Contents of issue 4 vol. XLVII

Polish Academy of Sciences
Institute of Fundamental Technological Research

Archives of Mechanics

Archiwum Mechaniki Stosowanej

volume 47 issue 4

Polish Scientific Publishers PWN
Warszawa 1995

ARCHIVES OF MECHANICS IS DEVOTED TO
Theory of elasticity and plasticity • Theory of nonclassical continua • Physics of continuous media • Mechanics of discrete media • Nonlinear mechanics • Rheology • Fluid gas-mechanics • Rarefied gas • Thermodynamics

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Printed in Poland, Editorial Office: Świętokrzyska 21,
00-049 Warszawa (Poland)

Arkuszy wydawniczych 12,25. Arkuszy drukarskich 11.25
Papier offset. kl. III 70 g. B1. Oddano do składania w lipcu 1995 r.
Druk ukończono w październiku 1995 r.
Druk i oprawa: Drukarnia Braci Grodzickich, Żabieniec ul. Przelotowa 7

Arch. Mech., 47, 4, pp. 647–659, Warszawa 1995

Instability of disturbed elongational flows of viscoelastic fluids

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IN THE PAPER we consider the instability or sensitivity problem to small external disturbances of steady elongational flows of viscoelastic fluids realized in the rotary clamp extensometers. To this end the flows are treated as the flows with dominating extension (FDEs) and the stationary, exponentially increasing in time and oscillatory types of instability are discussed in greater detail. It is shown that steady elongational flows may be unstable if only small external disturbances are applied in stationary or dynamic forms.

1. Introduction

THERE EXIST many papers devoted to various aspects of instability of the elongational flows of viscoelastic fluids. They are connected either with stability problems considered as the eigen-value problems with homogeneous boundary conditions, or with sensitivity problems to small internally or externally imposed disturbances. For more details we refer to the books of DENN [1], ZIABICKI [2], PETRIE [3] and PEARSON [4]. The essential differences between instability and sensitivity problems are also discussed by KASE [5]. It is worth mentioning that instability problems of extensional flows of the Rivlin–Ericksen viscoelastic fluids were already considered in my early paper [6].

In the recent paper [7], closely connected with the present one, the instability problems of steady elongational flows realized in the rotary clamp extensometers, as well as the sensitivity problems to small external disturbances, were only briefly outlined. In the present contribution we seek unsteady or dynamic solutions of the problem considered to discuss possible instabilities caused by small disturbances imposed on various kinematic or dynamic quantities.

In Sec. 1 we briefly quote the necessary information on real elongational flows, i.e. with inertia, drag and surface-tension effects, treated as the flows with dominating extension (FDEs) widely discussed elsewhere [8, 9]. As the next step, we discuss in greater detail possible solutions of the disturbed boundary problem considered. In what follows, the stationary, oscillatory and exponentially increasing in time types of stability loss are consecutively studied. To this end some harmonic oscillations superposed on the additional velocity fields are used.

It is shown that the steady elongational flows considered may always be unstable if small external disturbances occur.

2. Real elongational flows as the flows with dominating extension

In real steady elongational flows a sample subjected to the action of inertial, drag and surface-tension forces does not preserve its cylindrical shape, i.e. its radius varies along the axis, Fig. 1. The non-uniform velocity profiles across the sample, as well as the presence of shear stresses, are necessary to balance the acting forces and satisfy the boundary conditions.

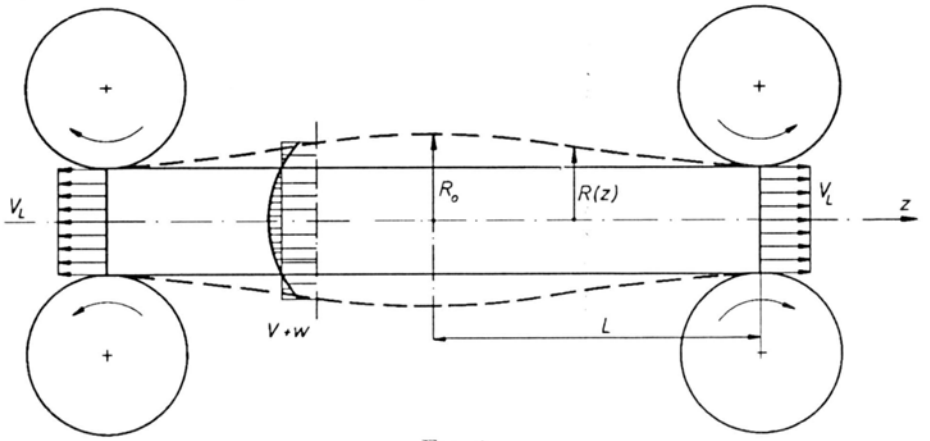

FIG. 1.

As shown in paper [7], the above requirements are met, by introducing the additional velocity field, with cylindrical components $(u, 0, w)$ resulting from the concept of flows with dominating extension (FDEs) (cf. [8, 9]). Thus, we start from the following velocity gradient:

$$(2.1)\qquad [\nabla \mathbf{V}^*] = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} V' + \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z} \\ 0 & \frac{u}{r} & 0 \\ \frac{\partial w}{\partial r} & 0 & \frac{\partial w}{\partial z} \end{bmatrix},$$

where the first matrix corresponds to a steady elongational flow with constant gradient V', while the second one denotes the additional gradients. If we assume, moreover, that in the flow considered the ratio of sample radius to its half-length is a small quantity (like in a thin layer) $\varepsilon = R_0/L \ll 1$, the perturbed constitutive equations can be used in the form:

$$(2.2)\qquad \mathbf{T}^* = -p\mathbf{1} + \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_1^+ + \beta_2(\mathbf{A}_1^2)^+ + \frac{d\beta_1}{dV'}(V')^+\mathbf{A}_1 + \frac{d\beta_2}{dV'}(V')^+\mathbf{A}_1^2,$$

where the crosess denote the increments of the Rivlin–Ericksen kinematic tensors and increments of the gradients V'. β_i $(i = 1, 2)$ are the V'-dependent material functions, while $d\beta_i/dV'$ denote the corresponding derivatives.

For the fundamental quasi-elongational flow we have

$$T^{33} - T^{11} = 3\beta V', \tag{2.3}$$

where the elongational viscosity η_E can be defined as

$$\eta_E \equiv 3\beta = \beta_1 + \beta_2 V'. \tag{2.4}$$

Similarly, the balance of axial forces (or momentum balance) leads to

$$F_0 \equiv 3\beta V' \pi R_0^2 = 3\beta V' \pi R^2 + F, \tag{2.5}$$

where R is the variable radius of a sample, R_0 – the radius in the middle plane of the sample i.e. for $z = 0$, and F denotes the force composed of inertia, drag and surface-tension forces, respectively (cf. [7]). Of course, we have for the fundamental velocities

$$V(0) = 0, \qquad V(L) = V_L = V'L, \tag{2.6}$$

where V_L denotes the uniform velocity at both ends of a sample (Fig. 1).

3. Governing equations for steady and unsteady flows

The constitutive equations (2.2) introduced into the equations of equilibrium expressed in cylindrical coordinates lead, after retaining the terms only of the highest order of magnitude with respect to ε (i.e. $\varepsilon^0 = 1$), to the following nonlinear differential equation (cf. [7, 9]):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{2}\frac{k}{V'}\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2 = \frac{C(z)}{\beta} + \frac{\varrho}{\beta}\frac{\partial w}{\partial t}, \tag{3.1}$$

where

$$k = \frac{1}{\beta}\left(\frac{d\beta_1}{dV'} + \frac{d\beta_2}{dV'}V'\right)V' \tag{3.2}$$

is the material parameter. The function $C(z)$ results from the relations:

$$C(z) = \frac{dp^*}{dz}, \qquad \frac{\partial p^*}{\partial r} = 0, \qquad p^* = p - T^{*11}. \tag{3.3}$$

The boundary conditions for the additional velocity field are applied in the following form:

$$\int_0^R w\, 2\pi r\, dr = 0, \tag{3.4}$$

and

$$R'\left(T^{*33}-T^{*11}\right)\Big|_{r=R}=T^{*13}\Big|_{r=R}, \tag{3.5}$$

where the prime denotes differentiation with respect to z, and

$$\begin{aligned} T^{*33}-T^{*11} &= 3\beta V'+\frac{1}{2}\beta\frac{k}{V'}\left(\frac{\partial w}{\partial r}\right)^2, \\ T^{*13} &= \beta\frac{\partial w}{\partial r}. \end{aligned} \tag{3.6}$$

The first condition (3.4) ensures that at both ends of a sample the average velocities are equal to V_L, while the second condition (3.5) expresses the balance of all forces acting on the free surface, and is exactly equivalent to that derived by KASE [10] (Fig. 1).

Under the Newtonian approximation ($k \equiv 0$), Eq. (3.1) takes the simplified form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_0}{\partial r}\right)=\frac{C_0(z)}{\beta_0}+\frac{\varrho}{\beta_0}\frac{\partial w_0}{\partial t}, \tag{3.7}$$

where the subscript 0 denotes Newtonian quantities.

For slightly non-Newtonian fluids, i.e. in the range in which parameter k is small enough (cf. [9]), we shall seek unsteady solutions of Eq. (3.1) in the form of the following series:

$$\begin{aligned} w &= w_0+kw_1+(\tilde{w}_0+k\tilde{w}_1)\exp\lambda(t-t_0), \\ C &= C_0+kC_1+(\tilde{C}_0+k\tilde{C}_1)\exp\lambda(t-t_0), \end{aligned} \tag{3.8}$$

where the subscript 1 denotes the first non-Newtonian terms, and the tildas refer to the corresponding dynamic amplitudes. The parameter λ is connected with the first frequency of harmonic oscillations imposed at the moment $t = t_0$; it may be real, imaginary or, in general, complex.

For steady-state solutions the boundary condition (3.5) takes the form:

$$R'3\beta_0 V'\Big|_{r=R}=\beta_0\frac{\partial w_0}{\partial r}\Big|_{r=R}, \tag{3.9}$$

in the case of a Newtonian fluid, and

$$R'\left(3\beta V'+\frac{9}{2}k\beta V'R'^2\right)\Big|_{r=R}=\beta\left(\frac{\partial w_0}{\partial r}+k\frac{\partial w_1}{\partial r}\right)\Big|_{r=R}, \tag{3.10}$$

in the case of a non-Newtonian one. The steady-state solutions of Eqs. (3.7) and (3.5) amount to

$$w_0=\frac{C_0}{4\beta_0}\left(r^2-\frac{R^2}{2}\right), \qquad C_0=6\beta_0\frac{R'}{R}V', \tag{3.11}$$

and

$$w_1 = \frac{C_1}{4\beta}\left(r^2 - \frac{R^2}{2}\right) - \frac{9}{32}V'\frac{\partial}{\partial z}\left(\frac{R'}{R}\right)^2\left(r^4 - \frac{R^4}{3}\right), \tag{3.12}$$

respectively. Finally, we arrive at (cf. [7])

$$w_1 = \frac{9}{4}V'R\left(\frac{R'}{R}\right)^2\left(r^2 - \frac{R^2}{R}\right) + \frac{9}{16}V'R^2\frac{\partial}{\partial z}\left(\frac{R'}{R}\right)^2\left(r^2 - \frac{R^2}{2}\right) - \frac{9}{32}V'\frac{\partial}{\partial z}\left(\frac{R'}{R}\right)^2\left(r^4 - \frac{R^4}{3}\right). \tag{3.13}$$

For unsteady solutions the boundary condition (3.5) leads to the following expressions:

$$R'\Delta T\Big|_{r=R} = \beta_0\frac{\partial\tilde{w}_0}{\partial r}\Big|_{r=R}, \tag{3.14}$$

and

$$R'\Delta_1 T\Big|_{r=R} = \beta\frac{\partial\tilde{w}_1}{\partial r}\Big|_{r=R}, \tag{3.15}$$

where by ΔT and $\Delta_1 T$ we have denoted the dynamic increments of the stress difference:

$$T^{*33} - T^{*11} = 3\beta V' + (\Delta T + k\Delta_1 T)\exp\lambda(t - t_0). \tag{3.16}$$

If the above increments are caused by disturbances of other quantities, on the basis of Eqs. (2.3), (2.5), we note that either

$$\Delta T = 3V'\Delta\beta + 3\beta\Delta V', \tag{3.17}$$

or

$$\Delta T = \frac{\Delta F_0}{\pi R_0^2} - 2\frac{\Delta R_0}{R_0}\left(T^{33} - T^{11}\right), \tag{3.18}$$

where $\Delta\beta$, ΔF_0 and ΔR_0 denote positive or negative increments of the viscosity, the velocity gradient, the external force and the radius at $z = 0$, respectively.

Formal solution of the differential equations (3.7) and (3.1), satisfying the boundary conditions (3.14) and (3.15), can be presented as

$$\tilde{w}_0 = \tilde{w}_1 = \frac{\tilde{C}_{0,1}}{\beta}R^2\left[\frac{1}{2x}\frac{I_0\left(x\frac{r}{R}\right)}{I_1(x)} - \frac{1}{x^2}\right], \tag{3.19}$$

where

$$\tilde{C}_0 = 2\Delta T \frac{R'}{R}, \qquad \tilde{C}_1 = 2\Delta_1 T \frac{R'}{R}, \qquad x = \sqrt{\frac{\varrho\lambda}{\beta}} R, \tag{3.20}$$

if only the first harmonics are taken into account (cf. Eqs. (3.8)) for a Newtonian and non-Newtonian case as well. The functions I_0 and I_1 are the modified Bessel functions of the first kind, of order 0 and 1, respectively.

It results immediately from Eqs. (3.19), (3.20) that for $\Delta T = \Delta_1 T = 0$, i.e. in the absence of external disturbances, any unsteady solutions in the form (3.8) are not possible at all.

4. Instability caused by various external disturbances

Unsteady solutions discussed previously will be used to investigate the instability problems or rather the sensitivity problems to various external disturbances imposed on steady elongational flows.

In the case of harmonic disturbances in the form (3.8) the flow will be unstable, if there exists at least one solution of the problem for which the amplitudes limitlessly increase with time. The above condition may be regarded as the sufficient condition of instability "in the small" (cf. [11]), independently of whether the boundary conditions are homogeneous or disturbed.

If we assume that in our considerations

$$\lambda = \lambda_1 - i\lambda_2, \tag{4.1}$$

where λ_1 and λ_2 are real (the minus sign has no essential meaning; it is a matter of convention), the sufficient condition of instability can be written as

$$\lambda_1 = \operatorname{Re}\lambda > 0. \tag{4.2}$$

We also take into account the so-called "marginal" or "neutral" stability which is possible, if $\operatorname{Re}\lambda = 0$ passing from negative to positive values. Depending on what value is taken by $\operatorname{Im}\lambda$, two types of the stability loss may occur: when $\operatorname{Im}\lambda = 0$ we have the stationary type of instability, and when $\operatorname{Im}\lambda \neq 0$ – the oscillatory type.

In our problem of sensitivity to external disturbances all the types of instability may occur independently, and "the principle of exchange of stabilities", valid in the case of viscous fluids (cf. [11]), cannot be proved. Thus, the case of stationary instability ($\lambda = 0$), the case of exponentially increasing amplitudes ($\lambda_1 > 0$, $\lambda_2 = 0$) and that of oscillatory instability ($\lambda_1 = 0$, $\lambda_2 \neq 0$) must be studied separately.

4.1. The general case of complex frequencies.

For the case of small harmonic disturbances the differential equation (3.1) can be presented in the linearized form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{w}}{\partial r}\right) - \frac{\varrho\lambda}{\beta}\tilde{w} = \frac{\tilde{C}(z)}{\beta}, \tag{4.3}$$

where the subscript 0 or 1 have been omitted, since the above equation is valid for Newtonian and non-Newtonian fluids as well. This equation, together with the boundary conditions (3.4), (3.5), defines the boundary-value problem for additional velocities depending on external disturbances.

If λ is of the form (4.1), we obtain

$$\tilde{w} = \tilde{D}J_0\left(\sqrt{\frac{-\varrho\lambda}{\beta}}\,r\right) - \frac{\tilde{C}}{\varrho\lambda}, \tag{4.4}$$

or

$$\tilde{w} = \tilde{D}J_0\left(Me^{i\varphi}\right) - \frac{\tilde{C}}{\varrho(\lambda_1 - i\lambda_2)}, \tag{4.5}$$

where $\tilde{C}$ is defined by Eqs. (3.20) and

$$J_0\left(Me^{i\varphi}\right) = \mathrm{Ber}(M,\varphi) + i\,\mathrm{Bei}(M,\varphi), \tag{4.6}$$

$$M^2 = \sqrt{\lambda_1^2 + \lambda_2^2}\,\frac{\varrho r^2}{\beta}, \tag{4.7}$$

$$\mathrm{tg}^2\varphi = \left(\pm\sqrt{\lambda_1^2 + \lambda_2^2} + \lambda_1\right) \Big/ \left(\pm\sqrt{\lambda_1^2 + \lambda_2^2} - \lambda_1\right).$$

Here Ber, Bei denote the Kelvin functions of the first kind, depending on the modulus M and argument φ.

The boundary conditions (3.4), (3.5) lead to

$$\tilde{D}J_1\left(\sqrt{\frac{-\varrho\lambda}{\beta}}\,R\right)\sqrt{\frac{-\varrho\lambda}{\beta}} = -\frac{\tilde{C}R}{2\beta}, \qquad \tilde{C} = 2\Delta T\frac{R'}{R} \tag{4.8}$$

or, since $\mathrm{Im}\,\tilde{C} = 0$, they lead finally to

$$\tilde{D}(a\cos\varphi - b\sin\varphi)J_1(M) = -\frac{\tilde{C}R^2}{2\beta}, \tag{4.9}$$

$$(b\cos\varphi + a\sin\varphi) = 0, \tag{4.10}$$

where

$$a^2 = \frac{\varrho r^2}{2\beta}\left(-\lambda_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2}\right), \qquad b^2 = \frac{\varrho r^2}{2\beta}\left(\lambda_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2}\right). \tag{4.11}$$

The condition (4.10) expresses the requirement that the imaginary part of Eq. (4.8) must be identically equal to zero. This is possible only for some combinations of real and imaginary parts of λ. Thus, certain solutions of the boundary-value problem are always admissible for $\lambda_1 > 0$.

In what follows we shall discuss solutions valid for particular forms of λ.

4.2. The case of neutral or stationary type of instability

For $\lambda = 0$ ($\lambda_1 = \lambda_2 = 0$) Eq. (4.3) leads to the following time-independent solution:

$$\tilde{w} = \frac{\tilde{C}}{4\beta}\left(r^2 - \frac{R^2}{2}\right), \qquad \tilde{C} = 2\Delta T \frac{R'}{R}. \tag{4.12}$$

This result means that some stationary fields of additional velocities may occur, leading to the phenomenon of local necking or swelling. The higher extensional viscosity β always reduces admissible amplitudes of disturbances.

4.3. The case of disturbances increasing exponentially in time

This case corresponds to $\lambda_1 > 0$, $\lambda_2 = 0$ in the solution (3.19)–(3.21) with λ replaced by λ_1. The dependence of solution on the parameter $x = \sqrt{\varrho\lambda_1/\beta}R$ is shown in Fig. 2. It is clearly seen that admissible disturbances may remarkably increase for small values of λ_1.

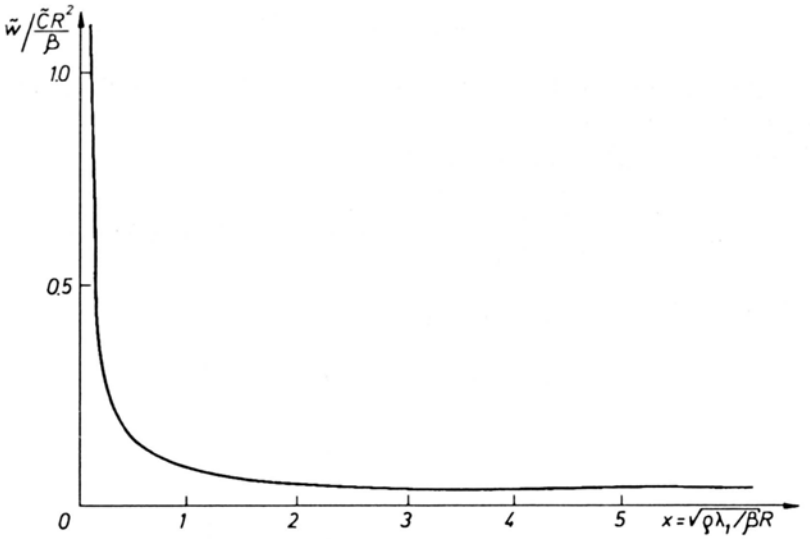

FIG. 2.

4.4. The case of oscillatory type of instability

This case described by $\lambda_1 = 0$, $\lambda = -i\lambda_2$ leads to the solution:

$$\tilde{w} = \tilde{D} J_0\left(\sqrt{\frac{i\varrho\lambda_2}{\beta}}\, r\right) + \frac{\tilde{C}}{i\varrho\lambda_2}, \tag{4.13}$$

and the boundary conditions (3.4), (3.5) give

$$\tilde{D} = -\tilde{C}R/2\beta\sqrt{\frac{i\varrho\lambda_2}{\beta}}\, J_1\left(\sqrt{\frac{i\varrho\lambda_2}{\beta}}\, R\right), \qquad \tilde{C} = 2\Delta T\frac{R'}{R}. \tag{4.14}$$

The requirement that $\tilde{C}$ is real leads to

$$\mathrm{bei}'z = 0, \qquad z = \sqrt{\varrho\lambda_2/\beta}\, R, \tag{4.15}$$

where

$$\mathrm{bei}'z = \frac{1}{\sqrt{2}}(\mathrm{bei}_1 z - \mathrm{ber}_1 z), \qquad \mathrm{ber}'z = \frac{1}{\sqrt{2}}(\mathrm{bei}_1 z + \mathrm{ber}_1 z), \tag{4.16}$$

are the corresponding combinations of the Kelvin functions of order 1. It can easily be observed that Eq. $(4.15)_1$ is satisfied only for the following values of z

$$z = 3.77,\ 8.28,\ 12.74,\ 17.19,\ 21.64,\ \ldots \tag{4.17}$$

i.e., for some small ranges of frequencies if the radius R varies from R_0 to R_L (Fig. 1).

The real and imaginary parts of the solution (4.13) can be presented as

$$\mathrm{Re}\,\tilde{w} = \frac{\tilde{C}R^2}{\beta}\left[\frac{1}{2z}\,\frac{\mathrm{ber}\,z\,\mathrm{ber}'z + \mathrm{bei}'z\,\mathrm{bei}\,z}{(\mathrm{ber}'z)^2 + (\mathrm{bei}'z)^2}\right], \tag{4.18}$$

$$\mathrm{Im}\,\tilde{w} = \frac{\tilde{C}R^2}{\beta}\left[\frac{1}{2z}\,\frac{\mathrm{ber}\,z\,\mathrm{bei}'z - \mathrm{bei}\,z\,\mathrm{ber}'z}{(\mathrm{ber}'z)^2 + (\mathrm{bei}'z)^2} - \frac{1}{z^2}\right], \tag{4.19}$$

and shown schematically in Fig. 3. It is seen that larger amplitudes of disturbances of the additional velocity fields are possible for relatively small frequencies λ_2. The increasing extensional viscosity β reduces the frequencies at which instability occurs.

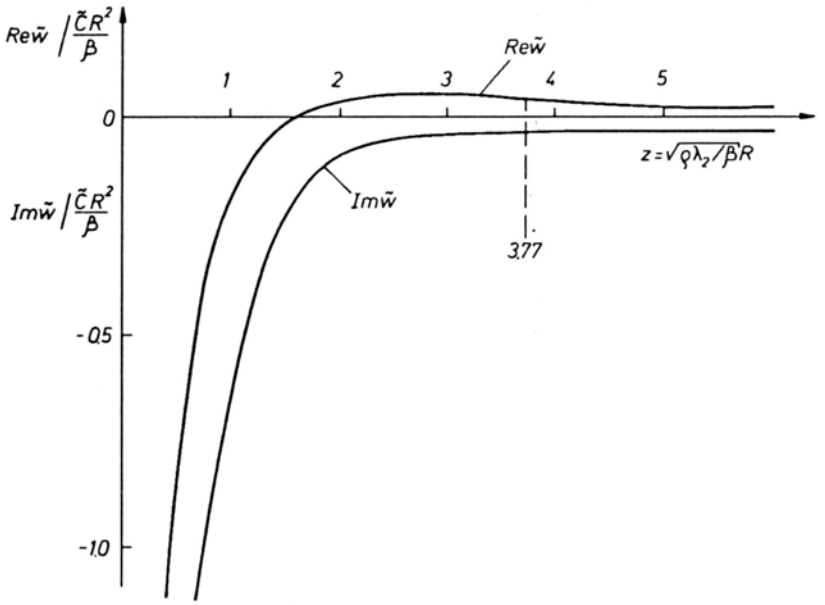

FIG. 3.

5. Instability of plane extensional flows

For rectangular cross-sections of samples subjected to plane extensional flows we obtain, instead of Eq. (3.1), the following differential equation (cf. [8, 12]):

$$\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\frac{k}{V'}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{C(x)}{\beta} + \frac{\varrho}{\beta}\frac{\partial w}{\partial t}, \tag{5.1}$$

where

$$k = \frac{1}{\beta}\frac{d\beta}{dV'}V', \qquad \beta \equiv \beta_1, \qquad C(x) = \frac{dp^*}{dx}, \tag{5.2}$$

and w denotes the additional velocity component along the sample axis. Now, the boundary conditions take the form:

$$\int_0^h wb\,dy = 0, \tag{5.3}$$

and

$$h'\left(T^{*11} - T^{*22}\right)_{y=h} = T^{*12}\Big|_{y=h}, \tag{5.4}$$

where b denotes the constant width and h – the variable height of a rectangular cross-section.

For unsteady solutions in the form of small-amplitude harmonic oscillations (cf. (3.8)), we arrive at

$$\frac{\partial^2 w}{\partial y^2} - \frac{\varrho\lambda}{\beta}\tilde{w} = \frac{\tilde{C}(x)}{\beta}, \tag{5.5}$$

where the tildas again denote the dynamic quantities. For Newtonian as well as non-Newtonian fluids, the corresponding boundary-value problem leads to the solution:

$$\tilde{w} = \tilde{D}\left[\exp\left(\sqrt{\frac{\varrho\lambda}{\beta}}y\right) + \exp\left(-\sqrt{\frac{\varrho\lambda}{\beta}}y\right)\right] - \frac{\tilde{C}}{\varrho\lambda}, \tag{5.6}$$

where $\tilde{D}$ is defined by Eq. (5.14) and

$$\tilde{C}(x) = \Delta T\frac{h'}{h}, \tag{5.7}$$

and ΔT denotes the first dynamic increment of the stress difference $T^{11} - T^{22} = 4\beta V'$.

In the case of stationary (neutral) type of instability ($\lambda_1 = \lambda_2 = 0$), we obtain

$$\tilde{w} = \frac{\tilde{C}}{2\beta}\left(y^2 - \frac{h^2}{3}\right). \tag{5.8}$$

Similarly, for the case of disturbances increasing exponentially in time ($\lambda_1 > 0$, $\lambda_2 = 0$), we arrive at

$$\tilde{w} = \frac{\tilde{C}h^2}{\beta}\left(\frac{1}{\zeta\,\mathrm{th}\,\zeta} - \frac{1}{\zeta^2}\right), \qquad \zeta = \sqrt{\frac{\varrho\lambda_1}{\beta}}h. \tag{5.9}$$

The plot of the above function looks very similar to that shown in Fig. 2 for axially symmetric elongation.

In the case of oscillatory type of instability ($\lambda_1 = 0$, $\lambda = -i\lambda_2$), we have

$$\mathrm{Re}\,\tilde{w} = \frac{\tilde{C}R^2}{\beta}\left[\frac{(\mathrm{th}\,\xi - \mathrm{tg}\,\xi) + \mathrm{th}\,\xi\,\mathrm{tg}\,\xi(\mathrm{th}\,\xi + \mathrm{tg}\,\xi)}{2\xi(\mathrm{th}^2\xi + \mathrm{tg}^2\,\xi)}\right], \tag{5.10}$$

$$\mathrm{Im}\,\tilde{w} = \frac{\tilde{C}R^2}{\beta}\left[\frac{(\mathrm{tg}\,\xi - \mathrm{th}\,\xi)\mathrm{th}\,\xi\,\mathrm{tg}\,\xi + (\mathrm{th}\,\xi + \mathrm{tg}\,\xi)}{2\xi(\mathrm{th}^2\xi + \mathrm{tg}^2\,\xi)} - \frac{1}{\xi^2}\right], \tag{5.11}$$

where

$$\xi = \sqrt{\varrho\lambda_2/\beta}h. \tag{5.12}$$

For sufficiently high values of the parameter ξ we can use the following limits:

$$\lim_{\xi\to\infty} \operatorname{Re}\tilde{w} = \frac{1}{2\xi} = 0, \qquad \lim_{\xi\to\infty} \operatorname{Im}\tilde{w} = \frac{1}{2\xi} - \frac{1}{2\xi^2} = 0. \tag{5.13}$$

The plots of the functions (5.10), (5.11) or (5.13) are very similar to those shown in Fig. 3 for axially symmetric flows.

The boundary conditions (5.3), (5.4) lead to

$$\tilde{D} = \frac{\tilde{C}h}{\varrho\lambda}\sqrt{\varrho\lambda}\beta\,(\exp\kappa - \exp(-\kappa))^{-1}, \tag{5.14}$$

where $\tilde{C}$ has been defined by Eq. (5.7). The requirement that imaginary part of $\tilde{C}$ is identically equal to zero gives the following condition for κ:

$$\cos\kappa \operatorname{sh}\kappa + \sin\kappa \operatorname{ch}\kappa = 0, \tag{5.15}$$

where

$$\kappa = \sqrt{\varrho\lambda_2/2\beta}h = \frac{1}{\sqrt{2}}\xi. \tag{5.16}$$

In the limit of $\kappa \to \infty$, we have instead of Eq. (5.15)

$$\cos\kappa + \sin\kappa = 0. \tag{5.17}$$

It can easily be observed that Eq. (5.15) is satisfied only for the following values of κ:

$$\kappa = 2.35,\ 5.50,\ 8.64,\ 11.78,\ 14.92,\ \ldots \tag{5.18}$$

i.e. for some small ranges of frequencies, if the height varies from h_0 to h_L.

6. Conclusions

The instability studies presented in the paper lead to the following general conclusions:

1. Steady elongational flows realized under real experimental conditions, i.e. with inertia, drag and surface-tension effects taken into account, may be unstable if only small external disturbances are imposed on the fundamental flow.

2. These external disturbances can be connected in a straightforward manner with the normal stress difference or caused by the corresponding increments of forces, velocities, viscosities, radii, etc.

3. The stationary, the exponentially increasing in time and the oscillatory types of stability loss are admissible and independent of each other.

4. The stationary or increasing in time types of instability may occur for any small disturbances, while the oscillatory type is possible for some discrete ranges of frequencies. Usually smaller amplitudes of disturbances and extensional viscosities lead to the effective instability for higher values of frequencies.

5. All the conclusions drawn for axially symmetric elongations remain valid for plane extensional flows of samples with rectangular cross-sections.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received September 16, 1994.

Arch. Mech., 47, 4, pp. 661–676, Warszawa 1995

On some unsteady motions of fluids of second grade

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WE FIRST STUDY the start-up flow of a fluid of second grade occupying the space above a plate which is impulsively set in motion and show that previous attempts to solve the problem using Laplace transform techniques are erroneous, and that the inversion of the Laplace transform is far from routine. We then study the problems in which the plate moves with a constant speed or acceleration or is subject to a constant shear. The differences between these solutions and the corresponding cases in the Navier–Stokes theory are delineated.

1. Introduction

STOKES' FIRST PROBLEM also known as Rayleigh's problem, namely the flow of a fluid past an impulsively started body of arbitrary shape, has received much attention by many researchers, in view of its practical importance. STOKES [1] solved the problem for a viscous fluid flow past an infinite plate. SOUNDALGEKAR [2] extended it to a fluid of second grade using a perturbation technique. TEIPEL [3] showed that for such a fluid, a strict similarity solution does not exist and provided a series solution. PURI [4] studied the problem and found a solution which does not satisfy the initial condition.

As the governing equations are linear, the method of Laplace transform is particularly well suited, at least in theory, for solving the problem. However, it is usually not a trivial matter to invert the Laplace transform. For the problems that are considered here, the method of separation of variables does not work, unlike in other problems in bounded domains and in some special problems in unbounded domains (cf. RAJAGOPAL [5]). We show that the method of Laplace transform does not work for the Rayleigh problem, in that the solution so obtained does not satisfy the initial condition. We show that this is due to an incompatibility between the prescribed data. We then proceed to show that there is no such problem when care is exercised in prescribing the initial data.

As early as 1963, TING [6] studied unsteady flows of fluids of second grade by appealing to methods in complex analysis. His elegant work showed that solutions to the problems he considered exist only if one of the material moduli that characterize the fluid, namely α_1, is positive. This result of his was pregnant with consequences and was not recognized at that time. Subsequently, it was shown by DUNN and FOSDICK [7] that if all motions of the fluid are to be compatible with the Clausius–Duhem inequality and with the assumption that the specific Helmholtz free energy is a minimum at equilibrium, then α_1 has to be non-negative. It has also been shown that $\alpha_1 < 0$, even purely within the context of mechanics, leads

to undesirable stability characteristics (cf. FOSDICK and RAJAGOPAL [8], GALDI, PADULA and RAJAGOPAL [9]). These results are in keeping with the seminal work of TING [6]. Recently, DUNN and RAJAGOPAL [10] have written an exhaustive article that discusses the issues of mechanical and thermal stability. Here also, not surprisingly, we find that if $\alpha_1 < 0$, the solution to the problems that we consider blows up in time.

The arrangement of the paper is as follows. In Sec. 2, we document the governing equations. This is followed by the study of the flow due to an impulsively accelerated plate in Sec. 3. In Sec. 4, we show that if $\alpha_1 < 0$, the solution to the problem becomes unbounded with time. The flow induced due to the sudden constant application of a shear stress at a plate is discussed in Sec. 5 and, finally, we study the problem of the flow induced due to a constantly accelerating plate in Sec. 6.

2. Governing equations

The incompressible fluid of grade two is characterized by the following constitutive equation (cf. [5]):

$$\mathbf{T} = -p\,\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are normal stress moduli, $-p\,\mathbf{I}$ denotes the indeterminate spherical stress and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = (\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T \tag{2.2}_1$$

and

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_1\,. \tag{2.2}_2$$

Here $\mathbf{v}$ is the velocity, grad the gradient operator and (d/dt) the material time derivative.

If we substitute (2.1) for the stress $\mathbf{T}$ in the balance of linear momentum

$$\operatorname{div}\mathbf{T} + \varrho\,\mathbf{b} = \varrho\frac{d\mathbf{v}}{dt}, \tag{2.3}$$

we obtain, in the case of a conservative body force field $\mathbf{b} = -\operatorname{grad}\phi$,

$$\mu\Delta\mathbf{v} + \alpha_1\Delta\mathbf{v}_t + \alpha_1(\Delta\mathbf{w}\times\mathbf{v}) + (\alpha_1+\alpha_2)\Big\{\mathbf{A}_1\Delta\mathbf{v} + 2\operatorname{div}\Big[(\operatorname{grad}\mathbf{v})(\operatorname{grad}\mathbf{v})^T\Big]\Big\} - \varrho(\mathbf{w}\times\mathbf{v}) - \varrho\mathbf{v}_t = \operatorname{grad}P, \tag{2.4}$$

where

$$P = p - \alpha_1(\mathbf{v}\cdot\Delta\mathbf{v}) - \frac{(2\alpha_1+\alpha_2)}{4}|\mathbf{A}_1|^2 + \frac{1}{2}\varrho|\mathbf{v}|^2 + \varrho\phi \tag{2.5}$$

and Δ is the Laplacian, the subscript t indicates partial differentiation with respect to time, $|\mathbf{A}_1|$ the trace norm of $\mathbf{A}_1$, and

$$\mathbf{w} = \operatorname{curl}\mathbf{v}. \tag{2.6}$$

We have used the assumption of incompressibility of the fluid which implies that it can undergo only isochoric motions, therefore

$$\operatorname{div}\mathbf{v} = 0. \tag{2.7}$$

We shall consider unidirectional flows of the form

$$\mathbf{v} = v(x,t)\mathbf{j}, \tag{2.8}$$

where $\mathbf{j}$ denotes a unit vector along the y-coordinate direction. Equation (2.4) reduces to

$$\mu\frac{\partial^2 v}{\partial x^2} + \alpha_1\frac{\partial^3 v}{\partial x^2\,\partial t} - \varrho\frac{\partial v}{\partial t} = \frac{\partial p}{\partial y}, \tag{2.9}$$

$$(2\alpha_1+\alpha_2)\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial p}{\partial x}, \tag{2.10}$$

$$0 = \frac{\partial p}{\partial z}. \tag{2.11}$$

In the absence of a pressure gradient in the y-direction, Eq. (2.9) reduces to the linear partial differential equation

$$\varrho\frac{\partial v}{\partial t} = \mu\frac{\partial^2 v}{\partial x^2} + \alpha_1\frac{\partial^3 v}{\partial t\,\partial x^2}. \tag{2.12}$$

Equation (2.12) can be solved in principle by several methods, but their effectiveness depends strictly on the domain of definition of (2.12): when the domain is the half or the whole space, the separation of variables method fails since it yields a solution which does not satisfy the boundary conditions.

Here we will use the Laplace transform to study solutions to (2.12); with this technique the initial condition is automatically satisfied and the application of the boundary conditions is easier. However, the inversion procedure for obtaining the solution is not always straightforward and often requires care and ingenuity.

3. Flow due to an impulsively accelerated plate

Let us consider the counterpart to a problem studied by Stokes within the context of the Navier-Stokes theory, namely the flow engendered due to the sudden imposition of a velocity at some instant, say $t = 0^+$, to an infinite flat plate that has been at rest for all times $t \leq 0$; the region above the plate being occupied by a fluid of second grade. We are interested in solutions to Eq. (2.12) when $\alpha_1 \neq 0$, for, if $\alpha_1 = 0$, Eq. (2.12) reduces to the heat equation and the solution to the problem under consideration is well known (cf. CARSLAW and JAEGER [11]).

Let us suppose that $\alpha_1 > 0$. We shall find it convenient to nondimensionalize (2.12). We observe that (α_1/μ) has units of time, while $(\alpha_1/\varrho)^{1/2}$ has units of length. Let us introduce the variables:

$$\tau := \left(\frac{\mu}{\alpha_1}\right) t, \tag{3.1}$$

$$\xi := \left(\frac{\varrho}{\alpha_1}\right)^{1/2} x. \tag{3.2}$$

It would be appropriate to caution that the solutions for the Navier-Stokes fluid cannot be obtained from that for fluids of grade two by letting $\alpha_1 \to 0$, in virtue of the transformations (3.1) and (3.2).

It follows from (2.12), (3.1) and (3.2) that

$$\partial_\xi^2 V + \partial_\tau \partial_\xi^2 V = \partial_\tau V. \tag{3.3}$$

In deriving (3.3), we have assumed that

$$V := \frac{v}{U}, \tag{3.4}$$

where U is some characteristic speed, which in the present case would be the speed imposed on the plate at $x = 0$ at time $t = 0^+$.

Since the fluid is at rest for all $t \leq 0$,

$$V(\xi, 0) = 0. \tag{3.5}$$

Also, as the plate is maintained at a constant speed U for all $\tau > 0$, we have

$$V(0, \tau) = 1 \quad \forall \tau > 0. \tag{3.6}$$

Finally, the fluid will be assumed to be quiescent far away from the plate, and thus

$$V(\xi, \tau) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{3.7}$$

We observe from (3.5) and (3.6) that

$$\lim_{\xi\to 0} V(\xi,0) \neq \lim_{\tau\to 0} V(0,\tau), \tag{3.8}$$

and we shall see that this incompatibility causes difficulties in establishing a smooth solution to the Rayleigh problem in the case of a fluid of grade two.

It follows from the triangle inequality

$$|V(\xi,0)-V(0,\tau)| \leq |V(\xi,0)-V(\xi,\tau)| + |V(\xi,\tau)-V(0,\tau)|. \tag{3.9}$$

We also observe that

$$V(\xi,0)-V(\xi,\tau) = \int_0^\tau \frac{\partial V(\xi,\alpha)}{\partial\alpha}d\alpha = \int_0^\tau\int_0^\xi \frac{\partial^2 V(\beta,\alpha)}{\partial\alpha\,\partial\beta}d\alpha\,d\beta + \int_0^\tau \frac{\partial V(0,\alpha)}{\partial\alpha}d\alpha, \tag{3.10}$$

and similarly

$$V(0,\tau)-V(\xi,\tau) = \int_0^\tau\int_0^\xi \frac{\partial^2 V(\beta,\alpha)}{\partial\alpha\,\partial\beta}d\alpha\,d\beta + \int_0^\xi \frac{\partial V(\beta,0)}{\partial\beta}d\beta. \tag{3.11}$$

Also, for $\gamma\in(0,\xi)$, we have

$$\int_0^\tau \frac{\partial V(0,\alpha)}{\partial\alpha}d\alpha = \int_0^\tau \frac{\partial V(\gamma,\alpha)}{\partial\alpha}d\alpha - \int_0^\gamma\int_0^\tau \frac{\partial^2 V(\beta,\alpha)}{\partial\alpha\,\partial\beta}d\alpha\,d\beta. \tag{3.12}$$

Substituting (3.12) in (3.10) and integrating γ from 0 to ξ, we obtain

$$|V(\xi,0)-V(\xi,\tau)| \leq 2\int_0^\tau\int_0^\xi \left|\frac{\partial^2 V}{\partial\alpha\,\partial\beta}\right| d\alpha\,d\beta + \frac{(\xi\tau)^{1/q}}{\xi}\left(\int_0^\tau\int_0^\xi \left|\frac{\partial V}{\partial\alpha}\right|^q d\alpha\,d\beta\right)^{1/q}. \tag{3.13}$$

Similarly

$$|V(0,\tau)-V(\xi,\tau)| \leq 2\int_0^\tau\int_0^\xi \left|\frac{\partial^2 V}{\partial\alpha\,\partial\beta}\right| d\alpha\,d\beta + \frac{(\xi\tau)^{1/q}}{\xi}\left(\int_0^\tau\int_0^\xi \left|\frac{\partial V}{\partial\beta}\right|^q d\alpha\,d\beta\right)^{1/q}. \tag{3.14}$$

Adding (3.13) and (3.14), using the triangle inequality (3.9), and setting $\xi = \tau = a$, we obtain

$$|V(a,0) - V(0,a)| \leq 4 \int_0^a \int_0^a \left| \frac{\partial^2 V}{\partial\alpha\,\partial\beta} \right| d\alpha\, d\beta + a^{\frac{2}{q}-1} \left\{ \left(\int_0^a \int_0^a \left| \frac{\partial V}{\partial\beta} \right|^q d\alpha\, d\beta \right)^{1/q} + \left(\int_0^a \int_0^a \left| \frac{\partial V}{\partial\alpha} \right|^q d\alpha\, d\beta \right)^{1/q} \right\}. \tag{3.15}$$

We immediately conclude from (3.15) that

$$|V(a,0) - V(0,a)| \to 0 \qquad \text{as} \quad a \to 0, \tag{3.16}$$

which implies that smooth solutions to (3.3) have to meet the compatibility expressed in (3.16) and, as the data prescribed in (3.8) do not meet this compatibility, solutions with $|(\partial^2 V)/(\partial\alpha\,\partial\beta)|$ that are integrable are not possible.

Applying the Laplace transform

$$L\{f(\xi,\tau)\} := \int_0^\infty e^{-p\tau} f(\xi,\tau)\, d\tau \tag{3.17}$$

to (3.3), and using (3.4), we obtain

$$pL\{V(\xi,\tau)\} = (1+p)[L\{V(\xi,\tau)\}]_{\xi\xi} = (1+p)L\{V_{\xi\xi}(\xi,\tau)\}, \tag{3.18}$$

where the suffix ξ denotes partial derivative with respect to ξ. Since

$$L\{V(0,\tau)\} = \frac{1}{p}, \tag{3.19}$$

$$L\{V(\infty,\tau)\} = 0, \tag{3.20}$$

it follows from (3.18)–(3.20) that

$$L\{V(\xi,\tau)\} = \frac{1}{p} \exp\left\{ -\left(\frac{p}{1+p} \right)^{1/2} \xi \right\}. \tag{3.21}$$

3.1. Solution

We shall now invert the Laplace transform (*) given in (3.21) by appealing to a fundamental theorem in Laplace transforms (†): if

$$F(\xi,p) = L\{f(\xi,\tau)\}, \tag{3.22}$$

(*) A procedure similar to that outlined in this section has been used to study the propagation of waves in viscoelastic rods by Morrison. The equations he studies are different in structure from those considered here (cf. J.A. MORRISON, [14]).

(†) The theorem is attributed to Efros (cf. A.M. EFROS and A.M. DANILEVSKY [15]) by LUIKOV [16].

and

$$\frac{1}{g(p)} \exp\{-\eta h(p)\} = L\{\varphi(\eta, \tau)\}, \tag{3.23}$$

then

$$\frac{1}{g(p)} F(\xi, h(p)) = L\left\{\int_0^\infty \varphi(\eta, \tau) f(\xi, \eta)\, d\eta\right\}, \tag{3.24}$$

provided f, g and h satisfy certain conditions (cf. CARSLAW and JAEGER, p. 259 [11]).

Now (cf. [17]) recall (‡)

$$\frac{1}{p} \exp\left(\frac{\eta}{p}\right) = L\{I_0(2\eta^{1/2}\tau^{1/2})\}, \tag{3.25}$$

where I_0 is the modified Bessel function of order zero. It follows from (3.25) and the property of Laplace transforms, that

$$\frac{1}{p} \exp\left[-\eta\left(1 - \frac{1}{p}\right)\right] = L\left\{\left[I_0(2\eta^{1/2}\tau^{1/2})\right] \exp(-\eta)\right\}. \tag{3.26}$$

Also (§)

$$\frac{1}{p} \exp\left(-\xi^{1/2} p^{1/2}\right) = L\left\{\operatorname{erfc}\left(\frac{1}{2}\xi^{1/2}\tau^{-1/2}\right)\right\}, \tag{3.27}$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function. It follows immediately from (3.27) that

$$\frac{1}{p} \exp\left(-\xi p^{1/2}\right) = L\left\{\operatorname{erfc}\left(\frac{1}{2}\xi\tau^{-1/2}\right)\right\}. \tag{3.28}$$

Now, defining

$$\begin{aligned} g(p) &:= p, \\ h(p) &:= 1 - \frac{1}{p}, \end{aligned} \tag{3.29}$$

and

$$F(\xi, p) := \frac{1}{p} \exp(-\xi p^{1/2}), \tag{3.30}$$

(‡) Op. cit., Bateman, p. 245, formula (35).
(§) Ibid., p. 245, formula (3).

we find that

$$\frac{1}{g(p)}F(\xi, h(p)) = \frac{1}{(p-1)}\exp\left[-\xi\left(\frac{p-1}{p}\right)^{1/2}\right]. \tag{3.31}$$

Thus, by virtue of (3.24)

$$\frac{1}{(p-1)}\exp\left[-\xi\left(\frac{p-1}{p}\right)^{1/2}\right] = L\left\{\int_0^\infty \exp(-\eta) I_0(2\eta^{1/2}\tau^{1/2})\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) d\eta\right\}. \tag{3.32}$$

Using the shift theorem, we find that

$$L^{-1}\left\{\frac{1}{p}\exp\left[-\xi\left(\frac{p}{1+p}\right)^{1/2}\right]\right\} = \exp(-\tau)\int_0^\infty \exp(-\eta) I_0(2\eta^{1/2}\tau^{1/2})\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) d\eta, \tag{3.33}$$

and thus

$$V(\xi,\tau) = \exp(-\tau)\int_0^\infty \exp(-\eta) I_0(2\eta^{1/2}\tau^{1/2})\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) d\eta. \tag{3.34}$$

We notice that

$$\lim_{t\to 0} v(x,t) \neq 0,$$

thereby not satisfying the initial condition. This is not surprising due to the incompatibility of the data prescribed in that

$$\lim_{x\to 0} v(x,0) \neq \lim_{t\to 0} v(0,t).$$

However, (3.34) satisfies the boundary conditions and the governing equation and holds for $\tau > 0$.

It follows from (3.1) and (3.2) that

$$v(x,t) = U\exp\left\{\left(-\frac{\mu}{\alpha_1}\right)t\right\}\int_0^\infty \exp(-\eta) I_0\left(2\left(\frac{\mu}{\alpha_1}\right)^{1/2}\eta^{1/2}t^{1/2}\right) \times \operatorname{erfc}\left(\frac{1}{2}\left(\frac{\varrho}{\alpha_1}\right)^{1/2} x\eta^{-1/2}\right) d\eta. \tag{3.35}$$

4. Flow due to an impulsively accelerating plate: case $\alpha_1 < 0$

So far we have assumed the modulus α_1 to be positive. Now we consider the case $\alpha_1 < 0$. A few remarks about the sign of the material modulus α_1 are in order. The constitutive model (2.1) is properly frame-invariant and qualifies to model a fluid, if an appropriate fluid is to be found in nature. If the model has to be an exact model for some fluid, as is the Navier-Stokes model for, say, water, then thermodynamical arguments lead to the conclusion that $\alpha_1 \geq 0$. Of course, no model is exact in the sense that it can model perfectly all motions of the material. Even so, the spirit in which the classical models, once posited, are used is under this context. We shall not get into a detailed discussion of the meaning and validity of the model here. An interested reader will find a detailed exposition of the same in the paper by DUNN and RAJAGOPAL [10]. We will show that if $\alpha_1 < 0$, the solution to (2.12) is unbounded. We will consider Rayleigh problem as an example.

Suppose that $\alpha_1 < 0$. We define the non-dimensional variables

$$\hat{\tau} := \frac{\mu}{|\alpha_1|} t, \qquad \hat{\xi} := \left(\frac{\varrho}{|\alpha_1|}\right)^{1/2} x, \qquad V = \frac{v}{U}. \tag{4.1}$$

From (2.12) and (4.1) it follows that the dimensionless governing equation is:

$$\partial_{\hat{\xi}}^2 V - \partial_{\hat{\tau}} \partial_{\hat{\xi}}^2 V = \partial_{\hat{\tau}} V, \tag{4.2}$$

whose initial and boundary conditions are expressed in (3.5)–(3.7).

After taking the Laplace transform of (4.2) and using the initial condition (3.5), we obtain

$$pL\{V(\hat{\xi}, \hat{\tau})\} = (1-p)\left[L\left\{V(\hat{\xi}, \hat{\tau})\right\}\right]_{\hat{\xi}\hat{\xi}} = (1-p)L\{V_{\hat{\xi}\hat{\xi}}(\hat{\xi}, \hat{\tau})\},$$

with the same boundary conditions as stated in (3.19)–(3.20). It immediately follows that

$$L\{V(\hat{\xi}, \hat{\tau})\} = \frac{1}{p} \exp\left\{-\left(\frac{p}{1-p}\right)^{1/2} \hat{\xi}\right\}. \tag{4.3}$$

Now (3.26) and a property of the Laplace transform lead to

$$\frac{1}{p} \exp\left\{\eta\left(\frac{1}{p} + 1\right)\right\} = L\left\{I_0\left(2\eta^{1/2}\hat{\tau}^{1/2}\right) \exp(\eta)\right\}. \tag{4.4}$$

If we set

$$\frac{1}{g(p)} := \frac{1}{p}, \tag{4.5}$$

$$h(p) := -\left(\frac{1}{p} + 1\right) = -\frac{1+p}{p}, \tag{4.6}$$

$$F(\hat{\xi}, p) := \frac{1}{p}\exp\left(-\hat{\xi}p^{1/2}\right), \tag{4.7}$$

then

$$\frac{1}{g(p)}F(\hat{\xi}, h(p)) = -\frac{1}{1+p}\exp\left\{-\hat{\xi}\left(-\frac{1+p}{p}\right)^{1/2}\right\}. \tag{4.8}$$

Thus, by virtue of the Efros theorem

$$\begin{aligned}\frac{1}{g(p)}F(\hat{\xi}, h(p)) &= -\frac{1}{1+p}\exp\left\{-\hat{\xi}\left(-\frac{1+p}{p}\right)^{1/2}\right\} \\ &= L\left\{\int_0^\infty \exp(\eta) I_0\left(2\eta^{1/2}\hat{\tau}^{1/2}\right)\operatorname{erfc}\left(\frac{1}{2}\hat{\xi}\eta^{-1/2}\right)d\eta\right\}.\end{aligned} \tag{4.9}$$

The shift theorem leads to

$$V(\hat{\xi}, \hat{\tau}) = -\exp(\hat{\tau})\int_0^\infty \exp(\eta) I_0\left(2\eta^{1/2}\hat{\tau}^{1/2}\right)\operatorname{erfc}\left(\frac{1}{2}\hat{\xi}\eta^{-1/2}\right)d\eta, \tag{4.10}$$

and, by (4.1),

$$\begin{aligned}v(x,t) = -U\exp\left(\frac{\mu}{|\alpha_1|}t\right)\int_0^\infty \exp(\eta) \\ \times I_0\left(2\left(\frac{\mu}{|\alpha_1|}\right)^{1/2}\eta^{1/2}t^{1/2}\right)\operatorname{erfc}\left(\frac{1}{2}\left(\frac{\varrho}{|\alpha_1|}\right)^{1/2}x\eta^{-1/2}\right)d\eta.\end{aligned} \tag{4.11}$$

Clearly the solution reported in (4.11) is not bounded.

5. Flow induced by a constantly accelerating plate

Suppose that the plate is subject, after time zero, to a constant acceleration A. Again for the Navier-Stokes fluid, the solution can be easily derived with a couple of integrations from the solution for the velocity for Rayleigh problem; but for a fluid of grade two, the analytical solution cannot be obtained in the same way.

With the same dimensionless variables defined in (3.1), (3.2), and (3.4), the mixed initial-boundary value problem in this case reads as in (3.3)–(3.7), with the only exception being the boundary condition at $x = 0$ which is now

$$V = a\tau \qquad \text{at} \quad \xi = 0, \quad \tau > 0, \tag{5.1}$$

where a is the dimensionless acceleration. The solution of the subsidiary boundary value problem obtained using the Laplace transform is, after applying the boundary conditions,

$$L\{V(\xi,\tau)\} = \frac{a}{p^2}\exp\left(-\sqrt{\frac{p}{p+1}}\xi\right). \tag{5.2}$$

We are going to illustrate two methods to invert (5.2).

METHOD 1

We know that (¶)

$$\frac{1}{p}f(p) = L\left\{\int_0^t F(\tau)\,d\tau\right\}, \tag{5.3}$$

where $f(p) = L\{F(t)\}$. If we define

$$f(p) = \frac{1}{p}\exp\left(-\sqrt{\frac{p}{p+1}}\xi\right),$$

we can use the result established in (3.33) in (5.3) to get

$$V(\xi,\tau) = a\exp(-\tau)\int_0^\infty \exp(-\eta)\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) \times\left(\int_0^\tau \exp(-s)I_0\left(2\eta^{1/2}s^{1/2}\right)ds\right)d\eta. \tag{5.4}$$

With the substitution

$$\frac{r^2}{2} = s \qquad \rightarrow \qquad r\,dr = ds,$$

we realize that

$$\exp(-\eta)\int_0^\tau \exp(-s)I_0\left(2\eta^{1/2}s^{1/2}\right)ds = \exp(-\eta)\int_0^{\sqrt{2\tau}} \exp\left(-\frac{r^2}{2}\right)I_0\left(\sqrt{2}r\eta^{1/2}\right)r\,dr, \tag{5.5}$$

(¶) Op. cit., CARSLAW and JAEGER.

which is the P function; in fact, recall the definition of the P function [18]

$$P\left(\frac{z}{\sigma}, \frac{R}{\sigma}\right) = \frac{\exp\left(\frac{-R^2}{2\sigma^2}\right)}{\sigma^2} \int_0^z \exp\left(-\frac{r^2}{2\sigma^2}\right) I_0\left(\frac{rR}{\sigma^2}\right) r\, dr. \tag{5.6}$$

If in (5.6) we set $\sigma = 1$, $z = \sqrt{2\tau}$, $R = \sqrt{2\eta}$, then

$$\exp(-\eta) \int_0^{\sqrt{2\tau}} \exp\left(-\frac{r^2}{2}\right) I_0(\sqrt{2\eta} r) r\, dr = P(\sqrt{2\tau}, \sqrt{2\eta}). \tag{5.7}$$

With the help of the last result, we can write (5.4) in a compact form:

$$V(\xi, \tau) = a \exp(-\tau) \int_0^\infty \operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) P\left(\sqrt{2\tau}, \sqrt{2\eta}\right) d\eta, \tag{5.8}$$

and, in a dimensional form,

$$v(x,t) = A \exp\left(-\frac{\mu}{\alpha_1} t\right) \int_0^\infty \operatorname{erfc}\left(\frac{1}{2}\left(\frac{\varrho}{\alpha_1}\right)^{1/2} x\eta^{-1/2}\right) \times P\left(\left(\frac{2\mu}{\alpha_1} t\right)^{1/2}, (2\eta)^{1/2}\right) d\eta. \tag{5.9}$$

METHOD 2

We know that (‖)

$$\frac{1}{p^2} \exp\left(-\xi^{1/2} p^{1/2}\right) = \left(\tau + \frac{1}{2}\xi\right) \operatorname{erfc}\left(\frac{1}{2}\xi^{1/2}\tau^{-1/2}\right) - \pi^{-1/2}\xi^{1/2}\tau^{1/2} \exp\left(-\frac{\xi}{4\tau}\right). \tag{5.10}$$

It follows immediately that

$$\frac{1}{p^2} \exp\left(-\xi p^{1/2}\right) = \left(\tau + \frac{1}{2}\xi^2\right) \operatorname{erfc}\left(\frac{1}{2}\xi\tau^{-1/2}\right) - \frac{1}{\sqrt{\pi}}\xi\tau^{1/2} \exp\left(-\frac{\xi^2}{4\tau}\right). \tag{5.11}$$

Now, defining

$$g(p) := \frac{1}{p^2}, \tag{5.12}$$

$$h(p) := 1 - \frac{1}{p}, \tag{5.13}$$

(‖) Op. cit., BATEMAN, p. 245, formula (4).

and

$$F(\xi, p) := \frac{1}{p^2} \exp(-\xi p^{1/2}), \tag{5.14}$$

we find that

$$\frac{1}{g(p)} F(\xi, h(p)) = \frac{1}{(p-1)^2} \exp\left[-\xi \left(\frac{p-1}{p}\right)^{1/2}\right]. \tag{5.15}$$

Thus on using the Efros theorem, (5.15) and (5.9), we obtain

$$\frac{1}{(p-1)^2} \exp\left[-\xi \left(\frac{p-1}{p}\right)^{1/2}\right] = L\left\{\int_0^\infty \exp(-\eta)\eta^{-1/2}\tau^{1/2} I_1\left(2\eta^{1/2}\tau^{1/2}\right) \cdot \left[\left(\eta + \frac{1}{2}\xi^2\right) \operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) - \frac{1}{\sqrt{\pi}}\xi\eta^{1/2} \exp\left(-\frac{\xi^2}{4\eta}\right)\right] d\eta\right\}. \tag{5.16}$$

By applying the shift theorem we find that

$$L\left\{\frac{1}{p^2} \exp\left[-\xi \left(\frac{p}{p+1}\right)^{1/2}\right]\right\} = \tau^{1/2} \exp(-\tau) \int_0^\infty \exp(-\eta) I_1\left(2\eta^{1/2}\tau^{1/2}\right) \times \left[\left(\eta + \frac{\xi^2}{2}\right) \eta^{-1/2} \operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right) - \frac{1}{\sqrt{\pi}}\xi \exp\left(-\frac{\xi^2}{4\eta}\right)\right] d\eta. \tag{5.17}$$

Finally,

$$v(x,t) = At^{1/2} \exp\left(-\frac{\mu}{\alpha_1} t\right) \int_0^\infty \exp(-\eta) I_1\left[2\left(\frac{\mu}{\alpha_1}\right)^{1/2} \eta^{1/2} t^{1/2}\right] \cdot \left[\left(\eta + \frac{1}{2}\frac{\varrho}{\alpha_1} x^2\right) \eta^{-1/2} \operatorname{erfc}\left(\frac{1}{2}\left(\frac{\varrho}{\alpha_1}\right)^{1/2} x\eta^{-1/2}\right) - \frac{1}{\sqrt{\pi}}\left(\frac{\varrho}{\alpha_1}\right)^{1/2} x \exp\left(-\frac{\varrho}{4\alpha_1}\frac{x}{\eta}\right)\right] d\eta. \tag{5.18}$$

6. Flow induced by a plate that applies a constant stress

Suppose that a constant shear is applied to the plate at time $t = 0^+$. The mathematical formulation of the problem remains the same as the one in Sec. 3, except for the boundary condition at $x = 0$ which now reads

$$\mu \frac{\partial v}{\partial x} + \alpha_1 \frac{\partial^2 v}{\partial t\, \partial x} = f \qquad \text{at} \quad x = 0, \quad t > 0, \tag{6.1}$$

where f is the shear applied to the plate. It is worthwhile to point out that in the Navier-Stokes theory the stress satisfies the same partial differential equation and boundary conditions as does the velocity in the Stokes' first problem. Thus, once the solution to the latter is available, the one for the suddenly applied shear is readily obtained by a simple integration. In the case of a fluid of grade two, the presence of the term representing the elastic forces in the momentum equation and the boundary condition modifies the approach to the solution.

We will adopt the same dimensionless parameters used so far, but since a natural velocity scale is missing, we will define a dimensionless speed on the basis of the time and space scales as follows

$$U = \frac{\mu}{\sqrt{\alpha_1 \varrho}}. \tag{6.2}$$

With these definitions the initial-boundary value problem becomes, in dimensionless form,

$$\partial_\xi^2 V + \partial_\tau \partial_\xi^2 V = \partial_\tau V, \tag{6.3}$$

with

$$V(\xi, 0) = 0, \tag{6.4}$$

$$V(\xi, \tau) \to 0 \qquad \text{as} \qquad \xi \to 0, \tag{6.5}$$

and

$$\partial_\xi V + \partial_\tau \partial_\xi V = \overline{f} \qquad \text{at} \quad \xi = 0, \tag{6.6}$$

where $\overline{f} = \dfrac{f\alpha_1}{\mu^2}$ is the non-dimensional shear. The solution of the subsidiary boundary value problem is

$$L\{V(\xi, \tau)\} = -\frac{\overline{f}}{p} \frac{1}{\sqrt{p(p+1)}} \exp\left(-\sqrt{\frac{p}{p+1}}\xi\right). \tag{6.7}$$

To invert (6.7) recall that (**)

$$\frac{1}{p^2} \exp\left(\frac{\eta}{p}\right) = \left\{\eta^{-1/2}\tau^{1/2} I_1\left(2\eta^{1/2}\tau^{1/2}\right)\right\}, \tag{6.8}$$

where I_1 is the modified Bessel function of order 1. From (6.8) and the property of Laplace transforms, it follows that

$$\frac{1}{p^2} \exp\left[-\eta\left(1 - \frac{1}{p}\right)\right] = L\left\{\left[\eta^{-1/2}\tau^{1/2} I_1\left(2\eta^{1/2}\tau^{1/2}\right)\right] \exp(-\eta)\right\}. \tag{6.9}$$

(**) Ibid.

The shift theorem applied to (6.9) yields

$$\frac{1}{(p+1)^2}\exp\left(-\eta\frac{p}{p+1}\right) = L\left\{\exp(-\tau)\exp(-\eta)\left[\eta^{-1/2}\tau^{1/2}I_1\left(2\eta^{1/2}\tau^{1/2}\right)\right]\right\}. \tag{6.10}$$

Also (††),

$$\frac{\exp(-\sqrt{p}\xi)}{p\sqrt{p}} = L\left\{2\pi^{-1/2}\tau^{1/2}\exp\left(-\frac{\xi^2}{4\tau}\right) - \xi\operatorname{erfc}\left(\frac{1}{2}\xi\tau^{-1/2}\right)\right\}. \tag{6.11}$$

After defining

$$h(p) := \frac{p}{p+1}, \qquad g(p) := \frac{1}{(p+1)^2}, \tag{6.12}$$

and

$$F(\xi, p) := \frac{\exp(-\sqrt{p}\xi)}{p\sqrt{p}}, \tag{6.13}$$

we get

$$\frac{1}{g(p)}F(\xi, h(p)) = \frac{1}{p\sqrt{p(p+1)}}\exp\left(-\sqrt{\frac{p}{p+1}}\xi\right), \tag{6.14}$$

which is the expression for $-\dfrac{L\{V(\xi,\tau)\}}{f}$. If we appeal again to the Efros theorem, we obtain

$$\begin{aligned}\frac{1}{p\sqrt{p(p+1)}}\exp\left(-\sqrt{\frac{p}{p+1}}\xi\right) &= L\left\{\exp(-\tau)\int_0^\infty \exp(-\eta)\eta^{-1/2}\tau^{1/2}I_1\left(2\eta^{1/2}\tau^{1/2}\right)\right.\\ &\left.\cdot\left[2\pi^{-1/2}\eta^{1/2}\exp\left(-\frac{\xi^2}{4\eta}\right) - \xi\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right)\right]d\eta\right\}\\ &= 2L\left\{\tau^{1/2}\exp(-\tau)\left[\frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(-\eta - \frac{\xi^2}{4\eta}\right)I_1\left(2\eta^{1/2}\tau^{1/2}\right)d\eta\right.\right.\\ &\left.\left.-\frac{1}{2}\xi\int_0^\infty \exp(-\eta)\eta^{-1/2}\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right)I_1\left(2\eta^{1/2}\tau^{1/2}\right)\right]d\eta\right\}.\end{aligned} \tag{6.15}$$

(††) Ibid., p. 247, formula (7).

Ultimately,

$$(6.16)\qquad v(x,t)=\frac{2f}{\mu}(\nu t)^{1/2}\exp\left(-\frac{\mu}{\alpha_1}t\right)\left[\frac{1}{2}\left(\frac{\varrho}{\alpha_1}\right)^{1/2}x\int_0^\infty \exp(-\eta)\eta^{-1/2}\right.$$
$$\times\operatorname{erfc}\left(\frac{1}{2}\left(\frac{\varrho}{\alpha_1}\right)^{1/2}x\eta^{-1/2}\right)I_1\left(2\eta^{1/2}\left(\frac{\mu}{\alpha_1}t\right)^{1/2}\right)d\eta$$
$$\left.-\frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(-\eta-\frac{\varrho x^2}{4\alpha_1\eta}\right)I_1\left(2\eta^{1/2}\left(\frac{\mu}{\alpha_1}t\right)^{1/2}\right)d\eta\right].$$

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Received September 20, 1994.

Arch. Mech., 47, 4, pp. 677–698, Warszawa 1995

Duality and the complementary energy principle for a class of nonlinear structures Part I. Five-parameter shell model

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Dedicated to the memory of Pawełek Telega, son of the second author

THE PRESENT contribution is a continuation of the papers [1, 2] and deals with the five-parameter shell model developed in [3]. An explicit form of the density of the complementary energy is derived. Two particular cases are studied: linear strain measures and a simplified nonlinear model. Minimax or Lagrangian approach is applied to the linear model. In the Appendix the Fenchel conjugate of a semidefinite quadratic function is derived.

Introduction

IN OUR PAPERS [1, 2] we have derived the general form of the complementary energy principle for the five-parameter model of geometrically nonlinear shells proposed by BAŞAR [3]. This model is governed by the kinematic assumption that a straight material fiber, orthogonal to the undeformed shell middle surface, remains straight during deformation and no change of length occurs in this direction. According to PIETRASZKIEWICZ [4], Başar's model can be derived from the general six-parameter theory of shells presented in [5]. From the mathematical point of view, however, the model studied in [3] cannot be treated just as a particular case of the six-parameter one. The five-parameter model is characterized by the kinematic constraint (1.2), which renders the functional of the total potential energy non-convex, even in the geometrically and physically linear case, cf. Sec. 2 of the present paper. The theory of duality, such as developed in Ref. [6], leads always to a convex dual problem, see the formulae (5.18)–(5.22) of the paper [1]. A crucial point now is to find the explicit form of the density of the complementary potential $g^* = W^* + W_1^*$, given by Eq. (3.4). The first term, or W^*, may be considered as "standard". The second term, however, or W_1^*, is implied by the nonlinear terms of the strain measures. We observe, that W_1^* disappears in the geometrically linear case, i.e. when the strain measures are linear: $\gamma(w) = \overset{\circ}{\gamma}(w)$, $\kappa(w, v) = \overset{\circ}{\kappa}(w, v)$ and $\varrho(w, v) = \overset{\circ}{\varrho}(w, v)$, cf. Eqs. (1.3)–(1.8).

To understand better the influence of geometrical nonlinearities on the form of the complementary energy principle, in Sec. 2 we present its formulation under the assumption that the strain measures and the constitutive equations are linear (physically and geometrically linear case). Next in Sec. 3, the explicit form of the

term $W_1^*(\mathbf{N}, \mathbf{M}, \mathbf{T}, \mathbf{Q}^\alpha, \mathbf{R}^\alpha, \mathbf{D})$ is derived. Toward this end we have to apply the explicit form of the Fenchel conjugate of a semidefinite quadratic function. In the available literature an explicit form of the relevant formula seems to be lacking. ROCKAFELLAR [7, p. 108] formulated the problem, but did not solve it. Therefore in the Appendix, by using some classical algebraic results, such a formula is derived. By assuming that the strain measures $\kappa(\mathbf{w}, \mathbf{v})$ and $\varrho(\mathbf{w}, \mathbf{v})$ are linear, the nonlinear terms in Eqs. (1.4) and (1.5) disappear. The resulting influence on the complementary energy principle is discussed in Sec. 4. Section 5 provides an application of the minimax (Lagrangian) approach to the formulation of the complementary energy principle in the geometrically and physically linear case. From this point of view the nonlinear model has been discussed in Ref. [1], cf. also [8].

Rockafellar's theory of duality expounded in [6] leads always to a convex dual problem. For problems involving nonlinear strain measures, that convexity imposes restrictions on a class of loadings to which a structure is subjected, cf. Secs. 3 and 4. For instance, compressed beams and compressed von Kármán plates are precluded by Rockafellar's theory of duality, cf. Ref. [9]. Therefore, to enlarge the class of admissible loadings, in the second part of our paper [10], we shall investigate, by using some ideas due to AUCHMUTY [11], the possibility of an application of the so-called anomalous dual variational principles to compressed nonlinear elastic beams. It is worth noting that though Rockafellar's theory of duality involves a linear operator Λ, yet it applies to geometrically nonlinear problems. Its range of applicability is then limited, yet non-trivial as we have mentioned before. Some authors pretend that it is completely useless for geometrically nonlinear problems. Such a statement is false. A *linear operator* Λ appearing in Rockafellar's formulation can always be deduced from a *nonlinear* strain measure. However, Λ is then not defined by a single operator, cf. Sec. 4 and Refs. [1, 2, 8, 9].

The reader interested in finding more information on the search for variational and extremum principles for geometrically nonlinear elastic shells should refer to [1, 12] and the references cited therein.

1. Some basic relations

To facilitate the reading of the paper, we first recall some fundamental notions and equations related to the five-parameter shell model [3]. Extensive bibliography on various shell models is provided by the book [5] and the review papers [13, 14, 15, 16], cf. also [1, 17].

Let $\omega \subset \mathbb{R}^2$ be a bounded sufficiently smooth 2-dimensional domain and $\Phi : \omega \mapsto S$, $\Phi \in C^2(\overline{\omega})^3$, where $\overline{\omega}$ denotes the closure of ω, $\overline{\omega} = \omega \cup \partial\omega$. Here S stands for the middle surface of the undeformed shell. The plane $\mathbb{R}^2$ containing ω is referred to the coordinates (ξ^α); Greek indices take values 1 and 2, with the

exception of the Appendix, and the summation convention is consequently used. We set

$\mathbf{a}_\alpha = \partial\Phi/\partial\xi^\alpha = \Phi_{,\alpha}$ base vectors,

$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ metric tensor,

$b_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ curvature tensor.

Here $\mathbf{n}$ stands for the unit normal to S.

The displacement vector $\mathbf{u}$ of an arbitrary point of the shell has the following form

$$\mathbf{u}(\xi^\alpha, \zeta) = \mathbf{w}(\xi^\alpha) + \zeta\mathbf{v}(\xi^\alpha), \tag{1.1}$$

where $\mathbf{w} = (w_\alpha, w)$, $\mathbf{v} = (v_\alpha, v)$. Here ζ is the normal coordinate to S. For shells of constant thickness h we have $\zeta \in [-h/2, h/2]$.

In the case of the model considered [3] the length of normals to S remains unchanged, hence the following constraint has to be imposed on $\mathbf{v}$

$$(\mathbf{v} + \mathbf{n}) \cdot (\mathbf{v} + \mathbf{n}) = 1. \tag{1.2}$$

Strain measures in this theory are given, provided that the some notations are employed as in [1], by

$$\gamma_{\alpha\beta}(\mathbf{w}) = \overset{\circ}{\gamma}_{\alpha\beta}(\mathbf{w}) + \frac{1}{2}\mathbf{w}_{,\alpha} \cdot \mathbf{w}_{,\beta}, \tag{1.3}$$

$$\kappa_{\alpha\beta}(\mathbf{w}, \mathbf{v}) = \overset{\circ}{\kappa}_{\alpha\beta}(\mathbf{w}, \mathbf{v}) + \frac{1}{2}(\mathbf{v}_{,\alpha} \cdot \mathbf{w}_{,\beta} + \mathbf{v}_{,\beta} \cdot \mathbf{w}_{,\alpha}), \tag{1.4}$$

$$\varrho_\alpha(\mathbf{w}, \mathbf{v}) = \overset{\circ}{\varrho}_\alpha(\mathbf{w}, \mathbf{v}) + \frac{1}{2}\mathbf{v} \cdot \mathbf{w}_{,\alpha}. \tag{1.5}$$

The linear parts $\overset{\circ}{\gamma}_{\alpha\beta}$, $\overset{\circ}{\kappa}_{\alpha\beta}$ and $\overset{\circ}{\varrho}_\alpha$ of the strain measures have the following form:

$$\overset{\circ}{\gamma}_{\alpha\beta}(\mathbf{w}) = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{w}_{,\beta} + \mathbf{a}_\beta \cdot \mathbf{w}_{,\alpha}), \tag{1.6}$$

$$\overset{\circ}{\kappa}_{\alpha\beta}(\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{a}_\beta \cdot \mathbf{v}_{,\alpha} + \mathbf{n}_{,\alpha} \cdot \mathbf{w}_{,\beta} + \mathbf{n}_{,\beta} \cdot \mathbf{w}_{,\alpha}), \tag{1.7}$$

$$\overset{\circ}{\varrho}_\alpha(\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mathbf{a}_\alpha \cdot \mathbf{v} + \mathbf{n} \cdot \mathbf{w}_{,\alpha}). \tag{1.8}$$

Let us denote by $\left(N^{\alpha\beta}\right)$, $\left(M^{\alpha\beta}\right)$ and $\left(T^\alpha\right)$ the generalized internal forces. For a hyperelastic shell the constitutive equations are

$$N^{\alpha\beta} = \frac{\partial W}{\partial\gamma_{\alpha\beta}}, \qquad M^{\alpha\beta} = \frac{\partial W}{\partial\kappa_{\alpha\beta}}, \qquad T^\alpha = \frac{\partial W}{\partial\varrho_\alpha}, \tag{1.9}$$

provided that W is the stored energy function.

By Γ we denote the boundary ∂S of S. We assume that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, where $\Gamma_0 > 0$. Along Γ_0 the shell is clamped, i.e. $\mathbf{w} = \mathbf{v} = 0$. On Γ_1 static boundary conditions are prescribed.

2. Linear model

We shall first study the linear model for which the strain measures are given by Eqs. (1.6)–(1.8). Then we obviously have $\gamma_{\alpha\beta} = \overset{\circ}{\gamma}_{\alpha\beta}$, $\kappa_{\alpha\beta} = \overset{\circ}{\kappa}_{\alpha\beta}$, $\varrho_\alpha = \overset{\circ}{\varrho}_\alpha$. The elastic potential W is quadratic and has the following form

$$W(\overset{\circ}{\boldsymbol{\gamma}}, \overset{\circ}{\boldsymbol{\kappa}}, \overset{\circ}{\boldsymbol{\varrho}}) = \frac{1}{2} B_0^{\alpha\beta\lambda\nu} \overset{\circ}{\gamma}_{\alpha\beta} \overset{\circ}{\gamma}_{\lambda\nu} + B_1^{\alpha\beta\lambda\nu} \overset{\circ}{\kappa}_{\alpha\beta} \overset{\circ}{\gamma}_{\lambda\nu} + \frac{1}{2} B_2^{\alpha\beta\lambda\nu} \overset{\circ}{\kappa}_{\alpha\beta} \overset{\circ}{\kappa}_{\lambda\nu} + 2 B_0^{\alpha 3 \beta 3} \overset{\circ}{\varrho}_\alpha \overset{\circ}{\varrho}_\beta , \tag{2.1}$$

where

$$B_0^{\alpha\beta\lambda\nu} = B_0^{\beta\alpha\lambda\nu} = B_0^{\lambda\nu\alpha\beta}, \qquad B_0^{\alpha 3 \beta 3} = B_0^{\beta 3 \alpha 3},$$

$$B_1^{\alpha\beta\lambda\nu} = B_1^{\beta\alpha\lambda\nu} = B_1^{\lambda\nu\alpha\beta}, \qquad B_2^{\alpha\beta\lambda\nu} = B_2^{\beta\alpha\lambda\nu} = B_2^{\lambda\nu\alpha\beta}.$$

Additionally we make the following assumptions

$$B_0^{\alpha\beta\lambda\nu} \in L^\infty(S), \quad B_0^{\alpha 3 \beta 3} \in L^\infty(S), \quad B_1^{\alpha\beta\lambda\nu} \in L^\infty(S), \quad B_2^{\alpha\beta\lambda\nu} \in L^\infty(S), \tag{2.2}$$

$$\exists\, c > 0, \quad W(\boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\varrho}) \geq c(\gamma_{\alpha\beta}\gamma_{\alpha\beta} + \kappa_{\alpha\beta}\kappa_{\alpha\beta} + \varrho_\alpha \varrho_\alpha), \tag{2.3}$$

for each $\left(\gamma_{\alpha\beta}\right) \in \mathbb{M}_s(\mathbb{R})$, $\left(\kappa_{\alpha\beta}\right) \in \mathbb{M}_s(\mathbb{R})$, $\left(\varrho_\alpha\right) \in \mathbb{R}^2$, where $\mathbb{M}_s(\mathbb{R})$ is the space of symmetric 2×2 matrices.

By $\mathcal{V}$ we denote the space to which kinematically admissible fields belong. It is defined as follows

$$\mathcal{V} = \left\{(\mathbf{w}, \mathbf{v}) \in H^1(S, \mathbb{R}^3) \times H^1(S, \mathbb{R}^3) \mid \mathbf{w} = 0 \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0\right\}. \tag{2.4}$$

Function spaces used in this paper are defined and examined in Ref. [18].

Let $I_K(\mathbf{v})$ be the indicator function of the constraints set $\mathcal{K}$ which are imposed by (1.2), i.e.:

$$I_K(\mathbf{v}) = \begin{cases} 0, & \text{if } \mathbf{v} \in \mathcal{K}, \\ \infty, & \text{otherwise}, \end{cases} \tag{2.5}$$

where

$$\mathcal{K} = \left\{\mathbf{v} \in H^1(S, \mathbb{R}^3) \mid (\mathbf{v}(x) + \mathbf{n}(x)) \cdot (\mathbf{v}(x) + \mathbf{n}(x)) = 1, \quad x \in S \cup \Gamma_1\right\}. \tag{2.6}$$

It is worth noting that this constraint set $\mathcal{K}$ is non-convex.

The external loading is given by forces $\left(P^\alpha, P\right)$ and moments $\left(\mathfrak{M}^\alpha, \mathfrak{M}\right)$ distributed over S as well as by the boundary loading $\left(P_0^\alpha, P_0\right)$ and moments $\left(\mathfrak{M}_0^\alpha, \mathfrak{M}_0\right)$ acting along Γ_1. We make the following assumptions

$$\begin{aligned} &\mathbf{P} = \left(P^\alpha, P\right) \in L^2(S, \mathbb{R}^3), \qquad \mathfrak{M} = \left(\mathfrak{M}^\alpha, \mathfrak{M}\right) \in L^2(S, \mathbb{R}^3), \\ &\mathbf{P}_0 = \left(P_0^\alpha, P_0\right) \in L^2(\Gamma_1, \mathbb{R}^3), \qquad \mathfrak{M}_0 = \left(\mathfrak{M}_0^\alpha, \mathfrak{M}_0\right) \in L^2(\Gamma_1, \mathbb{R}^3). \end{aligned} \tag{2.7}$$

The primal problem or the minimum principle of the total potential energy reads:

$$(\overset{\circ}{\mathcal{P}}) \quad \left| \begin{array}{l} \text{Find } (\mathbf{w}, \mathbf{v}) \in \mathcal{V} \text{ such that} \\ J(\tilde{\mathbf{w}}, \tilde{\mathbf{v}}) = \inf\left\{ J(\mathbf{w}, \mathbf{v}) \mid (\mathbf{w}, \mathbf{v}) \in \mathcal{V} \right\}, \end{array} \right. \tag{2.8}$$

where

$$\begin{aligned} J(\mathbf{w}, \mathbf{v}) &= \int_S W(\overset{\circ}{\boldsymbol{\gamma}}(\mathbf{w}), \overset{\circ}{\boldsymbol{\kappa}}(\mathbf{w}, \mathbf{v}), \overset{\circ}{\varrho}(\mathbf{w}, \mathbf{v}))\, dS + F(\mathbf{w}, \mathbf{v}), \\ F(\mathbf{w}, \mathbf{v}) &= -\int_S (\mathbf{P} \cdot \mathbf{w} + \mathfrak{M} \cdot \mathbf{v})\, ds - \int_{\Gamma_1} (\mathbf{P}_0 \cdot \mathbf{w} + \mathfrak{M}_0 \cdot \mathbf{v})\, d\Gamma + I_K(\mathbf{v}). \end{aligned} \tag{2.9}$$

Due to the presence of the non-convex functional I_K, the minimization problem (2.8) is obviously non-convex. To prove the existence of a solution $(\tilde{\mathbf{w}}, \tilde{\mathbf{v}}) \in \mathcal{V}$ one can use the general approach proposed by BAIOCCHI *et al.*, [19], suitable for a large class of non-convex problems.

To derive the dual problem $(\overset{\circ}{\mathcal{P}}{}^*)$ we first choose the linear operator Λ, appearing in Rockaffelar's theory of duality [1, 6].

Now it is natural to put

$$W\left[\overset{\circ}{\boldsymbol{\gamma}}(\mathbf{w}), \overset{\circ}{\boldsymbol{\kappa}}(\mathbf{w}, \mathbf{v}), \overset{\circ}{\varrho}(\mathbf{w}, \mathbf{v})\right] = W[\Lambda_1(\mathbf{w}), \Lambda_2(\mathbf{w}, \mathbf{v}), \Lambda_3(\mathbf{w}, \mathbf{v})] = W[\boldsymbol{\Lambda}(\mathbf{w}, \mathbf{v})],$$

where

$$\Lambda(\mathbf{w}, \mathbf{v}) = (\Lambda_1(\mathbf{w}), \Lambda_2(\mathbf{w}, \mathbf{v}), \Lambda_3(\mathbf{w}, \mathbf{v})),$$

and

$$\Lambda_1(\mathbf{w}) = \left(\overset{\circ}{\gamma}_{\alpha\beta}(\mathbf{w})\right), \qquad \Lambda_2(\mathbf{w}, \mathbf{v}) = \left(\overset{\circ}{\kappa}_{\alpha\beta}(\mathbf{w}, \mathbf{v})\right), \qquad \Lambda_3(\mathbf{w}, \mathbf{v}) = \left(\overset{\circ}{\varrho}_\alpha(\mathbf{w}, \mathbf{v})\right).$$

We have $\Lambda: \mathcal{V} \to \mathcal{H} = L^2(S, \mathbb{M}_s(\mathbb{R})) \times L^2(S, \mathbb{M}_s(\mathbb{R})) \times L^2(S, \mathbb{R}^2)$, and $\Lambda^*: \mathcal{H} \to \mathcal{V}^*$, where $\Lambda^* \mathbf{p}^* = \left(\overset{\circ}{A}{}^* \mathbf{p}^*, \overset{\circ}{B}{}^* \mathbf{p}^*\right)$, and $\mathbf{p}^* = (\mathbf{N}, \mathbf{M}, \mathbf{T}) \in \mathcal{H}^* = \mathcal{H}$, cf. the formulae

(5.10)–(5.12) in [1], and

$$\overset{\circ}{A}{}^{*}\mathbf{p}^{*} = -\left[(N^{\alpha\beta} - b^{\alpha}_{\gamma}M^{\gamma\beta})\mathbf{a}_{\alpha} + \frac{1}{2}T^{\beta}\mathbf{n}\right]_{|\beta}, \quad \text{in } S, \tag{2.10}$$

$$\overset{\circ}{A}{}^{*}\mathbf{p}^{*} = \left[(N^{\alpha\beta} - b^{\alpha}_{\gamma}M^{\gamma\beta})\mathbf{a}_{\alpha} + \frac{1}{2}T^{\beta}\mathbf{n}\right]\boldsymbol{\nu}_{\beta}, \quad \text{on } \Gamma_1, \tag{2.11}$$

$$\overset{\circ}{B}{}^{*}\mathbf{p}^{*} = -(M^{\alpha\beta}\mathbf{a}_{\alpha})_{|\beta} + \frac{1}{2}T^{\alpha}\mathbf{a}_{\alpha}, \quad \text{in } S, \tag{2.12}$$

$$\overset{\circ}{B}{}^{*}\mathbf{p}^{*} = M^{\alpha\beta}\mathbf{a}_{\alpha}\boldsymbol{\nu}_{\beta}, \quad \text{on } \Gamma_1. \tag{2.13}$$

Here $\boldsymbol{\nu}$ is the outward unit normal to $\Gamma = \partial S$.

To formulate the complementary energy principle we must first calculate the density of the complementary stored energy W^{*}. Now we have

$$W^{*}(\mathbf{N},\mathbf{M},\mathbf{T}) = \sup\Big\{N^{\alpha\beta}\,\overset{\circ}{\gamma}_{\alpha\beta} + M^{\alpha\beta}\,\overset{\circ}{\kappa}_{\alpha\beta} + T^{\alpha}\,\overset{\circ}{\varrho}_{\alpha} - W(\overset{\circ}{\boldsymbol{\gamma}},\overset{\circ}{\boldsymbol{\kappa}},\overset{\circ}{\boldsymbol{\varrho}}) \mid \overset{\circ}{\boldsymbol{\gamma}},\overset{\circ}{\boldsymbol{\kappa}} \in \mathbb{M}_s(\mathbb{R}),\ \overset{\circ}{\boldsymbol{\varrho}} \in \mathbb{R}^2\Big\}, \tag{2.14}$$

where W is given by (2.1).

After some algebraic calculations we finally obtain

$$W^{*}(\mathbf{N},\mathbf{M},\mathbf{T}) = \frac{1}{2}(A_0^{-1})_{\alpha\beta\gamma\delta}N^{\alpha\beta}N^{\gamma\delta} + (A_1^{-1})_{\alpha\beta\gamma\delta}N^{\alpha\beta}M^{\gamma\delta} + \frac{1}{2}(A_2^{-1})_{\alpha\beta\gamma\delta}M^{\alpha\beta}M^{\gamma\delta} + \frac{1}{2}(A^{-1})_{\alpha\beta}T^{\alpha}T^{\beta}, \tag{2.15}$$

where

$$\left((A_0^{-1})_{\alpha\beta\gamma\delta}\right), \quad \left((A_1^{-1})_{\alpha\beta\gamma\delta}\right), \quad \left((A_2^{-1})_{\alpha\beta\gamma\delta}\right), \quad \left((A^{-1})_{\alpha\beta}\right)$$

are the inverse matrices of the following ones:

$$A_0^{\alpha\beta\gamma\delta} = B_0^{\alpha\beta\gamma\delta} - B_1^{\alpha\beta\mu\nu}(B_2^{-1})_{\mu\nu\varrho\kappa}B_1^{\varrho\kappa\gamma\delta},$$
$$A_1^{\alpha\beta\gamma\delta} = B_1^{\alpha\beta\gamma\delta} - (B_0^{-1})^{\alpha\beta\mu\nu}(B_1^{-1})_{\mu\nu\varrho\kappa}B_2^{\varrho\kappa\gamma\delta},$$
$$A_2^{\alpha\beta\gamma\delta} = B_2^{\alpha\beta\gamma\delta} - B_1^{\alpha\beta\mu\nu}(B_0^{-1})_{\mu\nu\varrho\kappa}B_1^{\varrho\kappa\gamma\delta}, \qquad A^{\alpha\beta} = 4B^{\alpha3\beta3}, \quad \text{respectively.}$$

Statically admissible fields of generalized forces belong to $\overset{\circ}{\mathcal{B}}$, where

$$\overset{\circ}{\mathcal{B}} = \Big\{(\mathbf{N},\mathbf{M},\mathbf{T}) \mid \mathbf{N} \in L^2(S,\mathbb{M}_s(\mathbb{R})),\ N^{\alpha\beta}{}_{,\beta} \in L^2(S),\ \mathbf{M} \in L^2(S,\mathbb{M}_s(\mathbb{R})),\ M^{\alpha\beta}{}_{,\beta} \in L^2(S),\ T^{\alpha} \in L^2(S),\ T^{\alpha}{}_{,\alpha} \in L^2(S)\Big\}. \tag{2.16}$$

Since now the strain measures are linear and W_1^* defined by (3.5) to be given below disappears, therefore the complementary energy principle results immediately from (5.18)–(5.22) in [1] and reads:

$$
(\overset{\circ}{\mathcal{P}}^*)\quad
\left|\begin{array}{ll}
\text{Find} & \\
\sup\left\{-\int\limits_S W^*(\mathbf{N},\mathbf{M},\mathbf{T})\,dS - \widetilde{F}_0^*\left[-\overset{\circ}{B}{}^*(\mathbf{M},\mathbf{T})\right] \;\middle|\; (\mathbf{N},\mathbf{M},\mathbf{T}) \in \overset{\circ}{\mathcal{B}}\right\} & \\
\text{subject to} & \\
N^{\alpha\beta}{}_{|\beta} - (b_\lambda^\alpha M^{\lambda\beta})_{|\beta} - \frac{1}{2} b_\lambda^\alpha T^\lambda + P^\alpha = 0, & \text{in } S, \\
b_{\alpha\beta} N^{\alpha\beta} - b_\beta^\alpha b_{\alpha\lambda} M^{\lambda\beta} + \frac{1}{2} T^\alpha{}_{|\alpha} + P = 0, & \text{in } S, \\
b_\lambda^\alpha M^{\lambda\beta}\nu_\beta - N^{\alpha\beta}\nu_\beta + P_0^\alpha = 0, & \text{on } \Gamma_1, \\
-\frac{1}{2} T^\alpha \nu_\alpha + P_0 = 0, & \text{on } \Gamma_1,
\end{array}\right.
\tag{2.17}
$$

where

$$
\widetilde{F}_0^*(-\overset{\circ}{B}{}^*(\mathbf{M},\mathbf{T})) = \int\limits_S \Big[a_{\alpha\lambda}(M^{\alpha\beta}{}_{|\beta} - \frac{1}{2}T^\alpha + \mathfrak{M}^\alpha)(M^{\lambda\gamma}{}_{|\gamma} - \frac{1}{2}T^\lambda + \mathfrak{M}^\lambda)
$$

$$
+ (b_{\alpha\beta}M^{\alpha\beta} + \mathfrak{M})^2\Big]^{1/2} dS + \int\limits_{\Gamma_1} \Big[a_{\alpha\lambda}(-M^{\alpha\beta}\nu_\beta + \mathfrak{M}_0^\alpha)(-M^{\lambda\gamma}\nu_\gamma + \mathfrak{M}_0^\lambda) + (\mathfrak{M}_0)^2\Big]^{1/2} d\Gamma
$$

$$
- \int\limits_S (b_{\alpha\beta}M^{\alpha\beta} + \mathfrak{M})dS - \int\limits_{\Gamma_1} \mathfrak{M}_0\, d\Gamma.
$$

We observe that in such a geometrically linear setting, the geometrical constraint (1.2) enters the complementary energy principle $(\overset{\circ}{\mathcal{P}}^*)$ by means of the functional $\widetilde{F}_o^*$, somewhat simplified in comparison with the functional $\widetilde{F}^*$ given by Eq. (5.15) in [1]. Equations (2.17) constitute one part of the equilibrium equations and the remaining ones are "hidden" in the functional $\widetilde{F}_0^*$.

3. Derivation of W_1^* in the general case

Let us return now to the general case when the constitutive equations are given by (1.9) and the nonlinear strain measures are specified by (1.3)–(1.5). One notes that the considerations of the present section also apply to a quadratic stored energy function in $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\varrho}$.

We write

$$
g(\overset{\circ}{\boldsymbol{\gamma}}(\mathbf{w}), \overset{\circ}{\boldsymbol{\kappa}}(\mathbf{w},\mathbf{v}), \overset{\circ}{\boldsymbol{\varrho}}(\mathbf{w},\mathbf{v}), \mathbf{w}_{,\alpha}, \mathbf{v}_{,\alpha}, \mathbf{v}) := W(\boldsymbol{\gamma}(\mathbf{w}), \boldsymbol{\kappa}(\mathbf{w},\mathbf{v}), \boldsymbol{\varrho}(\mathbf{w},\mathbf{v})).
\tag{3.1}
$$

The crucial role in the complementary energy principle is played by the function g^* or the Fenchel conjugate of g, cf. [1, 2, 8, 9]. Now we have

$$(3.2)\qquad g^*(\mathbf{N},\mathbf{M},\mathbf{T},\mathbf{Q}^\alpha,\mathbf{R}^\alpha,\mathbf{D}) = \sup\Big\{ N^{\alpha\beta}\,\overset{\circ}{\gamma}_{\alpha\beta} + M^{\alpha\beta}\,\overset{\circ}{\kappa}_{\alpha\beta} + T^\alpha\,\overset{\circ}{\varrho}_\alpha + \mathbf{Q}^\alpha\cdot\mathbf{q}_\alpha + \mathbf{R}^\alpha\cdot\mathbf{r}_\alpha + \mathbf{D}\cdot\mathbf{d} - g(\overset{\circ}{\gamma},\overset{\circ}{\kappa},\overset{\circ}{\varrho},\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) \mid \overset{\circ}{\gamma},\overset{\circ}{\kappa}\in\mathbb{M}_s(\mathbb{R}),\ \overset{\circ}{\varrho}\in\mathbb{R}^2,\ \mathbf{q}_\alpha,\mathbf{r}_\alpha\in\mathbb{M}(\mathbb{R})\times\mathbb{R}^2,\ \mathbf{d}\in\mathbb{R}^3\Big\},$$

where $\mathbf{Q}^\alpha = Q^{\alpha\beta}\mathbf{a}_\beta + Q^\alpha\mathbf{n}$, $\mathbf{R}^\alpha = R^{\alpha\beta}\mathbf{a}_\beta + R^\alpha\mathbf{n}$, $\mathbf{D} = D^\beta\mathbf{a}_\beta + D\mathbf{n}$.

For the sake of simplicity of notation we use indifferently $\mathbf{N}$, $\mathbf{M}$, etc. for fields and their values at $x = \Phi(\xi^\alpha)$.

By setting

$$(3.3)\qquad W_1(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) = \frac{1}{2}N^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta + M^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{r}_\beta + \frac{1}{2}T^\alpha\mathbf{q}_\alpha\cdot\mathbf{d},$$

we write relation (3.2) in the following form

$$(3.4)\qquad g^*(\mathbf{N},\mathbf{M},\mathbf{T},\mathbf{Q}^\alpha,\mathbf{R}^\alpha,\mathbf{D})$$
$$= \sup\Big\{ N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\kappa_{\alpha\beta} + T^\alpha\varrho_\alpha - W(\boldsymbol{\gamma},\boldsymbol{\kappa},\boldsymbol{\varrho}) \mid \boldsymbol{\gamma},\boldsymbol{\kappa}\in\mathbb{M}_s(\mathbb{R}),\ \boldsymbol{\varrho}\in\mathbb{R}^2\Big\}$$
$$+ \sup\Big\{ \mathbf{Q}^\alpha\cdot\mathbf{q}_\alpha + \mathbf{R}^\alpha\cdot\mathbf{r}_\alpha + \mathbf{D}\cdot\mathbf{d} - W_1(\mathbf{N},\mathbf{M},\mathbf{T};\ \mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) \mid (\mathbf{q}_\alpha),(\mathbf{r}_\alpha)\in\mathbb{M}(\mathbb{R})\times\mathbb{R}^2,\ \mathbf{d}\in\mathbb{R}^3\Big\}$$
$$= W^*(\mathbf{N},\mathbf{M},\mathbf{T}) + W_1^*(\mathbf{N},\mathbf{M},\mathbf{T};\ \mathbf{Q}^\alpha,\mathbf{R}^\alpha,\mathbf{D}),$$

where

$$(3.5)\qquad W_1^*(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{Q}^\alpha,\mathbf{R}^\alpha,\mathbf{D}) = \sup\Big\{ \mathbf{Q}^\alpha\cdot\mathbf{q}_\alpha + \mathbf{R}^\alpha\cdot\mathbf{r}_\alpha + \mathbf{D}\cdot\mathbf{d} - W_1(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) \mid (\mathbf{q})_\alpha,(\mathbf{r}_\alpha)\in\mathbb{M}(\mathbb{R})\times\mathbb{R}^2,\ \mathbf{d}\in\mathbb{R}^3\Big\}.$$

An immediate remarkable conclusion resulting from (3.4) and (3.5) is that W_1^* disappears if the strain measures are linear, i.e. for $\boldsymbol{\gamma} = \overset{\circ}{\gamma}$, $\boldsymbol{\kappa} = \overset{\circ}{\kappa}$, $\boldsymbol{\varrho} = \overset{\circ}{\varrho}$.

Our purpose now is to find an explicit form of the function $W_1^*(N^{\alpha\beta}, M^{\alpha\beta}, T^\alpha; \cdot,\cdot,\cdot)$, provided that $\mathbf{N},\mathbf{M}\in\mathbb{M}_s(\mathbb{R})$ and $\mathbf{T}\in\mathbb{R}^2$ are treated as known. We may write

$$(3.6)\qquad W_1(N^{\alpha\beta}, M^{\alpha\beta}, T^\alpha;\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) = \frac{1}{2}A^{ij}\mathbf{p}_i\cdot\mathbf{p}_j, \qquad i,j = 1,\ldots,5,$$

where $\mathbf{A} = \left(A^{ij}\right)$ is the symmetric matrix given by

$$\mathbf{A} = \left(A^{ij}\right) = \begin{pmatrix} N^{11} & N^{12} & M^{11} & M^{12} & \frac{1}{2}T^1 \\ N^{12} & N^{22} & M^{12} & M^{22} & \frac{1}{2}T^2 \\ M^{11} & M^{12} & 0 & 0 & 0 \\ M^{12} & M^{22} & 0 & 0 & 0 \\ \frac{1}{2}T^1 & \frac{1}{2}T^2 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

and $\left(\mathbf{p}_i\right) = \left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}_1, \mathbf{r}_2, \mathbf{d}\right)$. Here $\mathbf{q}_\alpha \in \mathbb{R}^3$, $\mathbf{r}_\alpha \in \mathbb{R}^3$, $\mathbf{d} \in \mathbb{R}^3$, $\alpha = 1, 2$.

By using the results presented in the Appendix we shall find the explicit form of the function $W_1^*(N^{\alpha\beta}, M^{\alpha\beta}, T^\alpha; \cdot, \cdot, \cdot)$. Obviously, this form depends on the matrix $\mathbf{A}$ given by Eq. (3.7), and particularly on the rank of $\mathbf{A}$ and on its principal minors.

Since $\det\left(A^{ij}\right) = 0$, hence we infer that rank $\mathbf{A} < 4$. Let us denote by $\mathbf{A}(k)$ a matrix obtained from $\mathbf{A}$ by deleting a k-row and a k-column. The principal minors of the fourth order are:

$$\det \mathbf{A}(1) = 0, \qquad \det \mathbf{A}(2) = 0, \tag{3.8}$$

$$\det \mathbf{A}(3) = \frac{1}{4}(M^{12}T^2 - M^{22}T^1)^2, \tag{3.9}$$

$$\det \mathbf{A}(4) = \frac{1}{4}(M^{11}T^2 - M^{12}T^1)^2, \tag{3.10}$$

$$\det \mathbf{A}(5) = [M^{11}M^{22} - (M^{12})^2]^2. \tag{3.11}$$

None of the matrices $\mathbf{A}(1)$, $\mathbf{A}(2)$, ..., $\mathbf{A}(5)$ can be positive definite. For matrices $\mathbf{A}(1)$ and $\mathbf{A}(2)$ this is implied by (3.8). For matrices $\mathbf{A}(3)$, $\mathbf{A}(4)$ and $\mathbf{A}(5)$, under the assumption $N^{11} > 0$ and $N^{11}N^{22} - (N^{12})^2 > 0$, the corresponding minors of the third order appearing in the relations (A.9) are non-positive. For instance, in the case of the matrix $\mathbf{A}(3)$ we have

$$\begin{vmatrix} N^{11} & N^{12} & M^{12} \\ N^{12} & N^{22} & M^{22} \\ M^{12} & M^{22} & 0 \end{vmatrix} = 2N^{12}M^{22}M^{12} - N^{22}(M^{12})^2 - N^{11}(M^{22})^2$$

$$= \frac{1}{N_{11}}\left[-(N^{12}M^{12} - N^{11}M^{22})^2 - (M^{12})^2(N^{11}N^{22} - (N^{12})^2)\right] \leq 0.$$

Similarly, the matrices of the third order, determinants of which are principal minors of the matrix $\mathbf{A}$, cannot be positive definite.

If rank $\mathbf{A} = 2$ and $N^{11} > 0$ jointly with $N^{11}N^{22} - (N^{12})^2 > 0$, then necessarily $M^{\alpha\beta} = 0$, $T^{\alpha} = 0$, because the principal minors of the third order must be equal to zero.

If rank $\mathbf{A} = 1$ then $N^{11}N^{22} - (N^{12})^2 = 0$ and $M^{\alpha\beta} = 0$, $T^{\alpha} = 0$ (all the principal minors of the second order must be equal to zero).

Let us set $\mathbf{B} = \left(B_{\alpha\beta}\right) = \left(N^{\alpha\beta}\right)^{-1}$. By using the results of the above considerations and (A.11) we finally obtain

$$
\begin{aligned}
(3.12)\qquad & W_1^*(N^{\alpha\beta}, M^{\alpha\beta}, T^{\alpha}; \mathbf{Q}^{\alpha}, \mathbf{R}^{\alpha}, \mathbf{D}) \\
& = \begin{cases}
\dfrac{1}{2} B_{\alpha\beta}\mathbf{Q}^{\alpha}\cdot\mathbf{Q}^{\beta}, & \text{if } \mathbf{N} \text{ is positive definite and} \\
& \mathbf{R}^{\alpha} = 0,\ \mathbf{D} = 0,\ \mathbf{M} = 0,\ T = 0; \\
\dfrac{1}{2}\dfrac{1}{N^{\sigma\sigma}}\mathbf{Q}^{\sigma}\cdot\mathbf{Q}^{\sigma}, & \text{if rank } \mathbf{N} = 1 \text{ and } N^{\sigma\sigma} > 0, \\
& \mathbf{R}^{\alpha} = 0,\ \mathbf{D} = 0,\ \mathbf{M} = 0, T = 0, \\
& \mathbf{Q}^{3-\sigma} = \dfrac{N^{12}}{N_{\sigma\sigma}}\mathbf{Q}^{\sigma} \quad (\sigma \text{ not summed!}); \\
\infty, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

The function W_1^* so obtained is a convex function. The density of the complementary energy g^*, given by the formula (3.4) is the sum of convex functions W^* and W_1^*; consequently it is a convex function. Unfortunately, from the physical point of view, the restrictions appearing in Eq. (3.12) and rendering W_1^* convex are very strong, since $\mathbf{N}$ has to be positive semi-definite and $\mathbf{M} = 0$, $\mathbf{T} = 0$. We recall that these restrictions are a natural consequence of the nonlinearities of the strain measures. In such a convex setting, the dual formulation is applicable to stretched shells in membrane states ($\mathbf{M} = 0$) provided that transverse forces are negligible ($\mathbf{T} = 0$).

Similar dual approach is applicable to a broader class of shell loadings if, for instance, the strain measures $\boldsymbol{\kappa}$ and $\boldsymbol{\varrho}$ are linear, see the next Section.

REMARK 1. By using the extremality relations derived in [1], we may assume

$$
\begin{aligned}
(3.13)\qquad & \mathbf{Q}^{\alpha} = N^{\alpha\beta}\mathbf{q}_{\beta} + M^{\alpha\beta}\mathbf{r}_{\beta} + \frac{1}{2}T^{\alpha}\mathbf{d}, \\
& \mathbf{R}^{\alpha} = M^{\alpha\beta}\mathbf{q}_{\beta}, \qquad \mathbf{D} = \frac{1}{2}T^{\alpha}\mathbf{q}_{\alpha},
\end{aligned}
$$

where

$$
\mathbf{q}_{\alpha} = \mathbf{w}_{,\alpha}, \qquad \mathbf{r}_{\alpha} = \mathbf{v}_{,\alpha}, \qquad \mathbf{d} = \mathbf{v}, \qquad \mathbf{Q}^{\alpha} = Q^{\alpha\beta}\mathbf{a}_{\beta} + Q^{\alpha}\mathbf{n},
$$

$$
\mathbf{R}^{\alpha} = R^{\alpha\beta}\mathbf{a}_{\beta} + R^{\alpha}\mathbf{n}, \qquad \mathbf{D} = D^{\alpha}\mathbf{a}_{\alpha} + D\mathbf{n}.
$$

Consequently, the displacement-dependent density of the complementary energy denoted now by g_d^* is defined in the following way:

$$
\begin{aligned}
(3.14)\qquad g_d^*(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) &= g^*\Big(\mathbf{N},\mathbf{M},\mathbf{T},\ \mathbf{Q}^\alpha = N^{\alpha\beta}\mathbf{q}_\beta + M^{\alpha\beta}\mathbf{r}_\beta + +\frac{1}{2}T^\alpha\mathbf{d},\\
&\qquad \mathbf{R}^\alpha = M^{\alpha\beta}\mathbf{q}_\beta,\ \mathbf{D} = \frac{1}{2}T^\alpha\mathbf{q}_\alpha\Big)\\
&= W^*(\mathbf{N},\mathbf{M},\mathbf{T}) + W_{1d}^*(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}),
\end{aligned}
$$

where

$$
\begin{aligned}
(3.15)\qquad W_{1d}^*(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) &= W_1^*\Big(\mathbf{N},\mathbf{M},\mathbf{T},\ \mathbf{Q}^\alpha = N^{\alpha\beta}\mathbf{q}_\beta + M^{\alpha\beta}\mathbf{r}_\beta + \frac{1}{2}T^\alpha\mathbf{d},\\
&\qquad \mathbf{R}^\alpha = M^{\alpha\beta}\mathbf{q}_\beta,\ \mathbf{D} = \frac{1}{2}T^\alpha\mathbf{q}_\alpha\Big).
\end{aligned}
$$

Taking account of (3.3) and (3.5) we readily obtain

$$
(3.16)\qquad W_{1d}^*(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) = \begin{cases} \dfrac{1}{2}N^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta & \text{if } \mathbf{N} \text{ is positive semidefinite}\\ & \text{and } \mathbf{M}=0,\ \mathbf{N}=0;\\ \infty & \text{otherwise,}\end{cases}
$$

where $(\mathbf{q}_\alpha)\in\mathbb{M}(\mathbb{R})\times\mathbb{R}^2$.

Thus the function W_{1d}^* is convex if and only if $\mathbf{N}$ is positive semidefinite and $\mathbf{M}=0$, $\mathbf{T}=0$; the same conditions render W_1^* a convex function.

We shall now generalize (3.15) in order to obtain nonconvex displacement-dependent density of the complementary energy. Toward this purpose we define

$$
\begin{aligned}
(3.17)\qquad \widetilde{W}_{1d}(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) &= \mathbf{Q}^\alpha\cdot\mathbf{q}_\alpha + \mathbf{R}^\alpha\cdot\mathbf{r}_\alpha + \mathbf{D}\cdot\mathbf{d} - W_1(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}),
\end{aligned}
$$

where

$$
\mathbf{Q}^\alpha = N^{\alpha\beta}\mathbf{q}_\beta + M^{\alpha\beta}\mathbf{r}_\beta + \frac{1}{2}T^\alpha\mathbf{d}, \qquad \mathbf{R}^\alpha = M^{\alpha\beta}\mathbf{q}_\beta, \qquad \mathbf{D} = \frac{1}{2}T^\alpha\mathbf{q}_\alpha.
$$

Hence

$$
\begin{aligned}
(3.18)\qquad \widetilde{W}_{1d}(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}) &= \frac{1}{2}N^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta + M^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{r}_\beta + \frac{1}{2}T^\alpha\mathbf{q}_\alpha\cdot\mathbf{d} = W_1(\mathbf{N},\mathbf{M},\mathbf{T};\mathbf{q}_\alpha,\mathbf{r}_\alpha,\mathbf{d}).
\end{aligned}
$$

The part $\widetilde{W}_{1d}$ of the density of the complementary energy coincides with W^*_{1d} provided that N is positive semidefinite and $\mathbf{M} = 0$, $\mathbf{T} = 0$. Otherwise $\widetilde{W}_{1d}$ is nonconvex; (3.17) is not the Fenchel transformation which always yields a convex conjugate. General form of the displacement-dependent density of the complementary energy is given by

$$\widetilde{g}^*_d(\mathbf{N}, \mathbf{M}, \mathbf{T}; \mathbf{q}_\alpha, \mathbf{r}_\alpha, \mathbf{d}) = W^*(\mathbf{N}, \mathbf{M}, \mathbf{T}) + W_1(\mathbf{N}, \mathbf{M}, \mathbf{T}; \mathbf{q}_\alpha, \mathbf{r}_\alpha, \mathbf{d}). \tag{3.19}$$

Obviously, $\widetilde{g}^*_d$ is not convex, in general. It may be used in formulating the complementary energy principle, similarly as the problem $(\mathcal{P}^*_d)$ in [1]. In such a general case this principle should rather be formulated as a variational principle. It is not clear, when it becomes an extremum principle and what is its relation to the primal problem.

4. Simplified nonlinear case

Let us neglect the nonlinear terms appearing in Eqs. (1.4) and (1.5). For this simplified model the strain measures are given by Eq. (1.3) and

$$\kappa_{\alpha\beta}(\mathbf{w}, \mathbf{v}) = \overset{\circ}{\kappa}_{\alpha\beta}(\mathbf{w}, \mathbf{v}), \qquad \varrho_\alpha(\mathbf{w}, \mathbf{v}) = \overset{\circ}{\varrho}_\alpha(\mathbf{w}, \mathbf{v}). \tag{4.1}$$

It means that the terms $\mathbf{v} \cdot \mathbf{w}_{,\beta}$ and $\mathbf{v}_{,\alpha} \cdot \mathbf{w}_{,\beta}$ are assumed to be negligible.

The operator Λ is now chosen as follows,

$$\Lambda(\mathbf{w}, \mathbf{v}) = (\Lambda_1(\mathbf{w}), \Lambda_2(\mathbf{w}, \mathbf{v}), \Lambda_3(\mathbf{w}, \mathbf{v}), \Lambda_4(\mathbf{w})), \tag{4.2}$$

where

$$\begin{aligned} \Lambda_1(\mathbf{w}) &= \left(\overset{\circ}{\gamma}_{\alpha\beta}(\mathbf{w})\right), \qquad \Lambda_2(\mathbf{w}, \mathbf{v}) = \left(\overset{\circ}{\kappa}_{\alpha\beta}(\mathbf{w}, \mathbf{v})\right), \\ \Lambda_3(\mathbf{w}, \mathbf{v}) &= \left(\overset{\circ}{\varrho}_\alpha(\mathbf{w}, \mathbf{v})\right), \qquad \Lambda_4(\mathbf{w}) = \left(\mathbf{w}_{,\alpha}\right). \end{aligned} \tag{4.3}$$

Now the space $\mathcal{V}_1$, to which kinematical fields $(\mathbf{w}, \mathbf{v})$ belong, is defined by

$$\mathcal{V}_1 = \left\{(\mathbf{w}, \mathbf{v}) \in W^{1,p}(S, \mathbb{R}^3) \times H^1(S, \mathbb{R}^3) \mid \mathbf{w} = 0 \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0\right\}, \tag{4.4}$$

where $p > 2$.

The functional of the external loading is still given by (2.9), yet now

$$\begin{aligned} \left(P^\alpha, P\right) &\in L^q(S, \mathbb{R}^3), \qquad \left(\mathfrak{M}^\alpha, \mathfrak{M}\right) \in L^2(S, \mathbb{R}^3), \\ \left(P^\alpha_0, P_0\right) &\in L^q(\Gamma_1, \mathbb{R}^3), \qquad \left(\mathfrak{M}^\alpha_0, \mathfrak{M}_0\right) \in L^2(\Gamma_1, \mathbb{R}^3), \end{aligned} \tag{4.5}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

The minimum principle of the total potential energy reads:

$$(4.6)\quad (\mathcal{P}_1)\ \left|\begin{array}{l}\text{Find } (\tilde{\mathbf{w}},\tilde{\mathbf{v}})\in\mathcal{V}_1 \text{ such that}\\ J(\tilde{\mathbf{w}},\tilde{\mathbf{v}})=\inf\left\{\int_S W(\gamma(\mathbf{w}),\overset{\circ}{\kappa}(\mathbf{w},\mathbf{v}),\overset{\circ}{\varrho}(\mathbf{w},\mathbf{v}))\,dS+F(\mathbf{w},\mathbf{v})\,|\,(\mathbf{w},\mathbf{v})\in\mathcal{V}_1\right\},\end{array}\right.$$

where the set $\mathcal{K}$ involved in the definition of F is still given by Eq. (2.6).

Let us set $\mathbf{p}^* = (\mathbf{N},\mathbf{M},\mathbf{T},\mathbf{Q}^\alpha)\in\mathcal{B}_1$, where

$$\begin{aligned}\mathcal{B}_1=\Big\{(\mathbf{N},\mathbf{M},\mathbf{T},\mathbf{Q}^\alpha)\ |\ & N^{\alpha\beta}\in L^q(S),\ N^{\alpha\beta}{}_{,\beta}\in L^q(S),\ M^{\alpha\beta}\in L^2(S),\\ & M^{\alpha\beta}{}_{,\beta}\in L^2(S),\ T^\alpha\in L^2(S),\ T^\alpha{}_{,\alpha}\in L^2(S),\ Q^{\alpha\beta}\in L^q(S),\\ & Q^\alpha\in L^q(S),\ Q^{\alpha\beta}{}_{,\beta}\in L^q(S),\ Q^\alpha{}_{,\alpha}\in L^q(S)\Big\}.\end{aligned}$$

The operator Λ^*, the dual of Λ, can be written in the form $\Lambda^*\mathbf{p}^* = (A_1^*\mathbf{p}^*, B_1^*\mathbf{p}^*)$, where

$$(4.7)\quad A_1^*\mathbf{p}^* = \begin{cases} -\left[(N^{\alpha\beta}-b_\lambda^\alpha M^{\lambda\beta})\mathbf{a}_\alpha+\frac{1}{2}T^\beta\mathbf{n}+\mathbf{Q}^\beta\right]_{|\beta}, & \text{in } S,\\ \left[(N^{\alpha\beta}-b_\lambda^\alpha M^{\lambda\beta})\mathbf{a}_\alpha+\frac{1}{2}T^\beta\mathbf{n}+\mathbf{Q}^\beta\right]\nu_\beta, & \text{on } \Gamma_1,\end{cases}$$

$$(4.8)\quad \overset{\circ}{B}{}_1^*\mathbf{p}^* = \begin{cases} -(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta}+\frac{1}{2}T^\alpha\mathbf{a}_\alpha, & \text{in } S,\\ M^{\alpha\beta}\mathbf{a}_\alpha\nu_\beta, & \text{on } \Gamma_1.\end{cases}$$

The second term of the density of the complementary energy, now denoted by $\widetilde{W}_1^*$, depends on $\mathbf{N}$ and $\mathbf{Q}^\alpha$ only. It is given by

$$(4.9)\quad \widetilde{W}_1^*(\mathbf{N},\mathbf{Q}^\alpha) = \begin{cases} \frac{1}{2}B_{\alpha\beta}\mathbf{Q}^\alpha\cdot\mathbf{Q}^\beta, & \text{if } \mathbf{N} \text{ is positive definite;}\\ \frac{1}{2}\frac{1}{N^{\sigma\sigma}}\mathbf{Q}^\sigma\cdot\mathbf{Q}^\sigma, & \text{if rank } \mathbf{N}=1 \text{ and } N^{\sigma\sigma}>0,\\ & \mathbf{Q}^{3-\sigma}=\frac{N^{12}}{N^{\sigma\sigma}}\mathbf{Q}^\sigma,\ (\sigma \text{ not summed!});\\ \infty & \text{otherwise,}\end{cases}$$

where, as previously, $\mathbf{B} = \left(B_{\alpha\beta}\right) = \left(N^{\alpha\beta}\right)^{-1}$. It is worth noting that due to the linearity of the strain measures $\overset{\circ}{\kappa}$ and $\overset{\circ}{\varrho}$, the formula (4.9) is similar to the corresponding one for Kirchhoff shells undergoing moderately large rotations around tangents, see Ref. [20].

The complementary energy principle reads:

$$
(\mathcal{P}_1^*)\quad
\begin{vmatrix}
&\text{Find}\\
&\sup\Big\{-\int\limits_S \Big[W^*(\mathbf{N},\mathbf{M},\mathbf{T})+\widetilde{W}_1^*(\mathbf{N},\mathbf{Q}^\alpha)\Big]\,dS-\widetilde{F}_0^*(-\overset{\circ}{B}{}_1^*(\mathbf{M},\mathbf{T}))\ \mid\\
&(\mathbf{N},\mathbf{M},\mathbf{T})\in\mathcal{B}_1\Big\},\ \text{subject to}\\
&Q^{\alpha\beta}{}_{|\beta}+N^{\alpha\beta}{}_{|\beta}-(b_\lambda^\alpha M^{\lambda\beta})_{|\beta}-b_\beta^\alpha Q^\beta-\frac{1}{2}b_\lambda^\alpha T^\lambda+P^\alpha=0,\ \text{in } S,\\
&b_{\alpha\beta}N^{\alpha\beta}-b_\lambda^\alpha b_{\alpha\beta}M^{\lambda\beta}+\frac{1}{2}T^\alpha{}_{|\alpha}+b_{\alpha\beta}Q^{\alpha\beta}+Q^\alpha{}_{|\alpha}+P=0,\ \text{in } S,\\
&-Q^{\beta\alpha}\nu_\beta+b_\lambda^\alpha M^{\lambda\beta}\nu_\beta-N^{\alpha\beta}\nu_\beta+P_0^\alpha=0,\ \text{on } \Gamma_1,\\
&-\frac{1}{2}T^\alpha\nu_\alpha-Q^\alpha\nu_\alpha+P_0=0,\ \text{on } \Gamma_1,
\end{vmatrix}
\tag{4.10}
$$

where $\widetilde{F}_0^*(-\overset{\circ}{B}{}_1^*(\mathbf{M},\mathbf{T}))=\widetilde{F}_0^*(-\overset{\circ}{B}{}^*(\mathbf{M},\mathbf{T}))$, see Eq. (2.21).

REMARK 2. As in Sec. 3, we may consider the complementary energy principle involving static and kinematic fields. For the case studied in the present section we have, for $\mathbf{Q}^\alpha = N^{\alpha\beta}\mathbf{q}_\beta$:

$$
g_d^*(\mathbf{N},\mathbf{M},\mathbf{T},\mathbf{q}_\alpha)=W^*(\mathbf{N},\mathbf{M},\mathbf{T})+\widetilde{W}_1^*(\mathbf{N},\mathbf{Q}_\alpha).
\tag{4.11}
$$

Let us set $g_1^*(\mathbf{N},\mathbf{q}_\alpha)=\widetilde{W}_1^*(\mathbf{N},\mathbf{Q}^\alpha)$ if $\mathbf{Q}^\alpha = N^{\alpha\beta}\mathbf{q}_\beta$. Similarly as in Sec. 3, we obtain

$$
g_1^*(\mathbf{N},\mathbf{q}_\alpha)=\begin{cases}\dfrac{1}{2}N^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta, & \text{if } \mathbf{N} \text{ is positive semi-definite,}\\ \infty, & \text{otherwise.}\end{cases}
\tag{4.12}
$$

Hence we have $\inf\mathcal{P}_1=\sup\mathcal{P}_{1d}^*$, provided that a minimizer of the problem $\mathcal{P}_1$ exists and $\mathbf{N}$ is positive semi-definite.

Taking, instead of (4.12), the following expression

$$
\widetilde{g}_1^*(\mathbf{N},\mathbf{q}_\alpha)=\frac{1}{2}N^{\alpha\beta}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta,\qquad \mathbf{N}\in\mathbb{M}_s(\mathbb{R}),\quad (\mathbf{q}_\alpha)\in\mathbb{M}(\mathbb{R})\times\mathbb{R}^2,
\tag{4.13}
$$

we see that the function $\widetilde{g}_1^*$ is no longer convex. In such a case, a duality gap is generally to be expected, that is

$$
\inf\mathcal{P}_1\neq\sup\widetilde{\mathcal{P}}_1^*.
$$

The problem $(\widetilde{\mathcal{P}}_1^*)$ results immediately from $(\mathcal{P}_1^*)$ by replacing $\mathbf{Q}^\alpha$ with $N^{\alpha\beta}\mathbf{w}_{,\beta}$ and the supremum is now taken over $\mathbf{N}$, $\mathbf{M}$, $\mathbf{T}$ and $\mathbf{w}$, $\mathbf{w}=0$ on Γ_0. Whether the supremum is finite, remains an open question. To investigate local behaviour of the functional of the complementary energy, one has to find its critical points, that is to solve the variational problem of vanishing of the first variation of this functional subject to Eqs. (4.10).

5. Minimax approach to the linear model

In the paper [1] we have also discussed the minimax approach to a general nonlinear five-parameter shell model. Inherent difficulties have been exhibited. For the linear model, however, studied already in Sec. 2 of the present paper, such an approach is quite satisfactory. The essential results for the linear model will now be presented.

The functional J of the total potential energy in this case has the following form

$$J(\mathbf{w},\mathbf{v}) = \int_S W(\overset{\circ}{\gamma}(\mathbf{w}), \overset{\circ}{\kappa}(\mathbf{w},\mathbf{v}), \overset{\circ}{\varrho}(\mathbf{w},\mathbf{v}))\, dS + F(\mathbf{w},\mathbf{v}), \tag{5.1}$$

where the loading functional F is given by Eq. (2.9), while the formula (2.6) defines the non-convex set $\mathcal{K}$ of kinematical constraints.

A natural form of the perturbed functional J_p is

$$J_p(\mathbf{w},\mathbf{v};\mathbf{y},\mathbf{z},\mathbf{t}) = \int_S W(\overset{\circ}{\gamma}(\mathbf{w}) + \mathbf{y}, \overset{\circ}{\kappa}(\mathbf{w},\mathbf{v}) + \mathbf{z}, \overset{\circ}{\varrho}(\mathbf{w},\mathbf{v}) + \mathbf{t})\, dS + F(\mathbf{w},\mathbf{v}).$$

We see that $J_p(\mathbf{w},\mathbf{v};0,0,0) = J(\mathbf{w},\mathbf{v})$. Then, according to EKELAND and TEMAM [6], one can find the Lagrangian

$$\begin{aligned} -L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T}) = -F(\mathbf{w},\mathbf{v}) + \sup\Big\{ \int_S \Big[& N^{\alpha\beta} y_{\alpha\beta} + M^{\alpha\beta} z_{\alpha\beta} + T^{\alpha} t_{\alpha} \\ & - W(\overset{\circ}{\gamma}(\mathbf{w}) + \mathbf{y}, \overset{\circ}{\kappa}(\mathbf{w},\mathbf{v}) + \mathbf{z}, \overset{\circ}{\varrho}(\mathbf{w},\mathbf{v}) + \mathbf{t}) \Big] dS \;\Big| \\ & \Big| \; \mathbf{y} \in L^2(S, \mathbb{M}_s(\mathbb{R})),\ \mathbf{z} \in L^2(S, \mathbb{M}_s(\mathbb{R})),\ \mathbf{t} \in L^2(S, \mathbb{R}^2) \Big\}. \end{aligned}$$

Hence

$$\begin{aligned} L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T}) = {} & F(\mathbf{w},\mathbf{v}) \\ & + \int_S \Big[N^{\alpha\beta} \overset{\circ}{\gamma}_{\alpha\beta} + M^{\alpha\beta} \overset{\circ}{\kappa}_{\alpha\beta} + T^{\alpha} \overset{\circ}{\varrho}_{\alpha} - W^*(\mathbf{N},\mathbf{M},\mathbf{T}) \Big] dS, \end{aligned} \tag{5.2}$$

where W^* is given by (2.15). The primal problem may now be formulated as inf sup of L.

PROBLEM $(\mathcal{P})$

$$\Big| \begin{array}{l} \text{Find} \\ \inf\limits_{(\mathbf{w},\mathbf{v})\in\mathcal{V}} \sup\limits_{(\mathbf{N},\mathbf{M},\mathbf{T})\in\overset{\circ}{\mathcal{S}}} L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T}). \end{array}$$

This problem coincides with the Problem $(\overset{\circ}{\mathcal{P}})$ from Sec. 2.

Let us now pass to discussion of the dual problem. Simple calculation yields

$$\int_S \left[N^{\alpha\beta}\,\mathring{\gamma}_{\alpha\beta}(\mathbf{w}) + M^{\alpha\beta}\,\mathring{\kappa}_{\alpha\beta}(\mathbf{w},\mathbf{v}) + T^{\alpha}\,\mathring{\varrho}_{\alpha}(\mathbf{w},\mathbf{v})\right] dS$$

$$= \int_S \left\{\left[-(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta \mathbf{n}_{|\beta}\right]\cdot\mathbf{w}\right.$$

$$\left. + \left[-(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha\right]\cdot\mathbf{v}\right\} dS$$

$$+ \int_{\Gamma_1}\left[\left(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n}\right)\cdot\mathbf{w}\nu_\beta + M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha\cdot\mathbf{v}\right] d\Gamma,$$

since $\mathbf{w} = 0$ and $\mathbf{v} = 0$ on Γ_0.

Next we find

$$\inf_{(\mathbf{w},\mathbf{v})\in\mathcal{V}} L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T}) = -\int_S W^*(\mathbf{N},\mathbf{M},\mathbf{T})\,dS \tag{5.3}$$

$$+ \inf\left\{\int_S\left[-(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n})_{|\beta} - \mathbf{P}\right]\cdot\mathbf{w}\,dS\right.$$

$$+ \int_S\left[-(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha - \mathfrak{M}\right]\cdot\mathbf{v}\,dS$$

$$+ \int_{\Gamma_1}\left[\left(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n}\right)\nu_\beta - \mathbf{P}_0\right]\cdot\mathbf{w}\,d\Gamma$$

$$\left. + \int_{\Gamma_1}(M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0)\cdot\mathbf{v}\,d\Gamma \mid (\mathbf{w},\mathbf{v})\in\mathcal{V} \text{ and } \mathbf{v}\in\mathcal{K}\right\}.$$

We set

$$\mathbf{r} = -(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha - \mathfrak{M},$$

$$\mathbf{r}_0 = M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0.$$

Similarly to our paper [1] (p. 258, formula (5.7)) one finds

$$\inf_{\mathbf{v}\in\mathcal{K}}\left[\int_S \mathbf{r}\cdot\mathbf{v}\,dS + \int_{\Gamma_1}\mathbf{r}_0\cdot\mathbf{v}\,d\Gamma\right]$$

$$= \inf_{\mathbf{v}\in\mathcal{K}}\left[\int_S \mathbf{r}\cdot(\mathbf{v}+\mathbf{n})\,dS + \int_{\Gamma_1}\mathbf{r}_0\cdot(\mathbf{v}+\mathbf{n})\,d\Gamma - \int_S \mathbf{r}\cdot\mathbf{n}\,dS + \int_{\Gamma_1}\mathbf{r}_0\cdot\mathbf{n}\,d\Gamma\right]$$

[cont.]
$$= -\int_S \mathbf{r}\cdot\mathbf{n}\,dS + \int_{\Gamma_1} \mathbf{r}_0\cdot\mathbf{n}\,d\Gamma$$

$$+ \inf_{\mathbf{v}\in\mathcal{K}} \left[\int_S |\mathbf{r}(x)||\mathbf{v}(x)+\mathbf{n}(x)|\cos\varphi(x)\,dS + \int_{\Gamma_1} |\mathbf{r}_0(x)||\mathbf{v}(x)+\mathbf{n}(x)|\cos\varphi_0\,d\Gamma\right]$$

$$= -\int_S \mathbf{r}\cdot\mathbf{n}\,dS - \int_{\Gamma_1} \mathbf{r}_0\cdot\mathbf{n}\,d\Gamma - \int_S |\mathbf{r}|\,dS - \int_{\Gamma_1} |\mathbf{r}_0|\,d\Gamma,$$

where $\varphi(x)$ is the angle between $\mathbf{r}(x)$ and $(\mathbf{v}(x)+\mathbf{n}(x))$ at a point $x\in S$; similarly $\varphi_0(x)$ measures the angle between $\mathbf{r}_0(x)$ and $(\mathbf{v}(x)+\mathbf{n}(x))$ at $x\in\Gamma_1$.

Consequently, we may write

$$G(\mathbf{N},\mathbf{M},\mathbf{T}) := \inf_{(\mathbf{w},\mathbf{v})\in\mathcal{V}} L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T})$$

$$= \begin{cases} \tilde{G}(\mathbf{N},\mathbf{M},\mathbf{T}), & \text{if } (N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n})_{|\beta} + \mathbf{P} = 0, \quad \text{in } S, \\ & \left(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n}\right)\nu_\beta - \mathbf{P}_0 = 0, \quad \text{on } \Gamma_1; \\ -\infty, & \text{otherwise,} \end{cases}$$

where

$$\tilde{G}(\mathbf{N},\mathbf{M},\mathbf{T}) = -\int_S \left\{\left[-(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha - \mathfrak{M}\right]\cdot\mathbf{n} + \left|-(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha - \mathfrak{M}\right|\right\} dS$$

$$- \int_{\Gamma_1} \left\{(M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0)\cdot\mathbf{n} + |M^{\alpha\beta}\nu^\beta\mathbf{a}_\alpha - \mathfrak{M}_0|\right\}\Gamma - \int_S W^*(\mathbf{N},\mathbf{M},\mathbf{T})\,dS$$

$$= -\int_S \left[W^*(\mathbf{N},\mathbf{M},\mathbf{T}) - \mathfrak{M} + \left|\left(-M^{\alpha\beta}_{|\beta} + \frac{1}{2}T^\alpha\right)\mathbf{a}_\alpha + \mathfrak{M}\right|\right] dS$$

$$- \int_{\Gamma_1} \left[\left|M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0\right| - \mathfrak{M}_0\right] d\Gamma, \tag{5.4}$$

because

$$\left[-(M^{\alpha\beta}\mathbf{a}_\alpha)_{|\beta} + \frac{1}{2}T^\alpha\mathbf{a}_\alpha\right]\cdot\mathbf{n} = 0,$$

$$(M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0)\cdot\mathbf{n} = -\mathfrak{M}_0\cdot\mathbf{n} = -(\mathfrak{M}^\alpha_0\mathbf{a}_\alpha + \mathfrak{M}_0\mathbf{n})\cdot\mathbf{n} = -\mathfrak{M}_0.$$

The dual problem $(\mathcal{P}^*)$ is expressed by

$$\sup\left\{G(\mathbf{N},\mathbf{M},\mathbf{T}) \mid (\mathbf{N},\mathbf{M},\mathbf{T}) \in \mathring{\mathcal{B}}\right\}$$

$$= \sup\left\{\inf_{(\mathbf{w},\mathbf{v})\in\mathcal{V}} L(\mathbf{w},\mathbf{v},\mathbf{N},\mathbf{M},\mathbf{T}) \mid (\mathbf{N},\mathbf{M},\mathbf{T}) \in \mathring{\mathcal{B}}\right\}$$

$$= \sup\left\{\widetilde{G}(\mathbf{N},\mathbf{M},\mathbf{T}) \mid (\mathbf{N},\mathbf{M},\mathbf{T}) \in \mathring{\mathcal{B}}_s\right\},$$

where

$$\mathring{\mathcal{B}}_s = \left\{(\mathbf{N},\mathbf{M},\mathbf{T}) \in \mathring{\mathcal{B}} \mid \left(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n}\right)_{|\beta} + \mathbf{P} = 0, \text{ in } S;\right.$$

$$\left.\left(N^{\alpha\beta}\mathbf{a}_\alpha - b^\alpha_\lambda M^{\lambda\beta}\mathbf{a}_\alpha + \frac{1}{2}T^\beta\mathbf{n}\right)\nu_\beta - \mathbf{P}_0 = 0, \text{ on } \Gamma_1\right\}. \tag{5.5}$$

Here the space $\mathring{\mathcal{B}}$ is defined by (2.16). Thus, via the minimax approach the dual problem derived in Sec. 2 has been recovered, i.e.: $(\mathring{\mathcal{P}}^*) = (\mathcal{P}^*)$.

REMARK 3. The form (5.4) of the functional $\widetilde{G}$ involved in the dual problem (5.5) implies

$$\widetilde{G}(\mathbf{N},\mathbf{M},\mathbf{T}) = -\int_S \left\{W^*(\mathbf{N},\mathbf{M},\mathbf{T}) - \mathfrak{M} + \left|(-M^{\alpha\beta}{}_{|\beta} + \frac{1}{2}T^\alpha)\mathbf{a}_\alpha + \mathfrak{M}\right|\right\} dS$$

$$-\int_{\Gamma_1}\left[\left|M^{\alpha\beta}\nu_\beta\mathbf{a}_\alpha - \mathfrak{M}_0\right| - \mathfrak{M}_0\right] d\Gamma \leq -\int_S [W^*(\mathbf{N},\mathbf{M},\mathbf{T}) - \mathfrak{M}] + \int_{\Gamma_1}\mathfrak{M}_0\, d\Gamma.$$

6. Conclusions and final remarks

The methods of convex analysis have been applied to the formulation of the complementary energy principle for a nonlinear model of shells with an independent rotation vector [3]. By using Rockafellar's theory of duality [6], in our paper [1] we have derived the general form of this principle as a convex dual problem. The density of the complementary energy g^* is a sum of W^*, the conjugate of the stored energy function W, and of an additional term W_1^* defined by (3.5). The last term is convex under strong restrictions specified by (3.12) and implied by the nonlinearities of the strain measures. For g^* convex we have $\inf \mathcal{P} = \sup \mathcal{P}^*$, provided that the infimum in the primal problem $(\mathcal{P})$ is attained. Another form of g^*, denoted by g_d^*, is given in terms of generalized internal forces and kinematical fields, cf. Eqs. (3.14)–(3.16). Now g_d^* is a sum of W^* and W_{1d}^*.

We may relax the strong conditions: $\mathbf{N}$ positive semidefinite and $\mathbf{M} = 0, \mathbf{T} = 0$ by considering the complementary energy principle in which the complementary elastic potential $\tilde{g}_d^*$ appears. This principle is then a non-convex problem and

should rather be formulated as a variational principle. Thus a challenging problem, still unresolved, is to formulate an alternative theory of duality rendering the dual problem nonconvex and yielding $\inf \mathcal{P} = \sup \mathcal{P}^*$. Then the complementary energy principle would take the form of a sup problem.

Another approach has been suggested by PODIO-GUIDUGLI [22]: in the presence of a kinematical constraint, the potential energy functional may well be convex over the collection of admissible deformations; a "pure", and easy, variational problem could be solved first, and then the reactions that maintain the constraint determined with the use of the complete equilibrium equations. Such a programme, however, remains to be realized!

As we know, in the Rockafellar's theory of duality [6] a linear operator Λ is involved, cf. also [1, 2, 8, 9, 11]. In applications to solid and structural mechanics its form follows naturally from strain-displacement relations. For nonlinear strain measures, terms like W_1^* or W_{1d}^* in the expression for the complementary potential are inevitable. Just this term imposes restrictions on the applicability of the complementary energy principle. However, a step forward is done in Part II of our paper [10], though actually only for nonlinear compressed elastic beams.

Appendix. Determination of the conjugate function of a quadratic one in the general case

If $\mathbf{A} = \left(A^{ij}\right)$ is a positive definite, and thus invertible, $n \times n$ matrix, then it is easy to find the conjugate $g_{\mathbf{A}}^*$ of the quadratic function

$$g_{\mathbf{A}}(\mathbf{p}) = g(\mathbf{A}, \mathbf{p}) = \frac{1}{2} A^{ij} p_i p_j, \qquad \mathbf{p} \in \mathbb{R}^n, \tag{A.1}$$

provided that $\mathbf{A} \in \mathbb{M}_s^n(\mathbb{R})$ is held fixed. Here $\mathbb{M}_s^n(\mathbb{R})$ is the space of symmetric $n \times n$ matrices and the summation convention is consequently applied.

In such a case we readily obtain

$$g_{\mathbf{A}}^*(\mathbf{p}^*) = g^*(\mathbf{A}, \mathbf{p}^*) = \sup_{\mathbf{p} \in \mathbb{R}^n} \left\{ \mathbf{p}^* \cdot \mathbf{p} - \frac{1}{2} A^{ij} p_i p_j \right\} = \frac{1}{2} B^{ij} p^{*i} p^{*j}, \tag{A.2}$$

where $\mathbf{B} = \mathbf{A}^{-1}$.

The problem of determination of the conjugate function $g_{\mathbf{A}}^*$ becomes more complicated in the general case when rank $\mathbf{A} = r < n$. Surprisingly, but as far as we know, the general case has not been solved in the literature. The solution is now proposed. ROCKAFELLAR [7, p. 108] mentioned the problem without solving it.

Suppose that a matrix $\mathbf{A} \in \mathbb{M}_s^n(\mathbb{R})$ is such that rank $\mathbf{A} = r < n$. We assume that the indices i, j run over $1, \ldots, n$; α, β take values $1, \ldots, r$ and K, L run over $r+1, \ldots, n$. If rank $\mathbf{A} = r$ then there exists a principal minor of $\mathbf{A}$ of the order r,

different from zero. Let us assume that this minor is composed of the first r rows and columns of the matrix $\mathbf{A}$, i.e.

$$M_r = \det\left(A^{\alpha\beta}\right) = \begin{vmatrix} A^{11} & \cdots & A^{1r} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A^{r1} & \cdots & A^{rr} \end{vmatrix} \neq 0. \tag{A.3}$$

One can always achieve it by such a renumeration of the variables p_i that for the submatrix of rank r the principal minor composed of the first r rows and columns would be different from zero.

By using Sylvester's identity [21, p.376] we write the function $g_{\mathbf{A}}(\mathbf{p})$ in the following form:

$$g_{\mathbf{A}}(\mathbf{p}) = -\frac{1}{2M_r} \begin{vmatrix} A^{11} & \cdots & A^{1r} & g^1(\mathbf{p}) \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ A^{r1} & \cdots & A^{rr} & g^r(\mathbf{p}) \\ g^1(\mathbf{p}) & \cdots & g^r(\mathbf{p}) & 0 \end{vmatrix} = \frac{1}{2} B_{\alpha\beta} g^{\alpha}(\mathbf{p}) g^{\beta}(\mathbf{p}), \tag{A.4}$$

where

$$g^{\alpha}(\mathbf{p}) = A^{\alpha i} p_i, \qquad i = 1, \ldots, n, \qquad \alpha = 1, \ldots, r, \tag{A.5}$$

and $B_{\alpha\beta} A^{\beta\gamma} = \delta^{\gamma}_{\alpha}$. Now we define the following transformation ($\mathbb{R}^n \mapsto \mathbb{R}^n$)

$$\begin{aligned} q^{\alpha} &= A^{\alpha i} p_i = A^{\alpha\beta} p_{\beta} + A^{\alpha K} p_K, \qquad \alpha = 1, \ldots, r, \\ q^{K} &= \delta^{KL} p_L, \qquad K, L = r+1, \ldots, n. \end{aligned} \tag{A.6}$$

Under the assumption $\det\left(A^{\alpha\beta}\right) \neq 0$, the inverse transformation is given by

$$p_{\alpha} = B_{\alpha\beta} q^{\beta} - B_{\alpha\beta} A^{\beta K} \delta_{KL} q^L, \qquad p_K = \delta_{KL} q^L. \tag{A.7}$$

Further we have

$$\begin{aligned} p^{*i} p_i = p^{*\alpha} p_{\alpha} + p^{*K} p_K &= p^{*\alpha} \left(B_{\alpha\beta} q^{\beta} - B_{\alpha\beta} \delta_{KL} A^{\beta K} q^L \right) + p^{*K} \delta_{KL} q^L \\ &= B_{\alpha\beta} p^{*\alpha} q^{\beta} + \delta_{KL} \left(p^{*K} - B_{\alpha\beta} A^{\beta K} p^{*\alpha} \right) q^L. \end{aligned}$$

Recalling that

$$g_{\mathbf{A}}(\mathbf{p}) = \frac{1}{2} B_{\alpha\beta} g^{\alpha}(\mathbf{p}) g^{\beta}(\mathbf{p}),$$

we obtain

$$(A.8)\qquad g^*(\mathbf{A},\mathbf{p}^*) = \sup_{\mathbf{p}\in\mathbb{R}}\left\{\mathbf{p}^*\cdot\mathbf{p} - \frac{1}{2}A^{ij}p_ip_j\right\}$$

$$= \sup_{\mathbf{p}\in\mathbb{R}}\left\{\mathbf{p}^{*\alpha}B_{\alpha\beta}q^\beta - \frac{1}{2}B_{\alpha\beta}q^\alpha q^\beta + q^L\delta_{KL}(p^{*K} - B_{\alpha\beta}A^{\beta K}p^{*\alpha})\right\}$$

$$= \begin{cases} \dfrac{1}{2}B_{\alpha\beta}p^{*\alpha}p^{*\beta}, & \text{if rank } \mathbf{A} = r,\ \mathbf{A} \text{ is positive semi-definite} \\ & \text{and } p^{*K} = p^{*\alpha}B_{\alpha\beta}A^{\beta K}; \\ \infty, & \text{otherwise.} \end{cases}$$

The condition of positive definiteness of the matrix $\mathbf{A}$ is equivalent to the following inequalities

$$(A.9)\qquad A^{11} > 0 \quad \text{and} \quad \begin{vmatrix} A^{11} & \cdots & A^{1\alpha} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ A^{\alpha 1} & \cdots & A^{\alpha\alpha} \end{vmatrix} > 0, \quad \text{for } \alpha = 1,\ldots,r.$$

Let us consider now the case when instead of (A.1) we have

$$(A.10)\qquad g_\Lambda(\mathbf{p}) = \frac{1}{2}A^{ij}\mathbf{p}_i\cdot\mathbf{p}_j\,,$$

where $\mathbf{p}_i \in \mathbb{R}^3$; $i = 1,\ldots,n$. By (p_{i1},p_{i2},p_{i3}) we denote the components of a vector $\mathbf{p}_i \in \mathbb{R}^3$. Then we readily obtain

$$(A.11)\qquad g^*_{\mathbf{A}}(\mathbf{p}^*) = g^*(\mathbf{A},\mathbf{p}^*) = \sup\left\{\mathbf{p}^{*i}\cdot\mathbf{p}_i - \frac{1}{2}A^{ij}\mathbf{p}_i\cdot\mathbf{p}_j \mid \mathbf{p}_i\in\mathbb{R}^3,\ i=1,\ldots,n\right\}$$

$$= \sup\left\{p^{*i}_1\cdot p_{i1} - \frac{1}{2}A^{ij}p_{i1}p_{j1} + p^{*i}_2p^*_{i2} - \frac{1}{2}A^{ij}p_{i2}p_{j2} + p^{*i}_3p^*_{i3} - \frac{1}{2}A^{ij}p_{i3}p_{j3} \mid \right.$$

$$\left. \mid (p_{i1},p_{i2},p_{i3})\in\mathbb{R}^3\right\}$$

$$= \sup_{p_{i1}\in\mathbb{R}}\left\{p^{*i}_1\cdot p_{i1} - \frac{1}{2}A^{ij}p_{i1}p_{j1}\right\} + \sup_{p_{i2}\in\mathbb{R}}\left\{p^{*i}_2\cdot p_{i2} - \frac{1}{2}A^{ij}p_{i2}p_{j2}\right\}$$

$$+ \sup_{p_{i3}\in\mathbb{R}}\left\{p^{*i}_3\cdot p_{i3} - \frac{1}{2}A^{ij}p_{i3}p_{j3}\right\}$$

$$= \begin{cases} \dfrac{1}{2}B_{\alpha\beta}\mathbf{p}^{*\alpha}\cdot\mathbf{p}^{*\beta}, & \text{if rank } \mathbf{A} = r,\ \mathbf{A} \text{ is positive semi-definite} \\ & \text{and } p^{*K} = p^{*\alpha}B_{\alpha\beta}A^{\beta K}; \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathbf{B} = \left(A^{\alpha\beta}\right)^{-1}$.

Acknowledgement

The paper was supported by the State Committee for Scientific Research (Poland) under grant No 300199101, and partially through the grant No 3 P404 013 06.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received October 28, 1994.

Arch. Mech., 47, 4, pp. 699–724, Warszawa 1995

Duality and the complementary energy principle for a class of nonlinear structures Part II. Anomalous dual variational principles for compressed elastic beams

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Dedicated to the memory of Pawełek Telega, son of the second author

FOR A NONLINEAR ELASTIC beam, two boundary value problems are studied by using a variational approach based on suitably chosen Lagrangians. It is shown that the dual problem for the compressed simply supported beam has the form of the anomalous dual principle. General setting for a class of anomalous dual variational principles is also proposed. Two dual problems are formulated for the compressed beam fixed at one end; one has the form of the inf sup principle, while the other is the sup inf principle.

Introduction

IN THE FIRST PART of the paper [1] some consequences have been drawn from the general form of the complementary energy principle derived in [2]. It has been concluded that the theory of duality developed by EKELAND and TEMAM [3], when applied to problems involving nonlinear strain measures, precludes practically important class of loadings. For instance, actually one cannot formulate a dual extremum principle for compressed von Kármán plates.

The present contribution constitutes a first step towards variational study of compressed, geometrically nonlinear structures like beams, plates and shells by the duality approach. Here we focus on the elastic nonlinear beam model, which readily results from the von Kármán plate model [4, 5]. Our approach has been inspired by AUCHMUTY'S paper [6], in which the so-called anomalous dual variational principles have been introduced.

The plan of the paper is as follows. In Sec. 1 two primal problems are formulated. They differ in the imposed boundary conditions. In the paper [6] the operator Λ is always linear. For structures, its form is naturally implied by the deformation measures. For nonlinear structures like beams, it is convenient to use a nonlinear operator Λ. Therefore in Sec. 2 some of AUCHMUTY'S [6] results are extended so as to include a class of nonlinear operators Λ. The complementary energy principle for a compressed, simply supported beam is formulated in Sec. 3 as an anomalous dual variational principle. In Sec. 4 two dual principles are formulated for the beam fixed at one end; one of them has the form of the

inf sup principle, whereas the second is the sup inf principle. These two dual principle are derived from two different Lagrangians.

For another class of problems, interesting from the variational and mathematical point of view, the reader should refer to [7–10].

1. Basic relations and primal problems

Let the axis x be directed along the mid-line of a nonlinear elastic beam of length ℓ. The displacement fields is denoted by (u, w), where u is the horizontal displacement along x-axis and w is the transverse displacement [4].

The strain-displacement relations are given by

$$\varepsilon(u, w) = u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2, \tag{1.1}$$

$$\kappa(w) = w_{,xx}\,, \tag{1.2}$$

where $u_{,x} = du/dx$, etc. In the sequel we shell write $w_{,x}^2$ instead of $\left(w_{,x}\right)^2$.

By N and M we denote the normal force and bending moment, respectively.

The constitutive equations are given by

$$N = A\varepsilon\,, \qquad M = B\kappa\,, \tag{1.3}$$

where $A > 0$ and $B > 0$; for a beam with a constant cross-section A and B are constants. In the general case A and B are functions of $x \in [0, \ell]$. We make the following assumption:

$$A \in L^\infty(0, \ell), \qquad B \in L^\infty(0, \ell), \tag{1.4}$$

$$\exists A_0 > 0, \quad \exists B_0 > 0, \quad \forall x \in (0, \ell) \quad A(x) \geq A_0, \quad B(x) \geq B_0. \tag{1.5}$$

The constitutive relations (1.3) might be generalized in the following way. Suppose that $W(a, b)$ is a convex, differentiable function such that

$$\exists c > 0 \quad \forall (a, b) \in \mathbb{R} \times \mathbb{R}, \quad W(a, b) \geq c(|a|^s + |b|^{s_1}), \tag{1.6}$$

where $\mathbb{R}$ stands for the space of reals and $s \geq 2$, $s_1 \geq 2$. In this case the constitutive equations are

$$N = \frac{\partial W}{\partial \varepsilon}\,, \qquad M = \frac{\partial W}{\partial \kappa}\,. \tag{1.7}$$

If the elastic potential W is only subdifferentiable then (1.7) has to be replaced by

$$N \in \partial_1 W(\varepsilon, \kappa), \qquad M \in \partial_2 W(\varepsilon, \kappa), \tag{1.8}$$

where $\partial_1 W(\varepsilon, \kappa)$, $\partial_2 W(\varepsilon, \kappa)$ denote the subdifferentials of the functions $W(\cdot, \kappa)$ $W(\varepsilon, \cdot)$ at points ε and κ, respectively, cf. [11].

In the subsequent sections dual principles will be derived for the introduced nonlinear beam model. The boundary value problems which will be studied differ in the boundary conditions imposed at the left end of the beam, i.e. for $x = 0$:

$$\text{(1.9)} \qquad 1. \quad u(0) = 0, \quad w(0) = w(\ell) = 0, \quad N(\ell) = q, \quad M(0) = 0, \quad M(\ell) = -m,$$

$$\text{(1.10)} \qquad 2. \quad u(0) = 0, \quad w(0) = w(\ell) = 0, \quad w_{,x}(0) = 0, \quad N(\ell) = q, \quad M(\ell) = -m.$$

The quantities q and m are obviously prescribed. We see that under the conditions (1.9), the beam is simply supported while in the case (1.10) it is fixed at the end $x = 0$.

The spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of kinematically admissible displacement fields are defined by

$$\text{(1.11)} \qquad \mathcal{V}_1 = \left\{(u, w) \in H^1(0, \ell) \times H^2(0, \ell) \mid u(0) = w(0) = w(\ell) = 0\right\},$$

$$\text{(1.12)} \qquad \mathcal{V}_2 = \left\{(u, w) \in H^1(0, \ell) \times H^2(0, \ell) \mid u(0) = w(0) = w(\ell) = 0,\ w_{,x}(0) = 0\right\}.$$

In the more general case (1.6) the definitions of the spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ are to be changed in the usual way, provided that $s > 2$ and $s_1 > 2$. For the definitions and properties of the function spaces employed in the present paper the reader should refer to the book by ADAMS [12].

Two minimum principles of the total potential energy are now formulated.

PROBLEM $(\mathcal{P}_1)$

Find

$$J(\tilde{u}, \tilde{w}) = \inf\{J(u, w) \mid (u, w) \in \mathcal{V}_1\}.$$

PROBLEM $(\mathcal{P}_2)$

Find

$$J(\tilde{u}, \tilde{w}) = \inf\{J(u, w) \mid (u, w) \in \mathcal{V}_2\},$$

where

$$\text{(1.13)} \qquad J(u, w) = \int_0^\ell \{W[\varepsilon(u(x), w(x)), \kappa(w(x))] - p(x)w(x)\}\, dx - qu(\ell) + mw_{,x}(\ell),$$

and $p(x)$ denotes a distributed transverse loading.

Usually, the elastic potential $W(\varepsilon, \kappa)$ is assumed as a quadratic function; then

$$W[\varepsilon(u,w),\kappa(w)] = \frac{A}{2}[\varepsilon(u,w)]^2 + \frac{B}{2}[\kappa(w)]^2 = \frac{A}{2}\left(u_{,x} + \frac{1}{2}w_{,x}^2\right)^2 + \frac{B}{2}w_{,xx}^2 . \tag{1.14}$$

We note that the presence of the nonlinear strain measure $\varepsilon(u, w)$ renders the functional J, given by (1.13), non-convex.

Existence results for the problem $(\mathcal{P}_2)$ where obtained by AUBERT and TAHRAOUI [13], cf. also [14, 15]. The main result proved in [13] is stated as follows.

THEOREM 1. *Let W be given by* (1.14). *Under the assumption*

$$q + \lambda_1^2 B > 0, \tag{1.15}$$

where λ_1^2 is the smallest eigenvalue of the problem

$$f_{,xxxx} + \lambda^2 f_{,xx} = 0, \qquad f(0) = f(\ell) = f_{,x}(0) = f_{,xx}(\ell) = 0, \tag{1.16}$$

the problem $(\mathcal{P}_2)$ admits a unique solution $(\tilde{u}, \tilde{w})$. ∎

The Problem $(\mathcal{P}_1)$ has not been studied in [13]. However, a similar existence and uniqueness theorem can be formulated replacing by $(1.16)_2$ by $f(0) = 0$, $f(\ell) = 0$, $f_{,xx}(0) = f_{,xx}(\ell) = 0$.

2. General setting for a class of anomalous dual variational principles derived from Lagrangians of type I

Auchmuty introduced the notion of Lagrangians of type I and II as well as anomalous dual variational principles [6]. At this point it is worth noting that much more complicated situations may arise than those studied in [6]. Auchmuty's results can be extended in various directions. As we have already noted, the operator Λ involved in Auchmuty's considerations is always linear and continuous. From the point of view of applications to geometrically nonlinear problems such assumption is insufficient.

The aim of this section is mainly a generalization of Auchmuty's Lemma 4.1 to the case when Λ is a nonlinear operator. The setting proposed here is suitable for a class of nonlinear structures. For instance, it applies to the nonlinear, simply supported beams, see the next section.

Let $U = (U_1, U_2)$ and $Y = (Y_1, Y_2)$ be locally convex topological vector spaces, and $U^* = (U_1^*, U_2^*)$, $Y^* = (Y_1^*, Y_2^*)$ – their topological duals [16]. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between a space and its dual. Thus we have

$$\langle (u_1^*, u_2^*), (u_1, u_2)\rangle_{U^* \times U} = \langle u_1^*, u_1\rangle_{U_1^* \times U_1} + \langle u_2^*, u_2\rangle_{U_2^* \times U_2} . \tag{2.1}$$

In our subsequent considerations duality pairings will be evident and only the simple notation $\langle \cdot, \cdot \rangle$ will usually be used.

Suppose $J: U_1 \times U_2 \to \overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is a functional. The primal problem means evaluating

$$(2.2) \qquad (\mathcal{P}) \qquad \inf\{J(u_1, u_2) \mid (u_1, u_2) \in U_1 \times U_2\}.$$

A functional $L: (U_1, U_2) \times (Y_1^*, Y_2^*) \to \overline{\mathbb{R}}$ is said to be a Lagrangian of type I if, for any $(u_1, u_2) \in U_1 \times U_2$

$$(2.3) \qquad J(u_1, u_2) = \sup\{L(u_1, u_2, y_1^*, y_2^*) \mid (y_1^*, y_2^*) \in (Y_1^*, Y_2^*)\}.$$

The dual problem $(\mathcal{P}^*)$ is to find

$$(2.4) \qquad \sup_{(y_1^* y_2^*) \in Y^*} \inf_{(u_1, u_2) \in U} L(u_1, u_2, y_1^*, y_2^*).$$

An anomalous dual principle $(\mathcal{P}^{\otimes})$ means evaluating

$$(2.5) \qquad \inf\{K(y_1^*, y_2^*) \mid (y_1^*, y_2^*) \in Y^*\},$$

where

$$(2.6) \qquad K(y_1^*, y_2^*) = \sup_{(u_1, u_2) \in U} L(u_1, u_2, y_1^*, y_2^*).$$

We observe that the dual problem for the geometrically nonlinear beam is of the form (2.4) provided that $q \geq 0$. Simply supported compressed beams fall under (2.5), cf. Sec. 3. A clamped beam requires "mixed" Lagrangians, of the type I–II, not examined by Auchmuty, cf. Sec. 4.

Let

$$(2.7) \qquad J(u_1, u_2) = J_1(\Lambda(u_1, u_2)) - J_2(u_1, u_2),$$

where $\Lambda: (U_1, U_2) \to (Y_1, Y_2)$ is a continuous Gâteaux differentiable operator, not necessarily linear. We assume, for the sake of simplicity, that $D(\Lambda) = U$, where $D(\Lambda)$ denotes the domain of Λ.

Having in mind applications to geometrically nonlinear problems, the form of Λ is inferred from the strain measures involved in a problem considered. For the nonlinear beam problem studied in Sec. 3, we may assume

$$\Lambda(u, w) = (\Lambda_1(u, w), \Lambda_2(w)) = (\varepsilon(u, w), \kappa(w)).$$

The functionals J_1 and J_2 are assumed to be lower semicontinuous; moreover J_1 is a convex functional. Thus $\tilde{J}_1(u_1, u_2) = J_1(\Lambda(u_1, u_2))$ is not necessarily a convex functional. The Lagrangian is defined by

$$(2.8) \qquad L(u_1, u_2, y_1^*, y_2^*) = \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - J_1^*(y_1^*, y_2^*) - J_2(u_1, u_2).$$

Hence

$$J(u_1, u_2) = \sup_{(y_1^*, y_2^*) \in Y^*} L(u_1, u_2, y_1^*, y_2^*), \qquad (u_1, u_2) \in U. \tag{2.9}$$

Denote by $\widetilde{\Lambda}_{(u_1,u_2)}$ the linearized operator of Λ at a point $(u_1, u_2) \in U$. Its adjoint $\widetilde{\Lambda}^*_{(u_1,u_2)}$ is defined by

$$\langle \widetilde{\Lambda}_{(u_1,u_2)}(v_1, v_2), (y_1^*, y_2^*) \rangle_{Y \times Y^*} = \langle \widetilde{\Lambda}^*_{(u_1,u_2)}(y_1^*, y_2^*), (v_1, v_2) \rangle_{U^* \times U}, \tag{2.10}$$

where $(v_1, v_2) \in D(\widetilde{\Lambda}_{(u_1,u_2)})$, $(y_1^*, y_2^*) \in Y^*$.

Here our attention is focussed on nonlinear operators Λ such that

$$\Lambda(u_1, u_2) = \widetilde{\Lambda}_{(u_1,u_2)} \left(u_1, \frac{1}{2} u_2 \right). \tag{2.11}$$

The Hamiltonian associated to the Lagrangian L is now defined by

$$H(u_1, u_2, y_1^*, y_2^*) = \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - L(u_1, u_2, y_1^*, y_2^*). \tag{2.12}$$

On account of (2.11) we have

$$H(u_1, u_2, y_1^*, y_2^*) = \left\langle \widetilde{\Lambda}_{(u_1,u_2)} \left(u_1, \frac{1}{2} u_2 \right), (y_1^*, y_2^*) \right\rangle - L(u_1, u_2, y_1^*, y_2^*). \tag{2.13}$$

The Lagrangian is not necessarily given by (2.8). Particularly, for L given by (2.8) one has

$$H(u_1, u_2, y_1^*, y_2^*) = J_1^*(y_1^*, y_2^*) + J_2(u_1, u_2). \tag{2.14}$$

A point $(\hat{u}, \hat{y}^*) = [(\hat{u}_1, \hat{u}_2), (\hat{y}_1^*, \hat{y}_2^*)]$ is said to be an anomalous critical point of the Lagrangian L if

$$\Lambda(\hat{u}) \in \partial_2 H(\hat{u}, \hat{y}^*), \tag{2.15}$$

and

$$\widetilde{\Lambda}^*_{\hat{u}}(\hat{y}^*) \in \partial_1 H(\hat{u}, \hat{y}^*), \tag{2.16}$$

where ∂_1 and ∂_2 stand for the partial subdifferentiations of the functions $H(\cdot, \cdot, y_1^*, y_2^*)$ and $H(u_1, u_2, \cdot, \cdot)$, respectively; Λ is not necessarily of the form (2.11). When the operator Λ is linear, the formulae (4.11) and (4.12) of AUCHMUTY [6] are readily recovered.

The main result of this section is given by

LEMMA 1. Let H be the Hamiltonian associated with a Lagrangian of type I. Suppose that Λ satisfies (2.11). Then

a. $\Lambda(u_1, u_2) \in \partial_2 H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*)$ implies

$$J(u_1, u_2) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq K(\tilde{y}_1^*, \tilde{y}_2^*).$$

b. $\tilde{\Lambda}^*_{(\tilde{u}_1, \tilde{u}_2)}(y_1^*, y_2^*) \in \partial_1 H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*)$ implies

$$K(y_1^*, y_2^*) = L(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \leq J(\tilde{u}_1, \tilde{u}_2).$$

P r o o f.

a. If $\Lambda(u_1, u_2) \in \partial_2 H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*)$ then

$$H(u_1, u_2, y_1^*, y_2^*) - H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \geq \langle \Lambda(u_1, u_2), (y_1^* - \tilde{y}_1^*, y_2^* - \tilde{y}_2^*) \rangle, \quad \forall (y_1^*, y_2^*) \in Y_1^* \times Y_2^*.$$

Hence

$$L(u_1, u_2, y_1^*, y_2^*) = \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - H(u_1, u_2, y_1^*, y_2^*)$$
$$\leq \langle \Lambda(u_1, u_2), (\tilde{y}_1^*, \tilde{y}_2^*) \rangle - H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq K(\tilde{y}_1^*, \tilde{y}_2^*), \quad \forall (y_1^*, y_2^*) \in Y_1^* \times Y_2^*.$$

Taking the supremum over $(y_1^*, y_2^*) \in Y^*$ we obtain

$$J(u_1, u_2) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq K(\tilde{y}_1^*, \tilde{y}_2^*).$$

b. When $\tilde{\Lambda}^*_{(\tilde{u}_1, \tilde{u}_2)}(y_1^*, y_2^*) \in \partial_1 H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*)$ one has

$$H(u_1, u_2, y_1^*, y_2^*) - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \geq \langle \tilde{\Lambda}^*_{(\tilde{u}_1, \tilde{u}_2)}(y_1^*, y_2^*), (u_1 - \tilde{u}_1, u_2 - \tilde{u}_2) \rangle, \quad \forall (u_1, u_2) \in U_1 \times U_2.$$

Hence

$$\langle \tilde{\Lambda}^*_{(\tilde{u}_1, \tilde{u}_2)}(y_1^*, y_2^*), (u_1, u_2) \rangle - H(u_1, u_2, y_1^*, y_2^*)$$
$$\leq \langle \tilde{\Lambda}^*_{(\tilde{u}_1, \tilde{u}_2)}(y_1^*, y_2^*), (\tilde{u}_1, \tilde{u}_2) \rangle - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*).$$

Thus

$$\langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(u_1, u_2), (y_1^*, y_2^*) \rangle - H(u_1, u_2, y_1^*, y_2^*)$$
$$\leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(\tilde{u}_1, \tilde{u}_2), (y_1^*, y_2^*) \rangle - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*).$$

Because

$$\tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)} = \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}\left(u_1, \frac{1}{2}u_2\right) + \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}\left(0, \frac{1}{2}u_2\right),$$

therefore

$$\left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}u_2\right),(y_1^*,y_2^*)\right\rangle + \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(u_1,\frac{1}{2}u_2\right),(y_1^*,y_2^*)\right\rangle - H(u_1,u_2,y_1^*,y_2^*)$$
$$\leq \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle + \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(\tilde{u}_1,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle - H(\tilde{u}_1,\tilde{u}_2,y_1^*,y_2^*)$$
$$= \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle + L(\tilde{u}_1,\tilde{u}_2,y_1^*,y_2^*) \leq \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle$$
$$+J(\tilde{u}_1,\tilde{u}_2), \qquad \forall\,(u_1,u_2)\in U_1\times U_2.$$

Taking now the supremum over $(u_1,u_2)\in U$, we obtain

$$\left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle + \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(\tilde{u}_1,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle - H(\tilde{u}_1,\tilde{u}_2,y_1^*,y_2^*)$$
$$\leq \left\langle \tilde{\Lambda}_{(\tilde{u}_1,\tilde{u}_2)}\left(0,\frac{1}{2}\tilde{u}_2\right),(y_1^*,y_2^*)\right\rangle + J(\tilde{u}_1,\tilde{u}_2).$$

By virtue of (2.13) one has

$$K(y_1^*,y_2^*) = L(\tilde{u}_1,\tilde{u}_2,y_1^*,y_2^*) \leq J(\tilde{u}_1,\tilde{u}_2).$$

Thus the lemma is proved. ∎

COROLLARY 1. From (2.15), (2.16) and the above lemma we conclude that if $(\hat{u}_1,\hat{u}_2,\hat{y}_1^*,\hat{y}_2^*)$ is an anomalous critical point of a Lagrangian of the type I, then

$$J(\hat{u}_1,\hat{u}_2) = K(\hat{y}_1,\hat{y}_2) = L(\hat{u}_1,\hat{u}_2,\hat{y}_1^*,\hat{y}_2^*). \tag{2.17}$$

Under stronger assumptions Lemma 2.1 can be reversed.

LEMMA 2. Suppose that $(\hat{u}_1,\hat{u}_2)$ solves the primal problem (2.2), whereas $(\hat{y}_1^*,\hat{y}_2^*)$ is a solution to the anomalous dual problem (2.5), provided that J is given by (2.3). Let the Hamiltonian be defined by (2.12) where the operator Λ satisfies (2.11). If (2.17) is satisfied, then $(\hat{u}_1,\hat{u}_2,\hat{y}_1^*,\hat{y}_2^*)$ is an anomalous critical point of the Lagrangian L.

P r o o f.

According to the assumptions we may write

$$L(\hat{u}_1,\hat{u}_2,\hat{y}_1^*,\hat{y}_2^*) = J(\hat{u}_1,\hat{u}_2) = \inf_{(u_1,u_2)\in U} L(u_1,u_2,\hat{y}_1^*,\hat{y}_2^*) \leq L(u_1,u_2,\hat{y}_1^*,\hat{y}_2^*),$$
$$\forall\,(u_1,u_2)\in U,$$

and

$$L(\hat{u}_1,\hat{u}_2,\hat{y}_1^*,\hat{y}_2^*) = K(\hat{y}_1^*,\hat{y}_2^*) = \inf_{(y_1^*,y_2^*)\in Y^*} L(\hat{u}_1,\hat{u}_2,y_1^*,y_2^*) \leq L(\hat{u}_1,\hat{u}_2,y_1^*,y_2^*),$$
$$\forall\,(y_1^*,y_2^*)\in Y^*.$$

Thus $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$ is a ∂-critical point of L, cf. [6]. Hence we may write

$$0 \in \partial_1 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) \quad \text{and} \quad 0 \in \partial_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*). \tag{2.18}$$

Since the operator Λ is Gâteaux differentiable therefore $(2.18)_2$ gives

$$0 \in \Lambda(\hat{u}_1, \hat{u}_2) - \partial_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*).$$

Similarly, $(2.18)_1$ yields

$$0 \in \tilde{\Lambda}^*_{(\hat{u}_1, \hat{u}_2)}(\hat{y}_1^*, \hat{y}_2^*) - \partial_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*). \quad \blacksquare$$

The next lemma generalizes Auchmuty's Lemma 4.2. The nonlinear operator Λ, however, is not necessarily of the form (2.11).

LEMMA 3. Let L be a Lagrangian of type I and $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$ be an anomalous critical point of L. Suppose L is partially Gâteaux differentiable at $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$, then $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$ is a critical point of L.

P r o o f.

Let us denote by $D_1 L(\cdot, \cdot, \hat{y}_1^*, \hat{y}_2^*)$ and $D_2 L(\hat{u}_1, \hat{u}_2, \cdot, \cdot)$ the partial Gâteaux derivatives, cf [17]. From (2.12) one has

$$\langle D_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2)\rangle = \langle \Lambda(\hat{u}_1, \hat{u}_2), (h_1, h_2)\rangle - \langle D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2)\rangle,$$

for each $(h_1, h_2) \in (Y_1, Y_2)$. Since $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$ is an anomalous critical point, we write

$$\langle D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2)\rangle = \langle \Lambda(\hat{u}_1, \hat{u}_2) - D_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2)\rangle = 0, \quad \forall\, (h_1, h_2) \in Y.$$

Hence

$$D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = 0.$$

Similarly

$$\langle D_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (z_1, z_2)\rangle \left\langle \tilde{\Lambda}_{(\hat{u}_1, \hat{u}_2)}(z_1, z_2), (\hat{y}_1^*, \hat{y}_2^*)\right\rangle - \langle D_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (z_1, z_2)\rangle, \quad \forall\, (z_1, z_2) \in U,$$

and consequently

$$D_1 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = 0. \quad \blacksquare$$

REMARK 1. The last proof exploits Auchmuty's Lemma 3.6. It states that if $f: X \to \mathbb{R}$ is Gâteaux differentiable at $\hat{x}$ and $\xi \in \partial f(\hat{x})$, then $\xi \in Df(\hat{x})$, where $Df(\hat{x})$ stands for the Gâteaux derivative of f at $\hat{x}$.

3. Dual approach to the simply supported nonlinear elastic beam

The dual problems $(\mathcal{P}_1^*)$ and $(\mathcal{P}_1^{\otimes})$ will now be derived. It will become evident that the problem $(\mathcal{P}_1^*)$ is suitable for $q > 0$ only; $(\mathcal{P}_1^{\otimes})$-problem is applicable to compressed beams when $0 > q \geq -\pi^2 B/\ell^2$.

The perturbed functional Φ is assumed in the following form, cf. [3]

$$\Phi(u, w, p_1, p_2) = \int_0^\ell \left[\frac{A}{2}\left(u_{,x} + \frac{1}{2}w_{,x}^2 + p_1\right)^2 + \frac{B}{2}(w_{,xx} + p_2)^2 - pw \right] dx - qu(\ell) + mw_{,x}(\ell), \tag{3.1}$$

where $(u, w) \in \mathcal{V}_1$ and $(p_1, p_2) \in \mathcal{H} = L^2(0, \ell) \times L^2(0, \ell)$.

Hence

$$\Phi(u, w, 0, 0) = J(u, w). \tag{3.2}$$

The Lagrangian L is calculated from

$$-L(u, w, N, M) = \sup_{(p_1, p_2) \in \mathcal{H}} \left\{ \int_0^\ell (Np_1 + Mp_2)\, dx - \Phi(u, w, p_1, p_2) \right\},$$

where $(N, M) \in \mathcal{H}$. Finding the supremum one has

$$L(u, w, N, M) = \int_0^\ell \left[N(x)\varepsilon(u(x), w(x)) + M(x)\kappa(w(x)) - W^*(N(x), M(x)) - p(x)w(x) \right] dx - qu(\ell) + mw_{,x}(\ell), \tag{3.3}$$

where $(u, w) \in \mathcal{V}_1$, $(N, M) \in \mathcal{H}$ and

$$W^*(a, b) = \sup \left\{ a\,e + b\,d - \frac{1}{2}A\,e^2 - \frac{1}{2}B\,d^2 \mid e \in \mathbb{R},\ d \in \mathbb{R} \right\} = \frac{1}{2A}a^2 + \frac{1}{2B}b^2, \qquad (a, b) \in \mathbb{R}^2. \tag{3.4}$$

Here W^* denotes the density of the complementary energy. Performing standard calculations, from the variational equation $\delta L = 0$ one obtains

$$N_{,x} = 0, \qquad M_{,xx} - (Nw_{,x})_{,x} - p = 0, \qquad \text{in } (0, \ell), \tag{3.5}$$

$$N = A\varepsilon(u, w), \qquad M = B\kappa(w), \tag{3.6}$$

$$N(\ell) = q, \qquad M(0) = 0, \qquad M(\ell) = -m. \tag{3.7}$$

The equilibrium equations (3.5) are to be understood in the sense of distributions.

For $(N, M) \in \mathcal{H}$, $(\overline{N}, \overline{M}) \in \mathcal{H}$, and $\overline{N} = A\varepsilon(u, w)$, $\overline{M} = B\kappa(w)$, simple calculations give

$$L(u, w, N, M) - L(u, w, \overline{N}, \overline{M}) = -\int_0^\ell \left[\frac{1}{2A}(N - \overline{N})^2 + \frac{1}{2B}(M - \overline{M})^2\right] dx \leq 0.$$

Hence

$$\sup_{(N,M)\in\mathcal{H}} L(u, w, N, M) = L(u, w, A\varepsilon(u, w), B\kappa(w)).$$

Thus we have

$$J(u, w) = L(u, w, A\varepsilon(u, w), B\kappa(w)), \qquad \forall\, (u, w) \in \mathcal{V}_1\,.$$

Consequently, the problem $(\mathcal{P}_1)$ may be written as follows

$$(\mathcal{P}_1) \quad \left| \begin{array}{l} \text{Find} \\ \inf\limits_{(u,w)\in\mathcal{V}_1} \sup\limits_{(N,M)\in\mathcal{H}} L(u, w, N, M)\,. \end{array} \right.$$

Performing integration by parts we obtain

$$L(u, w, N, M) = \int_0^\ell \left[-N_{,x}u + (M_{,xx} - p)w + \frac{1}{2}N w_{,x}^2 - \left(\frac{1}{2A}N^2 + \frac{1}{2B}M^2\right)\right] dx$$
$$+(N(\ell) - q)u(\ell) + (M(\ell) + m)w_{,x}(0) - M(0)w_{,x}(0).$$

For $(u, w) \in \mathcal{V}_1$, $(\overline{u}, \overline{w}) \in \mathcal{V}_1$ and

$$\begin{aligned} &N_{,x} = 0, \qquad M_{,xx} - (N\overline{w}_{,x})_{,x} - p = 0, \qquad \text{in } (0, \ell), \\ &N(\ell) = q, \qquad M(0) = 0, \qquad M(\ell) = -m, \end{aligned} \tag{3.8}$$

one has

$$L(u, w, N, M) - L(\overline{u}, \overline{w}, N, M) = \frac{1}{2}\int_0^\ell N(w_{,x} - \overline{w}_{,x})^2\, dx. \tag{3.9}$$

We observe that (3.8) yields

$$\begin{aligned} &N = q, \qquad M_{,xx} - q\overline{w}_{,xx} - p = 0, \qquad \text{in } (0, \ell), \\ &M(0) = 0, \qquad M(\ell) = -m. \end{aligned} \tag{3.10}$$

Thus for $N > 0$, Eq. (3.9) gives

$$L(u, w, N, M) - L(\overline{u}, \overline{w}, N, M) \geq 0,$$

while for $N < 0$ we have

$$L(u, w, N, M) - L(\overline{u}, \overline{w}, N, M) \leq 0.$$

Integrating Eq. $(3.10)_2$ one finds

$$\overline{w}_{,x} = \frac{1}{q}\left(M_{,x} + \frac{m}{\ell} - R(x) + \frac{1}{\ell}\int_0^\ell R(x)dx\right) \tag{3.11}$$

provided that $q \neq 0$; here

$$R(x) = \int_0^x p(t)\,dt.$$

Let us pass now to the formulation of the dual problems for $N > 0$ and $N < 0$. For Lagrangian (3.3) we have

$$\inf_{(u,w)\in\mathcal{V}_1} L(u, w, N, M) = \begin{cases} G_1(M), & \text{if } N = q > 0, \quad M(0) = 0, \\ & M(\ell) = -m, \\ -\infty, & \text{otherwise;} \end{cases} \tag{3.12}$$

where

$$G_1(M) = L(\overline{u}, \overline{w}, q, M) = -\int_0^\ell \left(\frac{1}{2}q\overline{w}_{,x}^2 + \frac{1}{2A}q^2 + \frac{1}{2B}M^2\right)dx, \tag{3.13}$$

$$\overline{w}_{,x}^2 = \frac{1}{q^2}\left[M_{,x}^2 + 2M_{,x}Z(x) + Z^2(x)\right],$$

and

$$Z(x) = -R(x) + \frac{m}{\ell} + \frac{1}{\ell}\int_0^\ell R(x)\,dx. \tag{3.14}$$

Because

$$\int_0^\ell \overline{w}_{,x}^2\,dx = \frac{1}{q^2}\int_0^\ell (M_{,x}^2 + 2pM)\,dx + 2\frac{1}{q^2}M(\ell)Z(\ell) + \frac{1}{q^2}\int_0^\ell Z^2(x)\,dx,$$

one eventually has

$$G_1(M) = -\int_0^\ell \left[\frac{1}{2q}(M_{,x}^2 + 2pM) + \frac{1}{2B}M^2\right]dx \tag{3.15}$$

$$+\frac{1}{q}mZ(\ell) - \frac{\ell}{2A}q^2 - \frac{1}{2q}\int_0^\ell Z^2(x)\,dx.$$

G_1 is the functional of the total complementary energy. To assess its nature we calculate

$$G_1(M) - G_1(\overline{M}) = -\int_0^\ell \left\{ \frac{1}{2q}\left[M_{,x}^2 - \overline{M}_{,x}^2 + 2p(M-\overline{M})\right] + \frac{1}{2B}(M^2 - \overline{M}^2)\right\} dx\,. \tag{3.16}$$

For a moment field $\overline{M}$ satisfying

$$\overline{M}_{,xx} - \frac{q}{B}\overline{M} - p = 0, \qquad \text{in } (0,\ell),$$

$$\overline{M}(0) = 0, \qquad \overline{M}(\ell) = -m,$$

Eq. (3.16) gives

$$G_1(M) - G_1(\overline{M}) = -\int_0^\ell \left[\frac{1}{2q}(M_{,x} - \overline{M}_x)^2 + \frac{1}{2B}(M-\overline{M})^2\right] dx\,. \tag{3.17}$$

Thus for $q > 0$ we have

$$G_1(M) - G_1(\overline{M}) \le 0, \tag{3.18}$$

and

$$\sup\left\{G_1(M) \mid M \in H^1(0,\ell),\ M(0) = 0,\ M(\ell) = -m\right\} = G_1(\overline{M}). \tag{3.19}$$

Consequently, the dual problem $(\mathcal{P}_1^*)$ means evaluating

$$\sup_{M\in L^2(0,\ell)} \inf_{(u,w)\in\mathcal{V}_1} L(u,w,N,M), \qquad \text{where } N = q > 0.$$

Consider now the case $q < 0$. Towards this end we find

$$\sup_{(u,w)\in\mathcal{V}_1} L(u,w,N,M) = \begin{cases} G_1(M), & \text{if } N = q < 0,\ M(0) = 0,\ M(\ell) = -m, \\ \infty, & \text{otherwise.} \end{cases}$$

To assess the nature of the functional G_1 for $q < 0$ we use the Friedrichs' inequality, cf. [18, 19]

$$\forall\, v \in H_0^1(0,\ell) \quad \|v\|_{L^2(0,l)}^2 \le \frac{\ell^2}{\pi^2}\|\nabla v\|_{L^2(0,l)}^2, \tag{3.20}$$

where $\nabla v = dv/dx = v_{,x}$. Applying the inequality (3.20) to $v = M - \overline{M} \in H_0^1(0, \ell)$, from (3.17) one has

$$G_1(M) - G_1(\overline{M}) \geq -\int_0^\ell \left(\frac{\pi^2}{2q\ell^2} + \frac{1}{2B} \right) (M - \overline{M})^2 \, dx \geq 0, \tag{3.21}$$

provided that

$$0 > q \geq -\frac{\pi^2 B}{\ell^2}. \tag{3.22}$$

It is worth noting that $q_{\text{crit}} = \pi^2 B/\ell^2$ is the first critical force.

The anomalous dual variational principle $(\mathcal{P}_1^{\otimes})$ is to find

$$\left| \inf_{M \in L^2(0,\ell)} \sup_{(u,w) \in \mathcal{V}_1} L(u, w, N, M), \qquad \text{where } 0 > N = q \geq -\frac{\pi^B}{\ell^2}. \right.$$

Suppose that $(\tilde{u}, \tilde{w})$ solves the primal problem $(\mathcal{P}_1)$ whereas $(\tilde{N} = \tilde{q}, \widetilde{M})$ is a solution to $(\mathcal{P}_1^{\otimes})$. The considerations of the present section show that

$$\begin{aligned} L(\tilde{u}, \tilde{w}, q, \widetilde{M}) &\leq L(u, w, q, \widetilde{M}) \qquad \forall\, (u, w) \in \mathcal{V}_1, \\ L(\tilde{u}, \tilde{w}, q, \widetilde{M}) &\leq L(\tilde{u}, \tilde{w}, q, M) \qquad \forall\, M \in L^2(0, \ell). \end{aligned}$$

Since the Lagrangian L has partial Gâteaux derivatives $D_1 L$ and $D_2 L$, therefore $(\tilde{u}, \tilde{w}, \tilde{N} = q, \widetilde{M})$ is a critical point of L, which is also an anomalous critical point. In this case Corollary 2.1 implies

$$J(\tilde{u}, \tilde{w}) = L(\tilde{u}, \tilde{w}, q, \widetilde{M}) = K(q, \widetilde{M}), \tag{3.23}$$

where

$$K(q, M) = G_1(M), \quad \text{for } N = q < 0, \quad M(0) = 0, \quad M(\ell) = -m. \tag{3.24}$$

Let us investigate this anomalous critical point more closely. For the sake of generality, we assume that the stored energy function W is a convex and l.s.c. function. The condition (3.22) is then to be replaced by

$$0 > q \geq -q_{\text{crit}}. \tag{3.25}$$

Now q_{crit} depends upon the potential W. Further, we set

$$F(u, w) = -\int_0^\ell pw \, dx - qu(\ell) + mw_{,x}(\ell). \tag{3.26}$$

The operator Λ is given by

$$\Lambda(u,w) = (\varepsilon(u,w), \kappa(w)), \qquad \Lambda: \mathcal{V}_1 \to \mathcal{H} = Y = Y^*, \quad (u,w) \in \mathcal{V}_1 . \tag{3.27}$$

If $s > 2$, $s_1 > 2$, then $\mathcal{H} = L^s(0,\ell) \times L^{s_1}(0,\ell) = Y$, and $Y^* = L^{s/(s-1)}(0,\ell) \times L^{s_1/(s_1-1)}(0,\ell)$. Without loss of generality, we confine our attention to (3.27). Now

$$\begin{aligned} L(u,w,N,M) &= \langle \Lambda(u,w), (N,M)\rangle - \int_0^\ell W^*[N(x), M(x)]\,dx + F(u,w) \\ &= \int_0^\ell [N\varepsilon(u,w) + M\kappa(w) - W^*(N,M)]\,dx + F(u,w), \\ &\qquad (u,w) \in \mathcal{V}_1, \quad (N,M) \in \mathcal{H}, \end{aligned} \tag{3.28}$$

where

$$W^*(N,M) = \sup\Big\{ N\,e + M\,d - W(e,d) \mid e \in \mathbb{R}, \quad d \in \mathbb{R} \Big\}. \tag{3.29}$$

The Hamiltonian associated with L given by (3.28) has the form

$$\begin{aligned} H(u,w,N,M) &= \langle \Lambda(u,w), (N,M)\rangle - L(u,w,N,M) \\ &= \int_0^\ell W^*[N(x), M(x)]\,dx - F(u,w). \end{aligned} \tag{3.30}$$

Let $\tilde{\Lambda}_w$ denote the linearized operator of Λ at $(0,w) \in \mathcal{V}_1$. We have

$$\tilde{\Lambda}_w(u,z) = [\nabla u + (\nabla w)\nabla z, \nabla^2 z], \qquad z \in H_0^1(0,\ell),$$

where $\nabla^2 z = d^2z/dx^2 = z_{,xx}$.

To find the dual operator $(\tilde{\Lambda}_w)^*$ of $\tilde{\Lambda}_w$ we calculate

$$\begin{aligned} &\Big\langle \tilde{\Lambda}_w(u,z), (N,M)\Big\rangle \\ &\qquad = \int_0^\ell [N(x)(\nabla u(x) + \nabla w(x)\nabla z(x)) + M(x)\kappa(z(x))]\,dx \\ &\qquad = -\int_0^\ell N_{,x} u\,dx + \int_0^\ell [M_{,xx} - (N w_{,x})_{,x}] z\,dx + N(\ell)u(\ell) + M(\ell) z_{,x}(\ell) \\ &\qquad = \Big\langle (\tilde{\Lambda}_w)^*(N,M), (u,z)\Big\rangle . \end{aligned} \tag{3.31}$$

Hence

$$(\widetilde{\Lambda}_w)^*(N,M) = \begin{cases} -N_{,x}\,, & x \in (0,\ell), \\ M_{,xx} - (Nw_{,x})_{,x}, & x \in (0,\ell), \\ N(\ell), \\ M(0), \\ M(\ell). \end{cases} \tag{3.32}$$

The inclusion, cf. (2.15)

$$\Lambda(\tilde{u},\tilde{w}) \in \partial_2 H(\tilde{u},\tilde{w},\tilde{N},\widetilde{M}), \qquad \text{where } \tilde{N} = q\,, \tag{3.33}$$

yields the constitutive relationship

$$\Lambda(\tilde{u},\tilde{w})(x) \in \partial_2 H(\tilde{u}(x),\tilde{w}(x),\tilde{N},\widetilde{M}(x)), \qquad x \in (0,\ell), \tag{3.34}$$

or

$$\varepsilon(\tilde{u}(x),\tilde{w}(x)) \in \partial_1 W^*(\tilde{N},\widetilde{M}(x)), \qquad \kappa(w(x)) \in \partial_2 W^*(\tilde{N},\widetilde{M}(x)), \tag{3.35}$$
$$\text{where } \tilde{N} = q\,.$$

For W^* of class C^1, (3.35) takes more familiar form

$$\varepsilon(\tilde{u}(x),\tilde{w}(x)) = \frac{\partial W^*(\tilde{N},\widetilde{M}(x))}{\partial N}\,, \qquad \kappa(\tilde{w}(x)) = \frac{\partial W^*(\tilde{N},\widetilde{M}(x))}{\partial M}\,. \tag{3.36}$$

Next, the inclusion (2.16) is expressed now as follows

$$(\widetilde{\Lambda}_{\tilde{w}(x)})^*(\tilde{N},\widetilde{M}(x)) \in \partial_1 H(\tilde{u}(x),\tilde{w}(x),\tilde{N},\widetilde{M}(x)), \qquad \tilde{N} = q\,. \tag{3.37}$$

Now we have

$$\partial_1 H(\tilde{u}(x),\tilde{w}(x),\tilde{N},\widetilde{M}(x)) = -[DF(\tilde{u},\tilde{w})](x) = \begin{cases} p(x), & x \in (0,\ell), \\ q, & x = \ell, \\ -m, & x = \ell. \end{cases} \tag{3.38}$$

Finally, (3.32), (3.37) and (3.38) yield the equilibrium equations and the static boundary conditions

$$\begin{aligned} &\tilde{N}_{,x} = 0, \qquad \widetilde{M}_{,xx} - (\tilde{N}\nabla\tilde{w}(x))_{,x} - p = 0, \quad \text{in } (0,\ell), \\ &\tilde{N}(\ell) = q, \qquad \widetilde{M}(0) = 0, \qquad \widetilde{M}(\ell) = -m. \end{aligned}$$

4. Problem $(\mathcal{P}_2)$ and duality

If one of the ends of the beam is fixed, the dual formulation for $q < 0$ significantly complicates. Unfortunately, the anomalous dual principle (2.5) does not apply and actually no general variational framework is available. It will be shown, however, that the dual principle in the case of the boundary conditions (1.10) and $q < 0$ assumes the form of sup inf or inf sup principles, depending upon the choice of Lagrangians.

4.1. Firstly, we will formulate the dual problem as an sup inf principle. For $(u, w) \in \mathcal{V}_2$ we define the operator Λ

$$\begin{aligned} &\Lambda(u, w) = (\Lambda_1 u, \Lambda_2 w, \Lambda_3 w) = (\nabla u, \nabla w, \nabla^2 w), \\ &\Lambda: \mathcal{V}_2 \to Y = Y^* = L^2(0, \ell) \times L^2(0, l) \times L^2(0, \ell). \end{aligned} \tag{4.1}$$

We observe that now Λ is a linear operator. The functional J of the total potential energy (1.13) may be written as follows

$$J(u, w) = W_1(\Lambda_1 u, \Lambda_2 w) + W_2(\Lambda_3 w) + F(u, w), \qquad (u, w) \in \mathcal{V}_2, \tag{4.2}$$

where

$$W_1(a, b) = \int_0^\ell \frac{A}{2}\left(a + \frac{1}{2}b^2\right)^2 dx, \qquad (a, b) \in \mathcal{H}, \tag{4.3}$$

$$W_2(t) = \int_0^\ell \frac{B}{2} t^2 \, dx, \qquad t \in L^2(0, \ell), \tag{4.4}$$

provided that the stored energy function is given by (1.14).

The perturbed functional Φ is now assumed in the following form

$$\Phi(u, w, p_1, p_2, p_3) = W_1(\Lambda_1 u + p_1, \Lambda_2 w + p_2) + W_2(\Lambda_3 w + p_3) + F(u, w), \tag{4.5}$$
$$(u, w) \in \mathcal{V}_2, \qquad (p_1, p_2, p_3) \in Y.$$

According to the "standard" theory of duality [3] we determine the first Lagrangian

$$-L_1(u, w, N, Q, M) = \sup\left\{ \int_0^\ell (N p_1 + Q p_2 + M p_3)\, dx - \Phi(u, w, p_1, p_2, p_3) \;\middle|\; (p_1, p_2, p_3) \in Y \right\}. \tag{4.6}$$

After some calculations one obtains

$$
\begin{aligned}
&(4.7)\qquad L_1(u,w,N,Q,M)\\
&=\begin{cases}
F(u,w)+\displaystyle\int_0^\ell \left[Nu_{,x}+Qw_{,x}+Mw_{,xx}-\left(\frac{1}{2B}M^2+\frac{1}{2A}N^2+\frac{1}{2N}Q^2\right)\right]dx,\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N>0;\\
F(u,w)+\displaystyle\int_0^\ell \left(Mw_{,xx}-\frac{1}{2B}M^2\right)dx, \quad \text{if } N=0 \text{ and } Q=0;\\
-\infty, \quad \text{if } N<0 \text{ or } N=0 \text{ and } Q\neq 0.
\end{cases}
\end{aligned}
$$

Moreover

$$
(4.8)\qquad J(u,w)=\sup\left\{L_1(u,w,N,Q,M)\mid (N,Q,M)\in Y^*\right\}.
$$

From (4.7) and (4.8) we conclude that the Lagrangian L_1 is appropriate for $N\geq 0$, i.e. for $q\geq 0$. Then the dual problem $(\mathcal{P}_2^*)$ is to find

$$
\left|\ \sup_{(N,Q,M)\in Y^*}\ \inf_{(u,w)\in\mathcal{V}_2} L(u,w,N,Q,M).\right.
$$

In the sequel we will focus on the case $q<0$.

Let us introduce the following Lagrangian

$$
\begin{aligned}
&(4.9)\qquad -L_2(u,w,N,Q,M)\\
&\quad=\inf_{p_2\in L^2(0,\ell)}\ \sup_{(p_1,p_3)\in\mathcal{H}}\left\{\int_0^\ell (Np_1+Qp_2+Mp_3)dx-\Phi(u,w,p_1,p_2,p_3)\right\}.
\end{aligned}
$$

where Φ is still given by (4.5). Hence

$$
\begin{aligned}
-L_2(u,w,N,Q,M)&=-F(u,w)+W_2^*(M)-\int_0^\ell (Nu_{,x}+Qw_{,x})\,dx\\
&+\inf_{p_2\in L^2(0,\ell)}\ \sup_{p_1\in L^2(0,\ell)}\left[\int_0^\ell\left(Np_1-\frac{A}{2}p_1^2\right)dx+\int_0^\ell\left(Qp_2-\frac{1}{2}Np_2^2\right)dx\right],
\end{aligned}
$$

and finally

(4.10) $L_2(u, w, N, Q, M)$

$$= \begin{cases} F(u,w) + \int_0^\ell \left[N u_{,x} + Q w_{,x} + M w_{,xx} - \left(\frac{1}{2B} M^2 + \frac{1}{2A} N^2 + \frac{1}{2N} Q^2 \right) \right] dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N < 0; \\ F(u,w) + \int_0^\ell \left(M w_{,xx} - \frac{1}{2B} M^2 \right) dx, \quad \text{if } N = 0 \text{ and } Q = 0; \\ \infty, \quad \text{if } N > 0 \text{ or } N = 0 \text{ and } Q \neq 0. \end{cases}$$

Here

$$W_2^*(M) = \sup \left\{ \int_0^\ell \left(Mt - \frac{B}{2} t^2 \right) dx \mid t \in L^2(0, \ell) \right\} = \int_0^\ell \frac{1}{2B} M^2(x)\, dx\,.$$

Let

$$G_2(N, Q, M) := \sup_{(u,w) \in \mathcal{V}_2} L(u, w, N, Q, M). \tag{4.11}$$

Taking account of (4.10) we obtain

(4.12) $G_2(N, Q, M)$

$$= \begin{cases} -\int_0^\ell \left(\frac{1}{2B} M^2 + \frac{1}{2A} N^2 + \frac{1}{2N} Q^2 \right) dx, & \text{if } N < 0 \text{ and } N_{,x} = 0, \\ & N(\ell) = q, \quad M(\ell) = -m, \\ & M_{,xx} - Q_{,x} - p = 0; \\ -\int_0^\ell \frac{1}{2B} M^2 dx, & \text{if } N = 0, \quad Q = 0 \text{ and} \\ & M_{,xx} - p = 0, \quad M(\ell) = -m; \\ \infty, & \text{otherwise.} \end{cases}$$

The equilibrium equation $M_{,xx} - Q_{,x} - p = 0$ is formally equivalent to

$$Q(x) = \nabla M(x) - \nabla M(0) + Q(0) - R(x), \tag{4.13}$$

where R has been defined in the previous section.

It will now be shown that the dual problem

$$(\mathcal{P}_2^{\oplus}) \quad \Big| \quad \sup_{M \in \mathcal{M}} \inf_{(Q,N) \in \mathcal{H}} G_2(N, Q, M)$$

is well defined provided that

$$\frac{q}{B} + \frac{2}{\ell^2} > 0, \tag{4.14}$$

where

$$\mathcal{M} = \left\{ M \mid M_{,xx} - \frac{q}{B} M - p = 0, \quad \text{in } (0,\ell); \quad M(\ell) = -m \right\}. \tag{4.15}$$

To corroborate this statement, by taking account of (4.13), we find

$$\begin{aligned} &\inf_{Q \in L^2(0,\ell)} \inf_{N \in L^2(0\ell)} G_2(N,Q,M) \\ &= \inf_{Q(0)\in\mathbb{R}} \left\{ -\int_0^\ell \left[\frac{1}{2B} M^2 + \frac{1}{2A} q^2 + \frac{1}{2q} (M_{,x} - M_{,x}(0) + Q(0) - R)^2 \right] dx \right\} \\ &= -\int_0^\ell \left[\frac{1}{2B} M^2 + \frac{1}{2A} q^2 + \frac{1}{2q} \left(M_{,x} + \frac{M(0)}{\ell} + Z(x) \right)^2 \right] dx, \end{aligned} \tag{4.16}$$

since the infimum over $Q(0) \in \mathbb{R}$ is attained for

$$Q(0) = \frac{m}{l} + \frac{M(0)}{\ell} + M_{,x}(0) + \frac{1}{\ell} \int_0^\ell R(x)\, dx.$$

We recall that $Z(x)$ is given by (3.14).

Let us denote by g the functional on the r.h.s of (4.16), that is

$$g(M) := -\int_0^\ell \left[\frac{1}{2B} M^2 + \frac{1}{2A} q^2 + \frac{1}{2q} \left(M_{,x} + \frac{M(0)}{\ell} + Z(x) \right)^2 \right] dx\,. \tag{4.17}$$

By $\overline{M}$ we denote the bending moments field rendering the functional g stationary; hence

$$\begin{aligned} &\overline{M}_{,xx} - \frac{q}{B} \overline{M} - p = 0, \qquad \text{in } (0,\ell), \\ &\overline{M}_{,x}(0) + \overline{M}(0) + \int_0^\ell R(x)\, dx + m = 0, \qquad \overline{M}(\ell) = -m. \end{aligned} \tag{4.18}$$

For $M \in H^1(0,\ell)$, $M(\ell) = -m$, after lengthy calculation one obtains

$$\begin{aligned} g(M) - g(\overline{M}) = -\int_0^\ell \left[\frac{1}{2B} (M - \overline{M})^2 + \frac{1}{2q} (M_{,x} - \overline{M}_{,x})^2 \right] dx \\ + \frac{1}{2q\ell} (M(0) - \overline{M}(0))^2. \end{aligned} \tag{4.19}$$

The last step consists in proving that for bending moments satisfying

$$M_{,xx} - \frac{q}{B}M - p = 0, \qquad \text{in } (0,\ell); \qquad M(\ell) = -m, \tag{4.20}$$

one has

$$g(M) \leq g(\overline{M}). \tag{4.21}$$

Towards this end the Poincaré's inequality in the following form is employed

$$\int_0^\ell v^2(x)\,dx \leq \frac{\ell^2}{2}\int_0^\ell v_{,x}^2(x)\,dx + \frac{1}{\ell}\left[\int_0^\ell v(x)\,dx\right]^2. \tag{4.22}$$

It holds for any $v \in H^1(0,\ell)$, thus particularly for v such that $v(\ell) = 0$. By applying the Poincaré's inequality to $v(x) = \nabla M(x) - \nabla\overline{M}(x)$, $M(\ell) = -m$, $\overline{M}(\ell) = -m$ or $v(\ell) = M(\ell) - \overline{M}(\ell) = 0$, we readily obtain

$$\int_0^\ell (M_{,x} - \overline{M}_{,x})^2\,dx \leq \frac{\ell^2}{2}\int_0^\ell \frac{q^2}{B}(M - \overline{M})^2\,dx + \frac{1}{\ell}[M(0) - \overline{M}(0)]^2,$$

because $M_{,xx} - \overline{M}_{,xx} = \dfrac{q}{B}(M - \overline{M})$.

Thus we finally have

$$g(M) - g(\overline{M}) \leq -\int_0^\ell \left[\frac{1}{2B}(M - \overline{M})^2 + \frac{\ell^2 q}{4B^2}(M - \overline{M})^2\right] dx \leq 0,$$

provided that

$$0 > q \geq -\frac{2B}{\ell^2}. \tag{4.23}$$

REMARK 2. Comparing (4.23) with (3.22) we conclude that the range of compressive forces q is smaller for the beam fixed at $x = 0$. This surprising result follows directly from the comparison of the constant ℓ^2/π^2 in the Friedrichs' inequality (3.20) with the constant $\ell^2/2$ in the Poincaré's inequality (4.22). The former is better, hence the result. If, instead of $\ell^2/2$, a better constant for the Poincaré's inequality is found, then automatically the range of admissible compressive forces q for the beam fixed at $x = 0$ would be enlarged.

4.2. In the present subsection the dual problem will be formulated in the form of inf sup principle.

The perturbed functional is now given by

$$\Phi(u,w,p_1,p_2,p_3) = \int_0^\ell \left\{ \frac{A}{2}\left[u_{,x} + p_1 + \frac{1}{2}w_{,x}(w_{,x}+p_3)\right]^2 + \frac{B}{2}(w_{,xx}+p_2)^2 \right\} dx + F(u,w), \tag{4.24}$$

where $(u,w) \in \mathcal{V}_2$ and $(p_1,p_2,p_3) \in Y = [L^2(0,\ell)]^3$. One sees that

$$\Phi(u,w,0,0,0) = J(u,w), \qquad (u,w) \in \mathcal{V}_2 .$$

The Lagrangian is defined by

$$-L_3(u,w,N,M,Q) = \inf_{p_3 \in L^2(0,\ell)} \sup_{(p_1,p_2)\in\mathcal{H}} \left\{ \int_0^\ell (Np_1 + Mp_2 + Qp_3)\,dx - \int_0^\ell \left[\frac{A}{2}\left(u_{,x} + p_1 + \frac{1}{2}w_{,x}(w_{,x}+p_3)\right)^2 + \frac{B}{2}(w_{,xx}+p_2)^2 \right] dx - F(u,w) \right\}.$$

Hence

$$-L_3(u,w,N,M,Q) = \begin{cases} F(u,w) + \displaystyle\int_0^\ell \left[Nu_{,x} + Qw_{,x} + Mw_{,xx} - \left(\frac{1}{2B}M^2 + \frac{1}{2A}N^2\right)\right] dx, \\ \qquad\qquad \text{if } Q = \frac{1}{2}Nw_{,x}, \\ \infty, \qquad \text{if } Q \neq \frac{1}{2}Nw_{,x}. \end{cases} \tag{4.25}$$

We observe that by introducing the multiplier λ, the Lagrangian L_3 may be written in the form

$$L_3(u,w,N,M,Q) = \sup_{\lambda\in\mathbb{R}} L(u,w,N,M,Q,\lambda), \tag{4.26}$$

where

$$L(u,w,N,M,Q,\lambda) = F(u,w) + \int_0^\ell \left[Nu_{,x} + Qw_{,x} + Mw_{,xx} - \left(\frac{1}{2B}M^2 + \frac{1}{2A}N^2\right) + \lambda\left(Q - \frac{1}{2}Nw_{,x}\right)\right] dx . \tag{4.27}$$

It can be shown that

$$J(u,w) = \sup_{(N,M)\in\mathcal{H}} \inf_{Q\in L^2(0,\ell)} L_3(u,w,N,M,Q). \tag{4.28}$$

The second dual principle is formulated by using the functional

$$G_3(N,M,Q) = \inf_{(u,w)\in\mathcal{V}_2} L_3(u,w,N,M,Q) \tag{4.29}$$

$$= \begin{cases} G(M,Q), & \text{if } N = q,\ Q(0) = 0,\ \int_0^\ell Q\,dx = 0, \\ \infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell \frac{Q}{N}dx \neq 0, \\ -\infty, & \text{if } N \neq q,\ Q(0) = 0,\ \int_0^\ell \frac{Q}{N}dx = 0, \end{cases}$$

where

$$G(M,Q) = \int_0^\ell \left(\frac{2}{q}Q^2 + \frac{2}{q}MQ_{,x} + \frac{2}{q}RQ - \frac{1}{2B}M^2\right)dx + \frac{2}{q}mQ(\ell) - \frac{\ell}{2A}q^2. \tag{4.30}$$

The second dual problem, denoted by $(\mathcal{P}_2^{\odot})$, is to find

$$(\mathcal{P}_2^{\odot}) \quad \Big| \quad \inf_{Q\in H^1(0,\ell)} \sup_{(N,M)\in\mathcal{H}} G_3(N,M,Q).$$

Under an additional condition (see (4.40)), the last problem is well defined. To corroborate this statement we successively find

$$\sup_{N\in L^2(0,\ell)} G_3(N,M,Q) = \begin{cases} G(M,Q), & \text{if } Q(0) = 0,\ \int_0^\ell Q\,dx = 0, \\ \infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell \frac{Q}{N}dx \neq 0, \end{cases}$$

and

$$\sup_{M\in L^2(0,\ell)} \sup_{N\in L^2(0,\ell)} G_3(N,M,Q) = \begin{cases} f(Q), & \text{if } Q(0) = 0,\ \int_0^\ell Q\,dx = 0, \\ \infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell \frac{Q}{N}dx \neq 0, \end{cases}$$

where

$$f(Q) = \frac{2}{q}\int_0^{\ell}\left(Q^2 + RQ + \frac{B}{q}Q_{,x}^2\right)dx + \frac{2}{q}mQ(\ell) - \frac{\ell}{2A}q^2. \tag{4.31}$$

Further, we have

$$\inf_{Q\in H^1(0,\ell)}\sup_{(N,M)\in\mathcal{H}} G_3(N,M,Q)\inf\{f(Q) \mid Q\in\mathcal{W}\}, \tag{4.32}$$

where

$$\mathcal{W} = \left\{Q \mid Q(0) = 0, \int_0^{\ell} Q\,dx = 0\right\}. \tag{4.33}$$

For $Q\in\mathcal{W}$ and $\overline{Q}\in\mathcal{W}$ one has

$$f(Q) - f(\overline{Q}) = \frac{2}{q}\int_0^{\ell}\left[(Q-\overline{Q})^2 + \frac{B}{q}(Q_{,x}-\overline{Q}_{,x})^2\right]dx \tag{4.34}$$

$$+\frac{4}{q}\int_0^{\ell}(Q-\overline{Q})\left(\overline{Q} - \frac{B}{q}\overline{Q}_{,xx} + \frac{R}{2}\right)dx + \frac{2}{q}\left[Q(\ell)-\overline{Q}(\ell)\right]\left[m + \frac{2B}{q}\overline{Q}_{,x}(\ell)\right].$$

Let $\tilde{Q}\in\mathcal{W}$ be an element satisfying

$$\tilde{Q} - \frac{B}{q}\tilde{Q}_{,xx} + \frac{R}{2} = C, \quad \text{in } (0,\ell); \qquad \frac{2B}{q}\tilde{Q}_{,x}(\ell) + m = 0. \tag{4.35}$$

The constant C conforms with the fact that the second order Eq. $(4.35)_1$ has to satisfy three imposed conditions.

Then (4.34) reduces to

$$f(Q) - f(\tilde{Q}) = \frac{2}{q}\int_0^{\ell}\left[(Q-\tilde{Q})^2 + \frac{B}{q}(Q_{,x}-\tilde{Q}_{,x})^2\right]dx, \tag{4.36}$$

for any $Q\in\mathcal{W}$. Hence, if

$$\tilde{\lambda}_1^2 = \inf_{v\in V_0}\frac{\displaystyle\int_0^{\ell} v_{,x}^2\,dx}{\displaystyle\int_0^{\ell} v(x)\,dx}, \tag{4.37}$$

where

$$\mathcal{V}_0 = \left\{ v \in H^1(0,\ell) \mid v(0) = 0, \quad \int_0^\ell v(x)\,dx = 0 \right\}, \tag{4.38}$$

then

$$\forall\, Q \in \mathcal{V}_0, \qquad f(Q) - f(\tilde{Q}) \geq 0 \tag{4.39}$$

provided that

$$q < 0, \qquad \frac{q}{B} + \tilde{\lambda}_1^2 \geq 0. \tag{4.40}$$

REMARK 3. For $v \in \mathcal{V}_0$ the Poincaré's inequality (4.22) reduces to

$$\|v\|^2_{L^2(0,l)} \leq \frac{\ell}{2} \|\nabla v\|^2_{L^2(0,l)}.$$

In this case we have $\dfrac{1}{\tilde{\lambda}_1^2} \leq \dfrac{\ell^2}{2}$ and consequently

$$0 > q \geq -\frac{2B}{q} \geq -\tilde{\lambda}_1^2 B. \tag{4.41}$$

5. Concluding remarks

This paper was conceived as a first step towards formulation of dual principles for geometrically nonlinear structures, and not covered by the "standard" theory of duality presented in [3]. The results obtained for a model of nonlinear elastic beam exhibit the role of boundary conditions in the choice of Lagrangians from which dual principles are derived.

Some of the results proposed by Auchmuty in [6] have been generalized so as to include nonlinear operators Λ. However, the available theoretical results concerning anomalous dual variational principles are insufficient, even for the compressed beam fixed at one end. Therefore new developments in the spirit of Auchmuty's paper [6], our Sec. 2 and the dual principles $(\mathcal{P}_2^{\oplus})$ and $(\mathcal{P}_2^{\odot})$ are necessary in order to find, for instance, dual principles for compressed von Kármán plates.

Much remains to be done in the search for dual principles in nonlinear solid and structural mechanics. Most promissing seems to be an approach in which properly set up Lagrangians are assumed as a starting point.

Acknowledgement

The paper was partially supported by the State Committee for Scientific Research through the grant No 3 P404 013 06.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received October 28, 1994.

Arch. Mech., 47, 4, pp. 725–743, Warszawa 1995

Derivation of the variational inequalities of the incremental frictional elastic contact problems

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THE PAPER presents the local formulation and variational inequalities for the incremental Coulomb friction contact problem of two linear elastic bodies, assuming small deformations and allowing the slips to be large and contact region to change. The incremental description of the problem is based on the author's proposition of applying the non-incremental form of the elasticity equations and incremental contact mechanics equations. The general Hu - Washizu as well as Hellinger - Reissner, potential energy, and complementary energy functionals are employed to form the proper variational inequalities. Several representations known in the literature are contained as special cases.

1. Introduction

THE VARIATIONAL PRINCIPLES are the basis of many solution methods of continuum mechanics. The variational formulations of the contact problems have been of special interest for the last two decades. There are two main reasons of such an interest. Firstly, contact aspects have been very often included in the solutions of the real technological problems as they may play a considerable role. Secondly, the complexity of the contact problems caused mainly by inequality form of the contact mechanics formulation have attracted the attention of various researchers.

The variational principles of contact problems of linear elasticity correspond with either non-incremental [1–7] or incremental description [8–10]. Most of those papers present classical formulations, nowadays. The works [10, 11] of the author contribute to the incremental variational formulation of the contact problems, as they present the new proposition of applying the non-incremental linear elasticity equations and the incremental contact mechanics equations. Such an approach could be called semi-incremental. This formulation of the problem, which is also applied in this paper, is different from the predecessors' [8, 9] fully incremental approach with all the equations in the incremental form.

The main objective of the paper is formulation of the original variational inequality principles based on Hu - Washizu and Hellinger - Reissner functionals for the incremental frictional two-body contact problems of linear elasticity (the non-incremental problems can be obtained as simplified cases). Also, the principles based on the author's semi-incremental approach will be derived by means of the potential energy and complementary energy functionals. They will appear to correspond to the KALKER'S formulation [5, 6] based on the non-incremental description of the problem. Furthermore, some simpler predecessors' representa-

tions of the non-incremental frictional single body [3], and frictionless two-body contact problems [12] will be mentioned as special cases.

2. Local formulation of the contact problems of linear elasticity

2.1. Incremental description of the contact problems

Let us consider the incremental problem of two elastic bodies in contact. We have to determine the equilibrium path of the bodies as well as the corresponding stress and strain fields in the succeeding moments of time. The time instants increase with respect to the initial instant by the quantities Δt, $2\Delta t$, $3\Delta t$, ..., where Δt is the time increment. The solution is known in the interval from the time instant t_o to time t'. We look for the solution in the time moment $t = t' + \Delta t$. The load is increased at each of $k = 1, 2, \ldots, K$ moments of time by some value from zero at the initial instant up to the full load at the final moment of time. Hence the approach to the problem can be called incremental. It allows to find the equilibrium states of the bodies in a finite number of steps K. It should be stressed that the above assumption applied to the author's semi-incremental approach does not differ from the standard assumptions used for the fully incremental approach. Note that we assume a kineto-static problem. Therefore, the time stands only for a parameter (not for a variable) and thus it can be excluded from the problem equations.

The motion of each body is described in the inertial reference system with curvilinear coordinates ξ^i $(i = 1, 2, 3)$. It is assumed to be caused by small deformations in the reference system. The more general case of unlimited number of the bodies in the arbitrary motion and described in separate reference systems is presented in [10]. We apply three configurations of the bodies: initial, current and the sought ones which refer to times t_o, t' and $t' + \Delta t$, respectively. The latter is also the next current configuration. We apply material description of motion with the fixed reference configuration coinciding with the initial one for strains, stresses, displacements and gap (Fig. 1). The changing reference configuration

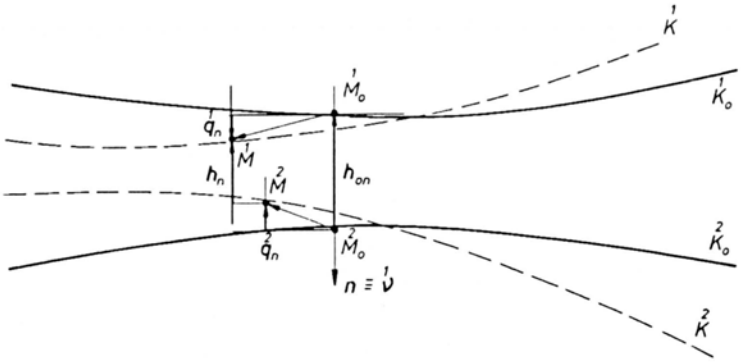

FIG. 1. Gap definition and its reference configuration.

coinciding with the current one is used for slip increments (Fig. 2). This makes the author's description distinct from the predecessors' ones.

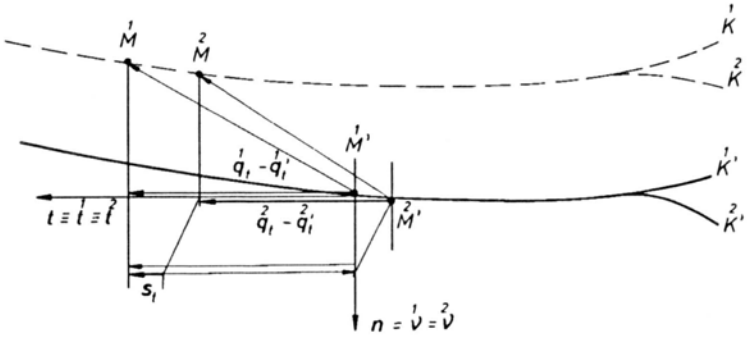

FIG. 2. Slip increment definition and its reference configuration.

2.2. Basic relations of the contact mechanics

Each of two elastic bodies a ($a = 1, 2$) is characterized by its elastic constant tensor $\overset{a}{D}{}^{ijkl}$. They are loaded by body forces $\overset{a}{f}{}^i$ in their volumes $\overset{a}{V}_o$ and by surface tractions $\overset{a}{p}{}^i$ on the parts $\overset{a}{P}_o$ of their surfaces in such a manner that a stable kineto-static contact equilibrium is possible. On the parts $\overset{a}{Q}_o$ of the body surfaces given displacements are assumed. The potential contact areas $\overset{a}{K}_o$ complete the total body surfaces. Note that for the problem under consideration, the common contact area is defined as $\overset{a}{K}_o = \overset{1}{K}_o \approx \overset{2}{K}_o$. The contact area K_o is updated in each incremental step and denoted K'. K is the sought contact area corresponding to the deformed state within the incremental step. The physical properties of the contact surface are characterized with the common Coulomb friction coefficient μ.

Deformation within the contact area is characterized by two quantities. The first one is the scalar h_n representing the gap between the body surfaces measured in the normal direction of $\mathbf{n}$ (Fig. 1) and defined as follows:

$$h_n = \overset{1}{q}_n - \overset{2}{q}_n + h_{o_n}, \tag{2.1}$$

where the $\overset{a}{q}_n$ are normal components of the displacement vectors of the bodies within the contact area, and h_{o_n} stands for the initial gap. Vector $\mathbf{n}$ is assumed to coincide with the exterior unit normal of the first body $\mathbf{n} = \overset{1}{\boldsymbol{\nu}} \approx -\overset{2}{\boldsymbol{\nu}}$. Note that the reference configuration for the gap is the initial configuration K_o of the contact surface.

The second quantity describing deformation within the body surfaces is the slip increment vector, components s_t of which are measured in any two orthogonal

tangent directions defining the common unit tangent vector **t** (Fig. 2). The slip increments equal

$$s_t = \overset{1}{q}_t - \overset{1}{q}'_t - \overset{2}{q}_t + \overset{2}{q}'_t = \Delta\overset{1}{q}_t - \Delta\overset{2}{q}_t , \tag{2.2}$$

where $\overset{a}{q}_t$, $\overset{a}{q}'_t$ and $\Delta\overset{a}{q}_t$ are tangent components of the sought displacements, of the given displacements from the previous incremental step and of the displacement increments. Note that reference configuration for slip increments is the current configuration K' of the contact surface.

The normal stress state within the contact area is defined by the normal component of the unknown surface reaction vectors $\overset{1}{r}_n$, $\overset{2}{r}_n$ (treated formally as scalar quantities as $n = 1$)

$$\overset{1}{r}_n = \overset{1}{\sigma}{}^{ij}\overset{1}{\nu}_j n_i , \qquad \overset{2}{r}_n = \overset{2}{\sigma}{}^{ij}\overset{2}{\nu}_j n_i , \tag{2.3}$$

where $\overset{a}{\sigma}{}^{ij}$ are stress tensors of each body and introduction of $\mathbf{n} \equiv \overset{1}{\mathbf{n}} \approx \overset{2}{\mathbf{n}}$ allows the third Newton's law to be fulfilled, $\overset{1}{r}_n = -\overset{2}{r}_n$.

The tangential stress state within the contact surface is defined by the components of the unknown tangent surface reaction vectors $\overset{1}{r}_t$, $\overset{2}{r}_t$ $(t = 1, 2)$

$$\overset{1}{\sigma}{}^{ij}\overset{1}{\nu}_j t_{is}\delta_{st} = \overset{1}{r}_t , \qquad \overset{2}{\sigma}{}^{ij}\overset{2}{\nu}_j t_{is}\delta_{st} = \overset{2}{r}_t , \tag{2.4}$$

where introduction of $\mathbf{t} \equiv \overset{1}{\mathbf{t}} \equiv \overset{2}{\mathbf{t}}$ allows the third Newton's law to be written as $\overset{1}{r}_t = -\overset{2}{r}_t$, and the term t_{is} is the pseudo-tensor defining two tangent directions $(s = 1, 2)$ in the global system of coordinates $(i = 1, 2, 3)$.

Two states of deformation and the corresponding stress in normal direction are possible. Either the gap between the bodies in contact exists (with normal stress vanishing) or not (the normal stress appears). Thus we have to write

$$h_n \leq 0, \qquad \overset{1}{r}_n \leq 0, \qquad h_n\overset{1}{r}_n = 0. \tag{2.5}$$

The above relations represent the kinematic, static and orthogonality normal contact conditions, respectively.

Three more conditions describing the frictional tangential slip have to be introduced in order to complete the contact problem formulation. As two states of tangential deformation and stress exist, we propose to construct the proper conditions in the way analogous to (2.5). As the slip either exists (with the tangential traction bound reached) or not (tangential traction is less than its bound), the kinematic, static and orthogonality conditions can be written in the form

$$|s_t| \geq 0, \qquad |\overset{1}{r}_t| + \mu p_n \leq 0, \qquad |s_t|\left(|\overset{1}{r}_t| + \mu p_n\right) = 0, \tag{2.6}$$

where the tangential traction bound μp_n is defined by the normal traction $p_n \leq 0$

and friction coefficient μ. Note that the first condition is always fulfilled. The tangential conditions are completed with the surface constitutive relation describing the Coulomb friction

$$\overset{1}{r}_t = \mu p_n s_t/|s_t|. \tag{2.7}$$

The form of the above relation assures that the directions of the stress and slip increment vectors are the same and their senses are opposite. Note that p_n in (2.6) and (2.7) is formally treated as given in advance. It may however be updated from increment to increment or iteratively within each increment in the case when $p_n \equiv r_n$. This question will be elucidated in Sec. 4.

2.3. Local equations of the semi-incremental contact problem

The semi-incremental contact problem of linear elasticity defined in the previous sections has to satisfy the elasticity equations consisting of the equations of equilibrium, equations of symmetry of the stress tensor (here assumed to be fulfilled in advance), strain-displacement relations and constitutive equations

$$\overset{a}{\sigma}{}^{ij}{}_{,j} + \overset{a}{f}{}^{i} = 0, \quad \overset{a}{\varepsilon}_{ij} = \overset{a}{q}_{(i,j)} = 1/2\left(\overset{a}{q}_{i,j} + \overset{a}{q}_{j,i}\right), \quad \overset{a}{\sigma}{}^{i,j} = \overset{a}{D}{}^{ijkl}\overset{a}{\varepsilon}_{kl}, \quad \xi \in \overset{a}{V}_o \tag{2.8}$$

in volumes $\overset{a}{V}_o$ of each body a ($a = 1, 2$), where $\overset{a}{\varepsilon}_{ij}$ are the strain tensors of the bodies.

This set is completed with the boundary conditions, either for stresses or displacements on parts $\overset{a}{P}_o$ and $\overset{a}{Q}_o$ of the body surfaces

$$\begin{aligned} &\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j = \overset{a}{p}{}^{i}, \qquad \xi \in \overset{a}{P}_o, \\ &\overset{a}{q}{}^{i} = \overset{a}{d}{}^{i}, \qquad \overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j = \overset{a}{r}{}^{i}, \qquad \xi \in \overset{a}{Q}_o, \end{aligned} \tag{2.9}$$

where the last relation defines the reaction stress vector.

Since the contact problem is considered, one has to add the contact mechanics equations on the common potential contact area K

$$\begin{aligned} &h_n = q_n + h_{o_n}, \qquad \sigma^{ij}\nu_j n_i = r_n, \qquad h_n \leq 0, \qquad r_n \leq 0, \\ &h_n r_n = 0, \qquad s_t = \Delta q_t, \qquad \sigma_{ij}\nu_j t_{is}\delta_{st} = r_t, \qquad |s_t| \geq 0, \\ &|r_t| + \mu p_n \leq 0, \quad |s_t|(|r_t| + \mu p_n) = 0, \quad r_t = \mu p_n s_t/|s_t|, \quad \xi \in K, \end{aligned} \tag{2.10}$$

where the simplified notation: $r_n = \overset{1}{r}_n = -\overset{2}{r}_n$, $r_t = \overset{1}{r}_t = -\overset{2}{r}_t$, $q_n = \overset{1}{q}_n - \overset{2}{q}_n$, $\Delta q_t = \Delta\overset{1}{q}_t - \Delta\overset{2}{q}_t$, $h_n = \overset{1}{h}_n - \overset{2}{h}_n$, $s_t = \overset{1}{s}_t - \overset{2}{s}_t$ is applied. The above equations are in fact the special kind of the boundary conditions. They contain all basic relations of semi-incremental contact mechanics introduced in the previous section.

The above set of Eqs. (2.8)–(2.10) plays a significant role in derivation of the variational inequalities in the next section, as it describes the solution quantities of the problem.

3. Variational inequalities of the semi-incremental contact problem

Let us consider variational inequality formulations of the kineto-static contact problem of two elastic bodies. The formulations under consideration correspond with the author's semi-incremental approach from Sec. 2. Four types of the most frequently used variational functionals are of our interest. The Hu-Washizu and Hellinger-Reissner functionals are employed in order to form the most general variational inequalities. The inequality principles based on the potential energy and complementary energy functionals are introduced as they can be used for FE formulations.

3.1. The inequality based on the Hu–Washizu functional

Let us derive the variational inequality based on the Hu-Washizu functional for the incremental frictional two-body contact problem of linear elasticity, which corresponds to the local formulation presented in the previous chapter. For this purpose we consider the set of arbitrary field quantities $\overset{*}{\overset{a}{q}}_i, \overset{*}{\overset{a}{\varepsilon}}_{ij}, \overset{*}{\overset{a}{\sigma}}^{ij}, \overset{*}{\overset{a}{r}}^i$ $(a = 1,2)$ and $\overset{*}{h}_i, \overset{*}{s}_i$ which need neither be kinematically nor statically admissible. The corresponding solution quantities $\overset{a}{q}_i, \overset{a}{\varepsilon}_{ij}, \overset{a}{\sigma}^{ij}, \overset{a}{r}^i$ $(a = 1,2)$ and h_i, s_i obeying relations (2.8)–(2.10) belong to the set $\mathcal{K}$ of kinematically admissible quantities $\overset{a}{\overline{q}}_i, \overset{a}{\overline{\varepsilon}}_{ij}$ $(a = 1,2)$ and $\overline{h}_i, \overline{s}_i$ and the set $\mathcal{S}$ of statically admissible quantities $\overset{a}{\overline{\sigma}}^{ij}, \overset{a}{\overline{r}}^i$ $(a = 1,2)$. These sets are defined as follows:

$$
\begin{aligned}
\mathcal{K} = \Big\{ & \overset{1}{\overline{q}}_i, \overset{2}{\overline{q}}_i, \overset{1}{\overline{\varepsilon}}_{ij}, \overset{2}{\overline{\varepsilon}}_{ij}, \overline{h}_i, \overline{s}_i \mid \overset{a}{\overline{\varepsilon}}_{ij} = \overset{a}{\overline{q}}_{(i,j)} \text{ in } \overset{a}{V}_o; \qquad \overset{a}{q}_i = \overset{a}{d}_i \text{ on } \overset{a}{Q}_o; \\
& \overset{1}{\overline{q}}_n + \overset{1}{h}_{o_n} - \overset{2}{\overline{q}}_n - \overset{2}{h}_{o_n} = \overline{h}_n, \qquad \overline{h}_n \leq 0 \text{ on } K; \\
& \Delta\overset{1}{\overline{q}}_t - \Delta\overset{2}{\overline{q}}_t = \overline{s}_t, \qquad |\overline{s}_t| \geq 0 \text{ on } K \Big\}, \qquad (3.1) \\
\mathcal{S} = \Big\{ & \overset{1}{\overline{\sigma}}^{ij}, \overset{2}{\overline{\sigma}}^{ij}, \overset{1}{\overline{r}}^i, \overset{2}{\overline{r}}^i \mid \overset{a}{\overline{\sigma}}^{ij} + \overset{a}{f}^i = 0 \text{ in } \overset{a}{V}; \qquad \overset{a}{\overline{\sigma}}^{ij}\overset{a}{\nu}_j = \overset{a}{\overline{p}}^i \text{ on } \overset{a}{P}_o; \\
& \overset{a}{\overline{\sigma}}^{ij}\overset{a}{\nu}_j = \overset{a}{\overline{r}}^i \text{ on } \overset{a}{Q}_o; \qquad \overline{\sigma}^{ij}\nu_j n_i = \overline{r}_n, \qquad \overline{r}_n \leq 0 \text{ on } K; \\
& \overline{\sigma}^{ij}\nu_j t_{is}\delta_{st} = \overline{r}_t, \qquad |\overline{r}_t| + \mu p_n \leq 0 \text{ on } K \Big\}.
\end{aligned}
$$

Then the two first fundamental inequalities

$$
\begin{aligned}
& r_n\left(\overset{*}{h}_n - h_n\right) = r_n\overset{*}{h}_n - r_n h_n = r_n\overset{*}{h}_n \lesssim 0, \\
& r_t\left(\overset{*}{s}_t - s_t\right) - \mu p_n\left(|\overset{*}{s}_t| - |s_t|\right) = r_t\overset{*}{s}_t - \mu p_n|\overset{*}{s}_t| + |r_t||s_t| + \mu p_n|s_t| \qquad (3.2) \\
& \qquad = r_t\overset{*}{s}_t - \mu p_n|\overset{*}{s}_t| + (|r_t| + \mu p_n)|s_t| = r_t\overset{*}{s}_t - \mu p_n|\overset{*}{s}_t| \lesssim 0
\end{aligned}
$$

hold as the following relations from (2.10) are satisfied: $r_n h_n = 0$, $(|r_t| + \mu p_n)|s_t| = 0$, and additionally $-r_t s_t = |r_t||s_t|$ arising from $r_t = \mu p_n s_t/|s_t|$. The above results (3.2) will be utilized later while defining the sign of the variational inequality.

Denoting the individual variations ($a = 1, 2$) by

$$\delta \overset{a}{q}_i = \overset{a}{\overset{*}{q}}_i - \overset{a}{q}_i, \qquad \delta \overset{a}{\varepsilon}_{ij} = \overset{a}{\overset{*}{\varepsilon}}_{ij} - \overset{a}{\varepsilon}_{ij}, \qquad \delta h_i = \overset{*}{h}_i - h_i, \qquad \delta s_i = \overset{*}{s}_i - s_i,$$

$$\delta \overset{a}{\sigma}^{ij} = \overset{a}{\overset{*}{\sigma}}^{ij} - \overset{a}{\sigma}^{ij}, \qquad \delta \overset{a}{r}^i = \overset{a}{\overset{*}{r}}^i - \overset{a}{r}^i,$$

variation of the Hu–Washizu functional for each body can be written in the form

$$\begin{aligned}(3.3)\qquad \delta \overset{a}{J}_{H-W} = {} & \int_{\overset{a}{V}_o} \overset{a}{D}^{klij} \overset{a}{\varepsilon}_{kl} \delta \overset{a}{\varepsilon}_{ij} \, d\overset{a}{V}_o - \int_{\overset{a}{V}_o} \delta \left[\left(\overset{a}{\varepsilon}_{ij} - \overset{a}{q}_{(i,j)} \right) \overset{a}{\sigma}^{ij} \right] d\overset{a}{V}_o \\ & - \int_{\overset{a}{V}_o} \overset{a}{f}^i \delta \overset{a}{q}_i \, d\overset{a}{V}_o - \int_{\overset{a}{P}_o} \overset{a}{p}^i \delta \overset{a}{q}_i \, d\overset{a}{P}_o - \int_{\overset{a}{Q}_o} \delta \left[\left(\overset{a}{q}_i - \overset{a}{d}_i \right) \overset{a}{r}^i \right] d\overset{a}{Q}_o \\ & - \int_{\overset{a}{K}} \delta \left[\left(\overset{a}{q}_n - \overset{a}{h}_n + \overset{a}{h}_{o_n} \right) \overset{a}{r}_n \right] d\overset{a}{K} - \int_{\overset{a}{K}} \overset{a}{d}_{st} \overset{a}{s}_s \delta \overset{a}{s}_t \, d\overset{a}{K} - \int_{\overset{a}{K}} \delta \left[\left(\Delta \overset{a}{q}_t - \overset{a}{s}_t \right) \overset{a}{r}_t \right] d\overset{a}{K},\end{aligned}$$

where the auxiliary surface constitutive tensor is given by $\overset{a}{d}_{st} = \mu \overset{a}{p}_n / |\overset{a}{s}_t|$ with $p_n \leq 0$. The form of the functional (3.3) is proposed to be obtained in the following way. The standard terms of the first two lines representing the volume integrals and surface integrals through the loaded $\overset{a}{P}_o$ and supported $\overset{a}{Q}_o$ parts of the body surface are taken from the well known Hu–Washizu functional for one elastic body. The non-standard terms representing the integrals through the contact surface $\overset{a}{K}$ are constructed with the postulate of formal similarity of two first volume integrals to the integrals over the contact surface $\overset{a}{K}$ (in tangent directions). Thus the last two surface integrals can be obtained by replacement of the volume quantities D^{ijkl}, $q_{(i,j)}$, ε_{ij} and σ^{ij} by the surface ones d_{st}, Δq_t, s_t and r_t, respectively. On the other hand, the first integral through the contact surface (in normal direction) is constructed in the same manner as applied to the integral through the surface $\overset{a}{Q}_o$ and to the second contact surface integral in the tangent directions.

Performing the operations of variation in (3.3) we can write the above functional in the equivalent form

$$(3.4)\qquad \int_{\overset{a}{V}_o} \left(\overset{a}{D}^{klij} \overset{a}{\varepsilon}_{kl} - \overset{a}{\sigma}^{ij} \right) \delta \overset{a}{\varepsilon}_{ij} \, d\overset{a}{V}_o - \int_{\overset{a}{V}_o} \left(\overset{a}{\varepsilon}_{ij} - \overset{a}{q}_{(i,j)} \right) \delta \overset{a}{\sigma}^{ij} \, d\overset{a}{V}_o$$

(3.4) [cont.)

$$+\int_{\overset{a}{V}_o}\overset{a}{\sigma}{}^{ij}\delta\overset{a}{q}_{(i,j)}\,d\overset{a}{V}_o-\int_{\overset{a}{V}_o}\overset{a}{f}{}^{i}\delta\overset{a}{q}_i\,d\overset{a}{V}_o-\int_{\overset{a}{P}_o}\overset{a}{p}{}^{i}\delta\overset{a}{q}_i\,d\overset{a}{P}_o-\int_{\overset{a}{Q}_o}\overset{a}{r}{}^{i}\delta\overset{a}{q}_i\,d\overset{a}{Q}_o$$

$$-\int_{\overset{a}{Q}_o}\left(\overset{a}{q}_i-\overset{a}{d}_i\right)\delta\overset{a}{r}{}^{i}\,d\overset{a}{Q}_o-\int_{\overset{a}{K}}\overset{a}{r}_n\delta\overset{a}{q}_n\,d\overset{a}{K}+\int_{\overset{a}{K}}\overset{a}{r}_n\delta\overset{a}{h}_n\,d\overset{a}{K}-\int_{\overset{a}{K}}\left(\overset{a}{q}_n-\overset{a}{h}_n+\overset{a}{h}_{o_n}\right)\delta\overset{a}{r}_n\,d\overset{a}{K}$$

$$-\int_{\overset{a}{K}}\overset{a}{r}_t\delta\overset{a}{q}_t\,d\overset{a}{K}+\int_{\overset{a}{K}}\left(\overset{a}{r}_t\delta\overset{a}{s}_t-\mu\overset{a}{p}_n\delta|\overset{a}{s}_t|\right)d\overset{a}{K}-\int_{\overset{a}{K}}\left(\Delta\overset{a}{q}_t-\overset{a}{s}_t\right)\delta\overset{a}{r}_t\,d\overset{a}{K}=\overset{a}{\delta}J_{H-W}.$$

Application of the Gauss theorem and integration by parts to the third volume integral of (3.4) leads to the following relation:

(3.5)

$$\int_{\overset{a}{V}_o}\left(\overset{a}{D}{}^{klij}\overset{a}{\varepsilon}_{kl}-\overset{a}{\sigma}{}^{ij}\right)\delta\overset{a}{\varepsilon}_{ij}\,d\overset{a}{V}_o-\int_{\overset{a}{V}_o}\left(\overset{a}{\varepsilon}_{ij}-\overset{a}{q}_{(i,j)}\right)\delta\overset{a}{\sigma}{}^{ij}\,d\overset{a}{V}_o$$

$$-\int_{\overset{a}{V}_o}\left(\overset{a}{\sigma}{}^{ij}_{,j}+\overset{a}{f}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{V}_o+\int_{\overset{a}{P}_o}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j-\overset{a}{p}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{P}_o+\int_{\overset{a}{Q}_o}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j-\overset{a}{r}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{Q}_o$$

$$-\int_{\overset{a}{Q}_o}\left(\overset{a}{q}_i-\overset{a}{d}_i\right)\delta\overset{a}{r}{}^{i}\,d\overset{a}{Q}_o+\int_{\overset{a}{K}}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j\overset{a}{n}_i-\overset{a}{r}_n\right)\delta\overset{a}{q}_n\,d\overset{a}{K}-\int_{\overset{a}{K}}\left(\overset{a}{q}_n-\overset{a}{h}_n+\overset{a}{h}_{o_n}\right)\delta\overset{a}{r}_n\,d\overset{a}{K}$$

$$+\int_{\overset{a}{K}}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j\overset{a}{t}_{is}\overset{a}{\delta}_{st}-\overset{a}{r}_t\right)\delta\overset{a}{q}_t\,d\overset{a}{K}-\int_{\overset{a}{K}}\left(\Delta\overset{a}{q}_t-\overset{a}{s}_t\right)\delta\overset{a}{r}_t\,d\overset{a}{K}$$

$$+\int_{\overset{a}{K}}\overset{a}{r}_n\delta\overset{a}{h}_n\,d\overset{a}{K}+\int_{\overset{a}{K}}\left(\overset{a}{r}_t\delta\overset{a}{s}_t-\mu\overset{a}{p}_n\delta|\overset{a}{s}_t|\right)d\overset{a}{K}=\overset{a}{\delta}J_{H-W}.$$

It is obvious that the solution quantities $\overset{a}{q}_i$, $\overset{a}{\varepsilon}_{ij}$, $\overset{a}{\sigma}{}^{ij}$, $\overset{a}{r}{}^{i}$, $\overset{a}{h}_i$, $\overset{a}{s}_i$ $(a=1,2)$ in (3.4) and (3.5) belong to the sets

(3.6)

$$\overset{a}{\mathcal{K}}=\Big\{\overset{a}{q}_i,\ \overset{a}{\varepsilon}_{ij},\ \overset{a}{h}_i,\ \overset{a}{s}_i\ |\ \overset{a}{\varepsilon}_{ij}=\overset{a}{q}_{(i,j)}\ \text{in}\ \overset{a}{V}_o;\qquad \overset{a}{q}_i=\overset{a}{d}_i\ \text{on}\ \overset{a}{Q}_o;$$

$$\overset{a}{q}_n+\overset{a}{h}_{o_n}=\overset{a}{h}_n,\quad \overset{a}{h}_n\le 0\ \text{on}\ \overset{a}{K};\qquad \Delta\overset{a}{q}_t=\overset{a}{s}_t,\quad |\overset{a}{s}_t|\ge 0\ \text{on}\ \overset{a}{K}\Big\},$$

$$\overset{a}{\mathcal{S}}=\Big\{\overset{a}{\sigma}{}^{ij},\ \overset{a}{r}{}^{i}\ |\ \overset{a}{\sigma}{}^{ij}_{,j}+\overset{a}{f}{}^{i}=0\ \text{in}\ \overset{a}{V}_o;\qquad \overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j=\overset{a}{p}{}^{i}\ \text{on}\ \overset{a}{P}_o;$$

$$\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j=\overset{a}{r}{}^{i}\ \text{on}\ \overset{a}{Q}_o;\qquad \overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j\overset{a}{n}_i=\overset{a}{r}_n,\quad \overset{a}{r}_n\le 0\ \text{on}\ \overset{a}{K};$$

$$\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j\overset{a}{t}_{is}\delta_{st}=\overset{a}{r}_t,\quad |\overset{a}{r}_t|+\mu\overset{a}{p}_n\le 0\ \text{on}\ \overset{a}{K}\Big\}.$$

Addition of Eqs. (3.5) and taking into account the simplified notation of contact stresses and displacements from Sec. 2.3 yield the variational principle of Hu – Washizu for semi-incremental description of frictional two-body contact problem of linear elasticity

$$
\begin{aligned}
(3.7)\qquad \sum_{a=1}^{2} \Bigg[& \int_{\overset{a}{V_o}} \left(\overset{a}{D}{}^{klij} \overset{a}{\varepsilon}_{kl} - \overset{a}{\sigma}{}^{ij} \right) \delta \overset{a}{\varepsilon}_{ij} \, d\overset{a}{V}_o - \int_{\overset{a}{V_o}} \left(\overset{a}{\varepsilon}_{ij} - \overset{a}{q}_{(i,j)} \right) \delta \overset{a}{\sigma}{}^{ij} \, d\overset{a}{V}_o \\
& - \int_{\overset{a}{V_o}} \left(\overset{a}{\sigma}{}^{ij}_{,j} + \overset{a}{f}{}^{i} \right) \delta \overset{a}{q}_i \, d\overset{a}{V}_o + \int_{\overset{a}{P_o}} \left(\overset{a}{\sigma}{}^{ij} \overset{a}{\nu}_j - \overset{a}{p}{}^{i} \right) \delta \overset{a}{q}_i \, d\overset{a}{P}_o + \int_{\overset{a}{Q_o}} \left(\overset{a}{\sigma}{}^{ij} \overset{a}{\nu}_j - \overset{a}{r}{}^{i} \right) \delta \overset{a}{q}_i \, d\overset{a}{Q}_o \\
& - \int_{\overset{a}{Q_o}} \left(\overset{a}{q}_i - \overset{a}{d}_i \right) \delta \overset{a}{r}{}^{i} \, d\overset{a}{Q}_o \Bigg] + \int_K (\sigma^{ij} \nu_j n_i - r_n) \delta q_n \, dK \\
& - \int_K (q_n - h_n + h_{o_n}) \delta r_n \, dK + \int_K (\sigma^{ij} \nu_j t_{is} \delta_{st} - r_t) \delta q_t \, dK \\
& - \int_K (\Delta q_t - s_t) \delta r_t \, dK + \int_K r_n \delta h_n \, dK + \int_K (r_t \delta s_t - \mu p_n \delta |s_t|) \, dK \lesssim 0.
\end{aligned}
$$

The sign of the principle is unknown due to fundamental inequalities (3.2). Note that two last terms in (3.7) are equivalent to the left-hand side terms of (3.2), and all other terms of (3.7) are equal to zero in order to satisfy local constitutive equations, local geometrical relations, local equations of motion, local stress and displacement boundary conditions as well as gap and slip increment definitions.

The above original result (3.7) can be utilized for derivation of the inequalities for simpler problems. For example, rejection of all the integrals through contact surface K in tangential directions leads to the principle for frictionless two-body contact. Furthermore, it follows from (3.5) that some reverse simplifications in (3.7) allows for obtaining frictional or frictionless single-body contact formulations. Also, the non-incremental problems can be described by (3.7) if we replace tangential displacement increments Δq_t by total displacements q_t, with s_t standing for slips instead of slip increments.

3.2. The inequality based on the Hellinger–Reissner functional

It can be demonstrated that Hellinger – Reissner potential energy (compare [10]) and complementary energy functionals for the contact problems can be easily obtained from the general Hu – Washizu functional assuming that some local relations hold *a priori*. This approach, however, is not suitable for derivation of the variational inequalities as it does not allow for determination of signs of these inequalities. Thus it cannot be applied in this paper.

In order to derive the variational inequality with the Hellinger–Reissner functional, let us consider the set of arbitrary field quantities $\overset{a}{*}{q}_i$, $\overset{a}{*}{\sigma}^{ij}$ $(a = 1,2)$ which need neither be kinematically nor statically admissible. The corresponding solution quantities $\overset{a}{q}_i$, $\overset{a}{\varepsilon}_{ij}$, $\overset{a}{\sigma}^{ij}$, $\overset{a}{r}^i$ $(a = 1,2)$ and h_i, s_i obeying relations (2.8)–(2.10) belong to the set $\mathcal{K}$ of kinematically admissible quantities, and the set $\mathcal{S}$ of statically admissible quantities. These sets are defined as follows:

$$
\begin{aligned}
\mathcal{K} = \Big\{ & \overset{1}{\bar{q}}_i, \overset{2}{\bar{q}}_i \mid \overset{a}{\bar{\varepsilon}}_{ij} = \overset{a}{\bar{q}}_{(i,j)} \text{ in } \overset{a}{V}_o; \qquad \overset{a}{q}_i = \overset{a}{d}_i \text{ on } \overset{a}{Q}_o; \\
& \overset{1}{\bar{q}}_n + \overset{1}{h}_{o_n} - \overset{2}{\bar{q}}_n - \overset{2}{h}_{o_n} = \bar{h}_n, \quad \bar{h}_n \leq 0 \text{ on } K; \\
& \Delta \overset{1}{\bar{q}}_t - \Delta \overset{2}{\bar{q}}_t = \bar{s}_t, \quad |\bar{s}_t| \geq 0 \text{ on } K \Big\}, \qquad (3.8) \\
\mathcal{S} = \Big\{ & \overset{1}{\bar{\sigma}}^{ij}, \overset{2}{\bar{\sigma}}^{ij} \mid \overset{a}{\bar{\sigma}}^{ij}{}_{,j} + \overset{a}{f}^i = 0 \text{ in } \overset{a}{V}_o; \qquad \overset{a}{\bar{\sigma}}^{ij} \overset{a}{\nu}_j = \overset{a}{p}^i \text{ on } \overset{a}{P}_o; \\
& \bar{\sigma}^{ij} \nu_j n_i = \bar{r}_n, \quad \bar{r}_n \leq 0 \text{ on } K; \qquad \bar{\sigma}^{ij} \nu_j t_{is} \delta_{st} = \bar{r}_t, \quad |\bar{r}_t| + \mu p_n \leq 0 \text{ on } K \Big\}.
\end{aligned}
$$

Then the two first fundamental inequalities (3.2) and two second fundamental inequalities in the form

$$
\begin{aligned}
& h_n \left(\overset{*}{r}_n - r_n \right) = h_n \overset{*}{r}_n - h_n r_n = h_n \overset{*}{r}_n \lesssim 0, \\
& s_t \left(\overset{*}{r}_t - r_t \right) = s_t \overset{*}{r}_t + |s_t||r_t| = s_t \overset{*}{r}_t + |s_t||r_t| + |s_t|\mu p_n - |s_t|\mu p_n \qquad (3.9) \\
& \qquad = s_t \overset{*}{r}_t - |s_t|\mu p_n + |s_t|(|r_t| + \mu p_n) = s_t \overset{*}{r}_t - |s_t|\mu p_n \lesssim 0
\end{aligned}
$$

hold. Note that the following relations from (2.10): $h_n r_n = 0$, $|s_t|(|r_t| + \mu p_n) = 0$ and additionally $-s_t r_t = |s_t||r_t|$ following from $r_t = \mu p_n s_t / |s_t|$ are introduced into (3.9). The above results will be utilized later while defining the sign of the variational inequality.

If we denote the individual variations $(a = 1,2)$ as

$$
\delta \overset{a}{q}_i = \overset{a}{*}{q}_i - \overset{a}{q}_i, \qquad \delta \overset{a}{\sigma}^{ij} = \overset{a}{*}{\sigma}^{ij} - \overset{a}{\sigma}^{ij},
$$

then variation of the Hellinger-Reissner functional for each body is

$$
\begin{aligned}
\delta \overset{a}{J}_R = & - \int_{\overset{a}{V}_o} \overset{a}{D}^{-1}_{klij} \overset{a}{\sigma}^{kl} \delta \overset{a}{\sigma}^{ij} \, d\overset{a}{V}_o + \int_{\overset{a}{V}_o} \delta \left(\overset{a}{q}_{(i,j)} \overset{a}{\sigma}^{ij} \right) d\overset{a}{V}_o - \int_{\overset{a}{V}_o} \overset{a}{f}^i \delta \overset{a}{q}_i \, d\overset{a}{V}_o \\
& - \int_{\overset{a}{P}_o} \overset{a}{p}^i \delta \overset{a}{q}_i \, d\overset{a}{P}_o - \int_{\overset{a}{Q}_o} \delta \left[\left(\overset{a}{q}_i - \overset{a}{d}_i \right) \overset{a}{\sigma}^i \right] d\overset{a}{Q}_o + \int_{\overset{a}{K}} \overset{a}{h}_n \delta \overset{a}{r}_n \, d\overset{a}{K} \\
& - \int_{\overset{a}{K}} \delta \left[\left(\overset{a}{q}_n + \overset{a}{h}_{o_n} \right) \overset{a}{\sigma}_n \right] d\overset{a}{K} + \int_{\overset{a}{K}} \overset{a}{d}^{-1}_{st} \overset{a}{r}_s \delta \overset{a}{r}_t \, d\overset{a}{K} - \int_{\overset{a}{K}} \delta \left(\Delta \overset{a}{q}_t \overset{a}{\sigma}_t \right) d\overset{a}{K}, \qquad (3.10)
\end{aligned}
$$

where $\overset{a}{d}{}^{-1}_{st}$ is the auxiliary inverse surface constitutive tensor such that $\overset{a}{s}_t = \overset{a}{d}{}^{-1}_{st}\overset{a}{r}_s$. Also, we introduce the auxiliary notation: σ^i, σ_n, σ_t standing for $\sigma^{ij}\nu_j$, $\sigma^{ij}\nu_j n_i$, $\sigma^{ij}\nu_j t_{is}\delta_{st}$, respectively. The form of the functional (3.10) is proposed to be obtained in the way partially similar to the case of Hu-Washizu functional. That means that the standard terms of the first two lines are taken from the well known Hellinger-Reissner functional for one elastic body. Moreover, the formal replacement of the volume quantities D^{-1}_{ijkl}, $q_{(i,j)} = \varepsilon_{ij}$, σ^{ij} by surface ones d^{-1}_{st}, Δq_t, $\sigma^{ij}\nu_j t_{is}\delta_{st} = r_t$ is performed while defining the non-standard integrals through the contact surface $\overset{a}{K}$ in tangent directions. The integrals through the contact surface in normal direction are constructed in the same manner as applied to the contact surface integrals in tangent directions.

Performing the operations of variation in (3.10), and applying the Gauss theorem and integration by parts leads to the form of (3.10) corresponding to relation (3.5) from Sec. 3.1. It is obvious that the solution quantities $\overset{a}{q}_i$ and $\overset{a}{\sigma}{}^{ij}$ $(a = 1, 2)$ in (3.12) belong to the kinematically and statically admissible sets $\overset{a}{\mathcal{K}}$ and $\overset{a}{\mathcal{S}}$ of the form similar to (3.6). These sets can be formally obtained from (3.8) in the way in which the sets (3.1) are obtained from (3.6).

Addition of Eqs. (3.10) in their equivalent form corresponding to (3.5) and taking into account the simplified notation of contact stresses and displacements from Sec. 2.3 yield the variational inequality of Hellinger-Reissner for semi-incremental frictional two-body contact problem of linear elasticity

$$
\begin{aligned}
(3.11)\qquad \sum_{a=1}^{2}\Bigg[& -\int_{\overset{a}{V}_o}\left(\overset{a}{D}{}^{-1}_{klij}\overset{a}{\sigma}{}^{kl} - \overset{a}{q}_{(i,j)}\right)\delta\overset{a}{\sigma}{}^{ij}\,d\overset{a}{V}_o - \int_{\overset{a}{V}_o}\left(\overset{a}{\sigma}{}^{ij} + \overset{a}{f}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{V}_o \\
& + \int_{\overset{a}{P}_o}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j - \overset{a}{p}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{P}_o - \int_{\overset{a}{Q}_o}\left(\overset{a}{q}_i - \overset{a}{d}_i\right)\delta\overset{a}{\sigma}{}^{i}\,d\overset{a}{Q}_o\Bigg] \\
& - \int_K (q_n + h_{o_n})\delta\sigma_n\,dK - \int_K \Delta q_t\delta\sigma_t\,dK = -\int_K h_n\delta r_n\,dK - \int_K s_t\delta r_t\,dK \lesseqgtr 0.
\end{aligned}
$$

The sign of the principle is unknown due to the second fundamental inequalities (3.9). Note that two terms of the right-hand side of (3.11) are equal to two last terms of the left-hand side of (3.11), and additionally are equivalent to the left-hand side terms of (3.9). All other terms of (3.11) are equal to zero in order to satisfy local constitutive equations associated with local geometrical relations, local equations of motion, local stress and displacement boundary conditions.

The above original result (3.11) can be utilized for derivation of the principles for simpler problems. For example, rejection of all the integrals through

contact surface K in tangential directions leads to the inequality describing the frictionless two-body contact. Furthermore, it follows from (3.10) that reverse simplifications in (3.11) allows for obtaining frictional or frictionless single-body contact formulations. Also, the corresponding non-incremental approaches are possible. In particular, the non-incremental frictionless two-body contact formulation proposed by BUFLER [12] can be obtained from (3.11).

The above formulation of the Hellinger-Reissner variational inequality could be called classical, since the contact terms are treated in the similar way to the volume terms. Such a formulation is not the only possible. Let us propose variation of the Hellinger-Reissner functional for each body a in the form equivalent to (3.10)

$$
\begin{aligned}
(3.12)\qquad \delta \overset{a}{J}_R = &- \int_{\overset{a}{V}_o} \overset{a}{D}{}^{-1}_{klij} \overset{a}{\sigma}{}^{kl} \delta \overset{a}{\sigma}{}^{ij} \, d\overset{a}{V}_o + \int_{\overset{a}{V}_o} \delta \left(\overset{a}{q}_{(i,j)} \overset{a}{\sigma}{}^{ij} \right) d\overset{a}{V}_o \\
&- \int_{\overset{a}{V}_o} \overset{a}{f}{}^i \delta \overset{a}{q}_i \, d\overset{a}{V}_o - \int_{\overset{a}{P}_o} \overset{a}{p}{}^i \delta \overset{a}{q}_i \, d\overset{a}{P}_o - \int_{\overset{a}{Q}_o} \delta \left[\left(\overset{a}{q}_i - \overset{a}{d}_i \right) \overset{a}{\sigma}{}^i \right] d\overset{a}{Q}_o \\
&- \int_{\overset{a}{K}} \overset{a}{r}_n \delta \overset{a}{h}_n \, d\overset{a}{K} - \int_{\overset{a}{K}} \overset{a}{r}_t \delta \overset{a}{s}_t \, d\overset{a}{K} - \int_{\overset{a}{K}} \mu p_n \delta |\overset{a}{\Delta q}_t| \, d\overset{a}{K} + \int_{\overset{a}{K}} \mu p_n \delta |\overset{a}{s}_t| \, d\overset{a}{K}.
\end{aligned}
$$

Note that the sum of two first integrals of (3.12) over the contact area $\overset{a}{K}$ is equivalent to the sum of four contact integrals of (3.10), while the sum of two last integrals of (3.12) equals zero.

Performing the operations of variation in (3.12), and application of the Gauss theorem and integration by parts allows to write (3.12) in the form corresponding to (3.5) from Sec. 3.1. Addition of (3.12) in this form and taking into account the simplified notation of contact stresses and displacements from Sec. 2.3 yield an original form of the variational inequality of Hellinger-Reissner for semi-incremental frictional two-body contact problem of linear elasticity

$$
\begin{aligned}
(3.13)\qquad \sum_{a=1}^{2} \Bigg[&- \int_{\overset{a}{V}_o} \left(\overset{a}{D}{}^{-1}_{klij} \overset{a}{\sigma}{}^{kl} - \overset{a}{q}_{(i,j)} \right) \delta \overset{a}{\sigma}{}^{ij} \, d\overset{a}{V}_o - \int_{\overset{a}{V}_o} \left(\overset{a}{\sigma}{}^{ij} + \overset{a}{f}{}^i \right) \delta \overset{a}{q}_i \, d\overset{a}{V}_o \\
&+ \int_{\overset{a}{P}_o} \left(\overset{a}{\sigma}{}^{ij} \overset{a}{\nu}_j - \overset{a}{p}{}^i \right) \delta \overset{a}{q}_i \, d\overset{a}{P}_o - \int_{\overset{a}{Q}_o} \left(\overset{a}{q}_i - \overset{a}{d}_i \right) \delta \overset{a}{\sigma}{}^i \, d\overset{a}{Q}_o \Bigg] + \int_K \sigma_n \delta q_n \, dK \\
&+ \int_K (\sigma_t \delta q_t - \mu p_n \delta |\Delta q_t|) \, dK = \int_K r_n \delta h_n \, dK + \int_K (r_t \delta s_t - \mu p_n \delta |s_t|) \, dK \lesssim 0,
\end{aligned}
$$

where $\delta q_t = \delta(\Delta q_t)$ as $\Delta q_t = q_t - q_t'$ with q_t' being the known displacement from the previous incremental step. The sign of the principle is unknown again due to the first fundamental inequalities (3.2). Note that two terms of the right-hand side of (3.13) are equal to two last terms of the left-hand side of (3.13) and, additionally, are equivalent to the left-hand side terms of (3.2). All other terms of (3.13) are equal to zero again. Again, formulations of simpler cases can be obtained from (3.13).

Two equivalent forms (3.11) and (3.13) of the variational inequality based on the Hellinger–Reissner functional are still not the only possible ones. Two more original representations are admissible for the two-body contact problem under consideration. The first one

$$
\begin{aligned}
(3.14)\qquad & \sum_{a=1}^{2}\left[-\int_{\overset{a}{V_o}}\left(\overset{a}{D}{}^{-1}_{klij}\overset{a}{\sigma}{}^{kl}-\overset{a}{q}_{(i,j)}\right)\delta\overset{a}{\sigma}{}^{ij}\,d\overset{a}{V}_o-\int_{\overset{a}{V_o}}\left(\overset{a}{\sigma}{}^{ij}_{,j}+\overset{a}{f}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{V}_o\right.\\
& \left.+\int_{\overset{a}{P_o}}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j-\overset{a}{p}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{P}_o-\int_{\overset{a}{Q_o}}\left(\overset{a}{q}_i-\overset{a}{d}_i\right)\delta\overset{a}{\sigma}{}^{i}\,d\overset{a}{Q}_o\right]-\int_K(q_n+h_{o_n})\delta\sigma_n\,dK\\
& +\int_K(\sigma_t\delta q_t-\mu p_n\delta|\Delta q_t|)\,dK=-\int_K h_n\delta r_n\,dK+\int_K(r_t\delta s_t-\mu p_n\delta|s_t|)\,dK\lesseqgtr 0
\end{aligned}
$$

can be obtained when the contact integrals in the normal direction are built in the way proposed for (3.11), while the integrals in the tangential directions are constructed in the way suitable for (3.13). The inequality sign cannot be determined due to the first of the second fundamental inequalities (3.9) and the second of the first fundamental inequalities (3.2).

If we apply the opposite approach, which consists in constructing normal and tangent contact integrals in the manner appropriate for (3.13) and (3.11), respectively, then we obtain

$$
\begin{aligned}
(3.15)\qquad & \sum_{a=1}^{2}\left[-\int_{\overset{a}{V_o}}\left(\overset{a}{D}{}^{-1}_{klij}\overset{a}{\sigma}{}^{kl}-\overset{a}{q}_{(i,j)}\right)\delta\overset{a}{\sigma}{}^{ij}\,d\overset{a}{V}_o-\int_{\overset{a}{V_o}}\left(\overset{a}{\sigma}{}^{ij}_{,j}+\overset{a}{f}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{V}_o\right.\\
& \left.+\int_{\overset{a}{P_o}}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j-\overset{a}{p}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{P}_o-\int_{\overset{a}{Q_o}}\left(\overset{a}{q}_i-\overset{a}{d}_i\right)\delta\overset{a}{\sigma}{}^{i}\,d\overset{a}{Q}_o\right]+\int_K\sigma_n\delta q_n\,dK\\
& -\int_K\Delta q_t\delta\sigma_t\,dK=\int_K r_n\delta h_n\,dK-\int_K s_t\delta r_t\,dK\lesseqgtr 0.
\end{aligned}
$$

The inequality sign is undetermined again due to the first of the first fundamental inequalities (3.2), and the second of the second fundamental inequalities (3.9). Note that simpler representations for frictionless two-body as well as frictional and frictionless single-body contact are possible for either (3.15) or (3.14). Also, the non-incremental formulations of (3.15) and (3.14) can be obtained.

3.3. The inequality based on the potential energy functional

Let us derive the variational inequality based on the potential energy functional of the incremental frictional two-body contact problem of linear elasticity which corresponds to the local semi-incremental formulation presented in the previous chapter. For this purpose we consider the set $\mathcal{K}$ of kinematically admissible quantities $\overset{a}{\overline{q}}_i$ $(a = 1, 2)$

$$\mathcal{K} = \Big\{ \overset{1}{\overline{q}}_i, \overset{2}{\overline{q}}_i \mid \overset{a}{\overline{\varepsilon}}_{ij} = \overset{a}{\overline{q}}_{(i,j)} \text{ in } \overset{a}{V}_o; \quad \overset{a}{\overline{q}}_i = \overset{a}{d}_i \text{ on } \overset{a}{Q}_o; \tag{3.16}$$

$$\overset{1}{\overline{q}}_n + \overset{1}{h}_{o_n} - \overset{2}{\overline{q}}_n - \overset{2}{h}_{o_n} = \overline{h}_n, \quad \overline{h}_n \leq 0 \text{ on } K;$$

$$\Delta\overset{1}{\overline{q}}_t - \Delta\overset{2}{\overline{q}}_t = \overline{s}_t, \quad |\overline{s}_t| \geq 0 \text{ on } K \Big\}$$

and the solution quantities $\overset{a}{q}_i$, $\overset{a}{\varepsilon}_{ij}$, $\overset{a}{\sigma}{}^{ij}$, $\overset{a}{r}{}^i$ $(a = 1, 2)$ and h_i, s_i obeying all static and kinematic relations (2.8)–(2.10). Then the two first fundamental inequalities can be uniquely defined as

$$\begin{aligned} & r_n(\overline{h}_n - h_n) = r_n\overline{h}_n - r_n h_n = r_n\overline{h}_n \geq 0, \\ & r_t(\overline{s}_t - s_t) - \mu p_n(|\overline{s}_t| - |s_t|) = r_t\overline{s}_t - \mu p_n|\overline{s}_t| + |r_t||s_t| + \mu p_n|s_t| \\ & \qquad \geq -|r_t||\overline{s}_t| - \mu p_n|\overline{s}_t| + (|r_t| + \mu p_n)|s_t| = -(|r_t| + \mu p_n)|\overline{s}_t| \geq 0. \end{aligned} \tag{3.17}$$

We applied the following relations (2.10) in (3.17): $r_n h_n = 0$, $(|r_t| + \mu p_n)|s_t| = 0$ and additionally $-r_t s_t = |r_t||s_t|$, arising from $r_t = \mu p_n s_t/|s_t|$. The inequalities: $\overline{s}_t r_t \geq -|\overline{s}_t||r_t|$ and $|r_t| + \mu p_n \leq 0$ were also employed. The above results (3.17) will be utilized later while defining the sign of the variational inequality.

Denoting the kinematically admissible variations $(a = 1, 2)$ by

$$\delta\overset{a}{q}_i = \overset{a}{\overline{q}}_i - \overset{a}{q}_i,$$

variation of the potential energy functional for each body can be written in the form

$$\begin{aligned} \delta\overset{a}{J}_P = & \int_{\overset{a}{V}_o} \overset{a}{D}{}^{klij}\overset{a}{q}_{(k,l)}\delta\overset{a}{q}_{(i,j)}\,d\overset{a}{V}_o - \int_{\overset{a}{V}_o} \overset{a}{f}{}^i\delta\overset{a}{q}_i\,d\overset{a}{V}_o - \int_{\overset{a}{P}_o} \overset{a}{p}{}^i\delta\overset{a}{q}_i\,d\overset{a}{P}_o \\ & - \int_{\overset{a}{K}} \overset{a}{r}_n\delta\overset{a}{h}_n\,d\overset{a}{K} - \int_{\overset{a}{K}} \overset{a}{r}_t\delta\overset{a}{s}_t\,d\overset{a}{K} - \int_{\overset{a}{K}} \mu p_n\delta|\Delta\overset{a}{q}_t|\,d\overset{a}{K} + \int_{\overset{a}{K}} \mu p_n\delta|\overset{a}{s}_t|\,d\overset{a}{K}. \end{aligned} \tag{3.18}$$

The form of the functional (3.18) is proposed to be obtained in the following way. The standard terms of the first two lines are taken from the well known potential energy functional of one elastic body. The non-standard terms representing the integrals over the contact surface $\overset{a}{K}$ are constructed in the same way as in the case of the Hellinger-Reissner functional in its form (3.12). Note that the first two integrals over the contact surface can be obtained in the way similar to the surface integral over the loaded part of the body surface, while the sum of two last integrals of (3.18) equals zero.

Application of the Gauss theorem and integration by parts to the volume integral of (3.18) leads to the equivalent form of (3.18) corresponding to relation (3.5) from Sec. 3.1. It is obvious that the solution quantities $\overset{a}{q}_i$ $(a = 1, 2)$ in (3.18) belong to the sets $\overset{a}{\mathcal{K}}$ of kinematically admissible displacements defined in Sec. 3.2.

Addition of equations (3.18) in their form corresponding to (3.5) and taking into account the simplified notation of contact stresses and displacements from Sec. 2.3 yield the inequality variational principle with the potential energy functional describing the semi-incremental frictional two-body contact problem of linear elasticity

$$
\begin{aligned}
(3.19)\qquad \sum_{a=1}^{2}\left[\int_{\overset{a}{V}_o}\left(\overset{a}{\sigma}{}^{ij}_{,j}+\overset{a}{f}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{V}_o+\int_{\overset{a}{P}_o}\left(\overset{a}{\sigma}{}^{ij}\overset{a}{\nu}_j-\overset{a}{p}{}^{i}\right)\delta\overset{a}{q}_i\,d\overset{a}{P}_o\right]& \\
+\int_K \sigma_n\delta q_n\,dK+\int_K(\sigma_t\delta q_t-\mu p_n\delta|\Delta q_t|)\,dK& \\
=\int_K r_n\delta h_n\,dK+\int_K(r_t\delta s_t-\mu p_n\delta|s_t|)\,dK\geq 0.&
\end{aligned}
$$

The sign of the principle can be uniquely determined due to the first fundamental inequalities (3.17). Note that two last right-hand terms of (3.19) are equal to the last left-hand side terms of (3.19) and, at the same time, are equivalent to the left-hand side terms of (3.17). Two other terms of (3.19) are equal to zero in order to satisfy local equations of motion and local stress boundary conditions.

Simpler problems can be also analyzed by (3.19). For example, rejection of all the integrals through contact surface K in tangential directions in (3.19) leads to the inequality for frictionless two-body contact problem. Furthermore, it arises from (3.18) that some reverse simplifications of (3.19) allows for obtaining frictional or frictionless single-body contact formulations. The corresponding non-incremental formulations of the two-body frictional, two-body frictionless and single-body frictional problems proposed by KALKER [5], BUFLER [12] and KIKUCHI and ODEN [3], respectively, can be obtained from (3.19).

3.4. The inequality based on the complementary energy functional

Let us derive the complementary energy variational inequality of the incremental frictional two-body contact problem of linear elasticity, which corresponds to the local semi-incremental formulation presented in the previous chapter. For this purpose we consider the set $\mathcal{S}$ of statically admissible quantities $\overset{a}{\overline{\sigma}}{}^{ij}$ $(a = 1,2)$

$$
\begin{aligned}
(3.20)\qquad \mathcal{S} = \Big\{ \overset{1}{\overline{\sigma}}{}^{ij}, \overset{2}{\overline{\sigma}}{}^{ij} \mid \overset{a}{\overline{\sigma}}{}^{ij}_{,j} + \overset{a}{f}{}^{i} = 0 \ \text{in}\ \overset{a}{V}_o; \qquad & \overset{a}{\overline{\sigma}}{}^{ij}\overset{a}{\nu}_j = \overset{a}{\overline{p}}{}^{i} \ \text{on}\ P_o; \\
& \overline{\sigma}^{ij}\nu_j n_i = \overline{r}_n, \quad \overline{r}_n \le 0 \ \text{on}\ K; \\
& \overline{\sigma}^{ij}\nu_j t_{is}\delta_{st} = \overline{r}_t, \quad |\overline{r}_t| + \mu p_n \le 0 \ \text{on}\ K \Big\}
\end{aligned}
$$

and the solution quantities $\overset{a}{q}_i, \overset{a}{\varepsilon}_{ij}, \overset{a}{\sigma}{}^{ij}, \overset{a}{r}{}^{i}$ $(a = 1,2)$ and h_i, s_i obeying all static and kinematic relations (2.8)–(2.10). Then the two second fundamental inequalities take the form

$$
\begin{aligned}
& h_n(\overline{r}_n - r_n) = h_n\overline{r}_n - h_n r_n = h_n\overline{r}_n \ge 0, \\
(3.21)\qquad & s_t(\overline{r}_t - r_t) = s_t\overline{r}_t + |s_t||r_t| = s_t\overline{r}_t + |s_t||r_t| - |s_t|\mu p_n + |s_t|\mu p_n \\
& \qquad \ge -|s_t||\overline{r}_t| - |s_t|\mu p_n + |s_t|(|r_t| + \mu p_n) = -|s_t|(|\overline{r}_t| + \mu p_n) \ge 0.
\end{aligned}
$$

Note that the following relations from (2.10): $h_n r_n = 0$, $s_t(|r_t| + \mu p_n) = 0$ and, additionally, $-s_t r_t = |s_t||r_t|$ arising from $r_t = \mu p_n s_t/|s_t|$ are introduced into (3.21). Also, the inequalities: $s_t\overline{r}_t \ge -|s_t||\overline{r}_t|$ and $|r_t| + \mu p_n \le 0$ are employed. The above results will be utilized later while defining the sign of the variational inequality.

Denoting the statically admissible variations $(a = 1,2)$ by

$$
\delta\overset{a}{\sigma}{}^{ij} = \overset{a}{\overline{\sigma}}{}^{ij} - \overset{a}{\sigma}{}^{ij},
$$

variation of the complementary energy functional for each body can be written in the form

$$
\begin{aligned}
(3.22)\qquad \delta\overset{a}{J}_C = & \int_{\overset{a}{V}_o} \overset{a}{D}{}^{-1}_{klij}\overset{a}{\sigma}{}^{kl}\delta\overset{a}{\sigma}{}^{ij}\,d\overset{a}{V}_o - \int_{\overset{a}{Q}_o} \overset{a}{d}_i\delta\overset{a}{\sigma}{}^{i}\,d\overset{a}{Q}_o \\
& - \int_{\overset{a}{K}} \overset{a}{h}_n\delta\overset{a}{r}_n\,d\overset{a}{K} + \int_{\overset{a}{K}} \overset{a}{h}_{o_n}\delta\overset{a}{\sigma}_n\,d\overset{a}{K} - \int_{\overset{a}{K}} \overset{a}{s}_t\delta\overset{a}{r}_t\,d\overset{a}{K} - \int_{\overset{a}{K}} \overset{a}{q}{}'_t\delta\overset{a}{\sigma}_t\,d\overset{a}{K},
\end{aligned}
$$

where $\Delta q_t = q_t - q'_t$ with q'_t being the displacement resulting from the previous incremental step. The functional (3.22) is proposed to be obtained in the following way. The standard terms of the first two lines representing the volume integral

and the surface integral over the supported part $\overset{a}{Q}_o$ of the body surface are taken from the well known complementary energy functional of one elastic body. The non-standard terms representing the integrals through the contact surface $\overset{a}{K}$ are constructed in the same way as for the case of the Hellinger-Reissner functional in its form (3.10). However, the second contact integrals in the normal and tangential directions take the same form as the integral over the supported part of the body surface.

Application of the Gauss theorem and integration by parts to the volume integral of Eq. (3.22) leads to its equivalent form which corresponds to relation (3.5) from Sec. 3.1. Obviously, the solution quantities $\overset{a}{\sigma}{}^{ij}$ $(a = 1, 2)$ in (3.22), or in its form corresponding to (3.5), belong to set $\overset{a}{S}$ of the statically admissible displacements defined in the same way as in Sec. 3.2.

Addition of equations (3.22) in their form corresponding to (3.5) and taking into account the simplified notation of contact stresses and displacements from Sec. 2.3 yield the inequality variational principle with the complementary energy functional for the semi-incremental frictional two-body contact problem of linear elasticity

$$
\begin{aligned}
(3.23)\qquad \sum_{a=1}^{2}\left[-\int_{\overset{a}{V}_o} \overset{a}{q}_i \delta \overset{a}{\sigma}{}^{ij}{}_{,j}\, d\overset{a}{V}_o + \int_{\overset{a}{Q}_o} (\overset{a}{q}_i - \overset{a}{d}_i)\delta \overset{a}{\sigma}{}^{i}\, d\overset{a}{Q}_o\right] \\
+ \int_K (q_n + h_{o_n})\delta\sigma_n\, dK + \int_K (q_t - q_t')\delta\sigma_t\, dK \\
= \int_K h_n \delta r_n\, dK + \int_K s_t \delta r_t\, dK \geq 0.
\end{aligned}
$$

The sign of the principle can be uniquely determined due to the second fundamental inequalities (3.21). Note that two last right-hand terms of (3.23) are equal to two last left-hand side terms of (3.23) and are equivalent to the left-hand side terms of (3.21). All other terms of (3.23) are equal to zero since $\delta \overset{a}{\sigma}{}^{ij}{}_{,j} = 0$ $(a = 1, 2)$ and in order to satisfy the local displacement boundary conditions.

Simpler problems can be also analyzed with (3.23). For example, rejection of all the integrals over contact surface K in tangential directions in (3.23) leads to the inequality for frictionless two-body contact problem. Furthermore, it follows from (3.22) that some reverse simplifications of (3.23) allows for obtaining frictional or frictionless single-body contact formulations. The corresponding non-incremental formulations of the two-body frictional, two-body frictionless and single-body frictional problems proposed by KALKER [5], BUFLER [12] and KIKUCHI and ODEN [3], respectively, can be obtained from (3.23).

4. Solution hints

The normal traction p_n entering the tangential traction terms in (2.6) and (2.7) is formally treated to be given in advance. Unfortunately, p_n is often not known beforehand. In such a case we have $p_n \equiv r_n$ and the value of p_n must be updated from increment to increment or better updated iteratively within each increment, in the solution process.

The form of all the functionals presented above corresponds to the proposed author's solution method [11], in which the normal traction p_n in the tangential traction terms is regarded as a known function of position. Note that a more general approach is also admissible, which takes advantage of the weaker holonomization assumption proposed by KALKER [5] who treats normal traction as a known function g of position and slip increment. Consequently, g has to be submitted to variation. Note that the author's method relaxes the variation δg purposefully in order to decouple, from increment to increment, the unilateral contact from the friction problem. The variation δg can, however, be included in all the above functionals without difficulty in the way analogous to that proposed in [5].

As the sign of the variational inequalities based on Hellinger-Reissner and Hu-Washizu functionals is unknown, it follows that the correct numerical procedure must be based on optimization methods applied to the potential energy and complementary energy which are characterized by extremum properties.

5. Conclusions

There exist the variational inequalities based on the Hu-Washizu and Hellinger-Reissner functionals for the incremental or non-incremental frictional two-body contact problem of linear elasticity.

There exist at least four equivalent formulations of the variational inequality employing the Hellinger-Reissner functional for the frictional two-body and single-body contact problems of linear elasticity.

The sign of the variational inequalities employing the above two functionals is indeterminate, while for the inequalities based on the potential energy and complementary energy functionals the sign can be uniquely defined.

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POLISH ACADEMY OF SCIENCES
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Received November 2, 1994.

Arch. Mech., 47, 4, pp. 745–753, Warszawa 1995

Asymptotic equations of geometrical optics for two models of hyperbolic heat waves (*)

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WE DISCUSS two models of hyperbolic heat waves propagation for rigid conductors from the point of asymptotic analysis of weakly nonlinear geometric optics. The evolution equations for the amplitudes of weakly nonlinear heat waves for both models are derived. These are uncoupled inviscid Burgers equations with linear dissipative terms.

1. Introduction

IN THIS PAPER two approaches to hyperbolic heat waves propagation for rigid conductors are presented and compared from the point of view of asymptotic analysis. The first, a phenomenological one, was developed by the second author in the series of papers [1–6]. It is based on the concept of semi-empirical temperature scale, different from the absolute one. The other approach was developed by Morro and Ruggeri in the framework of the extended thermodynamics [7, 8] and further applied to shock wave propagation in crystals at low temperatures [9]. These models describe the second sound effect. In both approaches the identification of constitutive functions can be done, knowing three experimental curves, namely the specific heat, heat conductivity coefficient and second sound speed, all given in an equlibrium state in terms of the absolute temperature.

In the first approach the governing system is composed of two scalar equations, one of which is of the first order in the absolute temperature θ, and of the second order in the semi-empirical temperature β. By the proper choice of new dependent variables the system can be written as a first order one.

In the second approach the governing system is of the first order in θ and the heat flux vector $\mathbf{q}$. In both models, under weak assumptions concerning the positiveness of the specific heat $c_v(\theta)$, the non-equilibrium heat conductivity $k = k(\theta)$, and the thermal relaxation time τ, the governing quasilinear first order equations are of a hyperbolic type.

The two models have already been compared by analyzing the structure of equations and constitutive functions appearing there (see [4]). Here, these models are being investigated by the method of weakly nonlinear geometric optics. Using this method, the asymptotic evolution equations for the amplitudes of weakly nonlinear waves are derived and then analysed. It turns out that asymptotic evolution equations for both models have similar structures. These are pairs

(*) Paper presented at the VII-th International Conference on Waves and Stability in Continuous Media, Bologna, October 4-9, 1993.

of inviscid Burgers equations with the linear dissipative terms. They represent waves travelling in the opposite directions. Under conditions giving the identification formulas for both models (see the formula (4.8) below or the paper [4]), the speeds of waves and the dissipative coefficients for both models are exactly the same, while the selfinteracting coefficients differ by a constant. However in general, without the identification assumptions, the coefficients in the asymptotic equations for both models differ from each other.

2. Two models

I. The first model of heat conduction with finite wave speed consists of the following equations (see papers [1–6] for the derivation and more detailed discussion of these equations):

a) the energy balance law for the rigid conductor of heat:

$$\varrho \frac{\partial e}{\partial t} + \operatorname{div} \mathbf{q} = 0, \tag{2.1}$$

b) the equation for the scalar internal state variable (called in [3] the semi-empirical temperature β):

$$\frac{\partial \beta}{\partial t} = f, \tag{2.2}$$

and c) the constitutive law relating the heat flux $\mathbf{q}$ to the spatial gradient of β:

$$\mathbf{q} = -k \operatorname{grad} \beta. \tag{2.3}$$

In formula (2.1) ϱ is a constant mass density, $e = e(\theta)$ – internal energy. In formula (2.2)

$$f = f(\theta, \beta) = \tau^{-1} (f_1(\theta) + f_2(\beta)) \tag{2.4}$$

with τ – relaxation time, f_1, f_2 – arbitrary differentiable functions, and $f_1' > 0$, where $'$ denotes a derivative with respect to θ. Finally, in (2.3) $k = k(\theta)$ is a non-equilibrium heat conductivity.

Inserting (2.3) into (2.1), we get a scalar equation mentioned in the Introduction:

$$\varrho\, e' \frac{\partial \theta}{\partial t} - k' \operatorname{grad} \theta \cdot \operatorname{grad} \beta - k \triangle \beta = 0. \tag{2.5}$$

To get a first order hyperbolic system, we eliminate θ from (2.2) and (2.4):

$$\theta = f_1^{-1} \left(\tau \frac{\partial \beta}{\partial t} - f_2(\beta) \right), \tag{2.6}$$

(provided that $f_1(\theta)$ is invertible), and introduce new dependent variables:

$$w = \frac{\partial \beta}{\partial t} \qquad \text{and} \qquad \mathbf{p} = \operatorname{grad} \beta. \tag{2.7}$$

Then, inserting (2.6) into (2.5) we obtain now the first order system which is composed of the following three equations:

$$\varrho e' \tau \frac{\partial w}{\partial t} - k f_1' \operatorname{div} \mathbf{p} - k' \tau \mathbf{p} \cdot \operatorname{grad} w + k' f_{2,\beta}\, \mathbf{p} \cdot \mathbf{p} - \varrho e' f_{2,\beta}\, w = 0, \tag{2.8}$$

equation $(2.7)_1$, and the consequence of Eq. $(2.7)_2$, namely

$$\frac{\partial \mathbf{p}}{\partial t} = \operatorname{grad} w. \tag{2.9}$$

II. The second model (introduced by MORRO and RUGGERI [7, 8]) is based on the Extended Thermodynamics Theory. It consists (again as in the first model) of the balance law of energy (Eq. (2.1)):

$$\varrho \frac{\partial e}{\partial t} + \operatorname{div} \mathbf{q} = 0,$$

and the new equation for the heat flux vector $\mathbf{q}$ of the form:

$$\frac{\partial (\alpha \mathbf{q})}{\partial t} + \operatorname{grad} \nu = -k^{-1} \nu' \mathbf{q}. \tag{2.10}$$

Here $\nu = \nu(\theta)$ and $\alpha = \alpha(\theta)$ are the constitutive scalar-valued functions (α plays the role of thermal inertia), and as before $'$ denotes a derivative with respect to θ. Let us notice that in this model we are faced with two unknown functions θ and $\mathbf{q}$.

3. Quasi-linear hyperbolic systems

For simplicity, in what follows we restrict our considerations to one spatial dimension, so that from now on all dependent variables are functions of time t and space x. Under natural assumptions the systems of equations representing both models are hyperbolic. These assumptions in case of the first model mean that the specific heat $c_v(\theta) \equiv e'(\theta)$, the thermal relaxation time τ, the density ϱ, the non-equilibrium heat conductivity $k = k(\theta)$, and the function f_1' are all positive. Similarly, for the second model we assume that ϱ, e', α, and ν' are positive. Obviously these are only sufficient conditions for the hyperbolicity.

Ad I. Using the formulas (2.6) and (2.7), we can express the absolute temperature θ as a function of the semi-emprical temperature β and its time derivative w: $\theta = \theta(w, \beta)$. Therefore we have also that $c_v(\theta) = c_v(\theta(w, \beta))$, $k(\theta) = k(\theta(w, \beta))$ and $f_1(\theta) = f_1(\theta(w, \beta))$.

Specified to one spatial dimension, Eqs. (2.8), $(2.7)_1$ and (2.9) take the following form:

$$
\begin{aligned}
&C(w,\beta)\frac{\partial w}{\partial t} - B(w,\beta)\, p\, \frac{\partial w}{\partial x} - A(w,\beta)\frac{\partial p}{\partial x} - H(w,\beta,p) = 0,\\
&\frac{\partial \beta}{\partial t} - w = 0,\\
&\frac{\partial p}{\partial t} - \frac{\partial w}{\partial x} = 0,
\end{aligned}
\tag{3.1}
$$

where the dependent variables are β, w and p. Here

$$
A(w,\beta) \equiv k(\theta(w,\beta))\, f_1'(\theta(w,\beta)),
$$

$$
C(w,\beta) \equiv \varrho\, c_v(\theta(w,\beta))\,\tau, \qquad B(w,\beta) \equiv \tau\, k'(\theta(w,\beta))
$$

and

$$
H(w,\beta,p) \equiv f_{2,\beta}(\beta)\,(\varrho\, c_v(\theta(w,\beta))\, w - k'(\theta(w,\beta))\, \mathbf{p}\cdot\mathbf{p}).
$$

Ad **II**. The equations of the second model written as a quasi-linear system in one spatial dimension look as follows:

$$
\begin{aligned}
&\varrho e'(\theta)\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial x} = 0,\\
&q\alpha'(\theta)\frac{\partial \theta}{\partial t} + \alpha(\theta)\frac{\partial q}{\partial t} + \nu'(\theta)\frac{\partial \theta}{\partial x} + \frac{\nu'(\theta)}{k(\theta)} q = 0.
\end{aligned}
\tag{3.2}
$$

Here the absolute temperature θ and the heat flux q are the dependent variables, while $e(\theta)$, $\nu(\theta)$, $k(\theta)$ and $\alpha(\theta)$ are material functions, ϱ is constant.

Under natural assumptions both systems (3.1) and (3.2) may be transformed to the following standard matrix form:

$$
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} + \mathbf{b}(\mathbf{u}) = \mathbf{0}. \tag{3.3}
$$

Ad **I**. In case of the first model we have $\mathbf{u}^s = [w, \beta, p]^T$

$$
\mathbf{A}^s(\mathbf{u}^s) = \begin{pmatrix} -\dfrac{B(w,\beta)\, p}{C(w,\beta)} & 0 & -\dfrac{A(w,\beta)}{C(w,\beta)} \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}^s(\mathbf{u}^s) = \begin{bmatrix} -\dfrac{H(w,\beta,p)}{C(w,\beta)} \\ -w \\ 0 \end{bmatrix}. \tag{3.4}
$$

Ad **II**. For the second model, we have $\mathbf{u}^e = [\theta, q]^T$

$$
\mathbf{A}^e(\mathbf{u}^e) = \begin{pmatrix} 0 & \dfrac{1}{\varrho\, e'(\theta)} \\ \dfrac{\nu'(\theta)}{\alpha(\theta)} & \dfrac{-q\,\alpha'(\theta)}{\varrho\, e'(\theta)\,\alpha(\theta)} \end{pmatrix}, \qquad \mathbf{b}^e(\mathbf{u}^e) = \begin{bmatrix} 0 \\ \dfrac{\nu'(\theta)\, q}{\alpha(\theta)\, k(\theta)} \end{bmatrix}. \tag{3.5}
$$

4. Asymptotic evolution equations

Let us present in short the procedure of derivation of asymptotic evolution equations for the amplitudes of weakly nonlinear solutions of the system (3.3) with initial conditions of the form:

$$\mathbf{u}_{|t=0} = \mathbf{u}_0 + \varepsilon \mathbf{v}\left(\frac{\phi_j}{\varepsilon}\right)_{|_{t=0}}, \tag{4.1}$$

where $\mathbf{u}_0$ = const, ε is a small parameter, and ϕ_j are the unknown phases. Assuming that the initial data $\mathbf{v}$ are of compact support, and $\mathbf{A}$ and $\mathbf{b}$ have Taylor expansions around the constant solution $\mathbf{u}_0$ of (3.3), we apply the method of weakly nonlinear geometrical optics. This is a high frequency small amplitude perturbation technique in which we are looking for the asymptotic solution of (3.3) in the form (see e.g. [10, 11]):

$$\mathbf{u}^{\varepsilon}(x,t) = \mathbf{u}_0 + \varepsilon \sum_j \sigma_j(x,t,\eta)\,\mathbf{r}_j + \mathcal{O}(\varepsilon^2), \tag{4.2}$$

where σ_j are the unknown amplitudes, $\mathbf{u}_0$ is a constant state such that $\mathbf{b}(\mathbf{u}_0) = 0$, $\mathbf{r_j}$ are right eigenvectors of the appropriate linearized eigenvalue problem for the matrix $\mathbf{A}(\mathbf{u}_0)$, and $\eta \equiv \phi_j/\varepsilon$.

The equations for the wave amplitudes σ_j are determined with the help of multiple scales analysis. The idea is to treat first (x,t,η) as independent variables and then evaluate at $\eta \equiv \phi_j/\varepsilon$. Applying this procedure [10, 11], we collect terms of like powers of ε. Taking the linear phases $\phi_j = x - \lambda_j t$ as the solutions of the eikonal equations, from the solvability condition we get evolution transport equations for the amplitude σ_j of each j wave. It turns out that these equations are of the form of uncoupled inviscid Burgers equations with linear dissipative terms:

$$\frac{\partial \sigma_j}{\partial t} + \lambda_j \frac{\partial \sigma_j}{\partial x} + \frac{1}{2}\Gamma_j \frac{\partial \sigma_j^2}{\partial \eta} + \Lambda_j \sigma_j = 0, \tag{4.3}$$

where λ_j and l_j are the eigenvalues and the left eigenvectors of the matrix $\mathbf{A}(\mathbf{u}_0)$, while Γ_j and Λ_j are self-interactive and dissipative coefficients, respectively. Here

$$\Gamma_j = \Gamma_j(\mathbf{u}_0) = \mathbf{l}_j \cdot (\nabla_{\mathbf{u}}\mathbf{A})_{|_{\mathbf{u}=\mathbf{u}_0}} \mathbf{r}_j \cdot \mathbf{r}_j, \tag{4.4}$$

$$\Lambda_j = \Lambda_j(\mathbf{u}_0) = \mathbf{l}_j \cdot (\nabla_{\mathbf{u}}\mathbf{b})_{|_{\mathbf{u}=\mathbf{u}_0}} \mathbf{r}_j. \tag{4.5}$$

For the two models we are discussing here, the evolution equations (4.3) are as follows:

Ad I. In the first model (3.1), let us take as a constant solution of (3.1) $\mathbf{u}_0^s = [w_0 = 0, \beta_0 \neq 0, p_0 = 0]^T$. This implies that $H(w_0, \beta_0, p_0) = 0$. Then the

asymptotic equations for the perturbed amplitudes in that model look as follows:

$$
\begin{aligned}
&\frac{\partial \sigma_1^s}{\partial t} - \sqrt{\frac{A}{C}}\frac{\partial \sigma_1^s}{\partial x} + \frac{1}{2C}\left(C_{,w}\frac{A}{C} - B - A_{,w}\right)\frac{\partial (\sigma_1^s)^2}{\partial \eta} - \frac{1}{2C}H_{,w}\,\sigma_1^s = 0,\\
&\frac{\partial \sigma_2^s}{\partial t} + \sqrt{\frac{A}{C}}\frac{\partial \sigma_1^s}{\partial x} + \frac{1}{2C}\left(C_{,w}\frac{A}{C} - B - A_{,w}\right)\frac{\partial (\sigma_2^s)^2}{\partial \eta} - \frac{1}{2C}H_{,w}\,\sigma_2^s = 0,\\
&\frac{\partial \sigma_3^s}{\partial t} = 0,
\end{aligned}
\tag{4.6}
$$

where $_{,w}$ denotes a partial derivative with respect to w. All coefficients in the equations (4.6) are being evaluated at $w_0 = p_0 = 0$, and $\beta_0 = \text{const}$.

Ad II. The asymptotic equations for the perturbed amplitudes in the second model (3.2) are (as the constant state here we choose $\mathbf{u}_0^e = [\theta_0 \neq 0, q_0 = 0]^T$):

$$
\begin{aligned}
&\frac{\partial \sigma_1^e}{\partial t} - \sqrt{\frac{\nu'}{\varrho e' \alpha}}\frac{\partial \sigma_1^e}{\partial x} + \frac{1}{2\varrho e'}\left(\frac{\nu''}{\nu'} - \frac{e''}{e'} - \frac{2\alpha'}{\alpha}\right)\frac{\partial (\sigma_1^e)^2}{\partial \eta} + \frac{\nu'}{2\alpha k}\sigma_1^e = 0,\\
&\frac{\partial \sigma_2^e}{\partial t} + \sqrt{\frac{\nu'}{\varrho e' \alpha}}\frac{\partial \sigma_2^e}{\partial x} + \frac{1}{2\varrho e'}\left(\frac{\nu''}{\nu'} - \frac{e''}{e'} - \frac{2\alpha'}{\alpha}\right)\frac{\partial (\sigma_2^e)^2}{\partial \eta} + \frac{\nu'}{2\alpha k}\sigma_2^e = 0,
\end{aligned}
\tag{4.7}
$$

where, as before, $'$ denotes derivative with respect to θ. All coefficients in these equations are being evaluated at $q_0 = 0$ and $\theta_0 = \text{const}$.

Let us compare the first two equations from (4.6) with Eqs. (4.7). All of them have the form of inviscid Burgers equation (4.3). We have for the model with a semi-empirical temperature:

$$
\lambda_j^s = \mp\sqrt{\frac{A}{C}} = \mp\sqrt{\frac{k\, f_1'}{\tau\, \varrho\, e'}} \qquad \text{and} \qquad \Lambda_j^s = -\frac{1}{2C}H_{,w} = -\frac{f_{2,\beta}}{2\tau}.
$$

Analogously, for the model based on the extended thermodynamics theory:

$$
\lambda_j^e = \mp\sqrt{\frac{\nu'}{\varrho e' \alpha}} \qquad \text{and} \qquad \Lambda_j^e = \frac{\nu'}{2\alpha k}.
$$

In order to compare these coefficients, let us recall the identifications (see [4]):

$$
f_1 \equiv \nu, \qquad \frac{\tau}{k} \equiv \alpha, \qquad f_{2,\beta} \equiv -\nu'. \tag{4.8}
$$

The identification formulas (4.8) imply some restrictions on both models. We will not discuss these restrictions here. Instead we will show that, under the

identifications formulas, the coefficients λ_j and Λ_j in the asymptotic equations for both models are the same. In fact, since using (4.8) we get

$$\sqrt{\frac{k\, f_1'}{\tau\, \varrho\, e'}} = \sqrt{\frac{\nu'}{\varrho e' \alpha}} \qquad \text{and} \qquad \frac{f_{2,\beta}}{2\tau} = \frac{-\nu'}{2\alpha k},$$

therefore $\lambda_j^s = \lambda_j^e$, and $\Lambda_j^s = \Lambda_j^e$, provided that the conditions (4.8) are satisfied. Now in that case we have also that

$$\Gamma_j^s = \delta\, \Gamma_j^e,$$

where

$$\delta = \frac{1 - \dfrac{\alpha'}{\alpha} - \dfrac{e''}{e'}}{2\,\dfrac{\alpha'}{\alpha} + \dfrac{e''}{e'}}.$$

5. Solution of the asymptotic evolution equation

Let us now briefly describe the method of solving the evolution equation (4.3). Using characteristic coordinate s for the operator $\partial/\partial t + \lambda_j(\partial/\partial x)$, we may write this equation as:

$$\frac{\partial \sigma}{\partial s} + \frac{1}{2}\,\Gamma\,\frac{\partial \sigma^2}{\partial \eta} + \Lambda\sigma = 0, \tag{5.1}$$

(here for the sake of brevity we have omitted the index j). To solve this equation under the initial condition $\sigma_{|s=0} = \sigma_0(\eta)$, first we get rid of the lower order term. This is accomplished by introducing the change of variables $h(s,\eta) = \exp(\Lambda s)\sigma(s,\eta)$. Then (5.1) becomes the inviscid Burgers equation:

$$\frac{\partial h}{\partial s} + \frac{1}{2}\,\Gamma e^{-\Lambda s}\,\frac{\partial h^2}{\partial \eta} = 0 \tag{5.2}$$

with the initial condition $h_{|s=0} = h_0(\eta) = \sigma_0(\eta)$. The solution of this initial value problem is given by the implicit formula

$$h(s,\eta) = h_0\left(\eta + \frac{\Gamma}{\Lambda}(e^{-\Lambda s} - 1)h\right). \tag{5.3}$$

Taking into account that $h(s,\eta) = \exp(\Lambda s)\sigma(s,\eta)$, and that $h_0(\eta) = \sigma_0(\eta)$, we easily get the formula for the solution σ of the initial value problem for the asymptotic evolution equation (5.1). The analysis of this evolution equation allows to get information about the qualitative behavior of solutions (e.g. blow-up time) of the original system.

6. Concluding remarks

Using the method of nonlinear geometric optics we have derived the asymptotic evolution equations for the amplitudes of weakly nonlinear heat waves for two models. It has turned out that the asymptotic equations for both models have the same form (4.3). Moreover, under some identification formulas the speeds of waves and dissipative coefficients in the asymptotic equations for both models are the same.

With the help of the method of characteristics the asymptotic evolution equations may be solved explicitly. Therefore the amplitudes and hence the asymptotic solutions of the original systems (2.1) and (2.2) can be found. To summarize, we show these solutions below:

Ad I:

$$(\mathbf{u}^s)^\varepsilon(x,t,\eta) = \begin{bmatrix} w_0 \\ \beta_0 \\ p_0 \end{bmatrix} + \varepsilon \left(\sigma_1^s \begin{bmatrix} \sqrt{\frac{A}{C}} \\ 0 \\ 1 \end{bmatrix} + \sigma_2^s \begin{bmatrix} -\sqrt{\frac{A}{C}} \\ 0 \\ 1 \end{bmatrix} + \sigma_3^s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + \mathcal{O}(\varepsilon^2),$$

where every $\sigma_j^s = \sigma_j^s(x,t,\eta)$ is the solution of the appropriate equation (4.6) with the initial condition $(\sigma_j^s)_{|t=0} = \mathbf{l}_j^s \cdot \mathbf{v}^s$; here $\mathbf{l}_j^s$ is the left eigenvector of the matrix $\mathbf{A}^s(\mathbf{u}_0^s)$ and $\mathbf{v}^s$ the function from the initial condition (4.1) for $\mathbf{u}^e{}_{|t=0}$.

Ad II :

$$(\mathbf{u}^e)^\varepsilon(x,t,\eta) = \begin{bmatrix} \theta_0 \\ q_0 \end{bmatrix} + \varepsilon \left(\sigma_1^e \begin{bmatrix} -\sqrt{\frac{\nu'}{\alpha\varrho e'}} \\ 1 \end{bmatrix} + \sigma_2^e \begin{bmatrix} \sqrt{\frac{\nu'}{\alpha\varrho e'}} \\ 1 \end{bmatrix} \right) + \mathcal{O}(\varepsilon^2);$$

here every $\sigma_j^e = \sigma_j^e(x,t,\eta)$ is the solution of the appropriate equation (4.7) with the initial condition $(\sigma_j^e)_{|t=0} = \mathbf{l}_j^e \cdot \mathbf{v}^e$, where $\mathbf{l}_j^e$ is the left eigenvector of the matrix $\mathbf{A}^e(\mathbf{u}_0^e)$ and $\mathbf{v}^e$ the function from the initial condition (4.1) for $\mathbf{u}^e{}_{|t=0}$.

The asymptotic method presented in this paper may be compared with the method of acceleration waves. Actually, the appropriate coefficients in the evolution asymptotic equations and Bernoulli-type equations (obtained for the acceleration waves), have the same form (cf. [7]). Therefore, both theories agree in their predictions on growth and decay of discontinuities of the first derivatives propagating along the wave front. However, the explicit form of the asymptotic solution obtained by the method of weakly nonlinear geometric optics gives more insight into the nature of the exact solution. A more detailed discussion of the solution of the asymptotic equations, as well as a comparison of the qualitative behavior of solutions for two models, will be given in a separate paper.

Acknowledgments

This paper was presented at VII-th International Conference on Waves and Stability in Continuous Media, Bologna, October 4-9, 1993. We would like to express our gratitude to the Organizers, particularly to Prof. Tommaso Ruggeri for the financial support and hospitality. Research on that paper has been conducted and travel expenses have been funded under the grant 0645/P4/93/05 from KBN.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received November 18, 1994.

Arch. Mech., 47, 4, pp. 755–771, Warszawa 1995

On the spatial decay estimates in certain time-dependent problems of continuum mechanics

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SPATIAL DECAY estimates of Saint–Venant type are established for the parabolic equation of transient linear and nonlinear heat conduction as well as for wave equation. It is shown that the end effects decay, at least for short times, very fast with the distance from the loaded end. The results are compared with previous mathematical research on the subject.

1. Introduction

THE SPATIAL DECAY of solutions of parabolic partial differential equations has been a subject of a considerable interest for a long time [1–11]. The studies were motivated by a desire to establish, for parabolic equations, the decay estimates analogous to those obtained for elliptic equations in the investigation of Saint-Venant's principle in linear theory of elasticity [12, 13]. For classical linear heat conduction, it was shown by KNOWLES [6] that end effects for the transient case decay spatially at least as rapidly as do their counterparts in the steady state case. This result was established by using arguments based on the first order differential inequalities for quadratic functionals and it was extended to nonlinear theory of heat conduction by NUNZIATO [8] (see also EDELSTEIN [7]). Further, in [11] it is shown that the spatial decay of end effects in the transient linear problem is faster than that for the steady state case. The explicit spatial decay estimates are obtained by using the comparison principle arguments involving solutions of the one-dimensional heat equation.

As regards the hyperbolic partial differential equations, it was remarked in [14, 15] that hyperbolic systems describing wave propagation did not appear to yield spatial decay results of the type discussed for elliptic and parabolic problems. Recently FLAVIN and KNOPS [16] have carried out an analysis of spatial decay for certain damped acoustic and elastodynamic problems in the low frequency range.

Our purpose in the present paper is to employ energy methods of the type described in [4–6] in order to establish stronger results, analogous to those of [11], proving that the spatial decay of end effects at time t in the transient linear heat conduction is faster than that for the steady state case. In fact, we consider initial-boundary value problems for linear parabolic equations in a region R whose boundary was, except for a plane part S_0, free of boundary data. We then show that the total energy (sum of thermal and dissipation energies) stored in the interval of time $[0, t]$ in that subregion R_z of R, all of whose points are of a distance from S_0 greater than or equal to z, decays exponentially with z,

what means that the spatial decay is controlled by the factor $\exp\{-z/\sqrt{t/2}\}$. A spatial decay estimate controlled by the same factor is established also for another energetic measure like that discussed above. An extension of the results to a nonlinear theory of heat conduction considered in [8] is also included.

On the other hand, we shall show that the energy methods allow us to establish spatial decay results of the type discussed above for the wave equation. Particularly, we show that the total energy (sum of kinetic and strain energies) stored in the region R_z over the time interval $[0, t]$, decays exponentially with z, for $z < t$, so that the decay rate is described by the factor $\exp\{-z/t\}$; while for $z > t$, the energy in question is vanishing.

2. Saint-Venant's principle for the linear thermal diffusion equation

Let R be a closed, bounded, regular region in three-dimensional space whose boundary ∂R includes a plane portion S_0. Choose Cartesian coordinates x_1, x_2, x_3 so that S_0 lies in the plane $x_3 = 0$, and suppose that R lies in the half-space $x_3 > 0$. Let $u(x,t) \equiv u(x_1, x_2, x_3, t)$ satisfy the heat equation

$$u_{,jj} - \dot{u} = 0 \qquad \text{on} \qquad R \times (0, t_0). \tag{2.1}$$

Here the dot stands for differentation with respect to t, a comma followed by a subscript indicates differentiation with respect to the corresponding coordinate, and the summation convention is employed for repeated subscripts. Assume further that u satisfies the boundary condition

$$u\left(\frac{\partial u}{\partial n} + \alpha u\right) = 0 \qquad \text{on} \quad (\partial R \setminus S_0) \times (0, t_0), \tag{2.2}$$

where $\partial u/\partial n$ represents the derivative of u in the direction of the outward unit normal n on $\partial R \setminus S_0$, and α is a nonnegative constant. In (2.2) several boundary conditions are written in the form of one equation. We also assume the initial condition

$$u(x, 0) = 0 \qquad \text{for} \quad x \in R. \tag{2.3}$$

If R_z denotes that portion of R in which $x_3 > z$, define

$$U(z,t) = \int_0^t \int_{R_z} \left(\dot{u}^2 + u_{,j}u_{,j}\right) dV\, ds, \qquad 0 \preceq z \preceq L, \quad t \succeq 0, \tag{2.4}$$

where L is the maximum value of x_3 on R. Then Edelstein's energetic analysis [4] (see also [5]) shows that

$$U(z,t) \preceq U(0,t) \exp\left\{-\frac{z}{c}\right\}, \tag{2.5}$$

where

$$c = \max\left[1, \frac{1}{2}\left(1 + \frac{1}{2}t^2\right)\right]. \tag{2.6}$$

On the other hand, in [6], Knowles defines

$$V(z,t) = \int_0^t \int_{R_z} u_{,j} u_{,j}\, dV\, ds + \frac{1}{2}\int_{R_z} u^2\, dV, \quad 0 \preceq z \preceq L, \quad t \succeq 0, \tag{2.7}$$

and, provided the boundary condition

$$u = 0 \qquad \text{on} \qquad (\partial R \setminus S_0) \times (0, t_0) \tag{2.8}$$

is considered, he shows that

$$V(z,t) \preceq V(0,t) \exp\{-2kz\}, \tag{2.9}$$

where

$$k = \min_{0 \preceq z \preceq L} \sqrt{\lambda_1(z)}\,, \tag{2.10}$$

and $\lambda_1(z)$ is the smallest eigenvalue of the two-dimensional clamped membrane problem for the intersection S_z of R with the plane $x_3 = z$. Thus the end effects for transient heat conduction decay spatially at least as rapidly as do their counterparts in the steady state case.

Later, by using a technique based on comparison principle arguments involving solutions of the one-dimensional heat equation, HORGAN, PAYNE and WHEELER [11] establish the following decay estimate:

$$\left(\int_{S_z} u^2\, dA\right)^{1/2} \preceq \left[\max_{[0,t]} \left(\int_{S_0} u^2\, dA\right)^{1/2}\right] \frac{2z(t/\pi)^{1/2} \exp(-\lambda_1 t)}{z^2 - 4\lambda_1 t^2} \exp\left(-\frac{z^2}{4t}\right), \tag{2.11}$$

which shows explicitly that, for fixed t and for $z > 2\sqrt{\lambda_1}\, t$, the spatial decay is ultimately controlled by the factor $\exp\{-z^2/4t\}$ rather than by the factor $\exp\{-\sqrt{\lambda_1}\, z\}$ found in the steady state case.

In what follows we shall show that it is possible to establish an estimate like (2.11) for two energetic measures associated with the solution $u(x,t)$ of equations (2.1) to (2.3). For such an estimate the spatial decay is controlled by the factor $\exp\{-z/\sqrt{t/2}\}$.

With a view toward proving this, we first associate with the solution $u(x,t)$ of (2.1) to (2.3) the following (nonnegative) functional

$$T(z,t) = \frac{1}{2}\int_0^t \int_{R_z} \left\{ u^2 + 2\int_0^s u_{,j}u_{,j}\, d\tau \right\} dV\, ds, \qquad 0 \preceq z \preceq L, \quad t \succeq 0. \tag{2.12}$$

Obviously, we have

$$T(z,t) = \int_0^t V(z,s)\, ds, \tag{2.13}$$

where $V(z,s)$ is defined by relation (2.7). It is easy to see that $T(z,t)$ represents the total energy (sum of thermal energy and dissipation energy) stored in the region R_z over the interval of time $[0,t]$.

The derivation of the spatial decay estimate for the energetic measure $T(z,t)$ follows the development of [4–6], with some modifications that allow us to find spatial decay at a rate which is, in general, faster than the exponential rate predicted by (2.9). Moreover, it allows us to consider the more general boundary condition (2.2).

THEOREM 1. *Let u satisfy* (2.1)–(2.3). *Then*

$$T(z,t) \preceq T(0,t)\exp\left\{-\frac{z}{\sqrt{t|2}}\right\}, \qquad 0 \preceq z \preceq L, \quad t \succeq 0. \tag{2.14}$$

P r o o f. First, we note that differentiation of (2.12) with respect to z leads to

$$\frac{\partial T}{\partial z}(z,t) = -\frac{1}{2}\int_0^t \int_{S_z} \left\{ u^2 + 2\int_0^s u_{,j}u_{,j}\, d\tau \right\} dA\, ds. \tag{2.15}$$

Then we multiply the equation (2.1) by u and integrate over $R_z \times [0,t]$. Further, we integrate by parts and invoke (2.2) and (2.3) in order to obtain

$$\frac{1}{2}\int_{R_z} \left\{ u^2 + 2\int_0^t u_{,j}u_{,j}\, ds \right\} dV + \alpha \int_0^t \int_{\partial R_z \setminus S_z} u^2\, dS\, ds = -\int_0^t \int_{S_z} uu_{,3}\, dA\, ds. \tag{2.16}$$

In view of the relation (2.12), from (2.16) we deduce

$$T(z,t) + \alpha \int_0^t \int_0^s \int_{\partial R_z \setminus S_z} u^2 \, dS \, d\tau \, ds = - \int_0^t \int_0^s \int_{S_z} u u_{,3} \, dA \, d\tau ds. \tag{2.17}$$

Since α is a nonnegative constant and by using the arithmetic-geometric mean inequality, from (2.17) we get

$$T(z,t) \preceq \frac{1}{2} \int_0^t \int_0^s \int_{S_z} \left(\varepsilon u^2 + \frac{1}{\varepsilon} u_{,3}^2 \right) dA \, d\tau \, ds, \tag{2.18}$$

where ε is a positive number to be chosen later.

By using an integration by parts, it is easy to see that

$$\int_0^t \int_0^s u^2 \, d\tau \, ds = \int_0^t (t - s) u^2 \, ds, \tag{2.19}$$

so that the inequality (2.18) leads to

$$T(z,t) \preceq \frac{1}{2} \int_0^t \int_{S_z} \left\{ \varepsilon t u^2 + \frac{1}{2\varepsilon} \int_0^s 2 u_{,3}^2 \, d\tau \right\} dA \, ds. \tag{2.20}$$

We may equate the coefficients of the two terms in the integral in (2.20), so that we set

$$\frac{1}{\varepsilon} = \sqrt{2t}, \tag{2.21}$$

and we get

$$T(z,t) \preceq \sqrt{\frac{t}{2}} \frac{1}{2} \int_0^t \int_{S_z} \left\{ u^2 + 2 \int_0^s u_{,j} u_{,j} \, d\tau \right\} dA \, ds. \tag{2.22}$$

An appeal to the relation (2.15) then leads at once to the following first order differential inequality:

$$\sqrt{\frac{t}{2}} \frac{\partial T}{\partial z}(z,t) + T(z,t) \preceq 0. \tag{2.23}$$

For $t > 0$, by an integration in (2.23), we get the relation (2.14), that is a spatial decay estimate like that described by (2.11). Thus, the proof is complete.

THEOREM 2. *Let u be a solution of the initial-boundary value problem defined by the relations* (2.1), (2.3) *and* (2.8). *Then we have*

$$T(z,t) \preceq T(0,t)\exp\{-2kz\}, \tag{2.24}$$

where

$$k = \min_{0\preceq z\preceq L} \sqrt{\lambda_1(z)}, \tag{2.25}$$

and $\lambda_1(z)$ is the smallest eigenvalue of the two-dimensional clamped membrane problem for the cross-section S_z.

P r o o f. In order to deduce the spatial decay estimate (2.24) we restrict our considerations to the boundary condition (2.8), so that we can use the inequality

$$\int_{S_z} u_{,\varrho}u_{,\varrho}\,dA \succeq \lambda_1(z)\int_{S_z} u^2\,dA, \tag{2.26}$$

in the estimate (2.18). Thus, we deduce

$$T(z,t) \preceq \frac{1}{2}\int_0^t\int_0^s\int_{S_z}\left\{\varepsilon\lambda_1^{-1}(z)u_{,\varrho}u_{,\varrho} + \frac{1}{\varepsilon}u_{,3}^2\right\}dA\,d\tau\,ds. \tag{2.27}$$

Choosing now $\varepsilon = \sqrt{\lambda_1(z)}$ and using the relations (2.15) and (2.25), we get the following differential inequality

$$\frac{\partial T}{\partial z}(z,t) + 2kT(z,t) \preceq 0. \tag{2.28}$$

By integration of the inequality (2.28) we obtain the spatial decay estimate (2.24), that is an estimate of the type described in (2.9).

Further, it is seen that the spatial decay estimates (2.14) and (2.24) are not written in an explicit form. In order to solve this question, we need a bound for $T(0,t)$ in terms of the boundary data on the end face S_0. For this reason we complete the initial-boundary value problem (2.1)–(2.3) by specifying the boundary data on S_0 as follows

$$u_{,3}(x_1,x_2,0,t) = u_0(x_1,x_2,t), \qquad (x_1,x_2)\in S_0, \quad t \succeq 0, \tag{2.29}$$

where $u_0(x_1,x_2,t)$ is a prescribed continuous function on $S_0\times[0,t_0)$.

THEOREM 3. *Let u be a solution of the initial-boundary value problem defined by* (2.1), (2.3), (2.29) *and* (2.2) (*or* (2.8)). *Then*

$$T(0,t) \preceq C\max[2t,\ 1]\int_0^t\int_0^s\int_{S_0} u_0^2\,dA\,d\tau\,ds, \tag{2.30}$$

where C is a positive constant depending only on R.

P r o o f. In order to obtain a bound for $T(0,t)$ in terms of the prescribed function $u_0(x_1,x_2,t)$ we note that the relation (2.17) implies

$$T(0,t)+\alpha\int_0^t\int_0^s\int_{\partial R\setminus S_0} u^2\,dS\,d\tau\,ds=-\int_0^t\int_0^s\int_{S_0} uu_0\,dA\,d\tau\,ds. \tag{2.31}$$

An application of Schwarz's inequality to the integral on the right-hand side then yields

$$T(0,t)+\alpha\int_0^t\int_0^s\int_{\partial R\setminus S_0} u^2\,dS\,d\tau\,ds \preceq \left(\int_0^t\int_0^s\int_{S_0} u^2\,dA\,d\tau\,ds\right)^{1/2}\left(\int_0^t\int_0^s\int_{S_0} u_0^2\,dA\,d\tau\,ds\right)^{1/2}. \tag{2.32}$$

Now we use the following inequality [17]

$$\int_{\partial R} u^2\,dS \preceq C\int_R \left(u^2+u_{,j}u_{,j}\right)\,dV, \tag{2.33}$$

where C is a positive constant depending only on R. Thus, we obtain

$$\begin{aligned}\int_0^t\int_0^s\int_{S_0} u^2\,dA\,d\tau\,ds &\preceq C\int_0^t\int_0^s\int_R (u^2+u_{,j}u_{,j})\,dV\,d\tau\,ds\\ &= \frac{C}{2}\int_0^t\int_R\left\{2(t-s)u^2+2\int_0^s u_{,j}u_{,j}\,d\tau\right\}dV\,ds\\ &\preceq \frac{C}{2}\max[2t,\,1]\int_0^t\int_R\left(u^2+2\int_0^s u_{,j}u_{,j}\,d\tau\right)dV\,ds.\end{aligned} \tag{2.34}$$

Clearly, by using the estimate (2.34) in (2.32) and by means of the relation (2.12), we obtain the estimate (2.30) and the proof is complete. Obviously, the estimate (2.30) makes the spatial decay estimates (2.14) and (2.24) more explicit.

In the remainder of this section we establish spatial decay estimates for the following energetic measure associated with the solution $u(x,t)$ of the Eqs. (2.1), (2.3),

$$T^*(z,t)=\int_0^t\int_{R_z}\left\{\frac{1}{2}u_{,j}u_{,j}+\int_0^s \dot{u}^2\,d\tau\right\}dV\,ds, \qquad 0\preceq z\preceq L, \quad t\succeq 0. \tag{2.35}$$

In fact we have

THEOREM 4. *Let us assume that u satisfies Eq.* (2.1), *the initial condition* (2.3) *and the boundary condition*

$$\dot{u}\left(\frac{\partial u}{\partial n}+\alpha u\right)=0 \qquad \text{on} \quad (\partial R\setminus S_0)\times(0,t_0), \tag{2.36}$$

where α is a nonnegative constant. Then

$$T^*(z,t)\preceq T^*(0,t)\exp\left\{-\frac{z}{\sqrt{t|2}}\right\}, \qquad 0\preceq z\preceq L, \quad t\succeq 0. \tag{2.37}$$

P r o o f. We first multiply (2.1) by $\dot{u}$ and then integrate over $R_z\times[0,t]$. Further we use the divergence theorem with the boundary condition (2.36) and the initial condition (2.3) in order to get

$$\int_0^t\int_{R_z}\dot{u}^2\,dV\,ds+\frac{1}{2}\int_{R_z}u_{,j}u_{,j}\,dV+\frac{\alpha}{2}\int_{\partial R_z\setminus S_z}u^2\,dS=-\int_0^t\int_{S_z}\dot{u}\,u_{,3}\,dA\,ds. \tag{2.38}$$

In view of the relation (2.35), (2.38) yields

$$T^*(z,t)+\frac{\alpha}{2}\int_0^t\int_{\partial R_z\setminus S_z}u^2\,dS\,ds=-\int_0^t\int_0^s\int_{S_z}\dot{u}\,u_{,3}\,dA\,d\tau\,ds. \tag{2.39}$$

Since α is a nonnegative constant, by means of the arithmetic-geometric mean inequality and integration by parts with respect to time variable, from (2.39) we deduce

$$T^*(z,t)\preceq\sqrt{\frac{t}{2}}\int_0^t\int_{S_z}\left\{\frac{1}{2}u_{,3}^2+\int_0^s\dot{u}^2\,d\tau\right\}dA\,ds. \tag{2.40}$$

On the other hand, from (2.35) we have

$$\frac{\partial T^*}{\partial z}(z,t)=-\int_0^t\int_{S_z}\left\{\frac{1}{2}u_{,j}u_{,j}+\int_0^s\dot{u}^2\,d\tau\right\}dA\,ds, \tag{2.41}$$

so that the relation (2.40) implies

$$\sqrt{\frac{t}{2}}\frac{\partial T^*}{\partial z}(z,t)+T^*(z,t)\preceq 0. \tag{2.42}$$

An integration of the differential inequality (2.42) leads to the spatial decay estimate (2.37) and the proof is complete.

We close this section by making the spatial decay estimate (2.37) more explicit. In the case of the original boundary condition (2.36), we are unable to treat the whole question posed above. An analysis of the various cases involved in (2.36) is necessary. Here we consider only the case when the input of heat across the boundary surface is prescribed. The other cases involved in (2.36) can be treated in a similar manner.

THEOREM 5. *Let u be a solution of the initial-boundary value problem defined by Eq.* (2.1), *the initial condition* (2.3) *and the boundary conditions*

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad (\partial R \setminus S_0) \times (0, t_0), \tag{2.43}$$

$$u_{,3} = f \qquad \text{on} \quad S_0 \times (0, t_0), \tag{2.44}$$

where $f(x_1, x_2, t)$ is a prescribed continuous function on $S_0 \times [0, t_0)$. Moreover, we assume that $\dot{f}$ is continuous on $S_0 \times (0, t_0)$ and satisfies the "self-equilibrated" condition

$$\int_{S_0} f \, dA = 0, \qquad t \in [0, t_0). \tag{2.45}$$

Then

$$T^*(0,t) \preceq 8C(m+1) \left[\int_0^t \left(\int_0^s \int_{S_0} \dot{f}^2 \, dA \, d\tau \right)^{1/2} ds \right]^2, \tag{2.46}$$

where C is the constant in (2.33) *and m is the constant in the Poincaré inequality.*

P r o o f. In order to establish (2.46), we first notice that integration of the equation (2.1) over R, followed by the application of the divergence theorem with the boundary conditions (2.43), (2.44) and (2.45) and the initial condition (2.3) give

$$\int_R u \, dV = 0. \tag{2.47}$$

On this basis we can use the Poincaré inequality

$$\int_R u^2 \, dV \preceq m \int_R u_{,j} u_{,j} \, dV, \qquad m > 0, \quad m - \text{positive constant}, \tag{2.48}$$

in the inequality (2.33), so that, in view of (2.35) we obtain

$$\int_0^t \int_{S_0} u^2 \, dS \, ds \preceq C(m+1) \int_0^t \int_R u_{,j} u_{,j} \, dV \, ds \preceq 2C(m+1) T^*(0,t). \tag{2.49}$$

Further, from (2.39) and (2.44), we deduce the following identity

$$T^*(0,t) = -\int_0^t \int_0^s \int_{S_0} \dot{u} f \, dA \, d\tau \, ds, \tag{2.50}$$

so that by an integration by parts we get

$$T^*(0,t) = -\int_0^t \int_{S_0} u f \, dA \, ds + \int_0^t \int_0^s \int_{S_0} u \dot{f} \, dA \, d\tau \, ds, \tag{2.51}$$

where use was made of the initial condition (2.3). Then, by means of the Schwarz's inequality we obtain

$$T^*(0,t) \preceq \left(\int_0^t \int_{S_0} u^2 \, dA \, ds \right)^{1/2} \left[\left(\int_0^t \int_{S_0} f^2 \, dA \, ds \right)^{1/2} + \int_0^t \left(\int_0^s \int_{S_0} \dot{f}^2 \, dA \, d\tau \right)^{1/2} ds \right]. \tag{2.52}$$

Due to the compatibility of the boundary condition (2.44) with the initial condition (2.3) it follows that

$$f(x_1, x_2, 0) = 0, \tag{2.53}$$

so that

$$\frac{d}{dt} \left\{ \int_0^t \int_{S_0} f^2 \, dA \, ds \right\} = 2 \int_0^t \int_{S_0} f \dot{f} \, dA \, ds \preceq 2 \left(\int_0^t \int_{S_0} f^2 \, dA \, ds \right)^{1/2} \left(\int_0^t \int_{S_0} \dot{f}^2 \, dA \, ds \right)^{1/2}. \tag{2.54}$$

By an integration we deduce

$$\left(\int_0^t \int_{S_0} f^2 \, dA \, ds\right)^{1/2} \preceq \int_0^t \left(\int_0^s \int_{S_0} \dot{f}^2 \, dA \, d\tau\right) ds. \tag{2.55}$$

Thus, from the relations (2.49), (2.55) and (2.52), we obtain the estimate (2.46) and the proof is complete.

As remarked by KNOWLES [6], it is clear that the estimates (2.14), (2.24) and (2.37) allow us to derive an interior pointwise estimate for functions u satisfying Eqs. (2.1) to (2.3), but we shall not pursue this here.

It is seen from the above analysis that the decay of end effects in the transient problem is indeed faster than that for the steady state case. This conclusion is in accordance with the observations made by BOLEY [3]. Moreover, we observe that the spatial decay estimates (2.14) and (2.37) are useful for appropriate short values of the time variable; while the estimate (2.24) is convenient for large values of the time variable.

At this moment we can compare our spatial decay estimates with those previously established and described by the relations (2.5), (2.9) and (2.11). It can be seen that our results are similar in nature with those described by (2.11).

3. Spatial decay estimates for the nonlinear theory of heat conduction

Our purpose in the present section is to extend the work of the above section to the general nonlinear theory of heat conduction considered by NUNZIATO [8]. Therefore, we shall consider the initial-boundary value problem [8] consisting of the energy equation

$$\dot{e} = -q_{i,i} \qquad \text{on} \quad R \times (0, t_0), \tag{3.1}$$

the constitutive equation

$$q_i = q_i(e, e_{,m}) \qquad \text{on} \quad R \times (0, t_0), \tag{3.2}$$

the boundary condition

$$e = 0 \qquad \text{on} \quad (\partial R \setminus S_0) \times (0, t_0), \tag{3.3}$$

and the initial condition

$$e(x, 0) = 0 \qquad \text{on} \quad R. \tag{3.4}$$

Here e is the internal energy and q_i are the components of the heat flux vector. The second law of thermodynamics requires that

$$q_i(e, e_{,m}) e_{,i} \preceq 0, \qquad q_i(e, 0) = 0. \tag{3.5}$$

Furthermore, we require that the response functions q_i satisfy the following constitutive assumption

$$\int_{S_z} q_3^2 \, dA \preceq -c_1 \int_{S_z} e_{,i} q_i \, dA, \qquad 0 \preceq z \preceq L, \quad t \succeq 0, \tag{3.6}$$

where c_1 is a positive constant. It should be noted that if the material obeys a form of Fourier's law, i.e., $q_i = -K_{ij}(e, e_{,m}) e_{,j}$, then the assumption (3.6) can be satisfied by imposing appropriate boundedness assumptions on K_{ij}. Moreover, it is satisfied by the class of materials studied by NUNZIATO [8].

THEOREM 6. *Let $e(x,t)$ be a solution of the initial-boundary value problem defined by* (3.1)–(3.4) *and assume that the heat flux satisfies the constitutive assumption* (3.6). *Then*

$$K(z,t) \preceq K(0,t) \exp\left\{-\sqrt{\frac{2}{c_1 t}}\, z\right\}, \qquad 0 \preceq z \preceq L, \quad t \succeq 0, \tag{3.7}$$

where

$$K(z,t) = \int_0^t \int_{R_z} \left\{ \frac{1}{2} e^2 - \int_0^s e_{,i} q_i \, d\tau \right\} dV \, ds. \tag{3.8}$$

P r o o f. Multiplying (3.1) by e and integrating over $R_z \times [0,t]$ we have

$$\frac{1}{2} \int_{R_z} e^2 \, dV = -\int_0^t \int_{R_z} q_{i,i} e \, dV \, ds, \tag{3.9}$$

where use was made of the initial condition (3.4). We then integrate the right-hand side by parts, use the boundary condition (3.3) and make use of (3.8) to obtain

$$K(z,t) = \int_0^t \int_0^s \int_{S_z} e q_3 \, dA \, d\tau \, ds. \tag{3.10}$$

On the basis of the Schwarz's inequality, from (3.10) we get

$$K(z,t) \preceq (2t)^{1/2} \left(\int_0^t \int_{S_z} \frac{1}{2} e^2 \, dA \, ds \right)^{1/2} \left(\int_0^t \int_0^s \int_{S_z} q_3^2 \, dA \, d\tau \, ds \right)^{1/2}. \tag{3.11}$$

Since

$$\frac{\partial K}{\partial z}(z,t) = -\int_0^t \int_{S_z} \left\{ \frac{1}{2} e^2 - \int_0^s e_{,i} q_i \, d\tau \right\} dA \, ds, \tag{3.12}$$

and by using the constitutive assumption (3.6), from (3.11) it follows

$$\sqrt{\frac{c_1 t}{2}}\,\frac{\partial K}{\partial z}(z,t) + K(z,t) \preceq 0, \tag{3.13}$$

so that by integration we obtain the estimate (3.7). Thus, the proof is complete.

It can be seen from (3.7) that the decay of end effects in the nonlinear theory of heat conduction is faster than that predicted by NUNZIATO [8].

REMARK. Let $e(x,t)$ be a solution of the initial-boundary value problem defined by (3.1)–(3.4) and assume now that the heat flux satisfies only the thermodynamic consequence (3.5). Then, for fixed $t > 0$, we have

$$K(z,t) \preceq \frac{K(0,t)}{1 + \dfrac{1}{2t^2}K(0,t)\displaystyle\int_0^z Q^{-1}(\zeta,t)d\zeta}, \qquad z \in (0, z_t), \tag{3.14}$$

and $e(x,t) = 0$ for $z \in [z_t, L]$, where z_t is the lowest value of $z \in [0, L]$ for which the integral

$$Q(z,t) = \int_0^t \int_{S_z} q_3^2\, dA\, ds \tag{3.15}$$

vanishes (eventually).

In order to prove this, we note that the inequality (3.5) when it is used in (3.11) and (3.12), allows us to obtain

$$2t^2 Q(z,t)\frac{\partial K}{\partial z}(z,t) + K^2(z,t) \preceq 0. \tag{3.16}$$

On the other hand, we notice that the relations (3.5) and (3.12) prove that the function $K(z,t)$ is nonincreasing with respect to z on $[0, L]$. Therefore, if there exists $z_t \in [0, L]$ so that $Q(z_t, t) = 0$, then from (3.16) it follows that $K(z,t) = 0$ on $[z_t, L]$, that is $e(x,t) = 0$ on $[z_t, L]$. Further, an integration of the differential inequality (3.16) leads to the spatial decay estimate (3.14).

Finally, we remark that the estimate (3.14) is established under mild constitutive assumptions. For this reason it does not give so explicit information about the decay rate of the end effects and, in order to know this, we need more information about the behaviour of $Q(z,t)$.

4. Saint–Venant's principle for the wave equation

In this section we shall establish spatial decay estimates appropriate to hyperbolic equations.

Let $u(x,t) \equiv u(x_1, x_2, x_3, t)$ satisfy the wave equation

$$u_{,jj} - \ddot{u} = 0 \qquad \text{on} \quad R \times (0, t_0), \tag{4.1}$$

the boundary condition

$$\dot{u}\left(\frac{\partial u}{\partial n} + \alpha u\right) = 0 \qquad \text{on} \quad (\partial R \setminus S_0) \times (0, t_0), \tag{4.2}$$

and the initial conditions

$$u(x,0) = 0, \qquad \dot{u}(x,0) = 0 \qquad \text{for} \quad x \in R, \tag{4.3}$$

where α is a given nonnegative constant.

To the function $u(x,t)$, solution of the initial-boundary value problem (4.1)–(4.3), we associate the following (nonnegative) functional $E(z,t)$ defined on $[0, L] \times [0, t_0)$ by

$$E(z,t) = \frac{1}{2}\int_0^t \int_{R_z} \left\{ \dot{u}^2 + u_{,j} u_{,j} \right\} dV\, ds, \qquad 0 \preceq z \preceq L, \quad t \succeq 0. \tag{4.4}$$

Obviously, $E(z,t)$ represents the sum of kinetic and strain energies stored in the portion R_z of R over the time-interval $[0,t]$.

Clearly, from (4.4) we get

$$\frac{\partial E}{\partial z}(z,t) = -\frac{1}{2}\int_0^t \int_{S_z} \left\{ \dot{u}^2 + u_{,j} u_{,j} \right\} dA\, ds. \tag{4.5}$$

We are now in a position to state and prove our main result concerning the wave equation (4.1).

THEOREM 7. *Let $u(x,t)$ be a solution of the initial-boundary value problem defined by* (4.1)–(4.3). *Then*

$$E(z,t) = 0, \qquad \text{for} \quad t < z \preceq L, \tag{4.6}$$

$$E(z,t) \preceq E(0,t) \exp\left\{-\frac{z}{t}\right\}, \qquad 0 \preceq z < t. \tag{4.7}$$

P r o o f. We multiply Eq. (4.1) by $\dot{u}$ and integrate over $R_z \times [0,t]$. Further, we integrate by parts and use the boundary conditions (4.2) and the initial conditions (4.3) in order to obtain

$$\frac{1}{2}\int_{R_z} \left\{ \dot{u}^2 + u_{,j} u_{,j} \right\} dV + \frac{\alpha}{2}\int_{\partial R_z \setminus S_z} u^2\, dS = -\int_0^t \int_{S_z} \dot{u}\, u_{,3}\, dA\, ds. \tag{4.8}$$

Thus, from relations (4.4) and (4.8), we get

$$E(z,t) + \frac{\alpha}{2}\int_0^t \int_{\partial R_z \setminus S_z} u^2\, dS\, ds = -\int_0^t \int_0^s \int_{S_z} \dot{u}\, u_{,3}\, dA\, d\tau\, ds. \tag{4.9}$$

Since α is a nonnegative constant and by using the arithmetic-geometric mean inequality, from (4.9) we deduce

$$\frac{\partial}{\partial t} E(z,t) \preceq \frac{1}{2}\int_0^t \int_{S_z} \left\{ \dot{u}^2 + u_{,3}^2 \right\} dA\, ds. \tag{4.10}$$

Clearly then the expressions (4.5) and (4.10) yield

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} \preceq 0. \tag{4.11}$$

On integrating (4.11) along the characteristic line $z = t$ in the (z,t) plane through $(0,0)$, we find that at $z = t \in [0, L]$ we have

$$E(t,t) \preceq E(0,0). \tag{4.12}$$

On the basis of the initial conditions (4.3) we observe that $E(0,0) = 0$. Moreover, since $E(z,t)$ is a nonincreasing function of z, we have

$$E(z,t) \preceq E(t,t), \qquad \text{for} \quad z \succeq t. \tag{4.13}$$

Thus, from (4.12) and (4.13), we deduce the relation (4.6).

Let us now suppose that $0 \leq z < t$. Then, by means of the arithmetic-geometric mean inequality and since α is a nonnegative constant, from (4.9) we get

$$E(z,t) \preceq \frac{1}{2}\int_0^t \int_0^s \int_{S_z} \{ \dot{u}^2 + u_{,3}^2 \} dA\, d\tau\, ds. \tag{4.14}$$

By an integration by parts we obtain

$$\frac{1}{2}\int_0^t \int_0^s \{ \dot{u}^2 + u_{,3}^2 \} d\tau\, ds = \frac{1}{2}\int_0^t (t-s)\{ \dot{u}^2 + u_{,3}^2 \} ds \preceq \frac{t}{2}\int_0^t \{ \dot{u}^2 + u_{,3}^2 \} ds. \tag{4.15}$$

If we use the inequality (4.15) into (4.14) we get

$$E(z,t) \preceq \frac{t}{2}\int_0^t \int_{S_z} \{ \dot{u}^2 + u_{,j} u_{,j} \} dA\, ds. \tag{4.16}$$

In view of the relation (4.5), the inequality (4.16) yields

$$t\frac{\partial E}{\partial z}(z,t) + E(z,t) \preceq 0. \tag{4.17}$$

For $0 < t < z$, an integration of the first order differential inequality (4.17) leads to the relation (4.7) and the proof is complete.

The above spatial decay estimate is genuinely dynamical. We don't know other spatial decay estimate for the wave equation in order to compare it with the above decay estimate. However, we remark that for short values of the time variable, the decay rate of the end effects in the wave equation problem is very fast. It is then attenuated in time.

On the other hand, by comparing the spatial decay estimates described by the Theorems 1 and 7, it is seen that, for appropriately short values of the time variable, the spatial decay of end effects in the wave equation problem is faster than that for the transient heat conduction.

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Received November 28, 1994.

Arch. Mech., 47, 4, pp. 773–791, Warszawa 1995

Microvoids formed inside the grains and at two, three and four-grain junctions

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NEW, MORE REALISTIC MODEL of metal and the additional possibility of nucleation of microvoids in the vicinity of four-grain junctions have been introduced. Beside the earlier microfracture criterion, extended by including the corner nucleation, an alternative criterion based on the non-equilibrium density of stable microvoids is presented.

1. Introduction

IN INCLUSION – free metals at least four types of microvoids are possible. Spherical shapes may occur inside the grains, whereas the remaining configurations appear at the grain boundaries and depend on the kind of grain junction. In developing the vacancy mechanism hypothesis of microfracture of metals the first three types of microvoids were already investigated [1–5]. It is the intention of the present paper to further extend this theory by introducing a spatial and more realistic model of polycrystalline metals, and supplementing the hitherto examined shapes of microvoids by the fourth one which may be nucleated at the four-grain junctions. Consequently, the previously proposed microfracture criterion is extended due to additional corner nucleation. Besides, application of the unified procedure permits to describe in a unique way all types of microvoids, and allows to obtain more transparent results. Moreover, an alternative criterion, based on the non-equilibrium distribution of stable microvoids, is presented.

2. The model of polycrystalline metals

A possibly consistent and convenient model of a polycrystalline structure should satisfy some topological and packing requirements [6–9]. Consequently, our model will be constructed under the following assumptions:

i. Only one type of polyhedron is used for the grains, creating the basic cell in an ordered array of identical grains.

ii. The space is filled by repetition of this typical average polyhedron without overlapping and gaps.

iii. The equilibrium of surface tensions, assumed to be isotropic, should be satisfied everywhere.

iv. The stack of such grains must conform to all physically feasible types of intergranular microvoids so that they may easily be nucleated at all possible grain junctions (cf. Sec. 3, Fig. 5).

v. The penetrability of vacancies across the boundaries of ignorable thickness is allowed.

vi. The other assumptions concerning the model of polycrystalline metals and the process considered are not changed [1, 5].

If the further restriction is made, that the side polygons have not more than ten edges, there are only 18 uniform polyhedra [6] which might be suitable to our purpose. The first five are known as regular or Platonic solids which are composed of one kind of plane, regular polygons. The next thirteen are semi-regular or Archimedean solids, made up of not more than three regular polygons. Unfortunately, no one of them fulfils exactly the first four aforesaid conditions. However, the closest to them is a regular octahedron truncated by a cube, and possessing fourteen faces (a cuboctahedron). Nearly all the above mentioned geometrical, topological, and physical limitations are fulfilled by this 14-sided body (tetrakaidecahedron) [7]. It will be adopted and used in this paper. It has 6 square and 8 hexagonal faces, 36 edges of equal length s, and 24 three-rayed vertices (in a stack they are four-rayed, Fig. 5); the boundary area $S_g = 3(1 + 2\sqrt{3})\delta^2/4$, the volume $V_g = \delta^3/2$, $s = \delta/2\sqrt{2}$, where δ is the distance between square faces (a "diameter" of the grain, Fig. 1). The dihedral angles in this uniform polyhedron are: between square faces and hexagonal faces 125°15′51.80″, between hexagonal faces 109°28′16.39″.

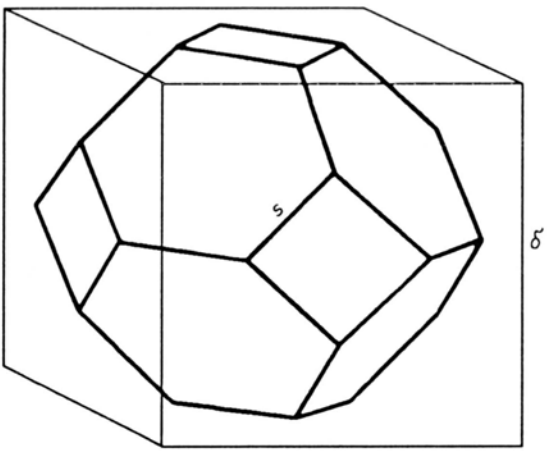

FIG. 1. Tetrakaidecahedron in the form of a truncated octahedron (cuboctahedron).

In three dimensions the topological requirements and those of surface-tension equilibrium can be satisfied when three surfaces meet each other along the edges, at angles of 120°, and four edges meet themselves in corners mutually at the angle

of 109°28′16.39″. This is the angle formed by straight lines joining the corners of a regular tetrahedron with a point equidistant from them all (center of gravity). However, in the space filled up by duplication of our adopted type of polyhedron (on a body-centered cubic packing), three neighbouring grains meet each other along the edges at angles slightly different from 120°. Moreover, the six polygons which meet at each corner, in a stack of uniform-sized tetrakaidecahedra (Fig. 5), consist of four hexagonal faces with angles of 120° and two square faces of 90°, so that the last two conditions are not fulfilled. These are the only accepted departures from the aforesaid restrictions in our model of polycrystalline metals.

3. Intergranular microvoid geometries

3.1. General remarks

Although heterogeneous nucleation of intergranular microvoids from vacancies follows the same coupled mechanisms everywhere at the grain boundaries, the voids can be of different shapes depending on the kind of grain junction. In this section we will consider the geometries of three types of microvoids (vacancy clusters) which may be formed at simple grain boundaries (two-grain junctions), three-grain junctions, and four-grain junctions. It is worth noting here that a five-grain corner can occur only as an unstable configuration.

It is assumed that all intergranular clusters have spherical surfaces of a radius r and the same dihedral angle θ, formed at each point of junction of the cluster and the grain boundary, as if they were determined by the minimum of the surface energy and the equilibrium of interface tensions. Consequently, the shape of these clusters (Fig. 2–4) is described by the radius r, the contact angle θ, and the kind of grain junction.

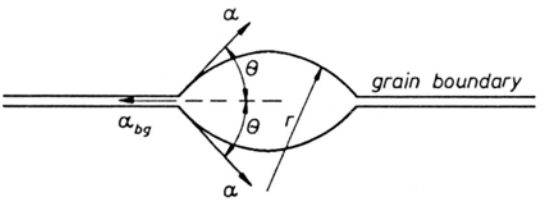

FIG. 2. Double-lens microvoid formed at two-grain junction with corresponding surface tension forces α, α_{bg}, and the contact angle θ.

Most of the geometrical relations for various types of clusters are given in Refs. [10, 11]. In the case of four-grain junctions they have, however, some essential errors.

Below, all the common features of the three considered microvoids are presented.

The contact angle θ is determined by the equilibrium between isotropic surface tension forces at the edge of each cluster (Fig. 2). Thus, we have for double-lens

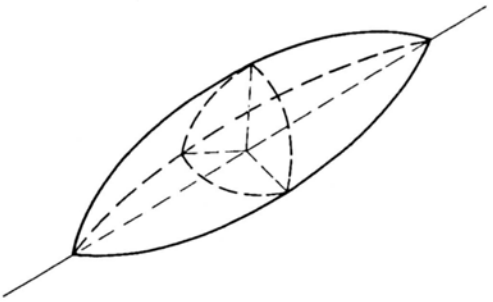

FIG. 3. Regular spherical trihedron microvoid formed at three-grain junction.

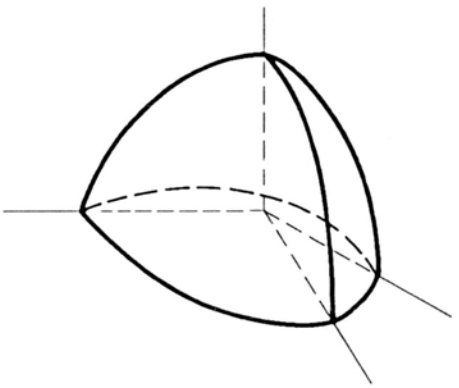

FIG. 4. Regular spherical tetrahedron microvoid formed at four-grain junction.

aggregate and for all other shapes

$$\alpha_{bg} = 2\alpha \cos\theta , \tag{3.1}$$

where α_{bg} is the specific free energy of the grain boundary (in pure metals $\alpha_{bg} \approx \alpha/2$ and $\theta \approx 75°$ [11]), α is the average surface tension of the crystal.

The grain boundary area B_M that is eliminated by a microvoid is

$$B_M = a_M(\theta) r^2 , \tag{3.2}$$

where the subscript M shows the type of the microvoid and takes on the Roman numerals II, III, IV, respectively.

It should be mentioned that vanishing of that part of the grain boundary which is contained within a microvoid decreases the grain boundary surface energy, thereby reducing the critical formation energy necessary for nucleation of the microcrack. The more grain boundary area is replaced by an originating microvoid, the smaller is its formation energy. In the case of double-lens cluster (Fig. 2) a part of one boundary is only eliminated, whereas spherical trihedron (Fig. 3) eliminates some parts of three adjacent boundaries and spherical tetrahedron cluster (Fig. 4) removes parts of six adjoining boundaries.

The surface area of the cluster S_M and its volume V_M are

$$S_M = b_M(\theta) r^2 , \tag{3.3}$$

$$V_M = c_M(\theta) r^3 . \tag{3.4}$$

The coefficients a_M, b_M, and c_M are connected by the relation

$$b_M - 2\kappa a_M = 3c_M , \tag{3.5}$$

where

$$\kappa = \cos\theta = \frac{\alpha_{bg}}{2\alpha} . \tag{3.6}$$

The length of the line of intersection of the cluster and the grain boundaries (the periphery of the cluster) reads

$$l_M = d_M(\theta) r . \tag{3.7}$$

Using the relation

$$i = \frac{V_M}{V_v} = \frac{c_M(\theta)}{V_v} r^3 , \tag{3.8}$$

where V_v is the volume of a vacancy, and i is the number of vacancies in the cluster, we can express the radius of the cluster surface as a function of the variable i

$$r = \left[\frac{V_v}{c_M(\theta)}\right]^{1/3} i^{1/3} , \tag{3.9}$$

so that B_M, S_M, V_M, and l_M become the functions of θ and i.

The nearly perfect conformability of the assumed basic grain shape with all the considered types of microvoids is illustrated in Fig. 5.

Apart from the above geometrical properties of the microvoids, there is another important factor affecting the process of their nucleation. This is the so-called nucleation zone describing the regions in which the intergranular microvoids may be formed. The surface area of the nucleation zone is z_M. At the two-grain junction nearly the whole grain boundary surface (shared with two grains) is accessible for nucleation of the double-lens vacancy clusters. Consequently, for two-grain junctions $z_{II} = S_g/2$ [1]. In case of three-grain junctions z_{III} determines the vicinities of each intersection line of the three grain boundaries in which spherical trihedron aggregates may be generated. For four-grain junctions z_{IV} gives the zones of nucleation of spherical tetrahedron clusters in the neighbourhood of each corner. The corresponding zones are given in the next subsections.

[1] The factor 1/2 was not taken into account in [1–5].

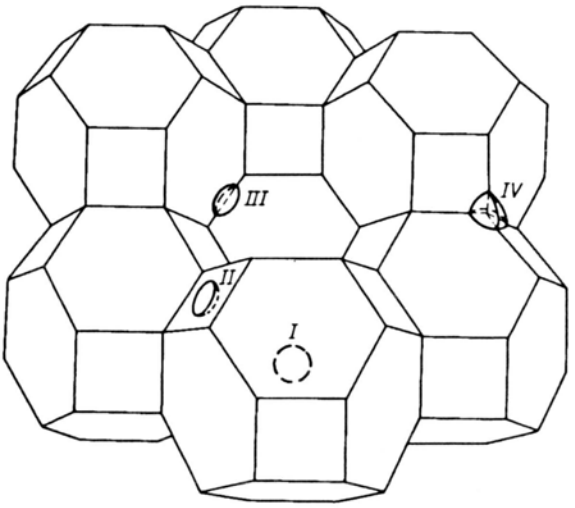

FIG. 5. The stacking of tetrakaidecahedra with four types of microvoids.

3.2. Two-grain junctions (symmetrical double-lens microvoid)

The area, volume and peripheral coefficients for double-lens cluster (Fig. 2) are

$$a_{II}(\theta) = \pi(1 - \kappa^2), \tag{3.10}$$

$$b_{II}(\theta) = 4\pi(1 - \kappa), \tag{3.11}$$

$$c_{II}(\theta) = \frac{2\pi}{3}(2 - 3\kappa + \kappa^3), \tag{3.12}$$

$$d_{II}(\theta) = 2\pi\sqrt{1 - \kappa^2}, \tag{3.13}$$

where

$$0 \leq \theta < 90^\circ, \qquad 1 \geq \kappa > 0. \tag{3.14}$$

The surface area of the nucleation zone is

$$z_{II} = \frac{S_g}{2}. \tag{3.15}$$

3.3. Three-grain junctions (regular spherical trihedron microvoid)

To satisfy the equilibrium between interfacial tensions, the newly originated microvoid at a three-grain junction should be bounded by three identical spherical surfaces forming a regular spherical trihedron (Fig. 3). Therefore, the geometrical coefficients are

$$a_{III}(\theta) = 3\beta(1 - \kappa^2) - \kappa\sqrt{3 - 4\kappa^2}, \tag{3.16}$$

$$b_{III}(\theta) = 12\left(\frac{\pi}{2} - \gamma - \kappa\beta\right), \tag{3.17}$$

$$c_{III}(\theta) = 2\left[\pi - 2\gamma + \frac{\kappa^2}{3}\sqrt{3 - 4\kappa^2} - \beta\kappa(3 - \kappa^2)\right], \tag{3.18}$$

$$d_{III}(\theta) = 6\beta\sqrt{1 - \kappa^2}, \tag{3.19}$$

where

$$\beta = \arccos\frac{\kappa}{\sqrt{3(1 - \kappa^2)}}, \tag{3.20}$$

$$\gamma = \arcsin\frac{1}{2\sqrt{1 - \kappa^2}}, \tag{3.21}$$

and

$$30° < \theta < 90°, \qquad 0.866 > \kappa > 0. \tag{3.22}$$

The nucleation zone is given by the size of the cluster [5], and by the length and number of edges in the grain. Since each edge is shared with three grains (2), we obtain

$$z_{III}(i) = 6\sqrt{6}\,\delta\left(\frac{V_v}{c_{III}}\right)^{1/3}\cos\left(\frac{2}{3}\pi - \theta\right) i^{1/3}. \tag{3.23}$$

3.4. Four-grain junctions (regular spherical tetrahedron microvoid)

The equilibrium conditions at the edges of the nucleating microvoid, at any of the four-grain junctions, demand that the microvoid should be bounded by four identical spherical surfaces. It is a regular spherical tetrahedron (Fig. 4) for which the area, volume, and peripheral coefficients are

$$a_{IV}(\theta) = 3\left[2\varphi(1 - \kappa^2) - \frac{\kappa}{3}(2\sqrt{3 - 4\kappa^2} - \kappa\sqrt{2})\right], \tag{3.24}$$

$$b_{IV}(\theta) = 24\left(\frac{\pi}{3} - \kappa\varphi - \gamma\right), \tag{3.25}$$

$$c_{IV}(\theta) = 2\left[4\left(\frac{\pi}{3} - \gamma\right) + \frac{\kappa^2}{3}(2\sqrt{3 - 4\kappa^2} - \kappa\sqrt{2}) - 2\kappa\varphi(3 - \kappa^2)\right], \tag{3.26}$$

$$d_{IV}(\theta) = 12\varphi\sqrt{1 - \kappa^2}, \tag{3.27}$$

where

$$\varphi = \arcsin\frac{\sqrt{2}\sqrt{3 - 4\kappa^2} - \kappa}{3\sqrt{1 - \kappa^2}}, \tag{3.28}$$

$$\gamma = \arcsin\frac{1}{2\sqrt{1 - \kappa^2}}, \tag{3.29}$$

(2) In (3.23) the factor 1/3 was introduced; it was not taken into account in [4, 5].

and

$$35°15'52'' < \theta < 90°, \qquad \left(\frac{2}{3}\right)^{1/2} = 0.8165 > \kappa > 0. \tag{3.30}$$

It is worth noting that the coefficients a_{IV} and c_{IV}, given in [10] and [11], are erroneously calculated.

The nucleation zone is determined here by the grain boundary area that is eliminated by spherical tetrahedron microvoid, and by the number of corners in the grain. Since each corner belongs to four grains, we have

$$z_{IV}(i) = \frac{1}{4} 24 a_{IV}(\theta) r^2 = 6 a_{IV} \left(\frac{V_v}{c_{IV}}\right)^{2/3} i^{2/3}. \tag{3.31}$$

4. The system and governing relations in the clustering process

We will consider a metal sample subject to tension and kept in a uniaxial state of strain. We further restrict our investigation to an internal region of the sample containing a grain with its neighbourhood. Such a domain, together with a set of all single vacancies, dispersed inside the grain and adsorbed on the grain boundary (and called advacancies), as well as the remaining sets of all kinds of vacancy clusters – presents our system. The assumptions regarding the advancing nucleation process are given in detail in Refs. [1, 5].

Below, the most important relations of the vacancy theory [1, 5] are shortly reminded.

The reduction of the activation energy of the spatial vacancy motion in the direction of the sample tension, caused by the linear strain ε

$$\Delta U(\varepsilon) = a R_0^{-m} \left\{1 - (1+\varepsilon)^{-m} - \frac{m}{n}[1 - (1+\varepsilon)^{-n}]\right\}, \tag{4.1}$$

where a, m, n are material constants, R_0 is the lattice constant.

The supersaturation of vacancies in the grain

$$S_{ho} = \frac{c}{c_0} = \frac{\mathcal{N} \exp\left[-\frac{(U_f - \Delta U)}{kT}\right]}{\mathcal{N} \exp\left(-\frac{U_f}{kT}\right)} = \exp\left(\frac{\Delta U}{kT}\right), \tag{4.2}$$

where c and c_0 are the actual ($\varepsilon > 0$) and initial ($\varepsilon = 0$) volume concentrations of vacancies, $\mathcal{N}$ is the total number of atomic sites per unit volume, U_f is the activation energy of a vacancy formation, k is Boltzman's constant, T is the absolute temperature.

The mean number of vacancies per unit time condensing on a unit area inside the grain

$$P_c = p_g \nu R_0 c_0 \exp\left[-\frac{(U_m - 2\Delta U)}{kT}\right], \tag{4.3}$$

where p_g is the probability that a vacancy will jump towards a cluster, ν is the atomic vibrational frequency ($\approx 10^{13}\mathrm{s}^{-1}$), U_m is the activation energy of vacancy motion.

The surface supersaturation of advacancies in the grain boundary

$$S_{he} = \frac{N(i=1)}{N^0(i=1)} = \frac{n}{n_0} = S_{ho} H(\varepsilon), \tag{4.4}$$

where $N^0(i)$ and $N(i)$ are equilibrium and non-equilibrium surface densities of intergranular aggregates of size i, respectively, and

$$H(\varepsilon) = \frac{\exp\left(-\frac{E_a}{kT}\right)(\mathcal{N} - c_0) + \frac{p_g \mathcal{N} R_0}{N_0} c_0}{\exp\left(-\frac{E_a}{kT}\right)(\mathcal{N} - c) + \frac{p_g \mathcal{N} R_0}{N_0} c}. \tag{4.5}$$

Here, E_a is the heat of adsorption, N_0 is the density of discrete adsorption sites per unit area of the grain boundary ($N_0 \approx 1/R_0^2 \approx 10^{15}\mathrm{cm}^{-2}$).

It should be noticed that the surface densities of aggregates are supplied further on with superscript M (showing the type of aggregates), and with two subscripts (indicating the appropriate nucleation mechanism).

The impingement rate of advacancies on a line of unit length lying on the grain boundary

$$\omega_c = \frac{p_b p_g \nu R_0^2 \mathcal{N} c_0}{\exp\left(-\frac{E_a}{kT}\right)(\mathcal{N} - c) + \frac{p_g \mathcal{N} R_0}{N_0} c} \exp\left(-\frac{U_{sd} - \Delta U)}{kT}\right), \tag{4.6}$$

where p_b is the probability that an advacancy will jump in the desirable direction, U_{sd} is the activation energy for surface autodiffusion.

5. Homogeneous microvoid nucleation in the grains

Within the grains spherical vacancy aggregations can be formed. Four most important nucleation characteristics appearing in homogeneous nucleation are recalled below [1].

Maximum of the size-dependent part of the free energy of formation of a spherical i-mer from isolated vacancies, reached at the critical aggregate ($i = i_g^*$) with equal probability of growth or disintegration

$$\Delta F_g^* = \Delta F_g(i_g^*) = \frac{4\pi\alpha^3 V_v^2}{3(\Delta U)^2}. \tag{5.1}$$

The number of vacancies in the critical aggregate

$$i_g^* = \frac{4\pi\alpha^3 V_v^2}{3(\Delta U)^3}. \tag{5.2}$$

The homogeneous nucleation rate (number of microvoid nuclei generated per unit volume and unit time inside the grain)

$$J_g = 2\mathcal{N} S_{ho} P_c V_v \left(\frac{\alpha}{kT}\right)^{1/2} \exp\left(-\frac{\Delta F_g^*}{kT}\right). \tag{5.3}$$

The non-equilibrium volume density of critical clusters in a steady state

$$A_g^* = A(i_g^*) = \frac{1}{2}\mathcal{N} S_{ho} \exp\left(-\frac{\Delta F_g^*}{kT}\right). \tag{5.4}$$

6. Heterogeneous nucleation by the isolated surface mechanism

In this section we consider a segment of the grain boundary perpendicular to the sample strain, and we assume that the addition of vacancies from the grains to the surface of the clusters is the dominant mechanism of growth.

The equilibrium surface distribution of aggregate sizes $N_{bg}^{Mo}(i)$ on the grain boundary has the same form for all types of aggregates

$$N_{bg}^{Mo}(i) = n_0 \exp\left[-\frac{\Delta F_{bg}^M(i) - kT \ln \frac{N_0}{n_0}}{kT}\right] = N_0 \exp\left[-\frac{\Delta F_{bg}^M(i)}{kT}\right], \tag{6.1}$$

where $-kT \ln(N_0/n_0)$ is the mixing entropy term, and $\Delta F_{bg}^M(i)$ is a size-dependent part of the free energy of formation of a cluster of i vacancies which is postulated in the form

$$\Delta F_{bg}^M(i) = W_M(i) - \frac{V_M(i)}{V_v}\left[kT \ln \frac{c}{c_0} + \Delta U(\varepsilon)\right], \qquad i > 1. \tag{6.2}$$

Here, $W_M(i)$ is the work spent in forming the surface of the cluster, and the second term represents the work gained in forming the new volume in the presence of strain [12].

Taking advantage of the geometric properties of the microvoids (3.3), (3.2), (3.1), (3.5), (3.9) we obtain

$$W_M(i) = S_M(i)\alpha - B_M(i)\alpha_{bg} = 3V_v^{2/3}c_M^{1/3}\alpha i^{2/3}. \tag{6.3}$$

Consequently, the function $\Delta F_{bg}^M(i)$ takes the form

$$\Delta F_{bg}^M(i) = \begin{cases} 3V_v^{2/3}c_M^{1/3}\alpha i^{2/3} - i[kT\ln S_{ho} + \Delta U(\varepsilon)] & \text{for } i > 1, \\ kT\ln\dfrac{N_0}{n_0} & \text{for } i = 1, \end{cases} \tag{6.4}$$

reaching a maximum

$$\Delta F_{bg}^{M*} = \Delta F_{bg}^M(i_{bg}^{M*}) = \frac{\alpha^3 V_v^2}{(\Delta U)^2}c_M \tag{6.5}$$

at the critical aggregate of i_{bg}^{M*} vacancies

$$i_{bg}^{M*} = \frac{\alpha^3 V_v^2}{(\Delta U)^3}c_M. \tag{6.6}$$

The surface nucleation rate equation of any kind of microvoids can be written in the form

$$J_{bg}^M(i,t) = N_{bg}^M(i-1,t)P_c S_M(i-1) - N_{bg}^M(i,t)P_e(i)S_M(i), \qquad i > 1, \tag{6.7}$$

where $N_{bg}^M(i,t)$ is the non-equilibrium surface density of heterogeneous aggregates of the M-type and of size i at a time t, $P_e(i)$ is the emission rate of vacancies per unit area of the surface $S_M(i)$ of an i-mer.

Following the procedure shown in [1, 2] one obtains

$$J_{bg}^M(i,t) = -P_c S_M(i)\left[\frac{N_{bg}^M(i,t)}{kT}\frac{\partial\Delta F_{bg}^M(i)}{\partial i} + \frac{\partial N_{bg}^M(i,t)}{\partial i}\right]. \tag{6.8}$$

Substituting (6.8) into the continuity equation and solving it under a steady state condition ($\partial N_{bg}^M(i,t)/\partial t = 0$), and approximating the integrals appearing in the solution, we attain ultimately to the isolated heterogeneous surface-growth nucleation rate of intergranular microvoids

$$J_{bg}^N = (3\pi)^{-1/2}N_0 S_{he}P_c V_v\left(\frac{\alpha}{kTc_M}\right)^{1/2} b_M \exp\left(-\frac{\Delta F_{bg}^{M*}}{kT}\right), \tag{6.9}$$

and the non-equilibrium surface density of critical clusters

$$N_{bg}^{M*} = N_{bg}^M(i_{bg}^{M*}) = \frac{1}{2}N_0 S_{he}\exp\left(-\frac{\Delta F_{bg}^{M*}}{kT}\right). \tag{6.10}$$

7. Heterogeneous nucleation by the isolated peripheral mechanism

We consider again the same segment of the grain boundary and we assume now that the attachment of advacancies to the periphery of each cluster is the only governing mechanism of growth.

The equilibrium distribution of all kinds of vacancy clusters has the usual form

$$N_{bs}^{Mo}(i) = N_0 \exp\left[-\frac{\Delta F_{bs}^M(i)}{kT}\right], \tag{7.1}$$

where the size part $\Delta F_{bs}^M(i)$ of the total formation energy of a cluster differs from Eqs. (6.2), (6.4) in the second term only, in which S_{ho} should be replaced by S_{he}.

Similarly as ΔF_g and ΔF_{bg}^M, the function ΔF_{bs}^M exhibits a maximum for the critical cluster size i_{bs}^{M*}

$$\Delta F_{bs}^{M*} = \Delta F_{bs}^M(i_{bs}^{M*}) = \frac{4\alpha^3 V_v^2}{(2\Delta U + kT \ln II)^2} c_M, \tag{7.2}$$

$$i_{bs}^{M*} = \frac{8\alpha^3 V_v^2}{(2\Delta U + kT \ln II)^3} c_M. \tag{7.3}$$

The growth mechanism of all clusters is confined here to the surface diffusion of advacancies. Consequently, the peripheral nucleation rate equation reads

$$J_{bs}^M(i,t) = N_{bs}^M(i-1,t)\omega_c l_M(i-1) - N_{bs}^M(i,t)\omega_e(i) l_M(i), \qquad i > 1, \tag{7.4}$$

where $\omega_e(i)$ is the emission rate of advacancies from a unit length of the periphery $l_M(i)$ of an i-cluster.

Similar approach as in Sec. 6 (cf. [1, 2]) yields the flux equation in the form

$$J_{bs}^M(i,t) = -\omega_c l_M(i)\left[\frac{N_{bs}^M(i,t)}{kT}\frac{\partial \Delta F_{bs}^M(i)}{\partial i} + \frac{\partial N_{bs}^M(i,t)}{\partial i}\right], \tag{7.5}$$

and after solving the equation of continuity in a steady state, the peripheral-growth nucleation rate of all microcracks assumes finally the form

$$J_{bs}^M = \frac{1}{2}(3\pi)^{-1/2} N_0 S_{he} \omega_c (2\Delta U + kT \ln II)(\alpha kT c_M)^{-1/2} d_M \times \exp\left(-\frac{\Delta F_{bs}^{M*}}{kT}\right). \tag{7.6}$$

The non-equilibrium density of critical clusters is also obtained

$$N_{bs}^{M*} = N_{bs}^M(i_{bs}^{M*}) = \frac{1}{2} N_0 S_{he} \exp\left(-\frac{\Delta F_{bs}^{M*}}{kT}\right). \tag{7.7}$$

8. Coupled heterogeneous nucleation

We assume now that both the surface and peripheral mechanisms of growth are coupled in the process of nucleation. However, in this case we do not know *a priori* the combined free energy of formation of clusters and we must, therefore, start at once from the kinetics writing the coupled heterogeneous nucleation rate equation in the form (cf. [13])

$$
\begin{aligned}
(8.1)\qquad J_{gs}^{M}(i,t) &= \hat{J}_{bg}^{M}(i,t,N_{gs}^{M}(i,t)) + \hat{J}_{bs}^{M}(i,t,N_{gs}^{M}(i,t)) \\
&= N_{gs}^{M}(i-1,t)P_c S_M(i-1) - N_{gs}^{M}(i,t)P_e(i)S_M(i) \\
&+ N_{gs}^{M}(i-1,t)\omega_c l_M(i-1) - N_{gs}^{M}(i,t)\omega_e(i)l_M(i), \qquad i>1,
\end{aligned}
$$

where $N_{gs}^{M}(i,t)$ is the non-equilibrium surface density of heterogeneous clusters of the M-type and of i vacancies at a time t in the coupled process, and the circumflex mark over J_{bg}^{M} and J_{bs}^{M} shows that these functions depend now on the density N_{gs}^{M} instead of N_{bg}^{M} and N_{bs}^{M}, respectively.

Using analogous treatment as in the last two sections, and particularly the isolated fluxes (6.8) and (7.5), we obtain

$$
\begin{aligned}
(8.2)\qquad J_{gs}^{M}(i,t) = -\Bigg\{ &[P_c S_M(i) + \omega_c l_M(i)]\frac{\partial N_{gs}^{M}(i,t)}{\partial i} + \Bigg[\frac{P_c S_M(i)}{kT}\frac{\partial \Delta F_{bg}^{M}(i)}{\partial i} \\
&+ \frac{\omega_c l_M(i)}{kT}\frac{\partial \Delta F_{bs}^{M}(i)}{\partial i}\Bigg] N_{gs}^{M}(i,t)\Bigg\}.
\end{aligned}
$$

The coupled flux (8.2) should satisfy the equation of continuity

$$
(8.3)\qquad \frac{\partial N_{gs}^{M}(i,t)}{\partial t} + \frac{\partial J_{gs}^{M}(i,t)}{\partial i} = 0,
$$

which under the steady state condition $\partial N_{gs}^{M}(i,t)/\partial t = 0$ delivers the differential equation for the function $N_{gs}^{M}(i)$

$$
(8.4)\qquad J_{gs}^{M}(i,t) = C,
$$

with a constant C.

The general solution of Eq. (8.4) reads

$$
\begin{aligned}
(8.5)\qquad N_{gs}^{M}(i) = \exp\left(-\frac{1}{kT}\int_{1}^{i} I_M(i)di\right)\Bigg[n + C\int_{1}^{i}\frac{1}{P_c S_M + \omega_c l_M} \\
\times \exp\left(\frac{1}{kT}\int_{1}^{i} I_M(i)di\right)di\Bigg],
\end{aligned}
$$

and the integral in the exponent can be evaluated to give

$$\int_1^i I_M(i)\,di = \widetilde{\Delta F}_{gs}^M(i) = \int_1^i \frac{P_c S_M \frac{\partial \Delta F_{bg}^M}{\partial i} + \omega_c l_M \frac{\partial \Delta F_{bs}^M}{\partial i}}{P_c S_M + \omega_c l_M} di = [\Delta F_{bg}^M(i) - L_M(i)]_1^i, \tag{8.6}$$

where

$$L_M(i) = \frac{3kT \ln H}{D_M^3}\left[\frac{D_M^2}{2} i^{2/3} - D_M i^{1/3} + \ln(D_M i^{1/3} + 1)\right], \tag{8.7}$$

with an auxiliary parameter, depending on the shape of cluster

$$D_M = \frac{P_c b_M}{\omega_c d_M}\left(\frac{V_v}{c_M}\right)^{1/3}. \tag{8.8}$$

In equilibrium $J_{gs}^M(i,t) = 0$, $N_{gs}^M(i) = N_{gs}^{Mo}(i)$, and Eq. (8.4) reduces to a homogeneous equation ($C = 0$) with a solution yielding the equilibrium distribution of clusters

$$N_{gs}^{Mo}(i) = n_0 \exp\left[-\frac{\widetilde{\Delta F}_{gs}^M(i)}{kT}\right]. \tag{8.9}$$

Here, $\widetilde{\Delta F}_{gs}^M(i)$, given by Eq. (8.6), is the total coupled free energy of formation of an aggregate of the M-type and of i vacancies. $\widetilde{\Delta F}_{gs}^M(i)$ is a sum of two terms

$$\widetilde{\Delta F}_{gs}^M(i) = \Delta F_{gs}^M(i) - \Delta F_{gs}^M(1). \tag{8.10}$$

The first term is a size-dependent part of the formation energy

$$\Delta F_{gs}^M(i) = \Delta F_{bg}^M(i) - L_M(i) \tag{8.11}$$

and the second one

$$\Delta F_{gs}^M(1) = \Delta F_{bg}^M(1) - L_M(1) \tag{8.12}$$

is the configurational entropy contribution (cf. Eq. $(6.4)_2$), and the effect of coupling. The total coupled energy of formation of the monomer itself is evidently zero, $\widetilde{\Delta F}_{gs}^M(1) = 0$.

By inserting Eqs. (8.10), (8.12) and $(6.4)_2$ into Eq. (8.9) the equilibrium distribution function $N_{gs}^{Mo}(i)$ is finally determined

$$N_{gs}^{Mo}(i) = N_0 \exp\left[-\frac{L_M(1) + \Delta F_{gs}^M(i)}{kT}\right]. \tag{8.13}$$

It can be shown [1, 2], that the function $\Delta F_{gs}^{M}(i)$ has a maximum

$$\Delta F_{gs}^{M*} = \Delta F_{gs}^{M}(i_{gs}^{M*}) = \Delta F_{bg}^{M}(i_{gs}^{M*}) - L_M(i_{gs}^{M*}) = 3V_V^{2/3}c_M^{1/3}\alpha i_{gs}^{M*2/3} - 2\Delta U i_{gs}^{M*} - L_M(i_{gs}^{M*}) \tag{8.14}$$

at the critical cluster composed of i_{gs}^{M*} vacancies

$$i_{gs}^{M*} = (2D_M)^{-3} i_{bg}^{M*} \left\{ D_M - i_{bs}^{M*-1/3} + \left[\left(D_M - i_{bs}^{M*-1/3} \right)^2 + 4 i_{bg}^{M*-1/3} D_M \right]^{1/2} \right\}^3 . \tag{8.15}$$

It should be noted that, in general, D_M turns out to be a very small number in relation to the values of the variable i, and since

$$\lim_{D_M \to 0} \frac{1}{D_M^3} \left[\frac{D_M^2}{2} i^{2/3} - D_M i^{1/3} + \ln(D_M i^{1/3} + 1) \right] = \frac{i}{3} , \tag{8.16}$$

$\Delta F_{gs}^{M}(i)$ and i_{gs}^{M*} assume asymptotic values

$$\lim_{D_M \to 0} \Delta F_{gs}^{M}(i) = \Delta F_{bs}^{M}(i) \tag{8.17}$$

and

$$\lim_{D_M \to 0} i_{gs}^{M*} = i_{bs}^{M*} . \tag{8.18}$$

Coming back to Eq. (8.5), imposing the boundary condition $N_{gs}^{M}(\infty) = 0$, and using Eqs. (8.6), (8.10), (8.12) and $(6.4)_2$, we are able to determine the constant C and attain to the particular solution for the non-equilibrium distribution function $N_{gs}^{M}(i)$

$$N_{gs}^{M}(i) = N_0 S_{he} \exp\left[-\frac{L_M(1) + \Delta F_{gs}^{M}(i)}{kT} \right] K_M(i) , \tag{8.19}$$

where

$$K_M(i) = 1 - \frac{\int_1^i \frac{1}{P_c S_M + \omega_c l_M} \exp\left[\frac{\Delta F_{gs}^{M}(i)}{kT} \right] di}{\int_1^\infty \frac{1}{P_c S_M + \omega_c l_M} \exp\left[\frac{\Delta F_{gs}^{M}(i)}{kT} \right] di} . \tag{8.20}$$

It should be observed that $K_M(i) \in [0, 1]$, and $K(1) = 1$.

Combining Eq. (8.19) with Eq. (8.13) yields the relation between $N_{gs}^{M}(i)$ and $N_{gs}^{Mo}(i)$

$$N_{gs}^{M}(i) = N_{gs}^{Mo}(i) S_{he} K_M(i) . \tag{8.21}$$

Expanding $\Delta F_{gs}^{M}(i)$ into Taylor series around the point of its maximum i_{gs}^{M*}, and replacing $S_M(i)$ and $l_M(i)$ by $S_M(i_{gs}^{M*})$ and $l_M(i_{gs}^{M*})$, both the integrals appearing in Eq. (8.20) can be approximated by the method shown in [14, 1, 2]. Eq. (8.19) is then substituted into Eq. (8.2), delivering the final form of the coupled heterogeneous nucleation rate of all types of microcracks on the grain boundaries

$$J_{gs}^{M} = (6\pi)^{-1/2} N_0 S_{he} \omega_c (kT)^{-1/2} \left(\frac{V_v}{c_M}\right)^{1/3} d_M \left[2 i_{bg}^{M*1/3} \Delta U \left(D_M + i_{bg}^{M*-1/3}\right)^2 - kT D_M \ln II\right]^{1/2} \exp\left[-\frac{L_M(1) + \Delta F_{gs}^{M*}}{kT}\right] . \tag{8.22}$$

The terminal shape of the surface distribution of critical clusters is directly obtained from Eq. (8.19) with the proper value of $K_M(i_{gs}^{M*})$

$$N_{gs}^{M*}(i) = N_{gs}^{M}(i_{gs}^{M*}) = \frac{1}{2} N_0 S_{he} \exp\left[-\frac{L_M(1) + \Delta F_{gs}^{M*}}{kT}\right] . \tag{8.23}$$

9. Global nucleation rate in the whole grain

In previous sections isolated processes have been considered with various mechanisms of cluster growth. Homogeneous, surface and peripheral heterogeneous nucleation, as well as the coupled one, have been treated separately as the local processes in the vicinities of four types of sites (one internal and three types of boundary sites). However, in any grain of a real polycrystalline metal all the fluxes are simultaneously operating in generation of different classes of vacancy aggregates. Consequently, if corner (four-grain junction) nucleation is additionally taken into account, the balance equation [13, 4, 5] for the steady state global nucleation rate J_0 in the whole internal grain must be supplemented by the fourth term, and results in the form

$$J_0 = J_g + \frac{z_M(i_{gs}^{M*})}{V_g} J_{gs}^{M} , \tag{9.1}$$

where appropriate nucleation zones are given in Sec. 3, J_g, J_{gs}^{M} by Eqs. (5.3), (8.22), and Einstein summation convention is used (M = II, III, IV).

10. Microfracture criteria

In earlier papers [1–5] a conventional microfracture criterion for uniaxially strained metals has been introduced: The global nucleation rate J_0^* yielding one arbitrary microcrack nucleus per second in the average grain is critical. Thus,

$$J_0^* = \frac{1}{V_{gav}} \frac{\text{nuclei}}{\text{cm}^3\text{s}}, \tag{10.1}$$

where V_{gav} is the volume of the average grain.

For this number of microcrack nuclei local critical strain ε_0^* should be found.

The aforesaid criterion is apparently not invariant since it depends on the shape and size of the grain and on assumption of the time unit. To get rid at least from the last inconvenience analogous of condition can be superimposed on the total number A_0 of all kinds of critical aggregates formed in the grain or at the grain boundary

$$A_0 = A_g^* + \frac{z_M(i_{gs}^{M*})}{V_g} N_{gs}^{M*}, \tag{10.2}$$

where A_g^*, N_{gs}^{M*} are given by Eqs. (5.4) and (8.23).

The microfracture condition, expressed by the distribution functions of the critical clusters, resolves itself into the following alternative criterion: The global number of critical aggregates A_0^* delivering one arbitrary microvoid nucleus in the average grain is critical

$$A_0^* = \frac{1}{V_{gav}} \frac{\text{nuclei}}{\text{volume unit}}. \tag{10.3}$$

As above, corresponding critical strain ε_0^* should be determined. Obviously, both criteria furnish different ε_0^*.

11. Concluding remarks

The improved, coherent model of a polycrystalline material permits to complete the already investigated types of microcracks with the corner microvoids. Moreover, the new approach introduced in the paper is constructed in such a way, that it embraces all the possible kinds of intergranular microvoids in a common procedure. Whereas the three types of boundary microvoids (double-lens, spherical trihedron, and spherical tetrahedron) show decreasing energy of formation, their nucleation zones are also decreasing. These two factors are opposing each other in boundary nucleation. In order to get an idea on quantitative results of the theory and to explore the contributions of all the particular nucleation rates ($J_g, J_{gs}^{\text{II}}, J_{gs}^{\text{III}}, J_{gs}^{\text{IV}}$) to the global nucleation rate J_0, some numerical estimates for two f.c.c. metals (aluminium and copper) and one b.c.c. metal

(α–iron) were computed. The equations of the theory, tetrakaidecahedral shape of grains (Fig. 1 and 5) with the diameters $\delta = 1, 10, 100\,\mu m$, $T = 300 - 1700\,K$, $\theta = 75°$ and other data given in Ref. [1] have been used. The first criterion (10.1) yields $\varepsilon_0^* \in [0.094 - 0.052]$, depending on the metal, grain size and temperature, and shows that the effect of the homogeneous flux J_g on the global nucleation rate J_0 is negligibly small and can be disregarded. From the other fluxes the dominant role plays J_{gs}^{III} (the trihedron microcracks) with increasing influence of J_{gs}^{II} (the lenticular microcracks) with the diameter δ, especially for copper. J_{gs}^{IV} (tetrahedron microcracks) is rather small (but not negligible), and decreases with increasing δ. This results from the fact that the surface area of the grain increases with δ while the number of vertices remain constant.

On the other hand, the second criterion (10.3) delivers $\varepsilon_0^* \in [0.108 - 0.063]$, and indicates that all the nucleation mechanisms are essential and no one can be disregarded. Since, however, the second criterion determines higher critical strains ε_0^*, as compared with the first one, it cannot be recommended for further use.

It is interesting to note that both the criteria reveal the decrease of the strength of the metal for microfracture with increasing grain size, in full agreement with the experimental data. Moreover, according to the first criterion, the size of the double-lens critical microcrack for α–iron in room temperature ($i_{ls}^{I*} \approx 3 - 4$ vacancies, depending on the grain size) coincides with the results given in Ref. [15]. This conformability proved to be true, despite the continuum approach applied, uncertain data used, and rough approximations made concerning the model of metal and the process itself. The sizes of all types of critical microcracks increase significantly with temperature, whereas the critical strain ε_0^* decreases.

In the above theory, the effect of grain boundary inclusions on the intergranular nucleation of vacancy microvoids was not taken into account. This important problem was already touched [11] but in a somewhat different aspect (intergranular fracture under creep conditions), and with the use of classical, uncoupled nucleation theory. More precise investigations are badly needed.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCIL

Received December 14, 1994.

Arch. Mech., 47, 4, pp. 793–812, Warszawa 1995

Analysis of interaction of impulse-like pressure wave in a fluid with deformable layer of porous material (*)

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THE PROBLEM of interaction of a plane impulse-like pressure wave in a barotropic fluid with elastic layer of porous material immersed in it is analysed for the case of normal incidence of the wave. The method of analytical solution of the problem is proposed. Considerations were realised on the assumption that the fluid is inviscid and the material of the porous layer is incompressible. This allowed us to reduce the number of necessary material constants in the problem to two quantities: velocity of a compressive wave in the nonsaturated skeleton and its velocity in the bulk fluid (without skeleton). The formulae for amplitudes of the first waves leaving the porous layer have been derived. They depend explicitly on both material constants, the parameters characterizing the layer pore structure and the densities of both physical constituents. Moreover, the problem of impulse-wave interaction with a halfspace made of porous material is solved.

1. Introduction

ANALYSIS OF INTERACTION of waves propagating in a fluid with fluid-filled porous solid is of great importance in many technical problems occuring, for example, in geological explorations, sea bed researches, architectural acoustics or machinery noise control [2, 20, 22]. Another area of importance of these problems is the determination of porous material constants by means of the dynamic methods [12–14]. In such investigations, the porous specimens in the form of layers are elements of the most fundamental measurement systems in which the impulse-like pressure waves (ultrasonic waves) are very often applied. Interaction of these waves with a porous layer is then the source of information about mechanical properties of the porous material.

Most of papers devoted to the study of interaction of waves with elastic fluid-filled porous solids concern the reflection and transmission of harmonic waves through the surface of a halfspace of porous medium [8, 11, 19, 21]; however, there is no theoretical analysis regarding the interaction of impulse-like waves with a fluid-filled deformable porous layer.

The purpose of this paper is to obtain the analytical solution of the problem of interaction of the plane impulse-like pressure waves propagating in a fluid with a deformable layer of porous material immersed in it. The case of normal incidence is considered. In the analysis, special attention is paid to obtain the explicit expressions relating the basic parameters characterizing the interaction of waves with porous layer, and the following quantities: parameters of skeleton pore

(*) The reported research was done under grant No. 300149101.

structure, material constants of both physical components, their mass densities and fluid pressure in the undisturbed state.

This paper is the second part of the analysis regarding interaction of impulse-like waves with a porous layer. The first part (see [7]) concerned the normal incidence of a plane wave on a rigid, fixed and permeable layer filled with fluid. In that part, the influence of the layer pore structure and viscous friction between the fluid and skeleton on the reflection and transmission of waves is discussed. It was shown that the amplitude of a wave penetrating the porous medium considerably increases as compared with the amplitude of the incident wave, and that this effect is caused by the skeleton pore structure.

In the present work, the considerations are based on the assumption that the motion of the fluid outside the porous layer is described by the Navier-Stockes equations, while the motion of the fluid-filled porous layer is given by equations of the theory of saturated deformable porous media [9, 10, 15–17]. In the framework of this theory the structure of the skeleton is characterized by two parameters: volume porosity and structural permeability. These two parameters enter the continuity and motion equations of the porous medium in an explicit form, and so do they in some of the compatibility conditions formulated as the continuity of the fluid mass flux on both faces of the layer.

The considerations are restricted to the purely elastic case, and the incompressibility of material of the skeleton is assumed. It makes it possible to reduce the number of material constants of the medium in a plane problem to two quantities: velocities of longitudinal waves propagating in the nonsaturated skeleton and in the bulk fluid (without skeleton), leading to very effective description of the medium.

The problem is solved in two steps. First, the interaction of the plane harmonic wave with a porous layer is considered, for which the explicit expressions describing the acoustical fields in particular regions of the system are obtained. These expressions are employed at the second step to construct a solution of the problem of interaction for an arbitrary profile of the plane incident wave.

The solution has been obtained in the analytical form, considering interaction of the impulse wave with the layer as the superposition of interactions of its harmonic components given by the Fourier transform.

2. Interaction of an acoustic impulse wave with a porous layer. Formulation of the problem

We analyse the problem of reflection and transmission of a plane pressure wave through a deformable layer of a porous isotropic material with thickness b, immersed in a fluid. The considerations are restricted to the case of purely elastic interactions of a wave falling perpendicularly on the surface of a porous layer (Fig. 1). It is assumed that the fluid is inviscid and barotropic, i.e. the effective

pressure p^f in the fluid is in one-to-one relation with its effective mass density ϱ^f ($p^f = p^f(\varrho^f)$), and that the porous skeleton of the layer is elastic.

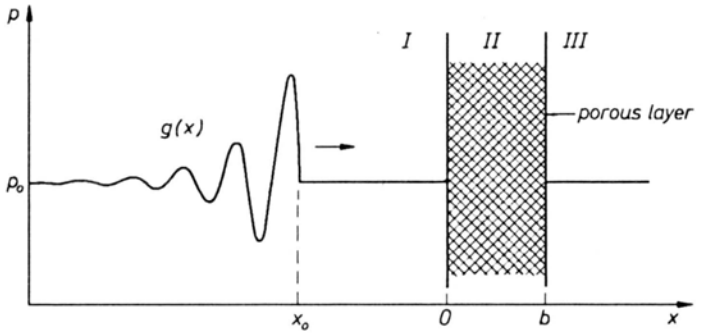

FIG. 1. Outline of the system: impulse wave – porous layer immersed in fluid.

In many cases of medium and greater values of the porosity it is observed that the elastic deformations of a porous skeleton take place at the expense of change of the pore volume. Therefore we assume that the material of the skeleton is incompressible. This assumption is the kinematical constraint imposed on the motion of the porous skeleton, and is equivalent to the assumption that the effective mass density ϱ^s of the skeleton is constant during the deformation process, i.e.

$$\varrho^s = \varrho^s_o, \tag{2.1}$$

where ϱ^s_o is the effective mass density of the skeleton in the undisturbed state of the porous layer.

It is assumed, moreover, that the vibration amplitude of fluid particles in the wave falling on the porous layer is considerably less than its characteristic length. It enables us to use the linear description of the problem and to avoid the necessity of imposing the compatibility conditions on the movable boundaries of deformable layer.

2.1. Basic equations

Under the above-mentioned assumptions, the propagation of disturbances with small amplitudes in the half-spaces $x < 0$ (region I) and $x > b$ (region III) is described by linear one-dimensional equations

$$\frac{\partial \varrho}{\partial t} + \frac{\partial v}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + \frac{p^f_o}{\varrho^f_o}\frac{\partial p}{\partial x} = 0, \tag{2.3}$$

$$\varrho = (\varrho^f - \varrho^f_o)/\varrho^f_o, \qquad p = (p^f - p^f_o)/p^f_o,$$

where v is the velocity of fluid particles and p_o^f, ϱ_o^f stand for its effective pressure and density in the undisturbed state, respectively.

The description of motion of the deformable layer filled with fluid (region II) will be based on the two-parameter continuum theory of fluid-filled deformable porous solid [9, 10, 15–17]. In that theory the pore structure of the skeleton is characterised by two parameters: volume porosity f_v and structural permeability λ (or, equivalently, by $\kappa = \lambda/f_v < 1$). Porosity f_v characterises the volume fraction of physical components of the porous medium and parameter λ is the measure of inhomogeneity of the fluid micro-velocity field in its motion relative to the skeleton, caused by the skeleton pore structure. Equations of this theory concern the so-called virtual components of medium, i.e. the porous skeleton together with the associated fluid moving at the velocity $\overset{1}{v}$ equal to the velocity of skeleton (first virtual component ($|^1$)) and the free fluid moving at the velocity $\overset{2}{v}$ (the second virtual component ($|^2$)).

In the case of incompressible material of the skeleton, the linear, one-dimensional equations of the two-parameter theory of porous medium take the form:

continuity equations

$$\frac{\partial f_v}{\partial t} - (1 - f_v^o)\frac{\partial \overset{1}{v}}{\partial x} = 0, \tag{2.4}$$

$$f_v^o \frac{\partial \varrho}{\partial t} + \kappa_o f_v^o \frac{\partial \overset{2}{v}}{\partial x} + (1 - \kappa_o f_v^o)\frac{\partial \overset{1}{v}}{\partial x} = 0; \tag{2.5}$$

motion equations

$$\overset{1}{\varrho}_o \frac{\partial \overset{1}{v}}{\partial t} = \frac{\partial \overset{1}{T}}{\partial x} - \kappa_o p_o^f \frac{\partial f_v}{\partial x}, \tag{2.6}$$

$$\overset{2}{\varrho}_o \frac{\partial \overset{2}{v}}{\partial t} = \frac{\partial \overset{2}{T}}{\partial x} + \kappa_o p_o^f \frac{\partial f_v}{\partial x}, \tag{2.7}$$

where $\overset{1}{\varrho}_o$, $\overset{2}{\varrho}_0$ stand for the partial densities of virtual components in the undisturbed state of the medium, and are related to the partial densities of physical components

$$\overline{\varrho}_o^s = (1 - f_v^o)\varrho_o^s, \qquad \overline{\varrho}_o^f = f_v^o \varrho_o^f \tag{2.8}$$

as follows:

$$\overset{1}{\varrho}_o = \overline{\varrho}_o^s + (1 - \kappa_o)\overline{\varrho}_o^f, \qquad \overset{2}{\varrho}_o = \kappa_o \overline{\varrho}_o^f. \tag{2.9}$$

The partial stresses $\overset{1}{T} = \overset{1}{T}_{xx}$, $\overset{2}{T} = \overset{2}{T}_{xx}$ of virtual components of the medium are related to the stresses $T^s = T^s_{xx}$, $T^f = T^f_{xx}$ of its physical components by the

expressions, [15],

$$\overset{1}{T} = T^s + (1 - \kappa_o)T^f, \qquad \overset{2}{T} = \kappa_o T^f, \tag{2.10}$$

where

$$T^f = -f_v^o p^f . \tag{2.11}$$

Constitutive relations are, [5],

$$\sigma^s = p^{\mathrm{II}} - \frac{\lambda_\varrho^* + 2\mu_\varrho^*}{p_o^f}\varepsilon, \tag{2.12}$$

$$p = \frac{a_o^2 \varrho_o^f}{p_o^f}\varrho \qquad \text{or} \qquad p^{\mathrm{II}} = \frac{a_o^2 \varrho_o^f}{p_o^f}\varrho^{\mathrm{II}}, \tag{2.13}$$

where $\varepsilon = \varepsilon_{xx}$ is the component of the small deformation tensor of the skeleton and

$$\sigma^s = -\frac{T^{*s} + p_o^f}{p_o^f}$$

represents the relative increment of the component of the effective skeleton stress tensor

$$T^{*s} = T^s/(1 - f_v). \tag{2.14}$$

This quantity in the undisturbed state of the medium (the reference configuration) takes the value

$$T^{*s} = -p_o^f .$$

The effective Lamé material constants λ_ϱ^*, μ_ϱ^* of a porous skeleton are measured at constant mass density ϱ of the skeleton and are related to the partial Lamé constants λ_ϱ and μ_ϱ by expressions

$$\lambda_\varrho = (1 - f_v^o)\lambda_\varrho^*, \qquad \mu_\varrho = (1 - f_v^o)\mu_\varrho^* . \tag{2.15}$$

The quantity

$$a_o = \left(\frac{\partial p^f}{\partial \varrho^f}\bigg|_{\varrho^f = \varrho_o^f} \right)^{1/2}$$

in the constitutive relation (2.13) for the fluid stands for the velocity of wave propagation in such a medium.

2.2. Compatibility conditions

The acoustic fields in particular regions of the system are coupled through the compatibility conditions imposed on both surfaces of the porous layer. For the small disturbances of the medium these conditions are [6]: continuity of the fluid mass flux, continuity of the effective pressure in a fluid, and the balance of forces on both surfaces of porous layer, i.e. the pressure in bulk fluid and total stress on the surface of porous medium.

Taking into account relations (2.10), (2.11) and (2.14), the compatibility conditions on the boundary $x = 0$ take the form

$$(v^{\mathrm{I}} - \overset{1}{v})|_{x=0-} = \kappa_o f_v^o (\overset{2}{v} - \overset{1}{v})|_{x=0+}, \tag{2.16}$$

$$p^{\mathrm{I}}|_{x=0-} = p^{\mathrm{II}}|_{x=0+}, \tag{2.17}$$

$$p^{\mathrm{I}}|_{x=0-} = \sigma^s|_{x=0+}. \tag{2.18}$$

Similarly, for the boundary $x = b$ we obtain

$$\kappa_o f_v^o (\overset{2}{v} - \overset{1}{v})|_{x=b-} = (v^{\mathrm{III}} - \overset{1}{v})|_{x=b+}, \tag{2.19}$$

$$p^{\mathrm{II}}|_{x=b-} = p^{\mathrm{III}}|_{x=b+}, \tag{2.20}$$

$$\sigma^s|_{x=b-} = p^{\mathrm{III}}|_{x=b+}, \tag{2.21}$$

where v^i $(i = \mathrm{I}, \mathrm{III})$, p^j $(j = \mathrm{I}, \mathrm{II}, \mathrm{III})$ are the resultant fields of the velocity and of the relative increments of fluid pressure in the corresponding regions of the system. The quantities γ^- and γ^+ denote the left-hand side and the right-hand side limiting values of the magnitude, respectively.

To solve the problem, a state of the system at the initial instant of the process must be assumed. The initial condition is assumed to be as follows

$$p(x,t)|_{t=0+} = \begin{cases} g(x) & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x \geq 0 \end{cases} \tag{2.22}$$

and $g(0^-) = 0$.

Equations (2.2)–(2.7), the constitutive relations (2.12) and (2.13) and the compatibility conditions (2.16)–(2.21) fully describe the reflection and transmission of an impulse wave in a fluid through the deformable layer of porous material. The form of equations (2.4)–(2.7) and the constitutive relation (2.12) show that the motion of porous skeleton and fluid filling its pores are strongly coupled, and this fact is directly connected with the pore structure of the layer.

The pore structure parameters appear in the continuity equations (2.4), (2.5) and in the motion equations (2.6) and (2.7). Their explicit presence in the compatibility conditions (2.16) and (2.19) shows that they influence the coupling of mechanical fields in particular regions of the system. The interaction of waves

with porous layer depends also on the mechanical properties of a fluid and the porous layer through the material constants appearing in the constitutive relations (2.12) and (2.13).

3. Solution of the problem

It follows from compatibility conditions (2.19) and (2.23), that on both faces bounding the porous layer the effective pressure of fluid is continuous, and also that there is a smooth, continuous transition from the pressure of fluid outside the layer to the effective stress of the porous skeleton. It concerns, as well, the relative increments of these quantities. Such a continuous transition enables the illustration of interaction of impulse-like pressure wave in a fluid with deformable porous layer. By this reason the acoustic fields in particular regions of the system will be represented by the following variables: the relative increment of fluid pressure p and the relative increment of the skeleton effective stress σ^s.

The above mentioned representation is motivated the more so since, as follows from (2.16) and (2.19), the velocity fields of particular constituents are discontinuous on both faces bounding the porous layer.

Using the constitutive relation (2.13), from equations (2.2) and (2.3) we obtain the wave equation

$$\frac{\partial^2 p}{\partial t^2} = a_0^2 \frac{\partial^2 p}{\partial x^2} \tag{3.1}$$

describing the propagation of disturbances in the bulk fluid occupying the region I and the region III of the system. At the same time, the propagation of disturbances in the porous layer filled with fluid (region II) is described by the system of two mutually coupled wave equations derived from (2.4)–(2.7), (2.12) and (2.13). These are:

$$\frac{\partial^2 \sigma^s}{\partial t^2} = \alpha_{11} \frac{\partial^2 \sigma^s}{\partial x^2} + \alpha_{12} \frac{\partial^2 p^{II}}{\partial x^2}, \tag{3.2}$$

$$\frac{\partial^2 p^{II}}{\partial t^2} = \alpha_{21} \frac{\partial^2 \sigma^s}{\partial x^2} + \alpha_{22} \frac{\partial^2 p^{II}}{\partial x^2}, \tag{3.3}$$

where

$$\begin{aligned}
\alpha_{11} &= \alpha_{21} + V_s^2 (1 - f_v^o) \varrho_o^s / \overset{1}{\varrho}_o, \\
\alpha_{12} &= \alpha_{22} + V_s^2 (1 - \kappa_o) f_v^o \varrho_o^s / \overset{1}{\varrho}_o, \\
\alpha_{21} &= c_o^2 (1 - f_v^o) \frac{(1 - \kappa_o f_v^o)}{\kappa_o f_v^o} \varrho_o^s / \overset{1}{\varrho}_o, \\
\alpha_{22} &= c_o^2 \left(1 + (1 - \kappa_o) f_v^o \frac{(1 - \kappa_o f_v^o)}{\kappa_o f_v^o} \varrho_o^s / \overset{1}{\varrho}_o \right),
\end{aligned} \tag{3.4}$$

and

$$V_s^2 = (\lambda_\varrho^* + 2\mu_\varrho^*)/\varrho_o^s = (\lambda_r + 2\mu_\varrho)/\overline{\varrho}_o^s \tag{3.5}$$

is the square of the wave velocity in nonsaturated (dried) porous skeleton, and c_o stands for the velocity of wave propagation in fluid filling the pores of rigid skeleton. The last quantity is related to the velocity a_o of wave propagation in the bulk fluid as follows, [4],

$$c_o^2 = \kappa_o a_o^2 . \tag{3.6}$$

Wave velocities V_s and c_o, together with the pore structure parameters f_v^o and κ_o and the effective densities ϱ_o^s and ϱ_o^f of physical components, describe explicitly the values of coefficients α_{ij} $(i, j = 1, 2)$ from Eqs. (3.2) and (3.3). It means that these quantities completely characterize the dynamic properties of the considered system.

The initial-boundary value problem of interaction of an impulse wave propagating in a fluid with a porous layer will be solved analytically in two steps. In the first step, the interaction of a plane harmonic wave will be considered to obtain the explicit analytical expressions for the acoustic fields in the particular regions of the system. Thus the frequency characteristics will be determined.

In the second step, an impulse wave falling on a porous layer will be dealt with. For this purpose, the incident wave will be decomposed into harmonic components with a continuous distribution of spectrum and, next, the knowledge of the interaction of harmonic waves with porous layer, gained at the first step, will be used to construct the solution for an arbitrary plane impulse wave.

3.1. Propagation of harmonic waves in a deformable porous medium filled with fluid

In order to describe the acoustic field in the porous layer, the knowledge of fundamental characteristics of wave propagation in a fluid-saturated deformable porous solid is necessary.

Such characteristics may be obtained by solving the system of Eqs. (3.2) and (3.3) for harmonic waves.

Assuming the solution of the system in the form

$$\sigma^s = \mathrm{Re}\left(A^\sigma e^{i\omega t} e^{-2\pi i k_p x}\right), \tag{3.7}$$

$$p^{\mathrm{II}} = \mathrm{Re}\left(A^p e^{i\omega t} e^{-2\pi i k_p x}\right), \tag{3.8}$$

$$i = \sqrt{-1}, \qquad \omega = 2\pi f,$$

from (3.2) and (3.3) we obtain the condition

$$\begin{bmatrix} \alpha_{11} - V^2 , & \alpha_{12} \\ \alpha_{21} , & \alpha_{22} - V^2 \end{bmatrix} \begin{bmatrix} A^\sigma \\ A^p \end{bmatrix} = 0, \tag{3.9}$$

where f is the wave frequency, k_p stands for the wave number and

$$V = f/k_p$$

is the velocity of harmonic wave.

The matrix equation (3.9) forms a homogeneous set of equations, with regard to the amplitudes A^σ and A^p. Its solution is not trivial, if the determinant of the matrix is equal to zero. Thus, we have biquadratic algebraic equation

$$V^4 - (\alpha_{11} + \alpha_{22})V^2 + \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 0, \tag{3.10}$$

the roots of which define velocities of waves propagation in the medium under consideration.

Since coefficients α_{12} and α_{21} are positive quantities, the discriminant of biquadratic trinomial (3.10) given in the form

$$\Delta = (\alpha_{11} + \alpha_{22})^2 - 4(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) = (\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21} \tag{3.11}$$

will be always positive.

Therefore, Eq. (3.10) has two real solutions for the square of velocities. They are

$$V_1^2 = \left(\alpha_{11} + \alpha_{22} + \sqrt{\Delta}\right)/2, \tag{3.12}$$

$$V_2^2 = \left(\alpha_{11} + \alpha_{22} - \sqrt{\Delta}\right)/2, \tag{3.13}$$

and, in view of the inequality

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = c_o^2 V_s^2 \varrho_o^s/\varrho_o^1 > 0,$$

both solutions are also positive. This proves that, in general, in a deformable porous solid filled with fluid, two longitudinal waves propagate and the set of equations (3.2) and (3.3) is the pure hyperbolic system.

The existence of two longitudinal volumetric waves in saturated porous medium was theoretically discovered by BIOT [1], and experimentally proved by PLONA [18].

Using the Biot's nomenclature, the wave propagating at greater velocity defined by (3.12) will be called the "fast" wave, and the wave moving at lower velocity defined by (3.13) will be called the "slow" wave. The corresponding stress state in the porous skeleton and fluid is related with each wave having amplitudes A_1^σ, A_1^p and A_2^σ, A_2^p, respectively.

Solving Eqs. (3.9) for the velocity of fast wave and the velocity of slow wave, we obtain expressions for the ratios of amplitudes as follows:

$$s_1 = A_1^\sigma/A_1^p = \alpha_{12}/(V_1^2 - \alpha_{11}) = (V_1^2 - \alpha_{22})/\alpha_{21} = (\alpha_{11} - \alpha_{22} + \sqrt{\Delta})/2\alpha_{21}, \tag{3.14}$$

$$s_2 = A_2^\sigma/A_2^p = \alpha_{12}/(V_2^2 - \alpha_{11}) = (V_2^2 - \alpha_{22})/\alpha_{21} = (\alpha_{11} - \alpha_{22} - \sqrt{\Delta})/2\alpha_{21}. \tag{3.15}$$

From the above relations it is seen that the ratio s_1 of amplitudes in the fast wave is always positive, while the ratio s_2 of amplitudes in the slow wave is always negative. This proves that the particles of porous skeleton and the pore fluid in the fast wave are moving in phase, while in the slow wave in antiphase. These are the characteristic features of both waves.

It is also worth to notice that, in general, none of the considered waves propagating in saturated porous solid can be connected with one constituent only as its carrier. Such simple situation takes place in the very particular limiting cases, where we consider the dynamic behaviour of a nonsaturated porous skeleton, or the dynamics of fluid filling pores of a rigid skeleton.

Taking into account the existence of two kinds of waves in saturated porous solids, the general solution of the set of equations (3.2) and (3.3) for harmonic waves can be written in the following form

$$\sigma^s = \mathrm{Re}\left\{e^{i\omega t}\left(A_1^p s_1 e^{-2\pi i k_1 x} + A_2^p s_2 e^{-2\pi i k_2 x}\right)\right\}, \tag{3.16}$$

$$p = \mathrm{Re}\left\{e^{i\omega t}\left(A_1^p e^{-2\pi i k_1 x} + A_2^p e^{-2\pi i k_2 x}\right)\right\}, \tag{3.17}$$

where

$$k_1 = f/V_1, \qquad k_2 = f/V_2$$

are the wave numbers of the fast and the slow wave, respectively.

3.2. Interaction of harmonic wave with porous layer

Interaction of a plane harmonic wave propagating in a fluid with porous layer placed perpendicularly to the direction of travelling wave results in the existence of plane acoustic fields in the particular region of the system.

In the region I the acoustic field consists of two waves: the incident wave of amplitude A_1 and frequency f ($\omega = 2\pi f$), and the wave of amplitude D_1 running in opposite direction, being the resultant of the reflected wave and the waves leaving the layer. These waves are described by expressions which are the solutions of Eq. (3.1), and the acoustic field in this region is

$$p^{\mathrm{I}}(\tilde{x}, \tilde{t}) = \mathrm{Re}\left\{A_1\left(e^{2\pi i\eta(\tilde{t}-\tilde{x})} + \overline{D}_1 e^{2\pi i\eta(\tilde{t}+\tilde{x})}\right)\right\}, \tag{3.18}$$

where $\overline{D}_1 = D_1/A_1$ is the relative amplitude and

$$\eta = kb, \qquad k = f/a_o.$$

In the expression (3.18), the non-dimensional coordinate $\tilde{x}$ and non-dimensional time $\tilde{t}$ are introduced. They are defined as follows

$$\tilde{x} = x/b, \qquad \tilde{t} = ta_o/b.$$

The space coordinate $\tilde{x}$ is the measure of distance from the origin of the reference frame expressed in terms of units referred to the thickness of the layer. The non-dimensional time $\tilde{t}$ is the measure of time passage expressed by means of units reffered to the time interval which is necessary for the wave in the bulk fluid to run the distance equal to the thickness of layer.

The resultant acoustic field in the region III is represented by one wave only, of the relative amplitude $\overline{A}_3$ being the sum of the waves leaving the layer. It satisfies Eq. (3.1) and can be written in the following form

$$p^{\mathrm{III}}(\tilde{x},\tilde{t}) = \mathrm{Re}\left\{A_1\overline{A}_3 e^{2\pi i\eta(\tilde{t}-\tilde{x})}\right\}. \tag{3.19}$$

The acoustic field in the layer (region II), consists of two coupled acoustic fields, the carriers for which are: the pore fluid and the deformable porous skeleton.

According to considerations given in Sec. 3.1, in the layer we have four pairs of coupled, resultant waves. Two pairs of waves are related with the fast and the slow waves propagating simultaneously in the pore fluid and the skeleton in the direction of increasing values of x coordinate, and two other pairs of waves travelling in the opposite direction.

These waves result from succeeding reflections of fast and slow waves from the contact surface of porous layer and fluid, where each of them generates a new fast and slow wave.

Taking expressions (3.16) and (3.17) for waves travelling in both opposite directions we obtain the acoustic field in the porous layer (region II),

$$\sigma^{\mathrm{II}}(\tilde{x},\tilde{t}) = \mathrm{Re}\left\{A_1\left(\overline{A}_1^p s_1 e^{2\pi i\eta(\tilde{t}-K_1\tilde{x})} + \overline{A}_2^p s_2 e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})} + \overline{D}_1^p s_1 e^{2\pi i\eta(\tilde{t}+K_1\tilde{x})} + \overline{D}_2^p s_2 e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})}\right)\right\}, \tag{3.20}$$

$$p^{\mathrm{II}}(\tilde{x},\tilde{t}) = \mathrm{Re}\left\{A_1\left(\overline{A}_1^p e^{2\pi i\eta(\tilde{t}-K_1\tilde{x})} + \overline{A}_2^p e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})} + \overline{D}_1^p e^{2\pi i\eta(\tilde{t}+K_1\tilde{x})} + \overline{D}_2^p e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})}\right)\right\}, \tag{3.21}$$

$$(\bar{}) = (\ )/A_1,$$

where $\overline{D}_1^p$, $\overline{D}_2^p$ are the relative amplitudes for the fluid pressure in the fast and the slow wave, respectively, running in the direction of decreasing values of x coordinate. Parameters K_1 and K_2 stand for the ratios of wave numbers

$$K_1 = k_1/k, \qquad K_2 = k_2/k. \tag{3.22}$$

The expressions (3.18)–(3.21) involve six unknown ratios of wave amplitudes. These can be determined by the use of six compatibility conditions (2.16)–(2.21). The conditions (2.17), (2.18) and (2.20), (2.21) can be directly satisfied by the expressions (3.18)–(3.21). To satisfy conditions (2.16) and (2.19) it is necessary to

interchange the dependent variables for stresses into the dependent variables for velocities. To do this, the continuity equations (2.2) and (2.5) and the constitutive relations (2.12) and (2.13) should be used.

Fulfilment of conditions (2.16)–(2.21) results in the following algebraic set of equations

$$
\begin{aligned}
&1 - \overline{D}_1 = f_v^o \left((\overline{A}_1^p - \overline{D}_1^p)/K_1 + (\overline{A}_2^p - \overline{D}_2^p)/K_2 \right), \\
&1 + \overline{D}_1 = \overline{A}_1^p + \overline{A}_2^p + \overline{D}_1^p + \overline{D}_2^p, \\
&1 + \overline{D}_1 = (\overline{A}_1^p + \overline{D}_1^p)s_1 + (\overline{A}_2^p + \overline{D}_2^p)s_2, \\
&f_v^o \left\{ \left(\overline{A}_1^p e^{-2\pi i \eta K_1} - \overline{D}_1^p e^{2\pi i \eta K_1} \right) / K_1 \right. \\
&\qquad \left. + \left(\overline{A}_2^p e^{-2\pi i \eta K_2} - \overline{D}_2^p e^{2\pi i \eta K_2} \right) / K_2 \right\} = \overline{A}_3 e^{-2\pi i \eta}, \\
&\overline{A}_1^p e^{-2\pi i \eta K_1} + \overline{D}_1^p e^{2\pi i \eta K_1} + \overline{A}_2^p e^{-2\pi i \eta K_2} + \overline{D}_2^p e^{2\pi i \eta K_2} = \overline{A}_3 e^{-2\pi i \eta}, \\
&\left(\overline{A}_1^p e^{-2\pi i \eta K_1} + \overline{D}_1^p e^{2\pi i \eta K_1} \right) s_1 + \left(\overline{A}_2^p e^{-2\pi i \eta K_2} + \overline{D}_2^p e^{2\pi i \eta K_2} \right) s_2 \\
&\qquad = \overline{A}_3 e^{-2\pi i \eta}.
\end{aligned} \tag{3.23}
$$

The above equations offer the possibilities of determination of the ratios of wave amplitudes in the system with respect to the incident wave amplitude in the form of explicit functions of the quantities characterizing the mechanical and dynamic properties of both physical constituents: the pore structure of the layer, and the macroscopic geometry of the system for various frequencies of the incident wave. In particular, they enable us to obtain the reflection and transmission coefficients for harmonic waves depending on these quantities.

From the set of equations (3.23) it follows

$$
\begin{aligned}
&\overline{D}_1 = \frac{r}{M} \left\{ (1 - e^{-z_1 - z_2})^2 - \frac{mn}{r} (e^{-z_1} - e^{-z_2})^2 \right\}, \\
&\overline{A}_3 e^{-2\pi i \eta} = \frac{(r+1)}{M} \left((m+1) e^{-z_1} \left(1 - e^{-2z_2} \right) \right. \\
&\qquad \left. + (n+1) e^{-z_2} \left(1 - e^{-2z_1} \right) \right),
\end{aligned}
$$

$$
\begin{aligned}
&\overline{A}_1^p = \frac{A_1^\infty}{M} \left(1 + n e^{-2z_2} - (n+1) e^{-z - z_2} \right), \\
&\overline{A}_2^p = \frac{A_2^\infty}{M} \left(1 + m e^{-2z_1} - (m+1) e^{-z_1 - z_2} \right), \\
&\overline{D}_1^p = \frac{A_1^\infty}{M} \left(m e^{-2z_1} + (n+1) e^{-z_1 - z_2} + r e^{-2(z_1 + z_2)} \right), \\
&\overline{D}_2^p = \frac{A_2^\infty}{M} \left(n e^{-2z_2} + (n+1) e^{-z_1 - z_2} + r e^{-2(z_1 + z_2)} \right),
\end{aligned} \tag{3.24}
$$

where

$$
M = (1 - r e^{-z_1 - z_2})^2 - (m e^{-z_1} - n e^{-z_2})^2, \tag{3.25}
$$

$$z_\alpha = 2\pi i K_\alpha \eta \qquad (\alpha = 1,2)$$

and

$$\begin{aligned} m+1 &= 2f_v^o V_1(s_2-1)/\mu_o\,, \\ n+1 &= -2f_v^o V_2(s_1-1)/\mu_o\,, \\ r+1 &= 2a_o(s_2-s_1)/\mu_o\,. \end{aligned} \tag{3.26}$$

In the above relations we have

$$\mu_o = f_v\,(V_1(s_2-1) - V_2(s_1-1)) + (s_2-s_1)a_o\,, \tag{3.27}$$

and

$$r + m + n + 1 = 0. \tag{3.28}$$

The coefficients A_1^∞ and A_2^∞ stand for the ratio of the amplitude of fast and slow waves, respectively, to the amplitude of the incident wave, in the case when the incident wave interacts with the porous halfspace (see Sec. 4). These quantities are

$$A_1^\infty = \frac{K_1}{f_v^o}(m+1), \qquad A_2^\infty = \frac{K_2}{f_v^o}(n+1).$$

Solving the simple problems of interaction of the harmonic waves, the fast wave and the slow wave, with the boundary of porous solid, we can easily find that the second power of the quantities r, m and n can be considered as: the reflection coefficient of the harmonic wave in fluid from the surface of porous medium, and the reflection coefficient of the fast wave and the slow wave from the contact surface of the porous body with the bulk fluid.

Amplitudes $\overline{D}_1$ and $\overline{A}_3$ given by expressions $(3.24)_{1,2}$ satisfy the identity

$$|\overline{D}_1|^2 + |\overline{A}_3|^2 = 1,$$

where $|\beta|$ is the magnitude of complex number β, resulting from the conservation of acoustic energy in non-dissipative system.

Relations (3.18)–(3.21) together with expressions (3.24) for wave amplitudes, fully determine the acoustic fields in particular regions of the system.

3.3. Interaction of an impulse wave with a porous layer

We use the response of the system under consideration to the harmonic input to describe the dynamic response to arbitrary impulse input. For this purpose the impulse wave travelling in the fluid towards a porous layer with the pressure distribution

$$p(\tilde{x},\tilde{t}) = f(\tilde{t}-\tilde{x}), \qquad \tilde{x} < 0,$$

satisfying the initial condition (2.22), will be decomposed into the harmonic components with the use of the Furier transform [3]. We obtain

$$f(\tilde{t} - \tilde{x}) = \int_{-\infty}^{\infty} F(u) e^{2\pi i u(\tilde{t}-\tilde{x})} du, \tag{3.29}$$

where $F(u)$ is a spectrum of a function $f(z)$ defined as its Fourier transform

$$F(u) = \int_{-\infty}^{\infty} f(z) e^{-2\pi i z u} du. \tag{3.30}$$

Taking into account the fact that $f(z)$ is a real function, its transform $F(u)$ becomes a Hermitian function satisfying the condition [3]

$$F^*(-u) = F(u),$$

where $F^*(u)$ is a complex function coupled with the function $F(u)$.

Thus, the real part of the function $F(u)$ is an even function and its imaginary part is an odd function.

Since $\exp(2\pi i u(\tilde{t}-\tilde{x}))$ is also a Hermitian function with respect to the variable u and the product of Hermitian functions remains a Hermitian function as well, the expansion (3.29) can be rearranged to become

$$f(\tilde{t} - \tilde{x}) = \int_{0}^{\infty} \operatorname{Re}\left\{2F(u) e^{2\pi i u(\tilde{t}-\tilde{x})}\right\} du. \tag{3.31}$$

Considering the expression (3.31) as a decomposition of an arbitrary wave $f(\tilde{t}-\tilde{x})$ into the harmonic components, the function $2F(u)$ appears to be a distribution density of the wave amplitude, while $2F(u)\,du$ is an amplitude of a wave with a frequency of the value from the range $<u, u + du>$.

The expression (3.31) allows us to consider the problem of interaction of an impulse wave with a porous layer as a superposition of interactions of the particular harmonic components of its expansion.

Assuming that

$$u = \eta, \qquad A_1 = 2F(\eta)\,d\eta,$$
$$\sigma^{II} = d\tilde{\sigma}^{II}, \qquad p^{\alpha} = d\tilde{p}^{\alpha} \qquad (\alpha = I,\ II,\ III),$$

the stress fields corresponding to the harmonic components of the impulse wave are given by the relations (3.18)–(3.21), while its resultant fields in the particular regions of the system can be obtained by integrating these relations over the whole frequency range for which the harmonic components exist, i.e. over $\eta \in <0, \infty)$. Taking into account the fact that all frequency-dependent quantities appearing in

(3.18)–(3.21) are Hermitian functions of $i\eta$, the resultant fields can be expressed in the form

$$\tilde{p}^{\mathrm{I}}(\tilde{x},\tilde{t}) = \int_{-\infty}^{\infty} F(\eta)\left(e^{2\pi i\eta(\tilde{t}-\tilde{x})} + \overline{D}_1 e^{2\pi i\eta(\tilde{t}-\tilde{x})}\right) d\eta \tag{3.32}$$

for $\tilde{x} \leq 0$;

$$\tilde{\sigma}^{\mathrm{II}}(\tilde{x},\tilde{t}) = \int_{-\infty}^{\infty} F(\eta)\left(\overline{A}_1^p s_1 e^{2\pi i\eta(\tilde{t}-K_1\tilde{x})} + \overline{A}_2^p s_2 e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})} + \overline{D}_1^p s_1 e^{2\pi i\eta(\tilde{t}+K_1\tilde{x})} + \overline{D}_2^p s_2 e^{2\pi i\eta(\tilde{t}+K_2\tilde{x})}\right) d\eta, \tag{3.33}$$

$$\tilde{p}^{\mathrm{II}}(\tilde{x},\tilde{t}) = \int_{-\infty}^{\infty} F(\eta)\left(\overline{A}_1^p e^{2\pi i\eta(\tilde{t}-K_1\tilde{x})} + \overline{A}_2^p e^{2\pi i\eta(\tilde{t}-K_2\tilde{x})} + \overline{D}_1^p e^{2\pi i\eta(\tilde{t}+K_1\tilde{x})} + \overline{D}_2^p e^{2\pi i\eta(\tilde{t}+K_2\tilde{x})}\right) d\eta, \tag{3.34}$$

for $0 \leq \tilde{x} \leq 1$, and

$$\tilde{p}^{\mathrm{III}}(\tilde{x},\tilde{t}) = \int_{-\infty}^{\infty} F(\eta)\overline{A}_3 e^{2\pi i\eta(\tilde{t}-\tilde{x})}\, d\eta, \tag{3.35}$$

for $\tilde{x} \geq 1$.

The expressions (3.32)–(3.35) satisfy the appropriate dynamic equations (3.1)–(3.3) and the compatibility conditions (2.16)–(2.21) given on both surfaces bounding the porous layer. These expressions are general solutions of the problem of interaction of an arbitrary plane pressure wave travelling in the bulk fluid with saturated deformable porous layer at normal incidence of the wave.

The solution (3.32)–(3.35) for a plane harmonic wave of the frequency η' and the amplitude A_1, the spectrum of which (the Fourier transform) is described by the Dirac delta function

$$F(\eta) = A_1\delta(\eta - \eta'),$$

reduces to the form given by the formulas (3.18)–(3.21).

In the case when the incident wave is an impulse-like pressure wave of the Dirac delta type with a unit area, its spectrum is a constant function equal to unity over the whole frequency range, i.e.

$$F(\eta) = 1.$$

Then the solution (3.32)–(3.35) takes the form which is called the impulse response of the system.

To obtain the explicit functional form of the solutions (3.32)–(3.35) it is necessary to invert the Fourier transforms involved. Such inversion is not difficult if the fraction $1/M$ in the expressions for waves amplitudes (see (3.24)) is expanded into the power series with respect to exponential functions. This leads, however, to solutions written by means of multiple series.

In most problems treating the interaction of an impulse-like wave with a porous layer, the knowledge of all kinds of impulse waves being formed in the system is not necessary; only the first waves leaving the layer on its both sides are important.

The explicit forms of these waves can be obtained by separation of the exponential functions of the lowest power from the equations for amplitudes of the reflected $(3.24)_1$ and penetrating $(3.24)_2$ wave.

We obtain

$$\overline{D}_1 = r + (1+r)\Big(m(m+1)e^{-2z_1} + n(n+1)e^{-2z_2} + 2(n+1)(m+1)e^{-z_1-z_2}\Big) + \overline{D}_1', \tag{3.36}$$

$$\overline{A}_3 = (1+r)\left((m+1)e^{-z_1} + (n+1)e^{-z_2}\right)e^{2\pi i\eta} + \overline{A}_3', \tag{3.37}$$

where

$$\overline{D}_1' = \frac{r+1}{M}\Big\{\Big(m(m+1)e^{-2z_1} + n(n+1)e^{-2z_2} + 2(n+1)(m+1)e^{-z_1-z_2}\Big)(1-M) + r(r+1)e^{-2(z_1+z_2)}\Big\},$$

$$\overline{A}_3' = \frac{r+1}{M}e^{2\pi i\eta}\Big\{(m+1)e^{-z_1}\left(1-M-e^{-2z_2}\right) + (n+1)e^{-z_2}\left(1-M-e^{-2z_1}\right)\Big\}.$$

Taking into account expressions (3.36) and (3.37), the solutions (3.32) and (3.35) for acoustic fields on both faces of the layer assume the form

$$\tilde{p}^{\mathrm{I}}(\tilde{x},\tilde{t}) = f(\tilde{t}-\tilde{x}) + rf(\tilde{t}+\tilde{x}) + (r+1)m(m+1)f(\tilde{t}+\tilde{x}-2K_1) + (r+1)n(n+1)f(\tilde{t}+\tilde{x}-2K_2) + 2(r+1)(m+1)(n+1)f(\tilde{t}+\tilde{x}-K_1-K_2) + \int_{-\infty}^{\infty} F(\eta)\overline{D}_1' e^{2\pi i\eta(\tilde{t}+\tilde{x})}\,d\eta, \tag{3.38}$$

$$\tilde{p}^{\mathrm{III}}(\tilde{x},\tilde{t}) = (r+1)(m+1)f(\tilde{t}-\tilde{x}-K_1+1) + (r+1)(n+1)f(\tilde{t}-\tilde{x}-K_2+1) + \int_{-\infty}^{\infty} F(\eta)\overline{A}_3' e^{2\pi i\eta(\tilde{t}-\tilde{x})}\,d\eta. \tag{3.39}$$

The coefficients associated with functions $f(\alpha)$ of different arguments are amplitudes of waves, their "history" of motion in the porous layer being defined

by the numbers occurring at quantities K_1 and K_2. These numbers show how many times a wave runs across the layer as the fast wave and the slow wave, respectively.

For example:

r is the amplitude of the reflected wave;

$(r+1)m(m+1)$ is the amplitude of the wave leaving the layer to region I, which has travelled across the layer twice as the fast wave;

$(r+1)(n+1)$ is the amplitude of the wave leaving the layer to the region III, which has travelled the layer as the slow wave.

Following a similar way, we can find amplitudes of the succeeding waves leaving the layer.

4. Interaction of an impulse wave with the halfspace of porous material

The analysis of interaction of the plane impulse wave in a fluid with the porous medium becomes much easier in the case when such a wave falls perpendicularly on the surface of the halfspace of the porous medium. Then the acoustic field in the bulk fluid is formed by the incident and the reflected waves, while such a field in the halfspace of porous medium is composed only of the fast and slow waves.

Solution of the problem of wave interaction with the porous halfspace can be obtained directly from expressions (3.32)–(3.34) and Eqs. $(3.23)_1$–$(3.23)_3$ regarding the fact that in the porous halfspace there are no waves propagating towards its surface. Putting

$$\overline{D}_1^p = \overline{D}_2^p = 0, \tag{4.1}$$

from equations $(3.23)_1$–$(3.23)_3$ we have

$$\overline{D}_1 = r, \qquad \overline{A}_1^p = A_1^\infty = \frac{K_1}{f_v^o}(m+1), \qquad \overline{A}_2^p = A_2^\infty = \frac{K_2}{f_v^o}(n+1), \tag{4.2}$$

where r, m, n, are given by (3.26).

Taking into account that amplitudes (4.2) do not depend on the frequency η, the Fourier transforms $F(\eta)$ in expressions (3.32)–(3.34) can be simply inverted. In this case the acoustical fields in both halfspaces take the forms

$$\tilde{p}^{\mathrm{I}}(\tilde{x},\tilde{t}) = f(\tilde{t}-\tilde{x}) + rf(\tilde{t}+\tilde{x}) \tag{4.3}$$

for $\tilde{x} \leq 0$, and

$$\tilde{p}^{\mathrm{II}}(\tilde{x},\tilde{t}) = A_1^\infty f(\tilde{t}-K_1\tilde{x}) + A_2^\infty f(\tilde{t}-K_2\tilde{x}), \tag{4.4}$$

$$\tilde{\sigma}^{\mathrm{II}}(\tilde{x},\tilde{t}) = A_1^\infty s_1 f(\tilde{t}-K_1\tilde{x}) + A_2^\infty s_2 f(\tilde{t}-K_2\tilde{x}) \tag{4.5}$$

for $\tilde{x} \geq 0$, where the dimensionless quantities $\tilde{x}$ and $\tilde{t}$ are defined by expressions

$$\tilde{x} = x/b_o, \qquad \tilde{t} = ta_o/b_o.$$

Here b_o is a certain constant reference legth.

Considering the explicit dependence of quantities A_α^∞, s_α, K_α ($\alpha = 1, 2$) on the parameters characterizing the pore structure of porous halfspace, the mechanical properties of both physical components of the system and the undisturbed state of the medium, equations (4.3)–(4.5) enable us to analyse the influence of these parameters on the reflection and transmission of an impulse wave through the surface of deformable porous material filled with fluid. An example of a course of this phenomenon, for the wave of initial profile given by the function (Fig. 1),

$$f(\tilde{x}, \tilde{t})\Big|_{\tilde{t}=0} = e^{\tau(\tilde{x}-\tilde{x}_o)}\,(H(\tilde{x}-\tilde{x}_o)-1)\sin[2\pi\eta^o(\tilde{x}-\tilde{x}_o)],$$

is shown in the Fig. 2. $H(\tilde{x})$ is the Heaviside function, η^o and τ are the dimen-

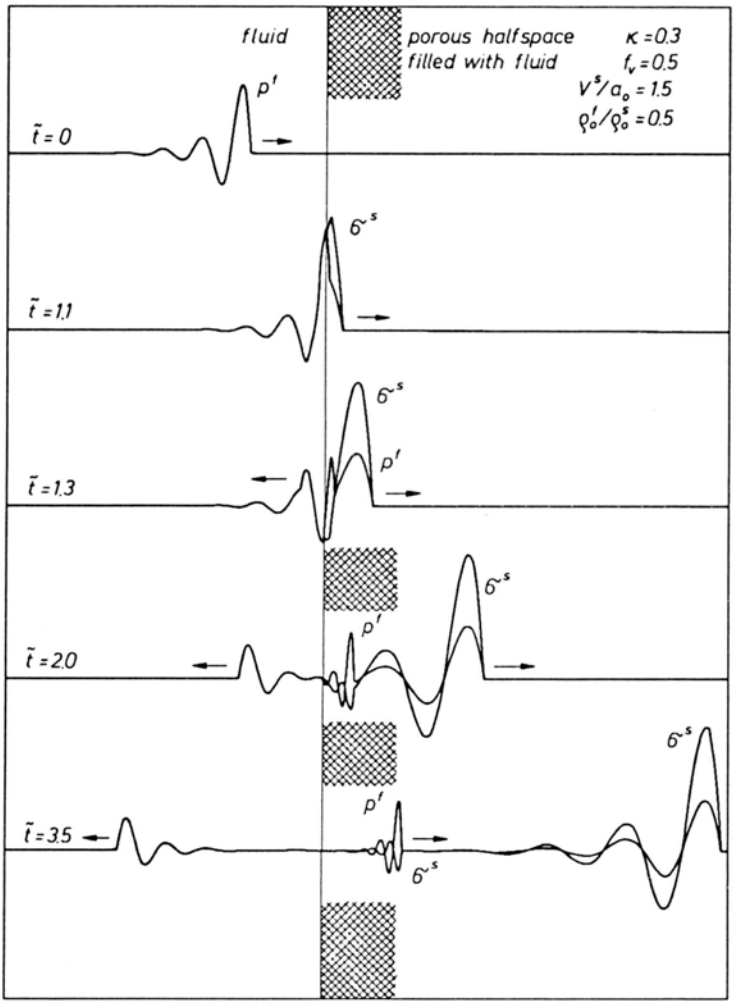

FIG. 2. Example of reflection and transmission of an impulse wave in fluid through the surface of deformable porous material.

sionless frequency and dimensionless coefficient of spatial fading of the wave. Coordinate $\tilde{x}_o$ stands for the initial position of the wave-front.

This figure is a good illustration of the process of formation of the fast and the slow waves during the wave penetration into the porous halfspace. It results from the form of these waves that the fast wave is the compressive type disturbance in both physical components of the medium, whereas the slow wave is of decompressive type in the skeleton and of compressive type in the fluid filling its pores. Such a behaviour of both waves agrees with the analytical results obtained in Sec. 3. Some doubts may arise from the fact that the skeleton stress amplitude in the fast wave is greater than the amplitude of the incident wave. This apparent discrepancy is conected with the fact that both stresses, in the skeleton and in the fluid filling its pores, illustrated in the figure, are effective, i.e. measured per unit surface of the skeleton and fluid, respectively. Additionally, in accordance to (2.12), the part of effective stress in the skeleton caused by the fluid pressure, due to the incompressibility of material of the skeleton, do not give rise to the energy of wave.

5. Final remarks

In the paper we have considered the problem of interaction of a plane impulse wave in a fluid with a deformable layer of porous material immersed in it. Basing on the two-parametric theory of fluid-filled porous solid, the purely elastic case of the medium with incompressible skeleton material has been analysed. It enabled us to reduce the number of necessary material constants in the plane problem to two quantities: the velocity of compressional wave in the nonsaturated skeleton and velocity in the bulk fluid (without skeleton), giving a very effective model of the problem.

The initial-boundary value problem was solved analytically and the explicit expressions relating the basic quantities which characterize the course of wave interaction with the pore structure parameters, material constants and densities of physical components have been obtained. Such relations make it possible to analyse the influence of these parameters on the impulse wave interaction with porous layer.

The obtained results, independently of their cognitive and practical importance for interpretation of the experimental data are also a good starting point for the considerations extended to the case when the fluid is viscous and the material of the skeleton is compressible.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received December 19, 1994.

Arch. Mech., 47, 4, pp. 813–824, Warszawa 1995

A direct boundary integral method for the two-dimensional lifting flow

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A REGULARIZED integral formulation in terms of tangential velocity for two-dimensional incompressible non-lifting and lifting flow is developed. In the case of lifting flow, a linear equality in terms of the tangential velocity is used as an equivalent of the equal-pressure Kutta condition. The numerical comparisons with some indirect panel metods show the superiority of the present method in the case of lifting flow.

1. Introduction

THE TWO-DIMENSIONAL FLOW is mainly studied by indirect or direct boundary integral methods.

The indirect methods (so-called singularity methods) expresses the velocity potential at any point of the flow as a combination of the velocity potential induced by a surface distribution of sources, doublets and/or vortices. An exhaustive list of such methods is presented in KATZ and PLOTKIN [8]. In DRAGOŞ [3] it is shown that for non-lifting bodies, the direct methods give better results than the indirect ones.

The direct methods express the main unknown of the problem by its values on the surface of the body. Such methods are the potential function method (BINDOLINO *et al.* [2], MORINO and CHING CHANG [10]), the stream function method (GRILO *et al.* [6]) and the kinetic field method (BASSANINI *et al.* [1], KOZLOV *et al.* [9], DRAGOŞ and DINU [4, 5]), the last method using the tangential velocity as unknown.

The method analyzed herein is an improved version of the direct kinetic method developed in [5]. The integral formulation is regularized, i.e. it is free of singular integrals, so we don't have to deal with the limit process from the domain to profile contour. Also, the deduction of the integral equation is made in a more systematic way than in [5]. Using the equivalence between equal pressures and tangential velocities, the Kutta condition is straightforwardly formulated in the same terms as the integral equation.

2. Statement of problem

The incompressible flow of an inviscid fluid in a two-dimensional domain D exterior to a given profile, is governed by the system of differential equations

$$\text{(2.1)} \qquad \operatorname{div}\mathbf{V} = 0, \qquad \operatorname{rot}\mathbf{V} = 0,$$

and by the boundary condition of tangential flow along the profile C

$$\mathbf{V} \cdot \mathbf{n} = 0. \tag{2.2}$$

Here, $\mathbf{V}$ has physical dimension.

We assume that the motion is a superposition of the onset flow and of a perturbation flow generated by the profile. In that case, the global velocity $\mathbf{V}$ may be written as

$$\mathbf{V} = U_\infty(\mathbf{i} + \mathbf{v}), \tag{2.3}$$

where $U_\infty \mathbf{i}$ is the velocity of the onset flow and $\mathbf{v}$ is the non-dimensional velocity of the perturbation flow. In terms of perturbation, the system (2.1) becomes

$$\operatorname{div} \mathbf{v} = 0, \qquad \operatorname{rot} \mathbf{v} = 0, \tag{2.4}$$

which is valid even when the function $\mathbf{v}$ is translated by any constant vector $\mathbf{c}$, i.e.

$$\operatorname{div}(\mathbf{v} - \mathbf{c}) = 0, \qquad \operatorname{rot}(\mathbf{v} - \mathbf{c}) = 0. \tag{2.5}$$

The boundary condition (2.2) is written in terms of perturbations as

$$\mathbf{v} \cdot \mathbf{n} = -n_x. \tag{2.6}$$

If we denote by $v_n \equiv \mathbf{v} \cdot \mathbf{n}$ and $v_s \equiv \mathbf{v} \cdot \mathbf{s}$ the normal and tangential components, Eq. (2.6) may be written as

$$v_n = -n_x \tag{2.6}'$$

and we will write

$$\mathbf{v} = -n_x \mathbf{n} + v_s \mathbf{s}. \tag{2.7}$$

It is also required that the perturbation velocity vanishes at infinity

$$\lim_{\mathbf{x} \to \infty} \mathbf{v}(\mathbf{x}) = 0. \tag{2.8}$$

Thus, the problem is to solve the differential equations system (2.5) with the supplementary conditions (2.8) and (2.6).

3. Non-lifting flow

3.1. Integral representation formula

In this section we will deduce a regularized representation formulae for the velocity at any point of $D + C$ in terms of the velocity on C.

Let $\mathbf{v}^* = (u^*, v^*)$ be the fundamental solution determined by the system

$$\operatorname{div} \mathbf{v}^* = \delta(\mathbf{x}), \qquad \operatorname{rot} \mathbf{v}^* = 0 \tag{3.1}$$

(δ is the Dirac's function) which is a source-type solution and is given by

$$\mathbf{v}^*(\mathbf{x}) = \operatorname{grad} \mathcal{E}, \qquad \mathcal{E}(\mathbf{x}) \equiv \frac{1}{2\pi} \ln R, \qquad R = \sqrt{x^2 + y^2}\,. \tag{3.2}$$

For any functions f and g we have the identity (due to Eq. (2.5))

$$\int_D [f \operatorname{div}(\mathbf{v} - \mathbf{c}) + (g\mathbf{k}) \cdot \operatorname{rot}(\mathbf{v} - \mathbf{c})]\, da = 0. \tag{3.3}$$

In order to apply the Gauss-type formulas (A.2), (A.3) on the unbounded domain D, we take a supplementary remote circle C_∞ of radius r_∞ (Fig. 1) and define

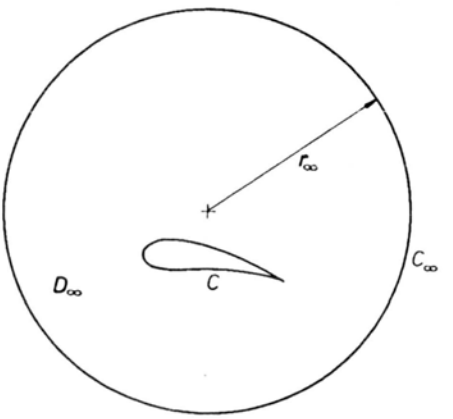


FIG. 1. The domain of integration.

D_∞, the domain bounded by C and C_∞. Now we are able to apply the Gauss formulas on the domain D_∞ and we obtain

$$\int_{D_\infty} (\mathbf{v} - \mathbf{c}) \cdot (\operatorname{grad} f - \operatorname{rot}(g\mathbf{k}))\, da = \int_{C+C_\infty} (\mathbf{v} - \mathbf{c}) \cdot (f\mathbf{n} - \mathbf{n} \times (g\mathbf{k}))\, ds, \tag{3.4}$$

where $\mathbf{n}$ is the normal unit vector directed outwards from the fluid (to the profile). For the following sets of functions

$$\begin{aligned} &(f, g) \to (u^*(\mathbf{x} - \mathbf{x}_0),\ -v^*(\mathbf{x} - \mathbf{x}_0)), \\ &(f, g) \to (v^*(\mathbf{x} - \mathbf{x}_0),\ u^*(\mathbf{x} - \mathbf{x}_0)), \qquad \mathbf{x}_0 \in D_\infty + \partial D_\infty, \end{aligned} \tag{3.5}$$

we get from (3.4) the equation

$$\int_{D_\infty} (\mathbf{v} - \mathbf{c}) \Delta \mathcal{E}\, da = \int_{C+C_\infty} [\mathbf{n} \cdot (\mathbf{v} - \mathbf{c}) \operatorname{grad} \mathcal{E} + (\mathbf{n} \times (\mathbf{v} - \mathbf{c})) \times \operatorname{grad} \mathcal{E}]\, ds. \tag{3.6}$$

Making $r_\infty \to \infty$ and using (3.1), (3.2), the domain integral is

$$\int_D (\mathbf{v} - \mathbf{c})\cdot\Delta\mathcal{E}\, da = \int_D (\mathbf{v} - \mathbf{c})(\mathbf{x})\cdot\delta(\mathbf{x} - \mathbf{x}_0)\, da = \mathbf{v}(\mathbf{x}_0) - \mathbf{c}.$$

Putting $\mathbf{c} \equiv \mathbf{v}(\mathbf{x}_0) \equiv \mathbf{v}_0$ and using the condition of vanishing (2.8), from (3.6) we obtain the representation formulae

$$\mathbf{v}_0 = \int_C^R [\mathbf{n}\cdot(\mathbf{v} - \mathbf{v}_0)\operatorname{grad}\mathcal{E} + (\mathbf{n} \times (\mathbf{v} - \mathbf{v}_0)) \times \operatorname{grad}\mathcal{E}]\, ds, \qquad \mathbf{x}_0 \in D + C. \tag{3.7}$$

In (3.7), the left-hand side results from the evaluation of the integral over C_∞ by using polar coordinates. This is a *regularized* integral representation formulae, which is valid both in the domain D and on the profile C. Although the function $\mathcal{E}$ has a singularity at $\mathbf{x} = \mathbf{x}_0 \in C$, the integral is of ordinary type, not Cauchy type, due to the fact that the term $\mathbf{v}(\mathbf{x}) - \mathbf{v}_0$ becomes zero when $\mathbf{x} \to \mathbf{x}_0 \in C$.

If we observe that on C we have

$$\mathbf{n} \times \mathbf{v} = (\mathbf{v} \cdot \mathbf{s})\mathbf{k} \equiv v_s\, \mathbf{k},$$

in Eq. (3.7) we use the boundary condition (2.6) and also the Eqs. (A.1) and (2.7), getting the representation formula for the perturbation velocity

$$\mathbf{v}_0 = \int_C^R \left\{ v_s \mathbf{k} \times \operatorname{grad}\mathcal{E} + v_s^0 \left(\mathbf{n}\frac{d\mathcal{E}}{ds_0} - \mathbf{s}_0 \frac{d\mathcal{E}}{dn} \right) - v_s(\mathbf{n} \cdot \mathbf{s}_0) \operatorname{grad}\mathcal{E} \right. \tag{3.8}$$

$$\left. + (-n_x + n_x^0 \mathbf{n} \cdot \mathbf{n}_0) \operatorname{grad}\mathcal{E} + n_x^0 \left(\mathbf{n}_0 \frac{d\mathcal{E}}{dn} - \mathbf{n}\frac{d\mathcal{E}}{dn_0} \right) \right\} ds.$$

3.2. Integral boundary equations

In order to get v_s outside the integral, we apply the cross-product $\mathbf{n}_0 \times$ (3.8) and we use the relation

$$\mathbf{n}_0 \times \mathbf{n} = (\mathbf{n} \cdot \mathbf{s}_0)\mathbf{k}.$$

By that, the one-dimensional boundary equation for the unknown v_s is obtained

$$v_s^0 - \int_C^R \left(v_s \cdot \frac{d\mathcal{E}}{dn^0} - v_s^0 \cdot \frac{d\mathcal{E}}{dn} \right) ds = \int_C^R \left(n_x^0 \cdot \frac{d\mathcal{E}}{ds} - n_x \cdot \frac{d\mathcal{E}}{ds_0} \right) ds. \tag{3.9}$$

Using the solution of (3.9), the perturbation $\mathbf{v}$ on C is obtained from the algebraic relation (2.7).

Now that we have the perturbation field $\mathbf{v}\big|_C$, the global velocity at any point $\mathbf{x}_0$ in th flow domain D may be determined from Eqs. (3.8) and (2.3).

4. The lifting flow

A lifting profile has a non-smooth contour with a sharp trailing edge (TE). In that case a suplementary condition must be satisfied by the velocity field: finite value at the TE, that is the Kutta–Joukowski condition.

From various properties related to the Kutta condition (HESS [7]) we have chosen the one that has the most obvious physical meaning: as the trailing edge is approached, the velocity magnitudes (pressures) on the upper and lower surface have a common limit, i.e.

$$\lim_{\substack{P\to P_{TE}\\ P\in \text{upper surface}}} |\mathbf{V}(P)| = \lim_{\substack{P\to P_{TE}\\ P\in \text{lower surface}}} |\mathbf{V}(P_{TE})|. \tag{4.1}$$

We recall that **V** is the velocity that has physical dimension.

In the two-dimensional case, the equal-pressure condition at TE may be written as

$$\mathbf{V}(P_{up})\cdot\mathbf{S} + \mathbf{V}(P_{low})\cdot\mathbf{S} \xrightarrow[P_{up},P_{low}\to P_{TE}]{} 0 \tag{4.2}$$

due to the fact that the dot products have the same magnitude (**S** being a unit vector) but opposite signs (Fig. 2). In terms of perturbation, Eq. (4.2) is written

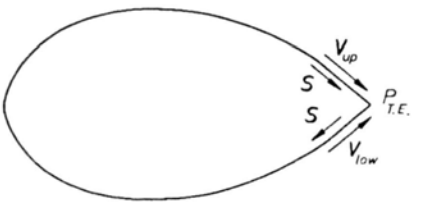


FIG. 2. Velocity direction at trailing edge.

as

$$(v_s^{up} - n_y^{up}) + (v_s^{low} - n_y^{low}) \to 0. \tag{4.3}$$

Obviously, in discrete (numerical) formulation of the problem with constant elements, Eq. (4.3) must be used in the form

$$v_s^{up} + v_s^{low} = n_y^{up} + n_y^{low}, \tag{4.4}$$

i.e. in terms of th unknown v_s from the integral equation (3.9).

Thus, the problem (3.9)+(4.4) determines the tangential perturbation velocity v_s on the profile C. This in an over-determined problem whose solution is obtained in the same way as the vortex method (see next section). Then, the global velocity in $D + C$ domain may be obtained (if desired) from Eqs. (3.8) and (2.3).

5. Numerical comparisons

In this section we make numerical comparisons between the present direct method and the indirect methods that use surface singularities. Both cases of lifting and non-lifting are considered for which analytic solutions are available. In the lifting case, the earlier direct method with regularizing variable is also considered.

We have solved the integral boundary equations by using the well-known method of collocation, and constant value elements.

5.1. Integral boundary equations

The first indirect method considered is called the *source method* [8] and it is applicable to non-lifting flow. The source method is based on the following integral boundary equations written in terms of perturbation:

representation formula

$$\mathbf{v}(\mathbf{x}_0) = \int_C \sigma(\mathbf{x}) \cdot \operatorname{grad} \mathcal{E}(\mathbf{x} - \mathbf{x}_0)\, ds, \quad \mathbf{x}_0 \in C + D,$$

boundary condition

$$\mathbf{v} \cdot \mathbf{n} = -n_x,$$

integral equation

$$\frac{1}{2}\sigma(\mathbf{x}_0) + \int_C \sigma(\mathbf{x}) \cdot \frac{d\mathcal{E}}{dn^0}(\mathbf{x} - \mathbf{x}_0)\, ds = -n_x(\mathbf{x}_0), \quad \mathbf{x}_0 \in C.$$

The second indirect method is called the *vortex method* [8] and it is applicable to both the lifting and non-lifting flows. Its particularity is that it uses an internal boundary condition [8, p. 328], i.e. the boundary condition is formulated at points inside the profile. The integral boundary equations are:

representation formulas

$$\mathbf{v}(\mathbf{x}_0) = -\mathbf{k} \times \int_C \gamma(\mathbf{x}) \cdot \operatorname{grad} \mathcal{E}(\mathbf{x} - \mathbf{x}_0)\, ds, \qquad \mathbf{x}_0 \in C + D,$$

internal zero-tangential boundary condition

$$\mathbf{v}(\mathbf{x}_0) \cdot \mathbf{s}(\mathbf{x}_0) = -s_x(\mathbf{x}_0), \qquad \mathbf{x}_0 \in \operatorname{int} C,$$

integral equation

$$\frac{1}{2}\gamma(\mathbf{x}_0) - \int_C \gamma(\mathbf{x}) \cdot \frac{d\mathcal{E}}{dn^0}(\mathbf{x} - \mathbf{x}_0)\, ds = -s_x(\mathbf{x}_0), \qquad \mathbf{x}_0 \in \operatorname{int} C,$$

Kutta condition

$$\gamma_{\text{up}} + \gamma_{\text{low}} = 0.$$

In numerical implementation, one of the algebraic equations that comes from discretisation must be replaced by the Kutta condition.

The *earlier direct method* [5] is based on the following equations:

representation formulas

$$\mathbf{v} = -n_x \mathbf{n} + v_s \mathbf{s} \quad \text{on } C,$$

$$\mathbf{v}(\mathbf{x}_0) = \int_C v_s(\mathbf{x}) \cdot \mathbf{k} \times \operatorname{grad} \mathcal{E}(\mathbf{x} - \mathbf{x}_0)\, ds - \int_C n_x \operatorname{grad} \mathcal{E}(\mathbf{x} - \mathbf{x}_0)\, ds, \quad \mathbf{x}_0 \in D,$$

integral equation

$$\frac{1}{2} v_s(\mathbf{x}_0) + \gamma - \int_C \frac{d\mathcal{E}}{dn^0}(\mathbf{x} - \mathbf{x}_0) \cdot v_s(\mathbf{x}_0)\, ds = -\int_C n_x \frac{d\mathcal{E}}{ds^0}(\mathbf{x} - \mathbf{x}_0)\, ds, \quad \mathbf{x}_0 \in C, \quad \mathbf{x}_0 \in \operatorname{int} C,$$

Kutta condition

$$v_s^{\text{up}} + v_s^{\text{low}} = n_y^{\text{up}} + n_y^{\text{low}}.$$

Here γ is a regularizing variable.

For completeness we recall also the equations for the *present direct method*:

representation formulas

$$\mathbf{v} = -n_x \mathbf{n} + v_s \mathbf{s} \quad \text{on } C,$$

$$\mathbf{v}_0 = \int_C^R \left\{ v_s \mathbf{k} \times \operatorname{grad} \mathcal{E} + v_s^0 \left(\mathbf{n} \frac{d\mathcal{E}}{ds_0} - \mathbf{s}_0 \frac{d\mathcal{E}}{dn} \right) - v_s (\mathbf{n} \cdot \mathbf{s}_0) \operatorname{grad} \mathcal{E} \right.$$

$$\left. + (-n_x + n_x^0 \mathbf{n} \cdot \mathbf{n}_0) \operatorname{grad} \mathcal{E} + n_x^0 \left(\mathbf{n}_0 \frac{d\mathcal{E}}{dn} - \mathbf{n} \frac{d\mathcal{E}}{dn_0} \right) \right\} ds \quad \text{on } D,$$

integral equation

$$v_s^0 - \int_C^R \left(v_s \cdot \frac{d\mathcal{E}}{dn_0} - v_s^0 \cdot \frac{d\mathcal{E}}{dn} \right) ds = \int_C^R \left(n_x^0 \cdot \frac{d\mathcal{E}}{ds} - n_x \cdot \frac{d\mathcal{E}}{ds_0} \right) ds,$$

Kutta condition

$$v_s^{\text{up}} + v_s^{\text{low}} = n_y^{\text{up}} + n_y^{\text{low}}.$$

In numerical implementation, one of the algebraic equations that comes from discretisation must be replaced by the Kutta condition.

At last we mention that all integrals in this section are singular except the one following from the present method which is non-singular.

5.2. Non-lifting flow

We have made numerical comparisons for a circle in free air (Fig. 3a) and in ground effect (equivalent to two circles in free air, Fig. 3b) and for a Van de Vooren profile of 17% thickness without incidence (Fig. 3c). The obtained results

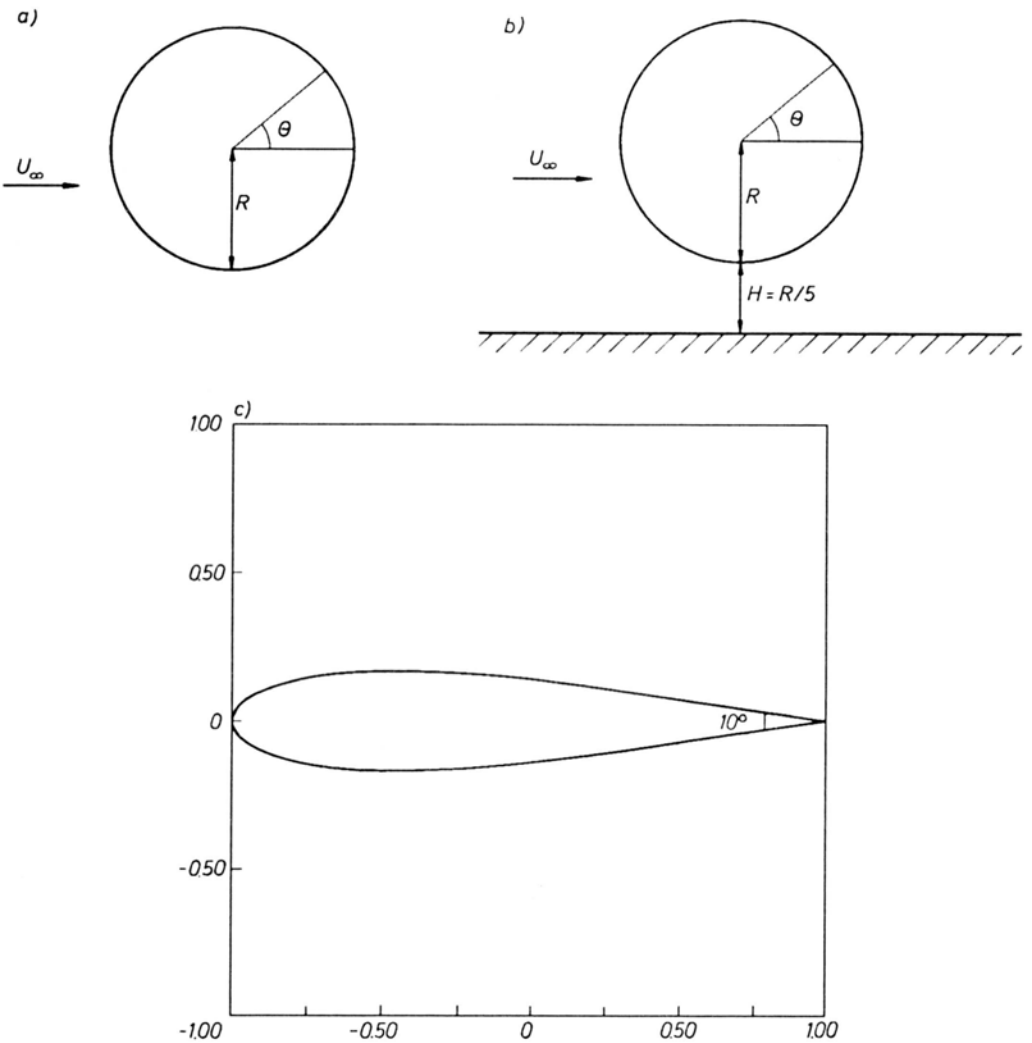
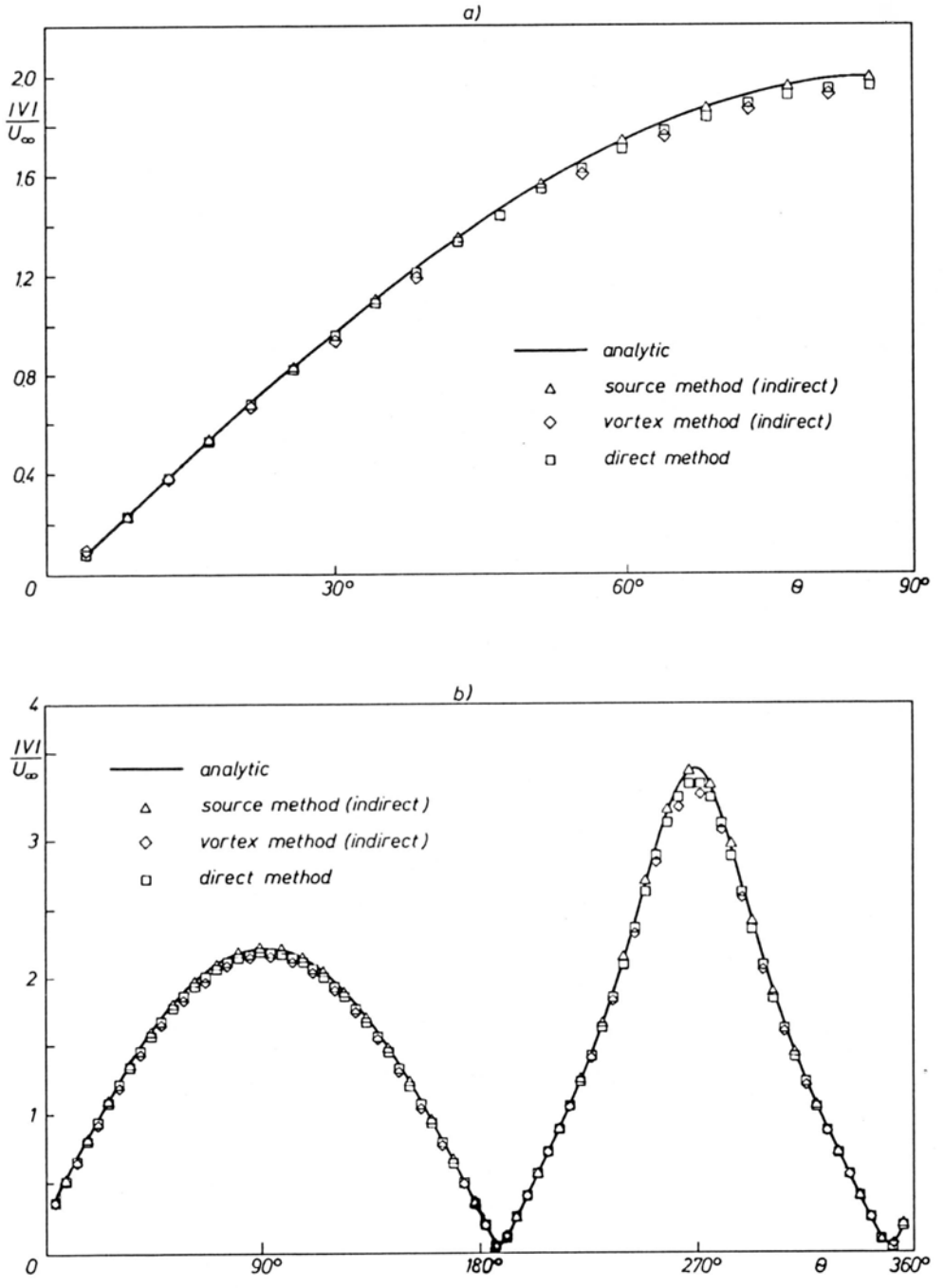


FIG. 3. Flow configurations; a) circle in free air, b) circle in ground effect, c) Van de Vooren airfoil 17% thickness.

are shown in Fig. 4abc. As it can be seen, the direct method is placed between the indirect methods: better than the vortex method in the case of the circle and better than the source method in the case of the thin profile. However, all three methods produce good results.



[FIG. 4]

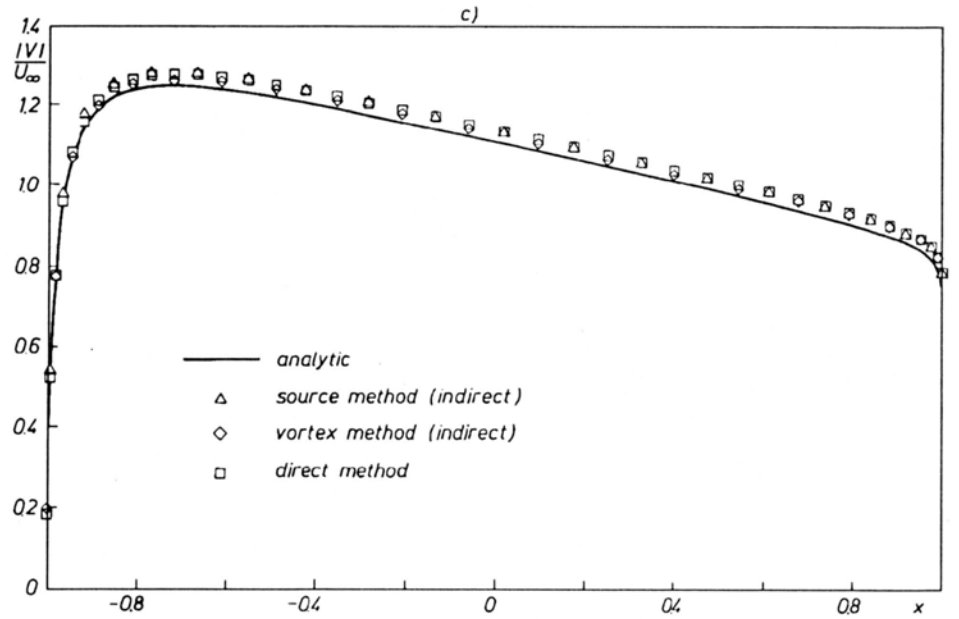


FIG. 4. Velocity distribution in case of non-lifting flow; a) circle in free air, b) circle in ground effect, c) Van de Vooren airfoil at 0° incidence.

5.3. Lifting flow

The lifting case have been studied for the Van de Vooren profile at 20° incidence (Fig. 5). We see that the present direct method improves the behaviour of the numerical solution, especially near the leading edge of the profile.

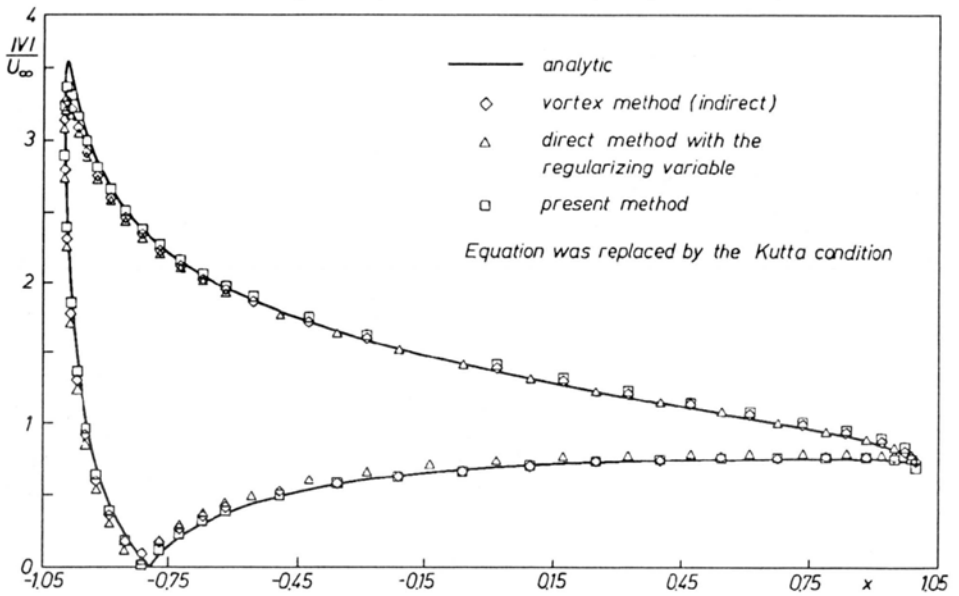


FIG. 5. Velocity distribution on Van de Vooren airfoil at 20° incidence.

6. Conclusions

The method is mainly recommended in the case of lifting flow where it had shown its superiority over the indirect methods.

The method may be used as an alternative to the indirect methods in case of non-lifting flow, giving at least the same precision and saving the computing effort, the tangential velocity being obtained directly from the integral equation.

Appendix

Vectorial formulas

$$(\mathbf{U} \times \mathbf{V}) \times \mathbf{W} = (\mathbf{U} \cdot \mathbf{W})\mathbf{V} - (\mathbf{V} \cdot \mathbf{W})\mathbf{U},$$

(A.1)

$$(\mathbf{U} \times \mathbf{V}) \times \mathbf{W} + (\mathbf{V} \times \mathbf{W}) \times \mathbf{U} + (\mathbf{W} \times \mathbf{U}) \times \mathbf{V} = 0.$$

Gauss-type formulas for a bounded domain D:

$$\int_D f \operatorname{div} \mathbf{V}\, da = -\int_D \mathbf{V} \cdot \operatorname{grad} f\, da + \int_C f \mathbf{n} \cdot \mathbf{V}\, ds, \tag{A.2}$$

$$\int_D \mathbf{V} \cdot \operatorname{rot} \mathbf{W}\, da = \int_D \mathbf{W} \cdot \operatorname{rot} \mathbf{V}\, da - \int_C \mathbf{W} \cdot (\mathbf{n} \times \mathbf{V})\, ds. \tag{A.3}$$

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Received December 21, 1994.