5. J. IGNACZAK, Arch. Mech. Stos., **23**, 789–800, 1971.
6. K. FRIEDRICHS, *Perturbation spectra in Hilbert space*, American Mathematical Society, Providence, Rhode Island 1965.
7. T. ROŻNOWSKI, Bull. Ac. Pol. Sci., Serie Sci. Tech., **25**, 67–77, 1977.
8. G. STRANG, *Linear algebra and its applications*, Academic Press, New York 1976 [in Russian: Nauka, Moskva 1980].

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Received August 18, 1995.

Arch. Mech., 48, 3, pp. 541–550, Warszawa 1996

Symmetrization of a heat conduction model for a rigid medium

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THE SYMMETRIZATION of the equations of a heat conduction model for a rigid medium in time and three space dimensions is performed. The general symmetrizability condition is formulated in terms of the entropy function. Examples of particular models (e.g. Debye's model) are discussed.

1. Introduction

MOST OF THE KNOWN DYNAMIC (non-equilibrium) problems in nonlinear continuum mechanics and thermodynamics lead to quasi-linear hyperbolic systems of partial differential equations. The problem of well-posedness, i.e. existence, uniqueness and continuous dependence (stability) of a solution on the initial data, is fundamental for any system of equations. It is well known [1, 2] that Cauchy's initial-value problem for symmetric hyperbolic quasi-linear system is locally well-posed in the Sobolev space H^s, with $s \geq n + 1$, where n is a number of space variables. The quasi-linear systems of continuum mechanics usually are not formulated in symmetric forms. To make use of the above well-posedness result, it is desirable to transform such systems into symmetric forms, by the appropriate change of the unknown variables.

The aim of this paper is to symmetrize the equations describing a non-equilibrium heat conduction problem in a rigid conductor governed by a modified Fourier law. The system of equations is of the second order in the scalar variable β, called internal state variable (or a semi-empirical temperature), and of the first order in the absolute temperature θ. In the general 3D case, this system can be transformed into the first order system in five unknowns. We symmetrize this system with the help of entropy function using some results of FRIEDRICHS, BOILLAT, RUGGERI and STRUMIA [3–5]. Instead of deriving the exact form of the entropy function from thermodynamics, we postulate the family of suitably chosen entropy-like functions that are then used to get the new dependent variables (the main fields).

In order to pick up the entropy from our family of postulated functions we formulate a general symmetrizability condition. It turns out that this condition is in fact the model compatibility condition which, on the other hand, can be obtained from the second law of thermodynamics. This symmetrizability condition can be easily fulfilled not only in the Debye's model, which we analyze in details, but also under some more general assumptions.

2. Model with semi-empirical temperature

Recently in a series of papers [6–9] a thermodynamic, phenomenological theory of heat conduction with finite wave speed has been developed and applied to thermal wave propagation problems, mostly 1D. The theory is based on the concept of a gradient generalization of the internal state variable approach, in which the gradient of a scalar internal state variable β (called a semi-empirical temperature) influences the response of the material at hand. The quantity β cannot be measured directly. Here it is considered as a potential, with the analogy to the classical heat conduction Fourier law. In the new model the heat flux is proportional to the gradient of β, instead of to the gradient of the classical absolute temperature θ.

In the model considered (cf. [7, 9]) we assume that the evolution of β is governed by the following equation:

$$\frac{\partial \beta}{\partial t} = f(\theta, \beta) = f_1^*(\theta) + f_2(\beta) \tag{2.1}$$

(with f_1^*, f_2 being real functions such that $df_2/d\beta < 0$), while the energy balance law reads:

$$\frac{\partial \varrho\varepsilon}{\partial t} + \operatorname{div} \mathbf{q}^* = \varrho\, r, \tag{2.2}$$

where (¹) ϱ is the mass density, ε – the specific internal energy, r – the body heat supply, and $\mathbf{q}^*$ is the heat flux vector. We also assume that the second law of thermodynamics

$$\frac{\partial \varrho\eta^*}{\partial t} + \operatorname{div} \frac{\mathbf{q}^*}{\theta} \geq \frac{\varrho\, r}{\theta} \tag{2.3}$$

is satisfied, with η^* being an entropy. Moreover, in our model we make the two additional simplifying assumptions:

(A.1) $\mathbf{q}^*$ depends linearily on $\nabla\beta$,

(A.2) ε is a function of θ only.

From the second law of thermodynamics (2.3), under the assumption (A.1) we can express the heat flux as:

$$\mathbf{q}^* = -\alpha^*(\theta)\, \nabla\beta, \tag{2.4}$$

where α^* is a positive function of dimension of the thermal conductivity coefficient. Also from (2.3) and from the assumptions (A.1), (A.2) we can derive the

(¹) Throughout this paper we use dimensionless variables. However, the following units have been assumed: temperature (θ and β) in K, length in cm, time in μs, speed in cm/μs, energy in J.

following form of the entropy function:

$$\eta^*(\theta, \nabla\beta) := \eta^*_e(\theta) - \frac{1}{2}c\,|\nabla\beta|^2, \tag{2.5}$$

with c being a positive constant.

3. Basic equations in a quasi-linear form

In order to express the system (2.1), (2.2), (2.4) in the conservative form we introduce the following vector of new dependent variables $\mathbf{u}$:

$$\mathbf{u}(x,t) = [e, \mathbf{q}, \beta], \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \quad \mathbf{q} = [q_1, q_2, q_3],$$

where $e = \varrho\varepsilon$ is internal energy and $\mathbf{q} = -\nabla\beta$ is the rescaled heat flux vector (cf. (2.4)). Moreover, we introduce the flux matrix $\mathbf{F}(\mathbf{u})$ and the vector of external influences $\mathbf{b}(\mathbf{u})$ as:

$$\mathbf{F}(\mathbf{u}) = \begin{bmatrix} \alpha(e)\,\mathbf{q} \\ f_1(e)\,\mathbf{I}_3 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{b}(\mathbf{u}) = \left[\varrho\, r, \frac{df_2}{d\beta}\mathbf{q}, f_1(e) + f_2(\beta)\right],$$

where $\mathbf{I}_3$ is the 3×3 identity matrix, α is a positive function of dimension of the thermal conductivity coefficient, and the function f_1 is f_1^* from (2.1) expressed as a function of e. In what follows we denote:

$$\mathbf{div\,A} = \nabla\mathbf{A}^T$$

with $\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right]$ and $\mathbf{A}$ being an arbitrary 3-column matrix. Now, after some calculation, we can describe the process (2.1), (2.2), (2.4) of the heat conduction in a rigid medium in the form of the following first order system of balance laws:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{div\,F}(\mathbf{u}) = \mathbf{b}(\mathbf{u}). \tag{3.1}$$

The quasi-linear form of this system is:

$$\frac{\partial\mathbf{u}}{\partial t} + \sum_{i=1}^{3} \mathbf{A}_i(\mathbf{u})\frac{\partial\mathbf{u}}{\partial x_i} = \mathbf{b}(\mathbf{u}) \tag{3.2}$$

with:

$$\mathbf{A}_i(\mathbf{u}) = \begin{bmatrix} \frac{d\alpha}{de}q_i & \alpha(e)\,\boldsymbol{\xi}_i \\ \frac{df_1}{de}\boldsymbol{\xi}_i^T & \mathbf{0}_4 \end{bmatrix}, \qquad i = 1, 2, 3,$$

where $\mathbf{0}_4$ is the 4×4 null matrix and $\boldsymbol{\xi}_1 = [1, 0, 0, 0]$, $\boldsymbol{\xi}_2 = [0, 1, 0, 0]$, $\boldsymbol{\xi}_3 = [0, 0, 1, 0]$.

4. New dependent variables

In order to symmetrize our quasi-linear system (3.2) we make use of a well known fact [10, 3, 5] that a system of hyperbolic first order balance laws can be symmetrized, provided that it is equipped with a convex entropy function satisfying supplementary conservation law. More precisely, such a system of balance laws becomes symmetric in the Friedrichs's sense when one takes the gradient components of the entropy function η as the new dependent variables (main fields) $\mathbf{v}$:

$$\mathbf{v} = \mathrm{grad}_{\mathbf{u}}\, \eta.$$

In the case of our system (3.1), having in mind the formula (2.5), we take as the candidate for the entropy η the family of functions that can be expressed in the following form:

$$\eta(e, \mathbf{q}) := \eta_e(e) + \frac{1}{2} c_1\, \mathbf{q} \cdot \mathbf{q}, \tag{4.1}$$

where $c_1 > 0$ and η_e is the so-called equilibrium entropy that will be detailed in the next section. Consequently, we obtain:

$$\mathrm{grad}_{\mathbf{u}}\, \eta = \left[\frac{d\eta_e}{de},\, c_1 q_1,\, c_1 q_2,\, c_1 q_3,\, 0\right].$$

Since semi-empirical temperature β is not involved in the divergence term in the quasi-linear system (3.2), we are free to put an arbitrary function as v_5 (e.g.: $v_5 = c_2 \beta$ with $c_2 = \mathrm{const}$). Hence, our main fields $\mathbf{v}$ are:

$$\mathbf{v} = [v_1, \ldots, v_5] = \left[\frac{d\eta_e}{de},\, c_1 q_1,\, c_1 q_2,\, c_1 q_3,\, c_2 \beta\right]. \tag{4.2}$$

Using the main fields (4.2) we obtain the symmetrizing matrix $\mathbf{H}$ for our quasi-linear system (3.2) in the form:

$$\mathbf{H} = \mathbf{grad}_{\mathbf{u}}\, \mathbf{v} = \mathrm{diag}\left[\frac{d^2\eta_e}{de^2},\, c_1,\, c_1,\, c_1,\, c_2\right], \tag{4.3}$$

where $\mathbf{grad}_{\mathbf{u}}\, \mathbf{v} = [\mathrm{grad}_{\mathbf{u}} v_1, \mathrm{grad}_{\mathbf{u}} v_2, \ldots, \mathrm{grad}_{\mathbf{u}} v_5]^T$ and diag$[\cdot]$ denotes a diagonal matrix with the diagonal $[\cdot]$. We can choose an appropriate sign of the constant c_2 to make our symmetrizing matrix $\mathbf{H}$ positive definite.

5. The symmetrizability condition

The matrix $\mathbf{H}$ of the form (4.3) symmetrizes our quasi-linear system (3.2) if and only if, by the definition, the following matrices $\mathbf{B}_i$, $i = 1, 2, 3$ are symmetric:

$$\mathbf{B}_i = \mathbf{H}\cdot\mathbf{A}_i = \begin{bmatrix} q_i \dfrac{d\alpha}{de}\dfrac{d^2\eta_e}{de^2} & \alpha \dfrac{d^2\eta_e}{de^2}\boldsymbol{\xi}_i \\ c_1 \dfrac{df_1}{de}\boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1, 2, 3. \tag{5.1}$$

Since the equalities of the corresponding off-diagonal elements of the matrices $\mathbf{B}_i$ do not depend on i, the condition (5.1) is reduced in fact to a single, general symmetrizability condition in the form:

$$c_1 = \alpha(e)\,\frac{d^2\eta_e}{de^2}\Big/\frac{df_1}{de}\,. \tag{5.2}$$

We remind that c_1 is a constant appearing in our family of functions (4.1). It can be shown that c_1 evaluated from (5.2) coincides with the constant c from (2.5) which, on the other hand, is evaluated on the basis of the thermodynamical considerations. It is also worth mentioning that under our assumptions the equilibrium entropy η_e is a convex function of e, provided that $df_1/de > 0$.

6. Specification of the equilibrium entropy

Under our assumptions the equilibrium entropy η_e^*, as a function of the classical temperature θ, is the derivative of the Helmholtz free energy ψ_1:

$$\eta_e^*(\theta) := -\frac{d\psi_1}{d\theta}\,, \tag{6.1}$$

where ψ_1 satisfies the following ordinary differential equation:

$$-\theta\,\frac{d\psi_1}{d\theta} + \psi_1 = \frac{1}{\varrho}\,\widehat{e}(\theta)$$

with $\widehat{e}$ being e as a function (²) of θ. Hence, ψ_1 takes the form:

$$\psi_1(\theta) = c_0\,\theta - \frac{\theta}{\varrho}\int_0^\theta \frac{\widehat{e}(s)}{s^2}\,ds, \qquad c_0 = \text{const.}$$

Substituting the solution ψ_1 into our postulate (6.1) we obtain the the equilibrium entropy as the following function of θ:

$$\eta_e^*(\theta) = \frac{\widehat{e}(\theta)}{\varrho\,\theta} + \frac{1}{\varrho}\int_0^\theta \frac{\widehat{e}(s)}{s^2}\,ds - c_0. \tag{6.2}$$

(²) In order to distinguish between a variable (e.g. e) and the same variable treated as a function of another variable (e.g. e as a function of θ), introduce the symbol $\widehat{\ }$ to denote the function (e.g. $\widehat{e}(\theta)$).

All that we need now is to express the equilibrium entropy as the function of the internal energy e only. To this end we introduce the specific heat c_v that relates, by the definition, e to θ in the following way:

$$c_v = \frac{1}{\varrho}\frac{d\widehat{e}(\theta)}{d\theta}. \tag{6.3}$$

Hence, e as a function of θ reads:

$$\widehat{e}(\theta) = \varrho \int c_v(\theta)\, d\theta.$$

Under the assumption that the specific heat c_v is a positive function of θ, so that $\widehat{e}(\theta)$ is monotonic, the inverse function

$$\widehat{\theta} : e \to \theta, \qquad \widehat{\theta}(e) = \widehat{e}^{-1}(e)$$

exists and the equilibrium entropy η_e as the function of the internal energy e takes the following form (cf. (6.2)):

$$\eta_e(e) = \eta_e^*(\widehat{\theta}(e)) = \frac{e}{\varrho\,\widehat{\theta}(e)} + \frac{1}{\varrho}\int_0^{\widehat{\theta}(e)} \frac{\widehat{e}(s)}{s^2}\, ds - c_0\,. \tag{6.4}$$

In terms of such $\eta_e(e)$, our general symmetrizability condition (5.2) takes the form:

$$c_1 = -\frac{\alpha(e)\dfrac{d\widehat{\theta}(e)}{de}}{\varrho\,(\widehat{\theta}(e))^2\dfrac{df_1}{de}}, \tag{6.5}$$

and the symmetrized matrices $\mathbf{B}_i$ are:

$$\mathbf{B}_i = -\frac{1}{\varrho\,\widehat{\theta}^2}\frac{d\widehat{\theta}}{de}\begin{bmatrix} q_i\dfrac{d\alpha}{de} & \alpha\,\boldsymbol{\xi}_i \\ \alpha\,\boldsymbol{\xi}_i^T & \mathbf{0}\end{bmatrix}, \qquad i = 1,2,3. \tag{6.6}$$

7. Specification of f_1 for various $\alpha(e)$

We may reformulate the symmetrizability condition (6.5) to obtain, after integration, the general form of the function f_1 such that it allows the symmetrization by our method. The function f_1 in this form is expressed in terms of $\alpha(e)$ and the constant c_1:

$$f_1(e) = \frac{\alpha(e)}{c_1\,\varrho\,\widehat{\theta}(e)} - \frac{1}{c_1\varrho}\int \frac{1}{\widehat{\theta}(e)}\frac{d\alpha(e)}{de}\, de\,. \tag{7.1}$$

Now we specify f_1 for two different functions $\alpha(e)$:

CASE 1

$$\alpha(e) = \alpha_0 \,(\widehat{\theta}(e))^2, \qquad \alpha_0 > 0. \tag{7.2}$$

Then the function f_1 has the form:

$$f_1(e) = -\frac{\alpha_0 \,\widehat{\theta}(e)}{c_1 \varrho}.$$

CASE 2

$$\alpha(e) = -\alpha_0 \,(\widehat{\theta}(e) - \theta_1)\,(\widehat{\theta}(e) - \theta_2), \qquad \alpha_0 > 0, \quad \theta_1\theta_2 < 0. \tag{7.3}$$

Then the function f_1 is the following:

$$f_1(e) = \frac{\alpha_0 \,\{\widehat{\theta}(e) - (\theta_1 + \theta_2) \ln(\widehat{\theta}(e))\}}{c_1 \varrho} - \frac{\alpha_0 \,\theta_1\theta_2}{c_1 \varrho \,\widehat{\theta}(e)}.$$

8. The example: Debye's model

8.1. Arbitrary $\alpha(e)$

Our general symmetrization formulas can be further specified if the explicit form of the θ-dependence of the specific heat c_v is assumed. For example, in Debye's model with

$$c_v = 4\, c_{v0}\, \theta^3, \qquad c_{v0} > 0, \tag{8.1}$$

the inverse function $\widehat{\theta}$ becomes:

$$\widehat{\theta}(e) = \left(\frac{e}{c_{v0}\, \varrho}\right)^{1/4}, \tag{8.2}$$

the symmetrizability condition (5.2), (6.5) reads:

$$c_1 = -\frac{\alpha(e)}{4 \,\dfrac{df_1}{de}} \left(\frac{c_{v0}}{e^5 \varrho^3}\right)^{1/4} \tag{8.3}$$

and the equilibrium entropy $\eta_e(e)$ takes the following form (cf. (6.4)):

$$\eta_e(e) = \left(\frac{4\, c_{v0}\, e^3}{3\, \varrho^3}\right)^{1/4} - (c_0 + c_{v0}/3). \tag{8.4}$$

8.2. Specified $\alpha(e)$

Let us recall that α is a positive function of dimension of the thermal conductivity coefficient. Now we specify the symmetrizability condition, symmetrized matrices $\mathbf{B}_i$, $i = 1,2,3$, and the function f_1 for two different $\alpha(e)$ taken from Sec. 7.

CASE 1 (cf. (7.2))

$$\alpha(e) = \alpha_0 (\widehat{\theta}(e))^2 = \alpha_0 \sqrt{\frac{e}{\varrho\, c_{v0}}}, \qquad \alpha_0 > 0. \tag{8.5}$$

Then the symmetrizability condition reads (cf. (5.2), (6.5), (8.3)):

$$c_1 = -\frac{\alpha_0}{4\,\dfrac{df_1}{de}} \left(\frac{1}{c_{v0}\, e^3\, \varrho^5}\right)^{1/4}, \tag{8.6}$$

the symmetrized matrices $\mathbf{B}_i$ are (cf. (6.6)):

$$\mathbf{B}_i = -\frac{\alpha_0}{8\,(c_{v0}\,\varrho^5)^{1/4}} \begin{bmatrix} q_i\, e^{-4/7} & 2\, e^{-4/3} \boldsymbol{\xi}_i \\ 2\, e^{-4/3} \boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1,2,3, \tag{8.7}$$

and the function f_1 has the form (cf. (7.1)):

$$\begin{aligned} f_1(e) &= -\alpha_0 \left(\frac{e}{c_{v0}\, c_1\, \varrho^5}\right)^{1/4}, \\ f_1^*(\theta) &= -\frac{\alpha_0\, \theta}{c_1\, \varrho}. \end{aligned} \tag{8.8}$$

CASE 2 (cf. (7.3), (8.2))

$$\alpha(e) = -\alpha_0 \left(\left(\frac{e}{c_{v0}\,\varrho}\right)^{1/4} - \theta_1\right)\left(\left(\frac{e}{c_{v0}\,\varrho}\right)^{1/4} - \theta_2\right), \quad \alpha_0 > 0, \quad \theta_1\theta_2 < 0. \tag{8.9}$$

Then the symmetrizability condition reads (cf. (5.2), (6.5), (8.3)):

$$c_1 = \frac{\alpha_0}{4\,\dfrac{df_1}{de}} \frac{(\theta_1 (c_{v0}\,\varrho)^{1/4} - e^{1/4})\,(\theta_2 (c_{v0}\,\varrho)^{1/4} - e^{1/4})}{(c_{v0}\, e^5\, \varrho^5)^{1/4}}, \tag{8.10}$$

the symmetrized matrices $\mathbf{B}_i$ are (cf. (6.6)):

$$\mathbf{B}_i = \frac{\alpha_0}{16\,(c_{v0}\,\varrho^5)^{1/4}} \begin{bmatrix} b_{11} & b_{12} \boldsymbol{\xi}_i \\ b_{12} \boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1,2,3, \tag{8.11}$$

where

$$b_{11} = -q_i\, e^{-2}\,\{(c_{v0}\,\varrho)^{1/4}(\theta_1+\theta_2) - 2e^{1/4}\},$$
$$b_{12} = 4e^{-5/4}(\theta_1(c_{v0}\,\varrho)^{1/4} - e^{1/4})\,(\theta_2(c_{v0}\,\varrho)^{1/4} - e^{1/4}),$$

and the function f_1 has the form (cf. (7.1)):

$$\begin{aligned} f_1(e) &= \frac{\alpha_0}{c_1}\left\{\left(\frac{e}{(c_{v0}\,\varrho^5)}\right)^{1/4} - \theta_1\,\theta_2\left(\frac{c_{v0}}{e\,\varrho^3}\right)^{1/4} - \frac{(\theta_1+\theta_2)\ln(e^{1/4})}{\varrho}\right\}, \\ f_1^*(\theta) &= \frac{\alpha_0}{\varrho\, c_1}\left\{\theta - \frac{\theta_1\,\theta_2}{\theta} - (\theta_1+\theta_2)\ln(\theta\sqrt[4]{\varrho\, c_{v0}})\right\}. \end{aligned} \tag{8.12}$$

9. Conclusions

The equations of a heat conduction model for a rigid medium in time and three space dimensions are analyzed. Using the internal energy, the heat flux vector and the semi-empirical temperature as the dependent variables, we formulate the conservative, and the quasi-linear hyperbolic forms of these equations.

We successfully symmetrize our quasi-linear system by introducing the family of suitably chosen entropy-like functions that are then used to obtain the new dependent variables, and by formulating additionally a general symmetrizability condition that allows us to specify the physically justified entropy function.

It turns out that this symmetrizability condition is in fact the model compatibility condition which, on the other hand, can be obtained from the second law of thermodynamics.

We illustrate our approach on a detailed example of the Debye's model with specified different forms of the thermal conductivity coefficients.

Our approach is effective when the classical temperature is an invertible function of the internal energy. Then we can always symmetrize our system of equations and the symmetrizing matrix is diagonal.

References

1. T. KATO, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rat. Mech. Anal., **58**, 181–205, 1975.
2. T.J.R. HUGHES, T. KATO and J. MARSDEN, *Well-posed quasi-linear second order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rat. Mech. Anal., **63**, 273–295, 1976.
3. K.O. FRIEDRICHS, *On the laws of relativistic electro-magneto-fluid dynamics*, Comm. Pure Appl. Math., **27**, 749–808, 1974.
4. G. BOILLAT, *Sur l'éxistence et la recherche d'équations de conservation supplementaire pour les systèmes hyperboliques*, C. R. Acad. Sci. Paris, **278 A**, 909–914, 1974.
5. T. RUGGERI and A. STRUMIA, *Main field and convex covariant density for quasi-linear hyperbolic systems. Relativistic fluid dynamics*, Ann. Inst. H. Poincaré, **34**,**A**, 65–84, 1981.
6. W. KOSIŃSKI, *Elastic waves in the presence of a new temperature scale*, [in:] Elastic Wave Propagation, M.F. MCCARTHY and M. HAYES [Eds.], Elsevier Science (North Holland), pp. 629–634, 1989.

7. V.A. Cimmelli and W. Kosiński, *Nonequilibrium semi-empirical temperature in materials with thermal relaxation*, Arch. Mech., **43**, 6, pp. 753–767, 1991.

8. V.A. Cimmelli, W. Kosiński and K. Saxton, *A new approach to the theory of heat conduction with finite wave speeds*, Proc. VI Intern. Conference on Waves and Stability in Continuous Media, Acireale, May 27 – June 1, 1991, S. Rionero, G. Mulone and F. Salemi [Eds.], Le Matematiche, **XLVI**, 1, 95–105, 1991.

9. V.A. Cimmelli, W. Kosiński and K. Saxton, *Modified Fourier law – comparison of two approaches*, Arch. Mech., **44**, 4, 409–415, 1992.

10. C.K. Godunov, *An interesting class of quasi-linear systems*, Sov. Math., **2**, pp. 947–949, 1961.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received September 25, 1995.

Arch. Mech., 48, 3, pp. 551–576, Warszawa 1996

On the extension of Newton's second law to theories of gravitation in curved space-time

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WE INVESTIGATE the possibility of extending Newton's second law to the general framework of theories in which special relativity is locally valid, and in which gravitation changes the flat Galilean space-time metric into a curved metric. This framework is first recalled, underlining the possibility to define uniquely a space metric and a local time in any given reference frame, hence to define velocity and momentum in terms of the local space and time standards. It is shown that a unique consistent definition can be given for the derivative of a vector (the momentum) along a trajectory. Then the possible form of the gravitation force is investigated. It is shown that, if the motion of free particles has to follow space-time geodesics, then the expression for the gravity acceleration is determined uniquely. It depends on the variation of the metric with space and time, and it involves the velocity of the particle.

1. Introduction

THIS WORK COMES from an attempt to explore the possibility of extending the "logic of absolute motion", which prevails in the Lorentz-Poincaré interpretation of special relativity [8–9, 15, 20–24], so as to obtain a consistent theory of gravitation. Thus, a theory with a preferred frame has been tentatively proposed [1–4]. Just like general relativity (GR), this theory endows the space-time with a curved metric. Just like in GR, special relativity (SR) holds true locally in this tentative theory. However, an extension of Newton's second law, or rather of its modified expression valid in SR, has been defined for a test particle (mass point or photon) in the most general situation within this investigated theory [4]. As it will be reported here, the way used in this theory to define Newton's second law in a "curved space-time" turns out to be both natural and general in its principle. Hence, it has been tried to find in the literature such a natural and general extension, but this quest has not been really successful. Apart from approximate equations occurring in "post-Newtonian" treatments, two exact extensions of Newton's second law to relativistic theories of gravitation can be found among well-known textbooks: LANDAU and LIFCHITZ [11, § 88] define this law for a *constant* gravitation field, and MØLLER [18, § 110] "tries to write [the equations of space-time geodesics] in the form of three-dimensional vector equations" in a general case but, as his sentence suggests, and as will be discussed below (note 1 and Sec. 4), his attempt is not fully satisfactory. JANTZEN *et al.* [10] review and unify the various attempts, including the important work of CATTANEO [6–7], to "split space-time into space plus time" and to rewrite the relativistic equations of motion with "spatial gravitational forces". It appears from their review that three different definitions have been introduced, by various authors, for the time-derivative of the momentum. These definitions will be examined in Sec. 4. It will appear that one does not obey Leibniz' rule, while none of the other two does involve *only* the *separate* ingredients "space metric" and "time metric" in a given reference frame, as should be true for a natural extension of Newton's second law. However, it seems that one has good reasons to search for such extension and hence to find this "missing link" [17] between classical and relativistic mechanics.

Indeed, the Lorentz-Poincaré construction of special relativity [15, 20–21], fully developed by JÁNOSSY [8–9] and PROKHOVNIK [22–24], obtains the "relativistic" effects as being all consequences of the "true" Lorentz contraction assumed to affect all bodies in motion with respect to the "ether". As it has been recently reestablished [27] against contrary statements, it is impossible to measure consistently the anisotropy in the *one-way* velocity of light. This makes the Lorentz-Poincaré version empirically undistinguishable from the Einstein version of SR [22]. The Lorentz-Poincaré interpretation allows to concile special relativity with our intuitive notion of distinct space and time, and thus with the most crucial concepts of classical mechanics. However, special relativity does not describe gravitation: for gravitation, general relativity is the current tool. But in GR, the laws of motion become a consequence of the space-time curvature, e.g. the "free" particles are assumed to follow the geodesic lines of the space-time metric. Thus, at least as long as the geodesic formulation of motion has not been derived from a generalization of Newton's second law, one is enforced to give a physical status to space-time in GR. On the other hand, despite the experimental success of GR, it leaves unsolved problems as regards gravitation. We may mention the problem of the singularity occurring with the gravitational collapse of very massive objects, and the need to postulate huge amounts of "dark matter" in order to explain stellar motion in galaxies. We should also mention the questions on the influence of the coordinate condition in GR, which were raised a long time ago (e.g. PAPAPETROU [19]), but that have been newly discussed by LOGUNOV *et al.* [13–14]. LOGUNOV *et al.* present detailed arguments against the usual agreement that, in GR, the choice of the coordinate condition has no physical consequence. It thus may be worth to investigate alternative, speculative theories and to ask questions on the formulation of motion.

In this paper, an extension of Newton's second law will be given for theories of gravitation in curved space-time in which SR is locally valid, *including* GR. In doing so, care will be taken to maintain *space* covariance in a given reference frame, in order that the force be properly defined. However, no attempt will be made to investigate the transformation of the force from one reference frame to another. Section 2 will be focused on the definition of the right-hand side of Newton's law, i.e. the time-derivative of the momentum: it will be shown that this may be defined from rather compelling principles, up to the same parameter λ as in the tentative theory [4], and which also must be $\lambda = 1/2$ if Leibniz' rule is to apply. In Sec. 3, it will be investigated which form of the gravitation force is compatible with Einstein motion (for "free" particles), i.e. the motion along space-time geodesics. In the first step, Leibniz' rule will not be imposed but it will be assumed, in analogy with the Newtonian theory, that the gravitation force depends linearly on the spatial derivatives of the metric and does not depend on its time-derivative. In the second step, Leibniz' rule will be assumed, but no restriction on the gravitation force will be imposed. In Sec. 4, the three anterior definitions of the time-derivative of a spatial vector, reviewed by Jantzen *et al.*, will be examined from the point of view of "consistency" (validity of Leibniz' rule), and "naturalness" (space plus time separation).

2. Definition of Newton's second law for a (pseudo-) Riemannian space-time metric

2.1. Some clarification on the kind of theories considered

We suppose that, according to some gravitation theory, the physical standards of space and time are influenced by a gravitation field, but that SR holds true locally (GR is the prototype of such gravitation theories, of course). It will be useful to recall in some detail what is meant by this, not the least because it will make clear that this framework does not preclude to consider a preferred-frame theory, nor does this framework imply that a fundamental physical meaning must be given to the mathematical concept of space-time. It will also give the way to separate the force into a gravitational force or rather a mass force, and a non-gravitational force.

i) According to a theory of this kind, our space and time measurements may be arranged so as to be described by a metric γ with (1,3) signature on a 4-dimensional, "space-time" manifold. This may be done as follows. Any possible *reference frame* $\mathcal{F}$, physically defined by a *spatial network of "observers"* (each one equipped with a ruler and a clock, all made in the same factory, say), allows one to define (in many ways, actually) an associated coordinate system (x^α) $(\alpha = 0, ..., 3)$, with x^0 the time coordinate and x^i $(i = 1, 2, 3)$ the space coordinates, so that *each observer has constant space coordinates*. Moreover, $t = x^0/c$ is the "formal date" assigned to an event occurring at a point specified by the space coordinates x^i (t has in general no immediate relation to real time-measurements made by the observer at this point). The observers in the same frame $\mathcal{F}$ are not necessarily at rest with each other, i.e. they may find that their mutual distances are not conserved (case of a deformable frame). The manifold structure of the space-time means simply that the same physical events will be given different space and time coordinates by different networks of observers, say (x^α) and (x'^α), and that the correspondence between (x^α) and (x'^α) is locally smooth (for smoothly deforming networks). So we have a space-time manifold M^4. The elements (points) of the spatial network cannot be identified with *points* in that manifold but with "world lines", thus with *lines* in space-time. Hence, from the point of view of "space-time", a reference frame is a 3-D differentiable manifold N whose each point is a (time-like) differentiable mapping from the real line onto the space-time M^4; moreover, N is diffeomorphic to any spatial section of M^4 (this is only the sketch of a rigorous definition; from the point of view of "space + time", a much simpler definition may be proposed [1]). Note that many new coordinate systems (x'^{α}) do *not* change the reference frame (network) specified by one system (x^{α}): the frame remains unaltered if and only if the change of the space coordinates does not depend on the time coordinate, i.e. $\partial x'^{i}/\partial x^0 = 0$. Up to this point, it seems that no physically restrictive assumption is involved (except, of course, for the fact that "classical" physics, not quantum physics, is envisaged here).

The assumption that SR applies locally is the one which allows to define a (1,3) space-time metric. This assumption means, in the first place, this: in any reference frame, the velocity of light, as measured on a to-and-fro path between infinitesimally distant positions, is always the same constant c. Under this condition, the link between physical space and time measurements and the metric γ may be described as in LANDAU and LIFCHITZ [11], it is based on using the Poincaré - Einstein synchronization convention for infinitesimally distant clocks. Thus the proper time along the trajectory of a mass point ("time-like" line in space-time), i.e. the time τ measured by a clock bound to the moving point, is directly given by metric γ:

$$ds^2 = c^2\, d\tau^2 = \gamma_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}. \tag{2.1}$$

Also, the distance dl between neighbouring observers (of a given frame $\mathcal{F}$, specified by a coordinate system), as they find by using their rulers, or by measuring the interval $d\tau$ of their proper time that it takes for a light signal to go forth and back, is expressed by a space metric tensor $\mathbf{h} = \mathbf{h}_{\mathcal{F}}$ (it depends on the frame $\mathcal{F}$):

$$dl^2 = (c\, d\tau/2)^2 = h_{ij}\, dx^i\, dx^j, \qquad h_{ij} = -\gamma_{ij} + (\gamma_{0i}\,\gamma_{0j}/\gamma_{00}). \tag{2.2}$$

Moreover, a synchronized local time $t_{\mathrm{x}}(\xi)$ may be defined along any open line in space-time (i.e. a piecewise differentiable and *one-to-one* mapping $\xi \to (x^{\alpha}(\xi))$ defined on a closed segment of the real line), such that its variation along the given trajectory is given by:

$$\frac{dt_{\mathrm{x}}}{d\xi} = \frac{\sqrt{\gamma_{00}}}{c}\left(\frac{dx^0}{d\xi} + \frac{\gamma_{0i}}{\gamma_{00}}\,\frac{dx^i}{d\xi}\right). \tag{2.3}$$

As emphasized by CATTANEO [6], the interval dt_{x} is invariant under any coordinate transformation that leaves the reference frame unchanged ("internal transformation") and has thus an objective physical meaning. If the γ_{0i} components $(i = 1, 2, 3)$ are identically equal to zero, the synchronization convention implies that events occurring at a given value of x^0 are simultaneous in the frame $\mathcal{F}$, independently of their spatial coordinates (this may be seen in Eq. (2.3)). Hence x^0 is a "universal time" in the frame $\mathcal{F}$. As a consequence, if one uses such coordinates (x^α), then the trajectory of any test particle may always be parametrized with the coordinate time t itself and, moreover, the local time has the simple expression

$$dt_{\mathbf{x}}/dt = \sqrt{\gamma_{00}} \equiv \beta. \tag{2.4}$$

The expression (2.4) of the local time has the immediate physical meaning of showing how clocks are affected by the gravitation field (usually they are slowed down, i.e. γ_{00} decreases towards the gravitational attraction). The property $\gamma_{0i} = 0$ holds true after any coordinate transformation of the form $x'^0 = \phi(x^0)$, $x'^i = \psi^i(x^1, x^2, x^3)$. Thus it is indeed a characteristic of a given frame $\mathcal{F}$. The restriction to space-independent transformation of time, $x'^0 = \phi(x^0)$, reflects simply the global synchronization. Using this time transformation, one may impose that the local time at a given point bound to the frame, $\mathbf{x}_0 = ({x_0}^i)$, coincides with the universal time (i.e. $\gamma_{00}(x^0, ({x_0}^i)) = 1 \ \ \forall x^0$), and then only a shift of x^0 is left free. The γ_{00} component is invariant under the remaining, purely spatial coordinate changes.

ii) The other assumption involved, in saying that SR applies locally, is that the laws of non-gravitational physics are "formally unaffected" by gravitation, in the following sense: in the absence of gravitation, any such law must (or should) be formulated in the frame of SR. Then, in the absence of gravitation, it may be expressed in a generally covariant form, in replacing the partial derivatives, valid in Galilean coordinates, by the covariant derivatives with respect to the *flat* space-time metric γ^0 (Galilean coordinates are the ones in which the flat metric γ^0 has the canonical diagonal form, ${g^0}_{\mu\nu} = \eta_{\mu\nu}$ with $(\eta_{\mu\nu}) \equiv \mathrm{diag}\,(1,\ -1,\ -1,\ -1)$). Now the *assumption* is that, in the presence of gravitation and hence (according to a theory of the class considered here) with a *curved* metric γ, *the expression of any such law is extended to this situation simply by substituting* γ *for* γ^0. This assumption is quite natural: physics must be described in terms of the local space and time standards which (cf. point (i)) are ruled by metric γ in the frame of SR. And at the local or rather at the infinitesimal scale, the presence or absence of curvature plays little or no rôle, i.e. any metric behaves (in many respects though not in all) as a flat metric in the infinitesimal. Some ambiguity may yet arise when trying to use this assumption, if differential expressions of order greater than one are involved: since Schwarz' theorem does not apply to covariant derivatives for a curved metric, different higher-order expressions may become identical for a flat metric and yet remain distinct for a curved one (e.g. WILL [26]). In a such case, a comparison with experiment may either decide between the possibilities, or show that they do not differ significantly. Such empirical procedure might lead, of course, to different choices for different gravitation theories, i.e. for different metrics γ in the same physical situation, and thus could create a bias when testing alternative theories.

2.2. Extended Newton law for a constant gravitation field

Let us first consider the static case, i.e. the case where a frame $\mathcal{F}$ exists, defined by a coordinate system (x^α), in which all components $\gamma_{\alpha\beta}$ of metric γ are independent of x^0, and moreover the γ_{0i} $(i = 1, 2, 3)$ components are zero. The first property holds true after any coordinate transformation of the form $x'^0 = ax^0 + \phi(x^1, x^2, x^3)$, $x'^i = \phi^i(x^1, x^2, x^3)$, thus in a different range for the time transformation than for the second property, discussed above. Then, the *right-hand side* of Newton's second law, valid for SR, i.e. $d\mathbf{P}/dt$ with $\mathbf{P}$ the momentum including the velocity-dependent mass, is easy to extend to any such theory of gravitation. The velocity $\mathbf{v}$ of a test particle (relative to the frame $\mathcal{F}$) is measured with the local time $t_\mathbf{x}$ of the momentarily coincident observer in the frame $\mathcal{F}$, and its modulus v is defined with the point-dependent (Riemannian) space metric h in the frame $\mathcal{F}$. Thus

$$(2.5) \qquad v^i \equiv dx^i/dt_\mathbf{x}, \qquad v \equiv [\mathsf{h}(\mathbf{v}, \mathbf{v})]^{1/2} = (h_{ij}\, v^i\, v^j)^{1/2}.$$

The momentum is hence for a time-like test particle (mass point):

$$(2.6) \qquad \mathbf{P} \equiv m(v)\mathbf{v}, \qquad m(v) \equiv m(v = 0)\cdot\gamma_v \equiv m(0)\cdot(1 - v^2/c^2)^{-1/2}$$

(using the mass-velocity relation of SR) ([1]). For a light-like test particle (photon), one substitutes the mass content of the energy for the inertial mass $m(v)$. Then we must define the derivative of the momentum with respect to the local time. Thus in general we have to define the derivative of a vector $\mathbf{w} = \mathbf{w}(\chi)$ attached to a point $\mathbf{x}(\chi) = (x^i(\chi))$ which moves, as a function of the real parameter χ, in some Riemannian space: here this space is the 3-D domain $N = N_\mathcal{F}$ constituted by the spatial network which defines the considered frame $\mathcal{F}$. Hence the points in N are specified by their constant space coordinates x^i, $i = 1, 2, 3$, and N is equipped with the space metric h. The derivative must be defined as the "absolute" derivative (e.g. BRILLOUIN [5], LICHNEROWICZ [12]), which is a space vector and accounts for the (merely spatial) variation of the space metric along the trajectory:

$$(2.7) \qquad \left(\frac{D\mathbf{w}}{D\chi}\right)^i \equiv \frac{dw^i}{d\chi} + \Gamma^i_{\ jk} w^j \frac{dx^k}{d\chi},$$

([1]) Equation (2.6) implicitly assumes that the rest mass $m(0)$ is the same constant m_0, independently of the gravitation field. This may be seen as an immediate consequence of defining the inertial mass m as the ratio $\mathbf{P}/\mathbf{v}$ $(= P^i/v^i)$ and assuming that the P^i are the spatial components of the 4-momentum, this being in turn assumed to have the form $P^\alpha = m_0\, dx^\alpha/d\tau$ with a constant m_0. This is consistent with LANDAU and LIFCHITZ [11]. On the other hand, MØLLER [18] defines the inertial mass as the ratio $m' = \mathbf{P}/\mathbf{v}_0$ with $\mathbf{v}_0 = d\mathbf{x}/dt$, thus $m' = m\, dt/dt_\mathbf{x}$, hence his rest mass $m'_0 = m'(\mathbf{v}_0 = 0) = m_0\, dt/dt_\mathbf{x}$ depends on the gravitation field. However, the definition of $\mathbf{v}_0$ and hence that of m'_0 depend on the chosen time coordinate t even in a given frame, while the velocity $\mathbf{v}$ used by Landau and Lifchitz (and used here) depends only on the reference frame, as it should.

where the $\Gamma^i{}_{jk}$ are the Christoffel symbols of metric h in coordinates (x^i). As shown in ref. [2], the use of Eq. (2.7) is *enforced* if one wants to know that Leibniz' rule applies, and that the derivative cancels for a vector $\mathbf{w}$ that is parallel-transported (relative to the space metric h) along the trajectory. This is considered to be important, because it means that Eq. (2.7) is not merely one possible formal rule to obtain a space-contravariant vector, but the unique consistent definition for the time-derivative of a vector along a trajectory, in the case of a time-independent metric. Now the *left-hand side* of Newton's second law is just the force. This may be decomposed into a "non-gravitational" force $\mathbf{F}_0$, which should have the same expression for any gravitation theory in the considered class (²), and a "gravitational" force $\mathbf{F}_g$ whose expression, of course, will depend on the theory. Note that $\mathbf{F}_g$ *will generally contain "inertial" forces as well* (since a general reference frame is considered here), hence "mass force" would be a more appropriate denomination [1]. Thus finally:

$$\mathbf{F}_0 + \mathbf{F}_g = D\mathbf{P}/Dt_{\mathbf{x}}\,. \tag{2.8}$$

Using the same equations (2.3) and (2.5) to (2.7), the same definition may and must be used in the *stationary* case, in which the $\gamma_{\alpha\beta}$'s remain time-independent, but the γ_{0i} components may be non-zero: although a synchronized local time cannot be defined in the frame $\mathcal{F}$ as a whole if the γ_{0i}'s are non-zero, what matters is that it is uniquely defined along the trajectory followed by the considered particle (provided that it follows an open line in space-time: a closed line would mean a travel back in time).

2.3. Extended Newton law for a general gravitation field

In the general case where the gravitation field is not constant in the frame $\mathcal{F}$, the new feature is that now the space-time metric γ depends also on x^0. Hence also the space metric h (Eq. (2.2)) varies, not only as a function of the space coordinates x^i (what is natural for a general Riemannian metric in a space depending on these coordinates), but also as a function of the time coordinate x^0. What is relevant for Newton's second law is, more precisely, the variation of h along a trajectory (of a test particle), i.e. the fact that *our spatial network N is equipped with a metric field h_χ that changes as the parameter χ evolves on the trajectory*, thus for any value of χ and at every point $X \in N$ we have a covariant tensor $\mathsf{h}_\chi(X)$. In our case, the variation of the metric field with χ is due to the variation of h with the point in space-time, thus in coordinates:

$$h_{\chi ij}[(x^k)_{k=1,2,3}] \equiv h_{ij}[x^0(\chi),\ (x^k)_{k=1,2,3}].$$

(²) The expression of $\mathbf{F}_0$ is taken from the situation without gravitation: thus, as recalled in point (ii) of Subsec. 2.1, it involves the field γ (in the place of the flat metric γ^0), and it depends on the non-gravitational fields; in practice, these are the electromagnetic field and/or thermomechanical fields (the nuclear fields are very microscopic matter fields and moreover, their current theory does not belong to classical physics, i.e. their influence cannot be described in terms of deterministic trajectories of mass points). A "*free*" particle is one which crosses a region free from matter and electromagnetic field: for such a particle, the force $\mathbf{F}_0$ will be zero *independently of the reference frame considered*.

Moreover, we have a preferred parameter $\chi = t_{\mathbf{x}}$ on the trajectory. It is easy to convince oneself that nothing needs to be changed in Eqs. (2.3), (2.5) and (2.6), because they involve only the local components of the metric (which now become its local and "current" components), not its variation. In order to define properly an extension of (2.7), let us list the properties that should be satisfied by this searched derivative of a vector on a trajectory in a manifold equipped with a variable metric:

a) It must be a (space) vector, i.e. it must be contravariant for any coordinate transformation of the form $x'^i = x'^i(x^j)$.

b) It must be linear in $\mathbf{w}$. More precisely, it must obviously have the form

$$(D\mathbf{w}/D\chi)^i = (dw^i/d\chi)_{\chi=\chi_0} + L^i{}_j w^j(\chi_0),$$

with χ_0 the point of the trajectory where the derivative is to be calculated, and where $L^i{}_j$ behave as a mixed second-order (space) tensor (transforming a (space) vector into another one), for linear coordinate transformations.

c) It must reduce to (2.7) if the metric field h_χ does *not* depend on χ.

d) It should account for the variation of metric h_χ as a function of χ.

e) It must be multiplied by $d\chi/d\zeta$ if χ is changed to $\zeta = \phi(\chi)$.

f) It must satisfy Leibniz' derivation rule for the derivative of a scalar product, i.e.

$$\text{(2.9)} \qquad \frac{d}{d\chi}(\mathsf{h}_\chi(\mathbf{w},\mathbf{z})) = \mathsf{h}_\chi\left(\mathbf{w},\frac{D\mathbf{z}}{D\chi}\right) + \mathsf{h}_\chi\left(\frac{D\mathbf{w}}{D\chi},\mathbf{z}\right),$$

in which it is understood that, on the left, the variation of metric h with x^0 is accounted for, as becomes obvious if one writes down explicitly the scalar product:

$$\text{(2.10)} \qquad \mathsf{h}_\chi(\mathbf{w},\mathbf{z}) = h_{ij}\,[(x^\alpha(\chi))_{\alpha=0,\ldots,3}]\, w^i(\chi) z^j(\chi).$$

(Hence, it is likely that (f) implies (d)).

First, we note that definition (2.7) still makes sense, and satisfies requirements (a), (b), (c) and (e). Of course, it is now specified that the Christoffel symbols of metric h are those at the relevant position and "*time*", thus in (2.7)

$$\text{(2.11)} \qquad \Gamma^i{}_{jk} = \Gamma_\chi{}^i{}_{jk}[(x^l)_{l=1,2,3}] = \Gamma^i{}_{jk}[(x^\alpha(\chi))_{\alpha=0,\ldots,3}].$$

The "candidate" thus defined by Eq. (2.7) will be now denoted by $D_0\mathbf{w}/D\chi$. It does not satisfy (d) (nor (f), in fact), for it amounts to substituting the metric h_{χ_0} of the "time" $a = x^0(\chi_0)$ for the variable metric h_χ. From (a), (b) and (c), it follows that we have to search an expression in the form

$$\text{(2.12)} \qquad D\mathbf{w}/D\chi = D_0\mathbf{w}/D\chi + \mathbf{t}\cdot\mathbf{w}(\chi_0),$$

in which $\mathbf{t}$ is a mixed second-order space tensor (indeed, the ordinary derivative $d\mathbf{w}/d\chi = (dw^i/d\chi)$ is already involved in $D_0\mathbf{w}/D\chi$, Eq. (2.7)). But to satisfy (d), it is hence necessary that this tensor should involve the variation of metric $\mathbf{h}_\chi$ with χ, due to the variation of $\mathbf{h}$ with x^0:

$$\frac{\partial h_{\chi ij}}{\partial \chi} = \frac{\partial h_{ij}}{\partial x^0}\frac{dx^0}{d\chi}.$$

Thus, tensor $\mathbf{t}$ must contain either $h_{ij,0}$ terms or $h^{ij}{}_{,0}$ ones, with (h^{ij}) the inverse matrix of (h_{ij}). In order to be a mixed tensor and satisfy (e), $\mathbf{t}$ should have the form

$$t^i{}_k = h^{ij}h_{jk,0}(dx^0/d\chi), \qquad \text{or} \qquad t'^i{}_k = h^{ij}{}_{,0}(dx^0/d\chi)h_{jk}, \tag{2.13}$$

or any linear combination of these two tensors. But since $h^{ij}h_{jk} = \delta^i{}_k$, we have $\mathbf{t}+\mathbf{t}' = 0$, so that, without imposing Leibniz' rule, we are left with a one-parameter family of candidates:

$$D_\lambda \mathbf{w}/D\chi \equiv D_0\mathbf{w}/D\chi + \lambda \mathbf{t}\cdot\mathbf{w}. \tag{2.14}$$

Finally, nearly the same short calculation as in Ref. [4] shows that *Leibniz' rule* (2.9) *imposes* $\lambda = 1/2$, hence only one definition of the derivative remains:

$$D\mathbf{w}/D\chi \equiv D_0\mathbf{w}/D\chi + (1/2)\mathbf{t}\cdot\mathbf{w}, \qquad \mathbf{t} \equiv \mathbf{h}_\chi{}^{-1}\cdot\frac{\partial \mathbf{h}_\chi}{\partial\chi} \equiv \mathbf{h}^{-1}\cdot\frac{\partial \mathbf{h}}{\partial x^0}\frac{dx^0}{d\chi}, \tag{2.15}$$

or in coordinates:

$$\left(\frac{D\mathbf{w}}{D\chi}\right)^i \equiv \frac{dw^i}{d\chi} + \Gamma^i_{jk}w^j\frac{dx^k}{d\chi} + \frac{1}{2}h^{ij}h_{jk,0}\frac{dx^0}{d\chi}w^k. \tag{2.16}$$

Thus, a theory of the kind considered should provide an expression for the mass force $\mathbf{F}_g$, and this expression would depend on what the theory considers as "the gravitation field" (this may include the space-time metric γ, in any case it must determine γ). Then one and only one "Newton law" can be consistently stated in such a theory: it is Eq. (2.8), where the momentum $\mathbf{P}$ is given by Eq. (2.6) and its derivative $D\mathbf{P}/Dt_\mathbf{x}$ is calculated using rule (2.16). The trajectory $\xi \to (x^\alpha(\xi))$ being defined with the help of an arbitrary parameter ξ, the variation of the local time $\chi = t_\mathbf{x}$ along the trajectory is given by Eq. (2.3).

2.4. Comments and link with the investigated preferred-frame theory

It is seen that the derivative of the momentum is defined in any possible reference frame (and it depends on the frame). Hence, if a theory gives a covariant expression for $\mathbf{F}_g$ and γ, the extended second Newton law does not restrict the covariance of the theory. On the other hand, a preferred-frame theory may give $\mathbf{F}_g$ and γ in one reference frame only; if one were able to calculate the transformation law of the derivative $D\mathbf{P}/Dt_{\mathbf{x}}$, then this same law would apply to the force, so the law of motion would be reexpressed in a covariant form.

The investigated ether theory [1–4], which is indeed non-covariant, starts from a heuristic interpretation of gravity as Archimedes' thrust in a perfectly fluid "micro-ether" (the rigid ether frame $\mathcal{E}$ considered by Lorentz and Poincaré would be defined by the average motion of this "micro-ether" at a very large scale). The transition to account for "relativistic" effects is based on a formulation of Einstein's equivalence principle, natural in this preferred-frame theory: the equivalence is stated to exist between the absolute metric effects of uniform motion and gravitation. This leads to postulate a gravitational contraction (resp. a dilation) of the space (resp. time) standards, depending on the field of the "ether pressure" p_e, thus getting a curved (Riemannian) space metric g and a local time $t_{\mathbf{x}}$ in the ether frame $\mathcal{E}$, which together build a curved space-time metric γ [2–3]. This theory gives $\mathbf{F}_{\mathrm{g}}$ and γ in the ether frame $\mathcal{E}$ only, as a function of the *scalar* gravitation field p_e, or the associated fields f and β with

$$f = \beta^2 = (p_e/p_e^{\infty})^2 \leq 1, \tag{2.17}$$

where $p_e^{\infty} = p_e^{\infty}(T)$ is the reference pressure (which, for an insular matter distribution, is asymptotically reached at large distance from the matter. Here, T is the "absolute time"). The gravitation force is assumed to be

$$\mathbf{F}_{\mathrm{g}} = m(v)\mathbf{g}, \tag{2.18}$$

with $\mathbf{g}$ the gravity acceleration, given by

$$\mathbf{g} = -c^2\frac{\mathrm{grad}_{\mathsf{g}} p_e}{p_e} = -c^2\frac{\mathrm{grad}_{\mathsf{g}}\beta}{\beta} = -\frac{c^2}{2}\mathrm{grad}_0 f\,, \tag{2.19}$$

where $\mathsf{g} = \mathsf{h}_{\mathcal{E}}$ is the physical space metric in the frame $\mathcal{E}$, and where $\mathrm{grad}_{\mathsf{g}}$ (resp. grad_0) is the gradient operator relative to metric g (resp. relative to the "natural" metric g^0, with constant curvature, of which the "ether" network (3-D manifold) $M = N_{\mathcal{E}}$ is assumed to be equipped with). And the line element of the space-time metric γ, affected by gravitational contraction of the space standards (relative to metric g^0) and by gravitational dilation of the time standards (relative to the "absolute time" T), has the form

$$ds^2 = \beta^2(dx^0)^2 - dl^2, \qquad x^0 = cT, \tag{2.20}$$

where dl^2 is the line element of metric g. This has the following simple expression in "isopotential" coordinates (y^{α}), i.e. coordinates such that, at a given time T, $y^1 = \text{const}$ (in space) is equivalent to $p_e = \text{const}$, and that the natural metric g^0 is diagonal, $(\mathsf{g}^0{}_{ij}) = \mathrm{diag}(a^0{}_i)$:

$$(\mathsf{g}_{ij}) = \mathrm{diag}(a_i) \qquad \text{with} \qquad a_1 = a^0{}_1/f, \quad a_2 = a^0{}_2\,, \quad a_3 = a^0{}_3\,. \tag{2.21}$$

For a time-dependent field p_e, such coordinates are *not* bound to the ether frame [4]. From Eq. (2.20), it follows that, if one selects any coordinates (x^α), with $x^0 = cT$, that *are* bound to the frame $\mathcal{E}$, then the components γ_{0i} are zero. Thus a simultaneity is defined for the frame $\mathcal{E}$ as a whole; in other words, the absolute time T is a universal time in the frame $\mathcal{E}$. For the important case of an insular matter distribution, the absolute time T is the local time measured at any point $\mathbf{x}_0$ which is bound to $\mathcal{E}$ and far enough from matter so that no gravitation field is felt there. Moreover, the global synchronization condition ($\gamma_{0i} = 0$) does not hold true in a frame that rotates rigidly with respect to $\mathcal{E}$, nor in general in a frame that moves uniformly with respect to $\mathcal{E}$ (³) (the condition $\gamma_{0i} = 0$ holds true for any frame in uniform translation, in the case that no gravitation field is present, thus for the flat metric $\gamma = \gamma^0$). These considerations justify the denomination "absolute time" for T. Hence, the ether frame $\mathcal{E}$, which is already a global inertial frame in the sense that the mass force in $\mathcal{E}$ (2.18)–(2.19) is purely gravitational, is really a physically privileged reference frame (according to this theory).

3. Extended Newton law and geodesic motion

3.1. A possible form for the gravitation force in a globally synchronized reference frame

We now investigate the possible form of the gravitation force. In order to make some meaningful induction from the Newtonian theory, it is very useful to work in a reference frame $\mathcal{F}$, in which the γ_{0i} components of metric γ are zero (Subsec. 2.1). The concept of global simultaneity is indeed so deeply involved in any Newtonian analysis, that any induction from the Newtonian theory to the general situation with curved space-time, where a simultaneity is defined only along a trajectory, would seem dangerous. Whereas, if one works in a frame such that $\gamma_{0i} = 0$, the only change in the time concept is that now the clocks go differently at different positions and times (Eq. (2.4)). We note that the existence of a frame $\mathcal{F}$, in which the γ_{0i} are zero, is not a physically restrictive assumption, since it breaks down only for rather pathological space-times: in "normal" space-times it is even possible to select a "synchronous" frame which not only enjoys this global synchronization, but in which the γ_{00} component is uniform, i.e. the local time flows uniformly (Landau and Lifchitz [11], Mavrides [16]). Thus there "normally" exist many different frames such that $\gamma_{0i} = 0$. Which form of the gravitation force could one consistently state in such a reference frame?

For the class of theories considered in Sec. 2, what is considered by any such theory as "the gravitation field", has been assumed to determine the space-time metric γ (for non-covariant theories, we should add that this has only to be true in some preferred reference frame which is like $\mathcal{E}$, i.e. such that $\gamma_{0i} = 0$). Here, we will assume, in a more restrictive way, that *the metric field γ contains the*

(³) Here, rigid rotation and uniform motion *can* be defined, at least if the metric manifold $(M, \mathbf{g}^0)$ has *zero* curvature, i.e. if it is Euclidean.

gravitation field (at least in the preferred frame). This is true in any reference frame for GR and for the "relativistic theory of gravitation" (RTG) proposed by LOGUNOV *et al.* [13–14], and this is true in the ether frame $\mathcal{E}$ in the tentatively proposed theory. On the other hand, in order that SR would hold true locally and that the inertial and (passive) gravitational mass might coincide, the gravitation force must have the form

$$\mathbf{F}_g = m(v)\mathbf{g}, \tag{3.1}$$

with $\mathbf{g}$ being a space vector in the considered frame. If we want the metric field to play the rôle of a potential, we must ask $\mathbf{g}$ to depend linearly on the first derivatives of γ, and bearing in mind the Newtonian theory we should add that only the *spatial* derivatives $\gamma_{\mu\nu,k}$ are allowed. But, in a frame where $\gamma_{0i} = 0$, we have $\gamma_{ij} = -h_{ij}$ with h denoting the space metric in this frame, i.e. the metric γ reduces to the joint data $\gamma = (f, \mathsf{h})$ with $f = \gamma_{00}$. Thus, we are looking for a space vector $\mathbf{g}$ depending linearly on the spatial derivatives of f and h. To be contravariant by a general space transformation, $\mathbf{g}$ must depend linearly on the *covariant derivatives* of f and h (with respect to the space metric h!). But, as is known, the covariant derivatives of metric h with respect to h itself are all zero (in other words, one may cancel all spatial derivatives $h_{ij,k}$ at any given point by a purely spatial coordinate transformation). Hence, $\mathbf{g}$ should have the form

$$\mathbf{g} = a(f, \mathsf{h})\, \mathrm{grad}_{\mathsf{h}} f, \tag{3.2}$$

where a must be a given function of the *values* of the metric fields at the considered point (x^α) in space-time, $f = f(x^\alpha)$ and $\mathsf{h} = \mathsf{h}(x^\alpha)$ in Eq. (3.2), thus $a(f, \mathsf{h})$ is completely independent of the *variation* of f and h with time and position.

Now we add the condition that *geodesic motion (Einstein's assumption) must apply to free particles* ($\mathbf{F}_0 = 0$) *for a static gravitation field*. This is exactly equivalent to assuming the following expression for the gravitation force *in the static case*:

$$\mathbf{F}_g = -m(v)c^2 \frac{\mathrm{grad}_{\mathsf{h}} \beta}{\beta} = m(v)\, \mathrm{grad}_{\mathsf{h}}(-c^2 \mathrm{Log}\, \beta), \qquad \text{where} \quad \beta \equiv \sqrt{\gamma_{00}}. \tag{3.3}$$

Indeed, it was already proved (and it will be proved again below, in a different way) that Eq. (3.3), which occurs naturally in the ether theory, implies geodesic motion for mass particles in the static case [2]; this is also true for photons [3], substituting in that case the mass content of the energy $e = h\nu$ for the inertial mass $m(v)$. Conversely, it is proved in LANDAU and LIFCHITZ [11] that geodesic motion implies the expression (3.3) for the force in the static case, defined as the derivative (2.7) of the momentum (2.6) ([4]). Thus the reason for assuming geodesic

([4]) Actually, LANDAU and LIFCHITZ [11, § 88] derived from geodesic assumption the expression of the force in the *stationary* case, using the same definition for the force (what is consistent with the present work, Subsec. 2.2). They found an expression involving an additional term which cancels if $\gamma_{0i} = 0$.

motion in the static case is that it is indeed so for the tentative ether theory as well as, of course (and in any situation) for the usual theories of gravitation with curved space-time, in particular GR and the RTG. So we must have, by Eqs. (3.1), (3.2) and (3.3):

$$\mathbf{g} = -c^2 \frac{\mathrm{grad}_{\mathbf{h}} \beta}{\beta} = -\frac{c^2}{2} \frac{\mathrm{grad}_{\mathbf{h}} f}{f}, \qquad \text{i.e.} \quad a(f, \mathbf{h}) = -\frac{c^2}{2f}, \tag{3.4}$$

when $f_{,0} = 0$ and $\mathbf{h}_{,0} = 0$. But since $a(f, \mathbf{h})$ depends only on the local values of f and $\mathbf{h}$, not on their variation, Eq. (3.2) implies then that $\mathbf{g}$ keeps the form (3.4) and thus Eq. (3.3) holds true in the most general situation.

3.2. Expression of the 4-acceleration for a "free" particle using the extended Newton law

In theories with a (pseudo-) Riemannian space-time metric, two well-known space-time vectors may be defined for a time-like test particle (i.e. a mass point). These are the 4-velocity $\mathbf{U}$, which is the velocity on the world line of the particle in space-time, when the world line is parametrized with the proper time τ of the particle,

$$U^\alpha \equiv dx^\alpha / d\tau, \tag{3.5}$$

and the 4-acceleration $\mathbf{A}$, which is the absolute derivative $\Delta \mathbf{U}/\Delta\tau$ of the former relative to the *space-time* metric γ. Thus

$$A^\alpha \equiv \left(\frac{\Delta \mathbf{U}}{\Delta \tau}\right)^\alpha \equiv \frac{dU^\alpha}{d\tau} + \Gamma'^\alpha_{\mu\nu} U^\mu \frac{dx^\nu}{d\tau} \equiv \frac{dU^\alpha}{d\tau} + \Gamma'^\alpha_{\mu\nu} U^\mu U^\nu, \tag{3.6}$$

symbols $\Gamma'^\alpha_{\mu\nu}$ being the Christoffel symbols of metric γ in coordinates (x^α).

i) Spatial components of the 4-acceleration in a globally synchronized reference frame.

It is recalled that we use coordinates (x^α) that are bound to a "globally synchronized" frame $\mathcal{F}$. Thus $\gamma_{0i} = 0$ $(i = 1, 2, 3)$, from which it follows immediately that:

$$h_{ij} = -\gamma_{ij}, \qquad \Gamma^i_{\ jk} = \Gamma'^i_{\ jk}, \tag{3.7}$$

hence

$$\left(\frac{\Delta \mathbf{U}}{\Delta \tau}\right)^i = \frac{dU^i}{d\tau} + \Gamma^i_{\ jk} U^j U^k + \Gamma'^i_{\ 00} (U^0)^2 + 2\Gamma'^i_{\ 0k} U^0 U^k. \tag{3.8}$$

In this equation, we note that, in view of Eq. $(3.7)_1$ (and since $h^{ij} = -\gamma^{ij}$ is always true):

$$\Gamma'^i_{\ 0k} U^0 U^k = \gamma^{i\alpha} \frac{(\gamma_{\alpha 0,k} + \gamma_{\alpha k,0} - \gamma_{0k,\alpha})}{2} U^0 U^k = \frac{1}{2} h^{ij} h_{jk,0} \frac{dx^0}{d\tau} U^k. \tag{3.9}$$

By (2.4) and (3.5) we get:

$$U^0 = (dx^0/dt_{\mathbf{x}})(dt_{\mathbf{x}}/d\tau) = c(dt_{\mathbf{x}}/d\tau)/\sqrt{\gamma_{00}}\,,$$

but, using Eqs. (2.1)–(2.3) and (2.5), it may be proved (cf. LANDAU and LIFCHITZ [11]) that, independently of the fact that $\gamma_{0i} = 0$, one has always:

$$\frac{dt_{\mathbf{x}}}{d\tau} = \gamma_v \tag{3.10}$$

as was already noted [2] for the tentative theory. Hence we obtain

$$U^0 = \frac{c\gamma_v}{\sqrt{\gamma_{00}}} = \frac{c\gamma_v}{\beta}\,, \tag{3.11}$$

so we reexpress another term in Eq. (3.8), calculating Γ'^i_{00} as for Γ'^i_{0k} in Eq. (3.9) and using again Eq. (2.4):

$$\Gamma'^i_{00}(U^0)^2 = h^{ij}\frac{\gamma_{00,j}}{2}\frac{c^2\gamma_v^2}{\beta^2} = h^{ij}\frac{2\beta\beta_{,j}}{2}\frac{c^2\gamma_v^2}{\beta^2} = \gamma_v^2 c^2\frac{(\mathrm{grad}_{\mathbf{h}}\beta)^i}{\beta}\,.$$

We recognize here the component g^i of the assumed gravity acceleration (Eq. (3.4)), thus

$$\Gamma'^i_{00}(U^0)^2 = -\gamma_v^2 g^i. \tag{3.12}$$

It is now possible to calculate $(\Delta\mathbf{U}/\Delta\tau)^i$ with the Newton law, for a "free" particle (Eq. (2.8) with $\mathbf{F}_0 = 0$ and with $\mathbf{F}_{\mathrm{g}}$ given by Eq. (3.1)). In a first step, let us calculate with the incompletely defined Newton law, which is obtained if one uses the derivative $D_\lambda\mathbf{P}/Dt_{\mathbf{x}}$ with the unspecified parameter λ (cf. Eq. (2.14)). Using (3.10), we may write this in terms of τ:

$$(D_\lambda\mathbf{P}/Dt_{\mathbf{x}})^i \equiv (D_\lambda\mathbf{P}/D\tau)^i/\gamma_v = m_0\gamma_v g^i\,,$$

and we have by Eqs. (2.5), (2.6) and (3.10):

$$P^i = m_0\gamma_v v^i = m_0\gamma_v\, dx^i/dt_{\mathbf{x}} = m_0\, dx^i/d\tau = m_0 U^i, \tag{3.13}$$

so the "unspecified" Newton law has the form

$$(D_\lambda\mathbf{u}'/D\tau)^i = \gamma_v^2 g^i, \tag{3.14}$$

where $\mathbf{u}' \equiv (U^i)$ means the spatial part of the 4-velocity $\mathbf{U}$. Applying definition (2.14) which involves terms given by Eqs. (2.13) and (2.7), we get

$$\left(\frac{D_\lambda\mathbf{u}'}{D\tau}\right) = \frac{dU^i}{d\tau} + \Gamma^i{}_{jk}U^jU^k + \lambda h^{ij}h_{jk,0}\frac{dx^0}{d\tau}U^k. \tag{3.15}$$

Hence, the unspecified Newton law imposes the following values to the spatial components (in coordinates bound to a globally synchronized frame $\mathcal{F}$) of the 4-acceleration of a free test particle (Eq. (3.8) with (3.9) and (3.12)), depending on the parameter λ:

$$\left(\frac{\Delta \mathbf{U}}{\Delta \tau}\right)^i = 2(1-\lambda)\Gamma'^i_{0k}U^0U^k = (1-\lambda)h^{ij}h_{jk,0}\frac{dx^0}{d\tau}U^k. \tag{3.16}$$

In particular, *the spatial part of the equation for space-time geodesics is satisfied for a variable gravitation field* ($h_{jk,0} \neq 0$) *if and only if the parameter* λ *has the value* $\lambda = 1$.

ii) Time component of the 4-acceleration in a globally synchronized frame

For the time component, we have simply

$$A^0 = \left(\frac{\Delta \mathbf{U}}{\Delta \tau}\right)^0 \equiv \frac{dU^0}{d\tau} + \Gamma'^0_{00}(U^0)^2 + 2\Gamma'^0_{0k}U^0U^k + \Gamma'^0_{ij}U^iU^j\,. \tag{3.17}$$

Using Eq. (3.7)$_1$ and the fact that $\gamma_{00} = \beta^2$ (Eq. (2.4)), the $\Gamma'^0_{\mu\nu}$ are easily calculated:

$$\Gamma'^0_{00} = \frac{\beta_{,0}}{\beta}, \qquad \Gamma'^0_{0k} = \frac{\beta_{,k}}{\beta}, \qquad \Gamma'^0_{ij} = \frac{h_{ij,0}}{2\beta^2}\,.$$

By Eq. (3.11), which implies also that $U^k = (\gamma_v/\beta)(dx^k/dt)$, one then rewrites (3.17) as

$$A^0\frac{\beta}{c\gamma_v} = \frac{d}{dt}\left(\frac{\gamma_v}{\beta}\right) + \frac{\gamma_v}{\beta^2}\frac{\partial\beta}{\partial t} + 2\frac{\gamma_v}{\beta^2}\beta_{,k}\frac{dx^k}{dt} + \frac{1}{2c^2\beta^2}\frac{\gamma_v}{\beta}\frac{\partial h_{ij}}{\partial t}\frac{dx^i}{dt}\frac{dx^j}{dt}\,. \tag{3.18}$$

At this point, we may insert the energy balance deduced from the "unspecified" Newton law for the free test particle (Eq. (4.21) in Ref. [4]):

$$\frac{d}{dt}(\beta\gamma_v) = \gamma_v\frac{\partial\beta}{\partial t} + \beta\gamma_v\frac{1-2\lambda}{2c^2}\frac{\partial \mathbf{h}}{\partial t}(\mathbf{v},\mathbf{v}) \tag{3.19}$$

with $v^i = (dx^i/dt)/\beta$ by Eqs. (2.4) and (2.5)(5). We have thus in Eq. (3.18):

$$\begin{aligned}\frac{d}{dt}\left(\frac{\gamma_v}{\beta}\right) &= \frac{d}{dt}\left(\frac{1}{\beta^2}\beta\gamma_v\right) = \frac{1}{\beta^2}\frac{d}{dt}(\beta\gamma_v) + \beta\gamma_v\left[\frac{\partial}{\partial t}\left(\frac{1}{\beta^2}\right) + \left(\frac{1}{\beta^2}\right)_{,k}\frac{dx^k}{dt}\right]\\ &= -\frac{\gamma_v}{\beta^2}\left(\frac{\partial\beta}{\partial t} + 2\beta_{,k}\frac{dx^k}{dt}\right) + \frac{\gamma_v}{\beta^3}\frac{1-2\lambda}{2c^2}\frac{\partial h_{ij}}{\partial t}\frac{dx^i}{dt}\frac{dx^j}{dt},\end{aligned}$$

(5) Equation (3.19) is derived using the fact that *some* derivation rule of a scalar product can be obtained even with the "unspecified" Newton law, although it does not obey the true Leibniz rule (Eq. (2.9)) unless $\lambda = 1/2$. However, if $\lambda \neq 1/2$, this balance equation cannot be rewritten as a true conservation equation, at least in the scalar theory [1–4].

so that some cancellation occurs in (3.18). We obtain finally:

$$(3.20)\qquad A^0 = \frac{\gamma_v^2}{c\beta^4}(1-\lambda)\frac{\partial h_{ij}}{\partial t}\frac{dx^i}{dt}\frac{dx^j}{dt} = \frac{(1-\lambda)}{\beta^2}h_{ij,0}U^iU^j = 2(1-\lambda)\Gamma'^0_{ij}U^iU^j.$$

In particular, the time part of the equation for space-time geodesics, as well as the spatial part, is satisfied for a variable gravitation field ($h_{ij,0} \neq 0$) if and only if the parameter λ has the value $\lambda = 1$. However, it is recalled that the value $\lambda = 1$ specifies the Newton law in an incorrect manner, since it means that Newton's second law is based on a vector time derivative which does not obey Leibniz' derivation rule.

Let us summarize the results of Subsecs. 3.1 and 3.2, which concern Newton's second law and geodesic motion:

(NGM1) *Consider a theory with curved space-time metric γ and locally valid SR, and assume that in some "globally synchronized" reference frame $\mathcal{F}$ ($\gamma_{0i} = 0$), the gravitation force* (3.1) *involves a space vector* $\mathbf{g}$ *depending only on the metric field γ. More precisely, assume that* $\mathbf{g}$ *does not depend on the time variation of γ and is linear with respect to the space variation of γ. In order that free particles would follow space-time geodesics in the static case ($\gamma_{\mu\nu,0} = 0$), it is necessary and sufficient that the general expression of vector* $\mathbf{g}$ *in the frame $\mathcal{F}$ should be*

$$(3.21)\qquad \mathbf{g} = -c^2\frac{\mathrm{grad}_{\mathbf{h}}\beta}{\beta} = -\frac{c^2}{2}\frac{\mathrm{grad}_{\mathbf{h}}f}{f}, \qquad f \equiv \gamma_{00} \equiv \beta^2,$$

with $\mathbf{h}$ *the space metric in $\mathcal{F}$. This expression implies Eqs.* (3.16) *and* (3.20) *for the 4-acceleration, thus it implies that, for a time-dependent field, geodesic motion corresponds exactly to the incorrect Newton law* ($\lambda = 1$).

3.3. Characteristic form of the gravitation force associated with geodesic motion

The assumption that the metric field γ plays the rôle of a potential for the gravity acceleration $\mathbf{g}$ seems quite natural, if one thinks of a "soft" generalization of Newtonian gravity. The foregoing result implies, among other things, that Einstein's assumption of a motion following space-time geodesics is not such a soft extension. But, after all, in Maxwell's theory the electric field involves also time derivatives of the electromagnetic potential, besides the usual space derivatives. Moreover, the Lorentz force depends on the velocity of the charged particle. A more general expression than we assumed for the gravity acceleration might hence be correct also, the more so as we now have empirical reasons to think that the gravity interaction indeed propagates, as does the electromagnetic field, and with the same velocity (TAYLOR and WEISBERG [25]). That gravitation propagates with the velocity of light was first envisaged by Poincaré in his "electromagnetic", Lorentz-invariant theory of gravitation [20–21] and, as is well known, it is predicted by Einstein's theory.

Thus we now investigate the possible form of the vector $\mathbf{g}$, subjected to the unique constraint that *geodesic motion should occur with the correct form of Newton's second law*, i.e. $\lambda = 1/2$. We continue to work in a globally synchronized reference frame and, in order to simplify the expressions, we take $\mathbf{g}$ in the form

$$\mathbf{g} = -c^2 \frac{\operatorname{grad}_{\mathsf{h}}\beta}{\beta} + \mathbf{g}' = -\frac{c^2}{2}\frac{\operatorname{grad}_{\mathsf{h}} f}{f} + \mathbf{g}', \qquad f \equiv \gamma_{00} \equiv \beta^2. \tag{3.22}$$

Starting from Eq. (3.6) as before, nothing changes until Eq. (3.12), which now becomes

$$\Gamma'^{i}_{00}(U^0)^2 = -\gamma_v^2(g^i - g'^i). \tag{3.23}$$

And again nothing changes until Eq. (3.16), which is modified into

$$A^i = \left(\frac{\Delta \mathrm{U}}{\Delta\tau}\right)^i = (1-\lambda)h^{ij}h_{jk,0}\frac{dx^0}{d\tau}U^k + \gamma_v^2 g'^i. \tag{3.24}$$

Hence, the spatial components of the 4-acceleration cancel with $\lambda = 1/2$, if and only if

$$g'^i = \frac{-1}{2\beta}h^{ij}\frac{\partial h_{jk}}{\partial t}v^k, \qquad \text{i.e.} \qquad \mathbf{g}' = \frac{-1}{2\beta}\mathsf{h}^{-1}\cdot\frac{\partial \mathsf{h}}{\partial t}\cdot\mathbf{v} = \frac{-1}{2}\mathsf{h}^{-1}\cdot\frac{\partial \mathsf{h}}{\partial t_{\mathbf{x}}}\cdot\mathbf{v} \tag{3.25}$$

But does this expression also cancel the time part of the 4-acceleration? To check this, one must reexamine the energy balance derived in Ref. [4]. Proceeding in the same way, we find easily that the energy balance resulting from the expression (3.22), (3.25) of $\mathbf{g}$ is (with $\lambda = 1/2$)

$$\frac{d}{dt}(\beta\gamma_v) = \gamma_v\frac{\partial\beta}{\partial t} - \frac{\beta\gamma_v}{2c^2}\frac{\partial\mathsf{h}}{\partial t}(\mathbf{v},\mathbf{v}), \tag{3.26}$$

instead of Eq. (3.19). Thus, with the correct Newton law ($\lambda = 1/2$), the same expression is now obtained as it was obtained before with the incorrect Newton law ($\lambda = 1$). Therefore, the time part of the geodesic equation, $A^0 = 0$, is satisfied for $\lambda = 1/2$, as it was previously for $\lambda = 1$. We have proved the following:

(NGM2) *Consider a theory with curved space-time metric* γ *and locally valid SR, and assume the correct time derivative* (2.15) *in the extension* (2.8) *of Newton's second law. In order that free particles* ($\mathbf{F}_0 = 0$ *in Eq.* (2.8)) *might follow space-time geodesics, it is necessary and sufficient that, in any globally synchronized reference frame* $\mathcal{F}$ ($\gamma_{0i} = 0$), *the gravitation force* (3.1) *should involve the following expression for the gravity acceleration (space vector* $\mathbf{g}$):

$$\mathbf{g}_{\text{geod}} = -c^2\frac{\operatorname{grad}_{\mathsf{h}}\beta}{\beta} - \frac{1}{2\beta}\mathsf{h}^{-1}\cdot\frac{\partial\mathsf{h}}{\partial t}\cdot\mathbf{v}, \qquad \beta \equiv \sqrt{\gamma_{00}}, \tag{3.27}$$

with h *being the space metric in* $\mathcal{F}$ *and* $\mathbf{v}$ *– the velocity vector* (Eq. (2.5)).

This result provides the general link between Newton's second law and Einstein's geodesic assumption.

4. Comparison with the literature

4.1. Møller's work and the relation between covariant and contravariant form of Newton's law

Among attempts to define Newton's second law in the case of a variable gravitation field, a well-known one is that of MØLLER [18]. However, Møller uses the absolute derivative with respect to the "frozen" space metric, thus $\lambda = 0$ in Eq. (2.14), so that Leibniz' rule is not satisfied with the actual, time-dependent metric. In connection with this, he notes that this derivative does not commute with raising or lowering the indices with respect to the space metric $\mathbf{h}$. As a consequence, when he rewrites the equations for space-time geodesics in the form of Newton's second law with gravitational forces, the latter look very different in covariant and in contravariant form. We show that this difficulty is absent with our definition.

Indeed, it is easy to adapt our line of reasoning so as to define the time-derivative of a spatial *covector* $\mathbf{w}^*$. One finds in exactly the same way that, apart from Leibniz' rule, a one-parameter family of time-derivatives may be defined as:

$$D_\lambda \mathbf{w}^*/D\chi \equiv D_0\mathbf{w}^*/D\chi - \lambda \mathbf{t}\cdot\mathbf{w}^*, \tag{4.1}$$

with

$$\begin{aligned}(\mathbf{t}\cdot\mathbf{w}^*)_i &\equiv h_{ij,0}(dx^0/d\chi)h^{jk}w^*{}_k \\ &\equiv (dx^0/d\chi)(\mathbf{h}_{,0}\cdot\mathbf{h}^{-1})_i{}^k w^*{}_k = (dx^0/d\chi)(\mathbf{h}^{-1}\cdot\mathbf{h}_{,0})^k{}_i w^*{}_k = t^k{}_i w^*{}_k\,,\end{aligned} \tag{4.2}$$

and where $D_0\mathbf{w}^*/D\chi$ is the absolute derivative using the "frozen" metric. And one finds that Leibniz' rule imposes $\lambda = 1/2$. It is also easy to verify that, for this correct value $\lambda = 1/2$ and, for a time-dependent metric $\mathbf{h}$, *only* for this value, the time-derivative $D_\lambda/D\chi$ *does commute* with raising or lowering the indices with respect to the space metric $\mathbf{h}$, that is

$$D_{1/2}(\mathbf{h}\cdot\mathbf{w})/D\chi = \mathbf{h}\cdot(D_{1/2}\mathbf{w}/D\chi). \tag{4.3}$$

Therefore, if one takes the covariant components of the momentum instead of the contravariant ones, thus substituting $\mathbf{P}^* = \mathbf{h}\cdot\mathbf{P}$ for $\mathbf{P}$, then the corresponding "covariant Newton law" will involve just the covariant components of the force, $\mathbf{F}^* = \mathbf{h}\cdot\mathbf{F} = \mathbf{h}\cdot(\mathbf{F}_0 + \mathbf{F}_g)$ in Eq. (2.8).

4.2. Newton's second law with the "Fermi–Walker" time-derivative

From now on, we will discuss the work on "Newton's second law in relativistic gravity" as reviewed and unified by JANTZEN *et al.* [10]. They define the equivalent of what we call a frame (spatial network) by a 4-velocity vector field $\mathbf{u}$, and they name it "observer congruence". What they call "observer-adapted frames" is a very different notion from that of adapted coordinates as defined by MØLLER

[18] and CATTANEO [6, 7]. Here we continue to work in adapted coordinates, i.e. such that the observers of the network (or congruence) have constant space coordinates. In such coordinates, the contravariant and covariant components of **u** are given by

$$(4.4)\qquad (u^\alpha) = \left(\frac{1}{\sqrt{-\gamma_{00}}}, 0, 0, 0\right), \qquad (u_\alpha) = \left(-\sqrt{-\gamma_{00}}, \left(\frac{\gamma_{0i}}{\sqrt{-\gamma_{00}}}\right)_{i=1,2,3}\right)$$

(we keep our notations, except for the fact that we set $u^\alpha \equiv dx^\alpha/ds$ and adopt the (3,1) signature as in Refs. [6–7] and [10], until the end of this Section). It follows that the spatial projection tensor $\mathbf{\Pi} = \mathbf{\Pi}(\mathbf{u})$ [7, 10], which is a space-time tensor defined in general by

$$\Pi^\mu{}_\nu \equiv \delta^\mu{}_\nu + u^\mu u_\nu \,,$$

has a simple expression:

$$(4.5)\qquad \Pi^i{}_j = \delta^i{}_j\,, \qquad \Pi^i{}_0 = 0, \qquad \Pi^0{}_j = -\gamma_{0j}/\gamma_{00}\,, \qquad \Pi^0{}_0 = 0.$$

It corresponds to the projection of the local tangent space to space-time onto the hyperplane which is γ-perpendicular to the local 4-velocity **u** of the observer congruence. In connection with this, what is called a "spatial tensor" by CATTANEO [7] and by JANTZEN *et al.* [10] is also a very different notion from that used by MØLLER [18] and in the rest of this paper. For us (and for Møller), a spatial tensor is just an element of a tensor space at the relevant point of the spatial network (3-D Riemannian manifold) N, thus its components depend on the three spatial (Latin) indices only, $i = 1, 2, 3$, in adapted coordinates. In Refs. [7, 10] and in the remainder of this section, a spatial tensor is a *space-time tensor* which is *equal to its projection*, the latter being generally defined by Eq. (2.2) of Ref. [10]. E.g. for a 4-vector (space-time vector) **X**, the projection reads:

$$(4.6)\qquad (\mathbf{\Pi}\cdot\mathbf{X})^\alpha = \Pi^\alpha{}_\mu X^\mu.$$

Hence in adapted coordinates, by (4.5):

$$(4.7)\qquad (\mathbf{\Pi}\cdot\mathbf{X})^i = X^i, \qquad (\mathbf{\Pi}\cdot\mathbf{X})^0 = -\gamma_{0j}X^j/\gamma_{00}\,,$$

so that the "time" component X^0 is *not* equal to zero for a "spatial vector" (except for a "normal congruence", i.e. the case where $\gamma_{0j} = 0$ in some adapted coordinates). We also note that the "rescaled time" $\tau_{(\mathbf{U},\mathbf{u})}$ considered in Ref. [10] (for a time-like test particle with 4-velocity **U**), as well as the "standard time" T considered in Refs. [6–7], is the same variable as our "local time" $t_\mathbf{x}$, synchronized along the trajectory of the test particle, with their $\gamma = \gamma_{(\mathbf{U},\mathbf{u})}$ being our γ_v (Eqs. (2.3) and (3.10) here). On the other hand, what is called the

"Fermi–Walker total spatial covariant derivative" (fw TSCD) in Ref. [10], has the following expression for an arbitrary parameter χ (although it is defined only for $\chi = \tau_{(\mathbf{U},\mathbf{u})} \equiv t_{\mathbf{x}}$ in Ref. [10]):

$$\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi} \equiv \mathbf{\Pi}\cdot\frac{\Delta\mathbf{X}}{\Delta\chi}. \tag{4.8}$$

We have thus in adapted coordinates, by Eq. (4.7):

$$\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^i \equiv \left(\frac{\Delta\mathbf{X}}{\Delta\chi}\right)^i \equiv \left(\frac{dX^i}{d\chi} + \Gamma'^i_{\ \mu\nu}X^\mu\frac{dx^\nu}{d\chi}\right), \qquad i = 1,\ 2,\ 3, \tag{4.9}$$

and the "time" part of the derivative is not independent of the "space" part:

$$\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^0 \equiv -\frac{\gamma_{0j}}{\gamma_{00}}\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^j. \tag{4.10}$$

What corresponds to Newton's second law in [10] is the evaluation of the spatial projection of the 4-acceleration $\mathbf{A}$ of the test particle. Apart from the different notation, it amounts almost exactly to Eq. (2.8) here, with the same definition (2.6) for the momentum, involving the same relative velocity (2.5), though with the derivative defined by Eq. (4.8) instead of Eq. (2.15). One difference is that the velocity $\mathbf{v}$ and momentum $\mathbf{P}$ are now spatial 4-vectors which turn out to be the respective projections of the 4-vectors $\mathbf{U}'$ and $\mathbf{P}'$, with $\mathbf{U}'$ the 4-velocity $\mathbf{U}$, rescaled to the local time, and $\mathbf{P}'$ the usual 4-momentum. Thus the spatial components of $\mathbf{v}$ and $\mathbf{P}$ are the same as in this work, and the "time" components obey the general rule for a spatial vector $\mathbf{X}$, i.e. such that $\mathbf{\Pi}\cdot\mathbf{X} = \mathbf{X}$:

$$X^0 = -\gamma_{0j}X^j/\gamma_{00}. \tag{4.11}$$

Another difference is that the gravitational force, which is the total force for a free particle, is necessarily deduced, in the frame of GR and other "metric theories", from the geodesic equation, i.e. $\mathbf{A} = 0$, whereas here geodesic motion is one possibility among others.

Having thus recognized that the spatial part (4.9) of the derivative (4.8) plays exactly the same rôle in Ref. [10] as the derivative (2.15) plays here, we may comment on the difference between the two derivatives. Since the spatial components (4.9) are just those of the space-time absolute derivative $\Delta\mathbf{X}/\Delta\chi$, the Fermi–Walker TSCD involves space-time coupling in a generally inextricable way, in that it cannot in general be defined in terms of only the spatial metric h and the local time $t_{\mathbf{x}}$. Hence, this derivative cannot be used in an arbitrary reference frame to define a "true" Newton law as it has been defined here, i.e. precisely a law involving only the separate space and time metrics in the given reference frame, thus allowing to "forget" the concept of space-time as long as one does not change the reference frame.

4.3. The "normal" and "corotational" Fermi–Walker derivatives obey Leibniz' rule

Surprisingly, the question whether the introduced time-derivatives satisfy the Leibniz rule is not investigated in Refs. [6, 7, 10]. However, it is not difficult to show that *the two Fermi–Walker derivatives do verify* Eq. (2.9) for spatial vectors. The spatial metric in those works is of course the same thing as here, except for the signature and the fact that it is now a *space-time* tensor (for a given observer congruence $\mathbf{u}$):

$$(4.12)\qquad \begin{aligned} &h_{\alpha\beta} \equiv \gamma_{\alpha\lambda}\Pi^{\lambda}{}_{\beta} \equiv \gamma_{\alpha\beta} + u_\alpha u_\beta \;\Rightarrow\; h_{ij} \equiv \gamma_{ij} - \frac{\gamma_{0i}\gamma_{0j}}{\gamma_{00}}, \\ &h_{0i} = h_{i0} = h_{00} = 0. \end{aligned}$$

Equation $(4.12)_1$ implies immediately that, for any two space-time vectors $\mathbf{X}$ and $\mathbf{Y}$:

$$(4.13)\qquad \mathsf{h}(\mathbf{X},\mathbf{Y}) = \gamma(\mathbf{X},\mathbf{\Pi}\cdot\mathbf{Y}) = \gamma(\mathbf{\Pi}\cdot\mathbf{X},\mathbf{Y}).$$

On the other hand, the absolute space-time derivative obeys the Leibniz rule:

$$(4.14)\qquad \frac{d}{d\chi}[\gamma(\mathbf{X},\mathbf{Y})] = \gamma\left(\mathbf{X},\frac{\Delta\mathbf{Y}}{\Delta\chi}\right) + \gamma\left(\frac{\Delta\mathbf{X}}{\Delta\chi},\mathbf{Y}\right).$$

Using Eq. (4.13), we rewrite Eq. (4.14), if *both* vectors $\mathbf{X}$ and $\mathbf{Y}$ are *spatial*, as:

$$\frac{d}{d\chi}[\mathsf{h}(\mathbf{X},\mathbf{Y})] = \mathsf{h}\left(\mathbf{X},\mathbf{\Pi}\cdot\frac{\Delta\mathbf{Y}}{\Delta\chi}\right) + \mathsf{h}\left(\mathbf{\Pi}\cdot\frac{\Delta\mathbf{X}}{\Delta\chi},\mathbf{Y}\right).$$

With the definition (4.8), this gives the Leibniz rule for the Fermi–Walker derivative:

$$(4.15)\qquad \frac{d}{d\chi}[\mathsf{h}(\mathbf{X},\mathbf{Y})] = \mathsf{h}\left(\mathbf{X},\frac{D_{(\mathrm{fw})}\mathbf{Y}}{D\chi}\right) + \mathsf{h}\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi},\mathbf{Y}\right).$$

The "corotational" Fermi–Walker (cfw) derivative, when acting on a spatial vector $\mathbf{X}$, is related to the "normal" Fermi–Walker derivative by [10]:

$$(4.16)\qquad \left(\frac{D_{(\mathrm{cfw})}\mathbf{X}}{D\chi}\right)^{\alpha} = \left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^{\alpha} + \omega^{\alpha}{}_{\mu}\frac{c\,dt_{\mathbf{x}}}{d\chi}X^{\mu}.$$

Here $\omega^{\alpha}{}_{\mu}$ are the mixed components of the "spin-rate" space-time tensor. This comes from the decomposition of the covariant "spatial 4-velocity gradient",

$$(4.17)\qquad \mathsf{k} = \mathsf{k}(\mathbf{u}) = -\mathbf{\Pi}\cdot\nabla^{(\gamma)}\mathbf{u}^{b}, \qquad \mathbf{u}^{b} \equiv (u_\alpha), \qquad k_{\alpha\beta} = -\Pi^{\lambda}{}_{\alpha}\,\Pi^{\mu}{}_{\beta}\,u_{\lambda;\mu},$$

into symmetric and antisymmetric part:

$$(4.18)\qquad k_{\alpha\beta} = -\theta_{\alpha\beta}+\omega_{\alpha\beta}, \qquad -\theta_{\alpha\beta} = (k_{\alpha\beta}+k_{\beta\alpha})/2, \qquad \omega_{\alpha\beta} = (k_{\alpha\beta}-k_{\beta\alpha})/2,$$

and the mixed components $\omega^{\alpha}{}_{\mu}$ are obtained by raising the index α with metric γ. It appears that, just like the ordinary one, the corotational Fermi–Walker derivative cannot in general be expressed in terms of the spatial metric h and the local time $t_{\mathbf{x}}$ only. Moreover, it is difficult here to refrain from asking the question: with respect to *what* does the "spin rate" ω measure the rate of relative spin of the considered reference fluid (network)? Already the understanding of the *strain* rate θ is difficult: without any preferred reference fluid, we may only define, so to speak, the "strain rate of the fluid with respect to itself" due to the evolution of the spatial metric h, and this is precisely what measures the $\mathsf{t} = \mathsf{h}^{-1}\cdot\mathsf{h}_{,0}\ (dx^0/dt_{\mathbf{x}})$ tensor in our derivative (2.15) (with $\chi = t_{\mathbf{x}}$) – but the tensors t and θ are two different objects.

As to Leibniz' rule, it applies to the cfw derivative, at least if *both* vectors $\mathbf{X}$ and $\mathbf{Y}$ are *spatial*. Indeed, due to the antisymmetry of the covariant tensor ω (Eq. (4.18)$_3$), the definition (4.16) gives

$$\gamma\left(\mathbf{X}, \frac{D_{(\mathrm{cfw})}\mathbf{Y}}{D\chi}\right) + \gamma\left(\frac{D_{(\mathrm{cfw})}\mathbf{X}}{D\chi}, \mathbf{Y}\right) - \gamma\left(\mathbf{X}, \frac{D_{(\mathrm{fw})}\mathbf{Y}}{D\chi}\right) - \gamma\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}, \mathbf{Y}\right)$$
$$= \frac{c\,dt_{\mathbf{x}}}{d\chi}\gamma_{\mu\nu}\left(\omega^{\mu}{}_{\varrho}X^{\varrho}Y^{\nu} + \omega^{\nu}{}_{\varrho}Y^{\varrho}X^{\mu}\right) = \frac{c\,dt_{\mathbf{x}}}{d\chi}\left(\omega_{\nu\varrho}X^{\varrho}Y^{\nu} + \omega_{\mu\varrho}Y^{\varrho}X^{\mu}\right) = 0.$$

The Leibniz rule follows from this by (4.13) and (4.15), the two vectors $\mathbf{X}$ and $\mathbf{Y}$ being assumed to be spatial vectors:

$$\text{(4.19)}\qquad \mathsf{h}\left(\mathbf{X}, \frac{D_{(\mathrm{cfw})}\mathbf{Y}}{D\chi}\right) + \mathsf{h}\left(\frac{D_{(\mathrm{cfw})}\mathbf{X}}{D\chi}, \mathbf{Y}\right)$$
$$= \mathsf{h}\left(\mathbf{X}, \frac{D_{(\mathrm{fw})}\mathbf{Y}}{D\chi}\right) + \mathsf{h}\left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}, \mathbf{Y}\right) = \frac{d}{d\chi}\left[\mathsf{h}(\mathbf{X}, \mathbf{Y})\right].$$

4.4. The case of a globally synchronized frame and the "Lie" time-derivative

We consider the particular case of a *globally synchronized frame* (or "normal congruence"), in which the γ_{0i} components of the space-time metric are zero in some adapted coordinates. Then the spatial projection tensor $\mathbf{\Pi}$ (Eq. (4.5)) is written simply

$$\text{(4.20)}\qquad (\Pi^{\mu}{}_{\nu}) = \mathrm{diag}\,(0, 1, 1, 1)$$

in such coordinates. Hence, in such coordinates, substituting its spatial projection $\mathbf{\Pi}(\mathbf{u})\cdot\mathbf{T}$ for a space-time tensor $\mathbf{T}$ amounts exactly to taking its space components only. In particular, the "time" component of a spatial vector $\mathbf{X}$ is now equal to zero. Moreover, the spatial Christoffel symbols of the space-time metric are equal to the Christoffel symbols of the spatial metric (Eq. (3.7)). This implies that the Fermi–Walker derivative coincides, for the case considered and for a *spatial* vector **X** (thus $X^0 = 0$), with the $D_{1/2}$ derivative. Indeed, using Eq. (3.9), we find:

$$(4.21)\qquad \left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^i \equiv \left(\frac{\Delta\mathbf{X}}{\Delta\chi}\right)^i \equiv \frac{dX^i}{d\chi} + \Gamma^i_{\ jk}X^j\frac{dx^k}{d\chi} + \Gamma'^i_{\ j0}X^j\frac{dx^0}{d\chi}$$
$$= \frac{dX^i}{d\chi} + \Gamma^i_{\ jk}X^j\frac{dx^k}{d\chi} + \frac{1}{2}h^{ik}h_{kj,0}X^j\frac{dx^0}{d\chi} = \left(\frac{D_{1/2}\mathbf{X}'}{D\chi}\right)^i,$$

with $\mathbf{X}' \equiv (X^i)$.

For the non-zero components of the **k** tensor (Eq. (4.17)), we obtain using Eqs. (4.20), (3.9) and (4.4) (and since $h_{jk} = \gamma_{jk}$ with the (3,1) signature):

$$(4.22)\qquad -k_{ij} = u_{i;j} = h_{ik}u^k_{\ ;j} = h_{ik}\Gamma'^k_{0j}u^0 = \frac{1}{2}h_{ij,0}u^0 = \frac{1}{2}h_{ij,0}\frac{dx^0}{c\,dt_{\mathbf{x}}}.$$

Therefore, the "spin-rate" tensor ω is nil for a normal congruence [6], so that the corotational Fermi–Walker derivative coincides, for spatial vectors, with the "normal" one, and thus with the proposed derivative, $D = D_{1/2}$. On the other hand, we have from (4.18) and (4.22):

$$\theta_{ij} = -k_{ij} = \frac{1}{2}h_{ij,0}\frac{dx^0}{c\,dt_{\mathbf{x}}}.$$

What is called "Lie" TSCD derivative in Ref. [10], is not a Lie derivative in the usual sense but the projection of a Lie derivative [10], and is defined in general by [10]:

$$(4.23)\qquad \left(\frac{D_{(\mathrm{lie})}\mathbf{X}}{D\chi}\right)^\alpha = \left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^\alpha + \frac{c\,dt_{\mathbf{x}}}{d\chi}\left(\omega^\alpha_{\ \mu}X^\mu - \theta^\alpha_{\ \mu}X^\mu\right)$$

(extending again the definition [10] to an arbitrary parameter χ). Hence, we have here:

$$(4.24)\qquad \left(\frac{D_{(\mathrm{lie})}\mathbf{X}}{D\chi}\right)^i = \left(\frac{D_{(\mathrm{fw})}\mathbf{X}}{D\chi}\right)^i - \frac{1}{2}\frac{c\,dt_{\mathbf{x}}}{d\chi}h^{ik}h_{kj,0}\frac{dx^0}{c\,dt_{\mathbf{x}}}X^j = \left(\frac{D_0\mathbf{X}}{D\chi}\right)^i.$$

In other words, the so-called "Lie" derivative coincides in that case with the absolute derivative with respect to the "frozen" spatial metric, and so does not obey Leibniz' rule.

5. Concluding remarks

1. From our bibliographical research, it would appear that it had not yet been proposed in the literature, as it is proposed here, to introduce a consistent definition of the time-derivative of a vector, in the following relevant situation:

the vector is moving along a trajectory in a manifold equipped with a metric field $\mathbf{h}_\chi$ (the *spatial* metric in a given reference frame) that changes with the parameter χ on the trajectory. Indeed, of the three different notions of frame-dependent time-derivatives that have been reviewed and unified by JANTZEN *et al.* [10], the two first ones (the Fermi - Walker derivatives) involve the whole *space-time* metric in an unseparable way, while the so-called "Lie" derivative does not obey Leibniz' rule. In our opinion, this would mean that no consistent *and* natural extension of Newton's second law to the case of a variable gravitation field in a general reference frame (in a theory with curved space-time as envisaged here) had yet been proposed either. It seems as if, from the orthodox relativistic point of view, it would be considered to be *a priori* impossible to define Newton's second law "really as before" – because the absolute priority is to maintain consistency with the notion that the 4-dimensional space-time is the essential physical reality. However, it turns out that the two Fermi - Walker derivatives coincide with the proposed derivative in the important case of a globally synchronized frame (or normal congruence).

2. We find that there is one and *only one natural* extension of Newton's second law to any theory with curved space-time metric, in the most general situation. In particular, one may *uniquely* identify that gravity acceleration $\mathbf{g}_{\text{geod}}$ which is necessary to obey Einstein's assumption, i.e. to obtain geodesic motion for free test particles. In doing so, we did not merely rewrite the three "spatial" equations for space-time geodesics as the space-vector relation "force = time-derivative of momentum": we also proved that the latter relation implies the "time" equation of geodesics, and this does not seem to have been done in earlier attempts. This "geodesic" gravity acceleration $\mathbf{g}_{\text{geod}}$ depends on the reference frame, as is natural in a "relativistic" theory (since the acceleration is not Lorentz-invariant). It may seem more surprising that $\mathbf{g}_{\text{geod}}$ depends on the velocity of the particle (Eq. (3.27)). However, this is also the case for the Lorentz force which a charged particle undergoes in an electromagnetic field. The striking difference is that the magnetic force does not work, whereas the velocity-dependent part of $\mathbf{g}_{\text{geod}}$ does work. In the investigated case of a normal congruence, it has the same form as the Newtonian inertial force that appears in a reference frame undergoing pure strain with respect to an inertial frame [1]. But here this "inertial" force comes from the straining of the reference frame "with respect to itself" (i.e. due to the fact that the spatial metric evolves with time) and it cannot in general be cancelled in a finite region by changing the reference frame. Thus, theories with geodesic motion inherently do not allow global inertial frames, although such global inertial frames do appear in their Newtonian limit. We also note that any velocity dependence of the gravity acceleration, $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{v})$, implies that the definition of the passive gravitational mass, i.e. $m_g \equiv \mathbf{F}_g/\mathbf{g}$ with $\mathbf{F}_g$ the gravitation force, becomes indissolubly mixed with that of the gravity acceleration itself: one may change $\mathbf{g}$ and m_g to $\alpha\mathbf{g}$ and m_g/α respectively, with α any scalar function of the velocity (e.g. $\alpha = \gamma_v{}^n$ where n is any real number), so that m_g is operationally

defined up to the arbitrary function α only. Hence, although Newton's second law *can* be defined in a "curved space-time" after all, the statement "m_g = inertial mass $m(v)$" still remains partly conventional. Indeed, the only testable statement is then the universality of the gravitation force (which is really a crucial point, of course).

3. The identity between inertial and gravitational mass would have a stronger meaning if **g** depended only on the position of a given test particle. However, for the kind of theories considered here, this could be true only in some preferred reference frame (this is, of course, in contrast with the Galilean situation). To check this identity, one might e.g. define **g** for particles at rest in the preferred reference frame, thus $\mathbf{g}(\mathbf{x}) \equiv \mathbf{F}_g(\mathbf{v} = 0)/m_0$, and check experimentally whether or not the gravitation force $\mathbf{F}_g$ is indeed equal to $m(v)\mathbf{g}$ for an arbitrary velocity. In the scalar ether theory which has been tentatively proposed [1–4], a vector **g** depending only on the position, Eq. (3.21), has been found to occur naturally, consistently with the notion that **g** should be determined by the local state of some substratum. Thus this theory predicts "strong identity" between inertial and gravitational mass and, in connection with this, geodesic motion does not hold true in the general case in this theory. If one were to modify this theory so as to obtain geodesic motion, one would have to postulate Eq. (3.27) instead of Eq. (3.21). Then, the modified **g**-field would still be determined (in the preferred frame $\mathcal{E}$) by the scalar field p_e or β (together with the particle velocity!) However, this would lead to the energy balance (3.26), which has been seen to be incompatible with the derivation of a true conservation equation for the energy in this scalar theory [4]. On the other hand, this theory could happen to predict unobserved post-Newtonian effects of absolute motion.

Acknowledgement

I am grateful to Prof. P. Guélin for his useful comments on the manuscript.

References

1. M. Arminjon, *A theory of gravity as a pressure force. I. Newtonian space and time*, Rev. Roum. Sci. Tech.-Méc. Appl., **38**, 3–24, 1993.
2. M. Arminjon, *A theory of gravity as a pressure force. II. Lorentz contraction and "relativistic" effects*, Rev. Roum. Sci. Tech.-Méc. Appl., **38**, 107–128, 1993.
3. M. Arminjon, *Scalar theory of gravity as a pressure force*, to appear in Rev. Roum. Sci. Tech.-Méc. Appl., **40**, 1995.
4. M. Arminjon, *Energy and equations of motion in a tentative theory of gravity with a privileged reference frame*, Arch. Mech., **48**, 1, 25–52, 1996.
5. L. Brillouin, *Les tenseurs en mécanique et en élasticité*, Masson, Paris 1949.
6. C. Cattaneo, *General relativity: relative standard mass, momentum, energy and gravitational field in a general system of reference*, Nuovo Cimento, **10**, 318–337, 1958; *On the energy equation for a gravitating test particle*, Nuovo Cimento, **11**, 733–735, 1959.

7. C. CATTANEO, *Problèmes d'interprétation en relativité générale*, [in:] Fluides et Champ Gravitationnel en Relativité Générale, A. LICHNEROWICZ and M.A. TONNELAT [Eds.], pp. 227-235, Ed. du CNRS, Paris 1969.
8. L. JÁNOSSY, *The Lorentz principle*, Acta Physica Polonica, **27**, 61–87, 1965.
9. L. JÁNOSSY, *Theory of relativity based on physical reality*, Akademiai Kiado, Budapest 1971.
10. R.T. JANTZEN, P. CARINI and D. BINI, *The many faces of gravitoelectromagnetism*, Ann. Physics (New York), **215**, 1–50, 1992.
11. L. LANDAU and E. LIFCHITZ, *Théorie des champs* (3rd French edition), Mir, Moscow 1970.
12. A. LICHNEROWICZ, *Eléments de calcul tensoriel*, Armand Colin, Paris 1950.
13. A.A. LOGUNOV, YU.M. LOSKUTOV and M.A. MESTVIRISHVILI, *The relativistic theory of gravitation and its consequences*, Sov. Phys. Usp., **31**, 581–596, 1988 (Transl. from Usp. Fiz. Nauk, **155**, 369–396, 1988).
14. A.A. LOGUNOV and M.A. MESTVIRISHVILI, *The relativistic theory of gravitation*, Mir, Moscow 1989.
15. H.A. LORENTZ, *The theory of electrons* (First edition 1909), Dover, New York 1952.
16. S. MAVRIDES, *L'univers relativiste*, Masson, Paris 1973.
17. P. MAZILU, *Missing link in the theory of gravity interaction*, Proc. 6th. Marcel Grossmann Meet. Gen. Relat., H. SATO and T. NAKAMURA [Eds.], World Scientific, Singapore 1992.
18. C. MØLLER, *The theory of relativity*, Clarendon Press, Oxford 1952.
19. A. PAPAPETROU, *Equations of motion in general relativity*, Proc. Phys. Soc., **A64**, 57–75, 1951.
20. H. POINCARÉ, *Sur la dynamique de l'électron*, C.R. Acad. Sci. Paris, **140**, 1504–1508, 1905.
21. H. POINCARÉ, *La dynamique de l'électron*, Rev. Gén. Sci. Pures et Appl., **19**, 386–402, 1908.
22. S.J. PROKHOVNIK, *The logic of special relativity*, Cambridge University Press, Cambridge 1967.
23. S.J. PROKHOVNIK, *Light in Einstein's universe*, Reidel, Dordrecht 1985.
24. S.J. PROKHOVNIK, *The physical interpretation of special relativity – a vindication of Hendrik Lorentz*, Z. Naturforsch., **48a**, 925–931, 1993.
25. J.H. TAYLOR and J.M. WEISBERG, *A new test of general relativity: gravitational radiation and the binary pulsar PSR 1913+16*, Astrophys. J., **253**, 908–920, 1982.
26. C.M. WILL, *Theory and experiment in gravitational physics*, 2nd ed., Cambridge University Press, Cambridge 1993.
27. Y.Z. ZHANG, *Test theories of special relativity*, Gen. Rel. Grav., **27**, 475–493, 1995.

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Received October 16, 1995.

Arch. Mech., 48, 3, pp. 577–582, Warszawa 1996

Non-uniform stagnant motions of materially non-uniform simple fluids

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NON-UNIFORM STAGNANT motions of materially non-uniform (inhomogeneous) incompressible fluids are reconsidered in greater detail. These motions may be used in many practical situations, such as fibre spinning and drawing processes. It is shown that the corresponding constitutive equations are very similar to those describing motions with constant stretch history or, in particular, steady extensional flows.

1. Introduction

THERE ARE AT LEAST three reasons for reconsidering non-uniform motions of materially non-uniform (inhomogeneous) simple fluids. The first reason is connected with pretty weak interest of the researchers involved either in the continuum theories or in the rheology of polymeric liquids. Existing references are rather devoted to what may be called inhomogeneities (dislocations, aeolotropy etc.) in materially uniform simple bodies (cf. [1]). The second reason results from serious needs for such considerations in the rheology of polymers when the material non-uniformity may be caused by a sensitivity of material properties to various temperature, viscosity, structure, etc. variations in the flows considered. The third reason, but not of minor importance, is the fact that the Referees of my previous papers on the necking phenomenon in fibre spinning processes [2, 3] had some doubts about the possibility of applying the constitutive equations in a form very similar to that describing uniform steady elongations of incompressible simple fluids [4].

In 1962 COLEMAN and NOLL discussed the class of substantially stagnant motions [5] or motions with constant stretch history (MCSH) [6].

According to Noll's definition, a motion is called a MCSH if, and only if, relative to a fixed reference configuration at time 0, the deformation gradient at any time τ is given by

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\exp(\tau\mathbf{M}), \qquad \mathbf{Q}(0) = \mathbf{1}, \tag{1.1}$$

where $\mathbf{Q}(\tau)$ is an orthogonal tensor and $\mathbf{M}$ is a constant tensor such that $\mathbf{M} = \kappa\mathbf{N}_0$, $|\mathbf{N}_0| = 1$, and κ a constant parameter. The above definition shows that in all MCSH, the history of the relative deformation tensor is one and the same function of $t - \tau$ for all current instants t.

Moreover, it results from WANG'S theorem [7] that in all MCSH, the extra-stress tensor can be expressed as an isotropic tensor function of at most first three

Rivlin–Ericksen kinematic tensors, i.e.

$$(1.2)\qquad \mathbf{T}_0(t) = \mathbf{h}(\mathbf{A}_1(t), \mathbf{A}_2(t), \mathbf{A}_3(t),), \qquad \operatorname{tr}\mathbf{T}_E = 0,$$

where by definition

$$(1.3)\qquad \mathbf{A}_1 = \mathbf{L}_1^T + \mathbf{L}_1, \qquad \mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + \mathbf{A}_n\mathbf{L}_1 + \mathbf{L}_1^T\mathbf{A}_n, \qquad n \geq 1,$$

and the velocity gradient amounts to

$$(1.4)\qquad \mathbf{L}_1(t) = \dot{\mathbf{F}}_0(t)\mathbf{F}_0^{-1}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) + \mathbf{Q}(t)\mathbf{M}\,\mathbf{Q}^T(t).$$

In the present paper we generalize the above results for the case of non-uniform stagnant motions (hereafter called NUSM) of materially non-uniform (inhomogeneous) incompressible simple fluids. It is shown that the corresponding constitutive equations are very similar in form to those valid for MCSH.

2. Non-uniform stagnant motions (NUSM)

Consider a more general class of motions for which the deformation gradient at any time τ, relative to a configuration at time 0 is of the form:

$$(2.1)\qquad \mathbf{F}_0(\mathbf{X}, \tau) = \mathbf{Q}(\mathbf{X}, \tau)\exp(\tau\mathbf{M}(\mathbf{X})), \qquad \mathbf{Q}(\mathbf{X}, 0) = \mathbf{1},$$

where $\mathbf{Q}(\mathbf{X}, \tau)$ is an orthogonal tensor, and $\mathbf{M}(\mathbf{X})$ depends only on the position $\mathbf{X}$ of a particle X in an arbitrarily chosen reference configuration $\boldsymbol{\kappa}$ (not necessarily at time 0). Thus, the non-uniformity of the quantities involved can be expressed either by $\mathbf{X}$ or X $(\mathbf{X} = \boldsymbol{\kappa}(X))$.

According to the definition (1.4), we obtain the following velocity gradient:

$$(2.2)\qquad \mathbf{L}_1(\mathbf{X}, t) = \dot{\mathbf{Q}}(\mathbf{X}, t)\mathbf{Q}^T(\mathbf{X}, t) + \mathbf{L}(\mathbf{X}, t),$$

where

$$(2.3)\qquad \mathbf{L}(\mathbf{X}, t) = \mathbf{Q}(\mathbf{X}, t)\mathbf{M}(\mathbf{X})\mathbf{Q}^T(\mathbf{X}, t),$$

is called the rotated parametric tensor (cf. [8]), and t denotes the current instant of time.

The deformation gradient, relative to a configuration at the current time t, amounts to

$$(2.4)\qquad \begin{gathered} \mathbf{F}_t(\mathbf{X}, t-s) = \mathbf{F}_0(\mathbf{X}, \tau)\mathbf{F}_0^{-1}(\mathbf{X}, t) = \mathbf{Q}(\mathbf{X}, t-s)\exp(-s\mathbf{M}(\mathbf{X}))\mathbf{Q}^T(\mathbf{X}, t), \\ \tau = t - s, \qquad 0 \leq s < \infty, \end{gathered}$$

what leads to the following history of the relative deformation tensor (cf. [8]):

$$\mathbf{C}_t^t(\mathbf{X}, s) \equiv \mathbf{C}_t(\mathbf{X}, t-s) = \mathbf{F}_t^T \mathbf{F}_t = \exp(-s\mathbf{L}^T(\mathbf{X}, t)) \exp(-s\mathbf{L}(\mathbf{X}, t)). \tag{2.5}$$

In full analogy to the case of MCSH, we may ask what will happen if $\mathbf{L}_1(\mathbf{X})$ defined through Eq. (2.2) is steady (independent of time t) but non-uniform in space? The answer results from the following differential equation based on Eq. (1.4):

$$\frac{d}{d\tau}\mathbf{F}_0(\mathbf{X}, \tau) = \mathbf{L}_1(\mathbf{X})\mathbf{F}_0(\mathbf{X}, \tau), \tag{2.6}$$

with the initial condition: $\mathbf{F}_0(\mathbf{X}, 0) = \mathbf{1}$. The corresponding solution can be written as

$$\mathbf{F}_0(\mathbf{X}, \tau) = \exp(\tau \mathbf{L}_1(\mathbf{X})). \tag{2.7}$$

The above expression evidently belongs to the class (2.1) with $\mathbf{Q} \equiv \mathbf{1}$. It is obvious that for steady flows in an Eulerian sense

$$\dot{\mathbf{L}}_1(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \cdot \nabla \mathbf{L}_1(\mathbf{x}), \tag{2.8}$$

where $\mathbf{V}$ is the velocity and ∇ denotes the gradient with respect to place $\mathbf{x}$.

It is worthwhile to mention that Noll's classification of MCSH based on the tensor $\mathbf{M}(\mathbf{X})$ (or $\mathbf{L}(\mathbf{X}, t)$) can be generalized to the case of NUSM. Therefore, in certain parts of a fluid, we may have the following classes of flows:

(I) non-uniform viscometric flow

$$\mathbf{M}^2 = \mathbf{0};$$

(II) non-uniform doubly-superposed viscometric flow

$$\mathbf{M}^2 \neq \mathbf{0}, \qquad \mathbf{M}^3 = \mathbf{0};$$

(III) non-uniform triply-superposed viscometric flow and extensional flow

$$\mathbf{M}^n \neq \mathbf{0} \qquad \text{for all} \quad n = 1, 2, \ldots .$$

The non-uniform extensional flows, because of their technological validity, will be discussed separately in Sec. 4.

3. Constitutive equations of materially non-uniform (inhomogeneous) simple fluids

As mentioned at the beginning, in many practical situations, instead of solving the usually complex problems, it is more useful to assume *a priori* that unknown temperature, viscosity, structure, etc. distributions lead to a material non-uniformity (inhomogeneity). In other words, such a non-uniformity means that the mechanical properties of a fluid vary from particle to particle.

The constitutive equations of materially non-uniform incompressible simple fluids can be written in the form (cf. [9]):

$$\mathbf{T}_E(\mathbf{X}, t) = \mathop{\mathcal{H}}_{s=0}^{\infty}(\mathbf{C}_t^t(\mathbf{X}, s); \mathbf{X}), \tag{3.1}$$

where $\mathbf{T}_E$ is the non-uniform extra-stress tensor, and $\mathcal{H}$ denotes a constitutive functional. Such a definition is not in contradiction with the principles of determinism and local action. Equations (3.1) also satisfy the principle of objectivity (invariance with respect to the reference frame) since all the tensors involved are objective (cf. [8]).

For non-uniform stagnant motions (NUSM) defined by Eq. (2.10), after introducing Eq. (2.5) into Eq. (3.1) and taking into account the properties of tensor exponentials,

$$\exp \mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n, \qquad (\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T)^n = \mathbf{Q}\,\mathbf{A}^n\mathbf{Q}^T, \tag{3.2}$$

we arrive at

$$\mathbf{T}_E(\mathbf{X}, t) = \mathbf{h}(\mathbf{L}(\mathbf{X}, t); \mathbf{X}), \tag{3.3}$$

where $\mathbf{h}$ is an isotropic function of the tensor argument. In particular, if the rotated parametric tensor $\mathbf{L}(\mathbf{X})$ is a steady one, the particle position $\mathbf{X}$ may be replaced by its place in space $\mathbf{x}$. This leads to

$$\mathbf{T}_E(\mathbf{x}) = \mathbf{k}(\mathbf{L}(\mathbf{x}); \mathbf{x}). \tag{3.4}$$

Since for the motions considered (NUSM) the following relations are also valid:

$$\mathbf{A}_1 = \mathbf{L}^T + \mathbf{L}, \qquad \mathbf{A}_{n+1} = \mathbf{A}_n\mathbf{L} + \mathbf{L}^T\mathbf{A}_n\,, \qquad n \geq 1, \tag{3.5}$$

the corresponding representation theorem analogous to that derived by Wang [7] can easily be proved (cf. [8]). Thus, it can be shown that the extra-stress tensor in the most general case amounts to

$$\mathbf{T}_E(\mathbf{X}, t) = \mathbf{f}(\mathbf{A}_1(\mathbf{X}, t), \mathbf{A}_2(\mathbf{X}, t), \mathbf{A}_3(\mathbf{X}, t); \mathbf{X}), \tag{3.6}$$

where all the quantities depend on the particle position $\mathbf{X}$. Similarly to the case of MCSH, a knowledge of the first two kinematic tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ is sufficient to determine $\mathbf{C}_t^t(\mathbf{X}, s)$ uniquely, if either $\mathbf{A}_1$ has three different eigenvalues, or two of them are equal but differ from the third one and, moreover, $[\mathbf{A}_2] = [\mathbf{A}_1^2]$ in the same basis in which $\mathbf{A}_1$ has a diagonal form. Such a generalization is possible since the proof of the theorem is based on the geometry of matrices involved, independently of whether they are functions of $\mathbf{X}$ or not.

4. The case of non-uniform steady extensional flows

The non-uniform steady extensional motions, under the assumption of quasi-elongational approximation (cf. [2, 3]), may be useful as applied to various fibre spinning and drawing processes [10]. For example, any temperature distribution may lead to observable material non-uniformity (inhomogeneity). We will show that the above motions are particular cases of those described by Eq. (2.1) (NUSM).

To this end, consider the following exponential deformation gradient at time τ

$$\mathbf{F}_0(\mathbf{X}, \tau) = \exp(\tau \mathbf{M}(\mathbf{X})), \tag{4.1}$$

where $\mathbf{X}$, like in Sec. 2, denotes the particle position at an arbitrary reference configuration, and the time-independent tensor $\mathbf{M}(\mathbf{X})$ is of a diagonal form. Instead of Eqs. (2.2), (2.4) and (2.5) we arrive at

$$\mathbf{L}_1(\mathbf{X}) = \frac{\partial}{\partial \tau}\mathbf{F}_t(\mathbf{X}, \tau)\Big|_{\tau=t} = \mathbf{L}(\mathbf{X}) = \mathbf{M}(\mathbf{X}), \tag{4.2}$$

$$\mathbf{F}_t(\mathbf{X}, t-s) = \exp(-s\mathbf{M}(\mathbf{X})), \qquad \tau = t-s, \qquad 0 \le s < \infty, \tag{4.3}$$

and

$$\mathbf{C}_t^t(\mathbf{X}, s) \equiv \mathbf{C}_t(\mathbf{X}, t-s) = \exp(-s\mathbf{L}^T(\mathbf{X}, t)) \exp(-s\mathbf{L}(\mathbf{X}, t)). \tag{4.4}$$

Therefore, for the flows considered, the velocity gradient $\mathbf{L}_1(\mathbf{X})$ is equal to the parametric tensor $\mathbf{L}(\mathbf{X})$ and also to $\mathbf{M}(\mathbf{X})$.

Now, the constitutive equations (3.1) lead to

$$\mathbf{T}_E(\mathbf{X}) = \mathbf{g}(\mathbf{L}(\mathbf{X}); \mathbf{X}), \tag{4.5}$$

where $\mathbf{g}$ is an isotropic function of the tensor argument, or to Eq. (3.4), if the spatial description of material non-uniformity is used.

Since for general extensional flows with diagonal $\mathbf{A}_1$ we have

$$\mathbf{A}_n = (\mathbf{A}_1)^n = (2\mathbf{L})^n, \qquad n \ge 1, \tag{4.6}$$

we can write instead of Eq. (4.5)

$$\mathbf{T}_E(\mathbf{X}) = \mathbf{k}(\mathbf{A}_1(\mathbf{X}); \mathbf{X}). \tag{4.7}$$

After taking into account the relevant representation of an isotropic tensor function of one symmetric tensor argument (cf. [8, 9]), we finally obtain

$$\mathbf{T}_E(\mathbf{X}) = \beta_1 \mathbf{A}_1(\mathbf{X}) + \beta_2 \mathbf{A}_1^2(\mathbf{X}), \qquad \operatorname{tr} \mathbf{A}_1 = 0, \tag{4.7}$$

where the material functions β_1 and β_2, depending on the invariants of $\mathbf{A}_1$ are also explicit functions of the position $\mathbf{X}$ (or the place $\mathbf{x}$ in steady flows).

5. Conclusions

Non-uniform stagnant motions (NUSM) are some generalization of the well known motions with constant stretch history (MCSH) defined by Coleman and Noll. In the case of materially non-uniform incompressible simple fluids, the constitutive equations take a form very similar to that valid for MCSH.

In the case of non-uniform steady extensional flows the corresponding constitutive equations simplify considerably and, of course, are independent of time. Those equations may be used in many practically important quasi-elongational flows such as fibre spinning and drawing processes.

References

1. W. Noll, *Materially uniform simple bodies with inhomogeneities*, Arch. Rat. Mech. Anal., **27**, 1, 1967.
2. S. Zahorski, *Necking in non-isothermal high-speed spinning with radial viscosity variation*, J. Non-Newtonian Fluid Mech., **50**, 65, 1993.
3. S. Zahorski, *Necking in non-isothermal high-speed spinning as a problem of sensitivity to external disturbances*, J. Non-Newtonian Fluid Mech., (to appear).
4. B.D. Coleman and W. Noll, *Steady extension of incompressible simple fluids*, Phys. Fluids, **5**, 840, 1962.
5. B.D. Coleman, *Kinematical concepts with applications in the mechanics and thermodynamics of incompressible viscoelastic liquids*, Arch. Rat. Mech. Anal., **9**, 273, 1962.
6. W. Noll, *Motions with constant stretch history*, Arch. Rat. Mech. Anal., **11**, 97, 1962.
7. C.C. Wang, *A representation theorem for the constitutive equation of a simple material in motions with constant stretch history*, Arch. Rat. Mech. Anal., **20**, 329, 1965.
8. S. Zahorski, *Mechanics of viscoelastic fluids*, Martinus Nijhoff, The Hague 1982.
9. C. Truesdell, *A first course in rational continuum mechanics*, The Johns Hopkins University, Baltimore 1982.
10. A. Ziabicki, *Fundamentals of fibre formation. The science of fibre spinning and drawing*, Wiley, London 1976.

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Received December 8, 1995.

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