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A thermodynamical model of solids undergoing martensitic phase transformations with shuffles

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AN APPROACH, connected with the material structures, to the thermodynamical description of materials undergoing martensitic phase transformations with shuffles is proposed in the paper. Shuffle displacements are seen as a micro-nonhomogeneous deformation in mechanical description. First, one considers thermodynamics of materials which undergo the micro-nonhomogeneous deformation only. Such a deformation is described by means of relative displacement vectors. Higher gradients of deformation are taken into account. The material structure of differential type for a body undergoing micro-nonhomogeneous deformations is considered. Next, a form of the free energy for materials which undergo martensitic transformations with shuffles is discussed. With the aid of the thermodynamics previously formulated, a thermodynamical description of these phase transformations is suggested. However, some modifications connected with additional variables have been carried out. In this case the material structure of differential type is also considered. The possibility of using internal state variables for describing the dissipation is discussed.

1. Introduction

THE MARTENSITIC transformation appears as a complicated phenomenon in materials. The predominant feature of this diffusionless transformation is a shear strain (COHEN, OLSON, CLAPP [1]). We can consider martensitic structures with mobile interfaces. Then, the shape memory effect can appear. Some martensitic structures have no mobile interfaces. Then, such a body can be seen as a composite of martensitic grains.

In the case of mobile martensitic structure we come across several complicated phenomena. Then, many different martensite variants appear. They can transform one into another. Each of them can rotate towards the habit plane in order to fit together with the parent phase. Martensitic transformation is predominated by shear strain. However, during the shear strain the shuffles can appear additionally. The shuffles are connected with complex crystal lattice and are deviations of atoms from the positions indicated by homogeneous deformation on a microscopic level. It happenes, for instance, in alloys CuAl, CuAlNi, OTSUKA, SAKAMOTO, SHIMIZU [9], NISHIYJAMA, KAJIWARA [22]. Shuffles are connected with determination of miscelaneous martensite variants (KACZMAREK [10]). They are also connected with two-path displacive phase transformation, KACZMAREK [8].

The mechanical description of the martensitic transformation is rather difficult. Miscellaneous approaches are considered in the literature. There, one considers qualitative one-dimensional models. They are discussed in papers of FALK [3, 4], FALK, SEIBEL [33], JAMES [5, 34]. Some three-dimensional considerations are given by JAMES [6].

Exhaustive application of thermodynamics to the description of martensitic transformation is given in papers by RANIECKI, LEXCELLENT, TANAKA [31], RANIECKI, TANAKA [32]. It is assumed there that the composition of martensite is averaged by means of variable x which is the mass fraction of martensite in the total mass of the system.

Statistical description of this transformation is studied by MÜLLER, WILMAŃSKI [2].

Descriptions more connected with a single crystal and detailed physical phenomena occurring in it are given by BARSCH, KRUMHANSL [7, 39, 40] and KACZMAREK [8, 10].

Shuffle displacements are considered in papers COUSINS [11], BARRON, GIBBONS, MUNN [12], SŁONCZEWSKI, THOMAS [37], CAO, BARSCH [38].

In the present paper one considers a description connected with a single crystal. The aim is to take into consideration whole complexity of this phenomenon as far as possible. I do hope that the complicated model obtained could be helpful for a more averaged description. Thus, in the paper are considered the shuffles which are modelled by means of relative displacement vectors, internal rotations which describe rotations of the martensitic structure toward the habit plane, higher gradients of deformation which are helpful in description of the geometry of mobile interfaces. Furthermore, one tries to describe dissipative effects with the help of material structures. The dissipative effects are inherently present in martensitic transformation.

In Sec. 2 the thermodynamics of materials undergoing micro-nonhomogeneous deformation only is introduced. There the theory with higher gradients of deformation is studied. The dissipation is considered in the frame of the material structure of the differential type. In Sec. 3 a model of thermodynamics connected with martensitic phase transformations with shuffles is discussed. There, the model of the free energy introduced in [10] is used. In this section one considers the material structure of differential type. The material structure with internal state variables is also suggested.

2. A thermodynamical model of solids which undergo micro-nonhomogeneous deformations

2.1. The balance of energy

In the paper we consider different kinds of deformation. Namely, one considers the macroscopic deformation. It relates to an area larger than the material point and is described by means of the Green tensor field. This field of deformation can be homogeneous or not. Next, we consider a micro-homogeneous deformation. It is a homogeneous deformation of the material point and it is described by the Green tensor in this point. Finally, the micro-nonhomogeneous deformation is considered. It is a nonhomogeneous deformation of the material point connected with shuffles.

Let us consider a crystal with N_λ sublattices. Such a crystal can undergo the micro-nonhomogeneous deformation. This deformation consist in deviation of sublattices from their positions indicated by micro-homogeneous deformation. Such a deformation can be described with the help of N_λ deformation functions.

Let us distinguish a crystal sublattice. The deformation of this sublattice can be described by the function

$$\mathbf{x} = \chi(\mathbf{X}, t)\,. \tag{2.1}$$

Deformations connected with the remaining sublattices can be described with the help of $N_\lambda - 1$ deformation functions

$$\mathbf{y}_\lambda = \mathbf{x} + \mathbf{R}\mathbf{w}_\lambda\,, \tag{2.2}$$

where $\mathbf{R}$ stands for a rotation tensor obtained with the aid of the polar decomposition of the deformation gradient tensor $\mathbf{F} = \mathbf{VR}$. The vector $\mathbf{w}_\lambda$ indicates the deviation of the sublattice from the position determined by a micro-homogeneous deformation. The $\mathbf{w}_\lambda$

will be called further the relative displacement vector. One can show [10] that the vector $w_\lambda = \mathbf{R}^{-1}(\mathbf{y}_\lambda - \mathbf{x})$ is an objective Lagrangean vector [29].

The deformation functions (2.1) and (2.2) introduced suggest that the body considered can be seen as a multicomponent body [26, 28]. Therefore, we introduce densities ϱ_h connected with the distinguished crystal lattice and ϱ_λ connected with the remaining sublattices. Furthermore, we have $\varrho_h + \sum \varrho_\lambda = \varrho$ where ϱ is the total mass density of the crystal.

In the paper the summation convention does not concern the index λ.

In order to describe the behaviour of the body we postulate the following form of the balance energy equation:

$$(2.3)\quad \int\limits_{\chi(\mathcal{P})} [\rho\dot{\varepsilon} + \rho\ddot{x}_i x_i + \sum_\lambda \rho_\lambda \ddot{w}_{\lambda N}\dot{w}_{\lambda N} + q_{j,j} - r_e - \rho b_i \dot{x}_i]d\nu$$

$$- \int\limits_{\partial\chi(\mathcal{P})} \sum_{\phi=1}^{S} t_{M_1\ldots M_{\phi-1}i}\dot{x}_{i,M_1\ldots M_{\phi-1}} + \sum_\lambda \sum_{\phi=1}^{T} f_{\lambda N_1\ldots N_{\phi-1}N}\dot{w}_{\lambda N,N_1\ldots N_{\phi-1}})da = 0,$$

where ε is the internal energy density, q_j are components of heat flux vector, r_e is the source of heat, b_i is a vector of the volume force.

In description of the martensitic phase transformation one considers higher gradients of the deformations [10]. Therefore in Eq. (2.3) two last summands have the form which takes into account the theory with the higher gradients of the deformation [19, 20, 21]. The forces $t_{M_1\ldots M_{\phi-1}i}$ and $f_{\lambda N_1\ldots N_{\phi-1}N}$ are connected respectively with higher gradients of deformation given by the Green strain tensor and higher gradients of relative displacement vectors.

Equation (2.3) can be introduced starting from the multicomponent balance energy equation similar to that from [27], and taking into account the deformation functions (2.1), (2.2) and some simplifications.

The form of the kinetic energy assumed in Eq. (2.3) means that the influences of inertia of the mass connected with relative displacement vectors on the motion of the body, and of acceleration of the body on the motion of the relative displacement vectors are neglected. Furthermore, the influence of the rotational velocity and acceleration effects on the motion of variables $w_{\lambda N}$ is neglected.

Let us assume that the free energy density ψ depends on variables $\psi = \psi(x_{i,M_1}, \ldots$ $\ldots, x_{i,M_1\ldots M_L}, w_{\lambda N}, w_{\lambda N,N_1}, \ldots, w_{\lambda N,N_1\ldots N_M}, T)$, where T is absolute temperature. Taking it into account we obtain

$$(2.4)\quad \int\limits_{\chi(\mathcal{P})} \dot{\psi}d\nu = \int\limits_{\chi(\mathcal{P})} \left[\frac{\partial\psi}{\partial T}\dot{T} + \sum_{\delta=1}^{s}(-1)^\delta \left(\frac{\partial\psi}{\partial x_{i,M_1\ldots M_\delta}}\right)_{,M_\delta\ldots M_1} \dot{x}_i \right.$$

$$\left. + \sum_{\nu=0}^{T} \sum_\lambda (-1)^\nu \left(\frac{\partial\psi}{\partial w_{\lambda N,N_1\ldots N_\nu}}\right)_{,N_\nu\ldots N_1} \dot{w}_{\lambda N}\right] d\nu$$

$$+ \int\limits_{\partial\chi(\mathcal{P})} \left(\sum_{\phi=1}^{s}\sum_{\delta=\phi}^{s}(-1)^{\delta+1}\left(\frac{\partial\psi}{\partial x_{i,M_1\ldots M_\delta}}\right)_{,M_\delta\ldots M_{\phi+1}} (x_{k,M_\phi}n_k)\dot{x}_{i,M_1\ldots M_{\phi-1}}\right.$$

$$(2.4)\ [\text{cont.}] \quad + \sum_{\phi=1}^{T} \sum_{\nu=\phi\lambda}^{T} \sum (-1)^{\nu+1} \left(\frac{\partial \psi}{\partial w_{\lambda N, N_1 \ldots N_\nu}} \right)_{, N_\nu \ldots N_{\phi+1}} x_{k, N_\phi} n_k \dot{w}_{\lambda N, N_1 \ldots N_{\phi-1}} \Big) da$$

$$= \int_{\chi(\mathcal{P})} \left[\frac{\partial \psi}{\partial T} \dot{T} + A_i \dot{x}_i + \sum_{\lambda} B_{\lambda N} \dot{w}_{\lambda N} \right] dv + \int_{\partial\chi(\mathcal{P})} \Big[\sum_{\phi=1}^{s} A_{\phi i} \dot{x}_{i, M_1 \ldots M_{\phi-1}}$$

$$+ \sum_{\phi=1}^{T} \sum_{\lambda} B_{\phi\lambda N} \dot{w}_{\lambda N, N_1 \ldots N_{\phi-1}} \Big] da .$$

New quantities A_i, $A_{\phi i}$, $B_{\lambda N}$, $B_{\phi\lambda N}$ have been defined during the transformation of this identity. Furthermore we introduce a quantity A_{iM} with the help of the formula $A_{iM_1, M_1} = -A_i$ and the third term in the first line of Eq. (2.4). For $\nu = 0$ the derivative order is equal to zero in the third summand on the right-hand side of the first equality sign in Eq. (2.4). Derivatives of order equal to zero for $\dot{x}_i$ and $\dot{w}_{\lambda N}$ appear there for $\phi = 1$. If $\delta = \phi$ and $\nu = \phi$, then the derivatives connected with indices M_δ and N_ν vanish in the fourth and the fifth summands on the right-hand side in (2.4). $A_{\phi i}$ and $B_{\phi\lambda N}$ are equal to zero for $\delta < \phi$ and $\nu < \phi$.

Let us assume that dissipation can take place in the body. Then we assume that the dissipation occurs with the help of a stress tensor t^d_{iM} and some dissipative forces $f^d_{\lambda N}$. We introduce these quantities into the balance equation (2.3) with the aid of the formula

$$(2.5) \quad t^d_{iM} \dot{x}_{i,M} + \sum_{\lambda} f^d_{\lambda N} \dot{w}_{\lambda N} - t^d_{iM} \dot{x}_{i,M} + \sum_{\lambda} f^d_{\lambda N} \dot{w}_{\lambda N} = -t^d_{iM,M} \dot{x}_i + \sum_{\lambda} f^d_{\lambda N} \dot{w}_{\lambda N}$$

$$+ (t^d_{iM} \dot{x}_i)_{,M} - t^d_{iM} \dot{x}_{i,M} - \sum_{\lambda} f^d_{\lambda N} \dot{w}_{\lambda N} .$$

In view of Eqs. (2.4), (2.5), the assumed potential relation $\partial\psi/\partial T = -s$ and $\dot{\varepsilon} = \dot{\psi} + T\dot{s} + \dot{T}s$, where s is the entropy density, we can rewrite Eq. (2.3) in the following form:

$$(2.6) \quad \int_{\chi_i(\mathcal{P})} ((\rho A_i - t^d_{iM,M} - \rho b_i + \rho \ddot{x}_i)\dot{x}_i + \sum_{\lambda} [\rho B_{\lambda N} + f^d_{\lambda N} + \rho_\lambda \ddot{w}_{\lambda N}] \dot{w}_{\lambda N}$$

$$+ \rho T \dot{s} - t^d_{iM} \dot{x}_{i,M} - \sum_{\lambda} f^d_{\lambda N} \dot{w}_{\lambda N} + q_{j,j} - p_e - \rho r_e) dv$$

$$+ \int_{\partial\chi_i(\mathcal{P})} \Big(\sum_{\phi=1}^{s} [\rho A_{\phi i} - t_{M_1 \ldots M_{\phi-1} i}] \dot{x}_{i, M_1 \ldots M_{\phi-1}} + t^d_{iM_1} \dot{x}_i x_{k, M_1} n_k$$

$$+ \sum_{\phi=1}^{T} [\rho B_{\phi\lambda N} - f_{\lambda N_1 \ldots N_{\phi-1} N}] \dot{w}_{\lambda N, N_1 \ldots N_{\phi-1}} \Big) da .$$

The balance equation (2.6) leads to the following balance momentum equations:

$$(2.7) \quad -(\rho A_{iM} + t^d_{iM})_{,M} - \rho b_i + \rho \ddot{x}_i = 0 ,$$

and

$$(2.8) \quad \rho B_{\lambda N} + f^d_{\lambda N} + \rho_\lambda \ddot{w}_{\lambda N} = 0 .$$

The boundary conditions for Eqs. (2.7) and (2.8) are determined also with the aid of Eq. (2.6).

The local form of balance of energy will be introduced in view of Eqs. (2.6), (2.7) and (2.8). The following equation

$$\int_{\chi_t(\mathcal{P})} \left(\rho T\dot{s} - t^d_{iM}x_{i,M} - \sum_{\lambda} f^d_{\lambda N}\dot{w}_{\lambda N} + q_{j,j} - p_e - \rho r_e\right)dv = 0 \tag{2.9}$$

is assumed as a starting point.

Let us consider $T\dot{s} = \dot{\varepsilon} - \dot{\psi} - s\dot{T} = \dot{\varepsilon} - \left(\frac{\partial\psi}{\partial T}\dot{T} + s\dot{T}\right) - \left(\dot{\psi} - \frac{\partial\psi}{\partial T}\dot{T}\right) = \dot{\varepsilon} - \left(\dot{\psi} - \frac{\partial\psi}{\partial T}\dot{T}\right)$. With the help of this we obtain

$$\begin{aligned}\rho\dot{\varepsilon} - \left(\rho\frac{\partial\psi}{\partial x_{i,N}} + t^d_{iN}\right)\dot{x}_{i,N} - \sum_{\delta=1}^{s}\rho\frac{\partial\psi}{\partial x_{i,NP_1\ldots P_\delta}}\dot{x}_{i,NP_1\ldots P_\delta} \\ - \sum_{\lambda}\left(\rho\frac{\partial\psi}{\partial w_{\lambda N}} + f^d_{\lambda N}\right)\dot{w}_{\lambda N} \\ - \sum_{\lambda}\sum_{\nu=1}^{T}\rho\frac{\partial\psi}{\partial w_{\lambda N,Q_1\ldots Q_\nu}}\dot{w}_{\lambda N,Q_1\ldots Q_\nu} + q_{j,j} - \rho r_e = 0.\end{aligned} \tag{2.10}$$

The local forms of balance equations obtained here will be further helpful in postulating the forms of local balance equations in connection with the material structures.

2.2. The material structure for a body which undergoes the micro-nonhomogeneous deformation

In this section the material structure for a body which undergoes micro-nonhomogeneous deformation is suggested. First, one postulates local form of balance momentum and energy equations. Thus, in view of previously considered equations (2.7), (2.8), one postulates the balance momentum equations

$$-t_{iM,M} - \rho b_i + \rho\ddot{x}_i = 0 \tag{2.11}$$

and

$$f_{\lambda N} + \rho_\lambda\ddot{w}_{\lambda N} = 0, \tag{2.12}$$

where t_{iM} is the first Piola–Kirchhoff stress tensor and $f_{\lambda N}$ is a force connected with the relative displacement vector.

The balance energy equation is suggested with the help of Eq. (2.10) in the form

$$\begin{aligned}\rho\dot{\varepsilon} - \bar{t}_{iN}\dot{x}_{i,N} - \sum_{\delta=1}^{s}\bar{t}_{iNP_1\ldots P_\delta}\dot{x}_{i,NP_1\ldots P_\delta} - \sum_{\lambda}\bar{f}_{\lambda N}\dot{w}_{\lambda N} \\ - \sum_{\lambda}\sum_{\nu=1}^{T}\bar{f}_{\lambda NQ_1\ldots Q_\nu}\dot{w}_{\lambda N,Q_1\ldots Q_\nu} + q_{j,j} - \rho r_e = 0,\end{aligned} \tag{2.13}$$

where $\bar{t}_{iN}, \bar{t}_{iNP_1\ldots P_\delta}, \bar{f}_{\lambda N}, \bar{f}_{\lambda NQ_1\ldots Q_\nu}$ are some stresses and forces. They are connected with the previously introduced stress tensor t_{iM} and forces $f_{\lambda N}$, and will be considered in detail after formulas (2.27), (2.30) and in Eqs. (2.26), (2.27), (2.29). This large number

of stress tensors follows from the fact that the theory considered contains higher gradients of deformation. Furthermore, as it will be shown, they give a possibility of taking into account the dissipation. On the other hand, our aim is to consider one stress tensor t_{iM} only. It will be attained further by plotting t_{iM} with other introduced tensors.

The theory of the material structures has been suggested in papers of PERZYNA, KOSIŃSKI [13], KOSIŃSKI, PERZYNA [14], PERZYNA [15, 16], FRISCHMUTH, KOSIŃSKI, PERZYNA [17] as an alternative approach to the mathematical theory of materials given by NOLL [18].

A system $(\mathcal{G}, \Phi, R)$ is called a body element with material structure, where $\mathcal{G}$ is the configuration space, Φ is a space which contains processes P with values in $\mathcal{G}$. $P : [0, d_p] \to \mathcal{G}$, where d_p is a duration of P. R is a constitutive relation. In general this relation is multivalued. The relation R assigns reaction processes $T \in \mathcal{Z}_P$ to the process $P \in \Phi$. In order to obtain a one-to-one mapping between processes P and Z, the method of preparation space was introduced [13].

At present we consider the material structure connected with the body which undergoes micro-nonhomogeneous deformations. Let us introduce the following symbols $\mathbf{e}^G = (e_{MN,P_1}, \ldots, e_{MN,P_1 \ldots P_S})$, $\mathbf{e} = (e_{MN})$, $\mathbf{w}_\lambda^G = (w_{\lambda N,Q_1}, \ldots, w_{\lambda N,Q_1 \ldots Q_T})$, $\mathbf{w}_\lambda = (w_{\lambda N})$, $\lambda = 1, \ldots, N_{\lambda-1}$.

DEFINITION 1. *The collection of quantities*

$$g(t) = (\mathbf{e}, \mathbf{e}^G, \mathbf{w}_\lambda, \mathbf{w}_\lambda^G, T, T_{,j})(t) \tag{2.14}$$

we shall call the actual deformation-temperature configuration of a particle X at the moment $t \in [0, d_P]$. ■

G stands for all such configurations and will be called the configuration space. Let us introduce also the space $\mathcal{G}_B$ of subconfigurations $(\mathbf{e}, \mathbf{w}_\lambda)$. Furthermore, let $B : \mathcal{G} \to \mathcal{G}_B$ be a projection from $\mathcal{G}$ to its subconfiguration space $\mathcal{G}_B$.

DEFINITION 2. *The collection of quantities*

$$s(t) = (\psi, t_{iM}, f_{\lambda N}, s, q_j)(t)\,, \tag{2.15}$$

where ψ *is the free energy density,* s *is the entropy density,* q_j *is the heat flux vector and* t_{iM}, $f_{\lambda N}$ *are given by Eqs.* (2.11), (2.12), *will be called the actual reaction of a particle X at the moment* $t \in [0, d_P]$. ■

The space $\mathcal{S}$ consist of all reactions. We can consider process spaces Φ and $Z, \Phi = \{P : P : [0, d_P] \to \mathcal{G}\}$, $\mathcal{Z} = \{Z : Z : [0, d_P] \to \mathcal{S}\}$. Furthermore it is assumed that each of processes $(P, Z) \in \Phi \times Z$ satisfies the balance equations (2.11)–(2.13). P_B stands for a process with values in $\mathcal{G}_B$.

DEFINITION 3. *The process* $(P, Z) \in \Phi \times Z$, $\operatorname{Dom} P = \operatorname{Dom} Z$ *which satisfies the balance equations* (2.11)–(2.13) *and the Clausius–Duhem inequality given later by Eq.* (2.25), *will be called the local thermodynamic process.* ■

Following the papers [14], [16] we introduce the material structure of the differential type. In this case the method of preparation space and the intrinsic state space will be introduced with the help of the following sets:

$$\mathcal{B} = \left\{ \frac{d}{d\tau} P_B(\tau)\Big|_{\tau=t} : P_B \in \Phi_B,\ t \in \operatorname{Dom} P_B \right\}, \tag{2.16}$$

$$\mathcal{B}_g = \left\{ \frac{d}{d\tau} P_B(\tau)\Big|_{\tau=t} : P_B \in \Phi_B,\ t \in \operatorname{Dom} P_B,\ P_B(t) = Bg \in \mathcal{G}_B \right\}. \tag{2.17}$$

Thus, we can define the method of preparation space as $K = B$ and the intrinsic state space as $\Sigma = \{(g,k) : g \in \mathcal{G},\ k \in \mathcal{B}_g\}$. Furthermore, we assume that the detailed conditions suitable for differential type structures [14, 16] are satisfied. Then we can postulate the constitutive mapping

$$\mathcal{R}(g,\cdot,g(0)) : \mathcal{B}_g \to \mathcal{Z}_{g(0)} \tag{2.18}$$

for processes with $\operatorname{Dom} P = \{0\}$ in the form

$$\psi = \psi(g), \tag{2.19}$$

$$t_{iM} = t^{\#}_{iM}(g) + t^{d}_{iM}(\cdot), \tag{2.20}$$

$$f_{\lambda N} = f^{\#}_{\lambda N}(g) + f^{d}_{\lambda N}(\cdot), \tag{2.21}$$

$$s = -\frac{\partial \psi}{\partial T}, \tag{2.22}$$

$$q_i = -\lambda_{ij} T_{,j}, \tag{2.23}$$

where $t^{\#}_{iM}$, $f^{\#}_{\lambda N}$ are equilibrium stress and forces and t^{d}_{iM}, $f_{\lambda} N^{d}$ are dissipative quantities. The suggested form of constitutive equations will be considered in more detail during investigations of constraints imposed by the Clausius–Duhem inequality. The constitutive mapping (2.18) determines the constitutive relation R previously mentioned.

The evolution function [16] of intrinsic states e takes the form

$$e(\sigma, P) = (P^f, P^f_B) \tag{2.24}$$

for $\sigma \in (\Sigma \times \Phi)^* = \{(\sigma, P) \in \Sigma \times \Phi | \sigma = (P^i, P^i_B)\}$. Symbols i and f stand for initial and final values of the process.

The foregoing considerations determine the structure of a material of the differential type [14], [16].

Let us consider the Clausius–Duhem unequality

$$\rho \dot{s} - \frac{\rho r_e}{T} + \left(\frac{q_j}{T}\right)_{,j} \geq 0. \tag{2.25}$$

Let us assume that the stresses and forces can be represented as a sum of equilibrium and dissipative parts

$$\bar{t}_{iN} = \bar{t}^{\#}_{iN} + t^{d}_{iN}, \tag{2.26}$$

$$\bar{f}_{\lambda N} = \bar{f}^{\#}_{\lambda N} + f^{d}_{\lambda N}, \tag{2.27}$$

where the symbols # and d are connected with equilibrium and dissipative quantities. The remaining stresses and forces are assumed to be equilibrium only. Thus $\bar{t}_{iNP_1 \ldots P_\delta} = \bar{t}^{\#}_{iNP_1 \ldots P_\delta}$, $\bar{f}_{\lambda N Q_1 \ldots Q_\nu} = \bar{f}^{\#}_{\lambda N Q_1 \ldots Q_\nu}$.

We can calculate ρr_e from Eq. (2.13). In view of $\dot{\varepsilon} = \dot{\psi} + s\dot{T} + \dot{s}T$, Eqs. (2.26), (2.27), the inequality (2.25) takes the form

$$\left(\bar{t}^{\#}_{iN} - \rho \frac{\partial \psi}{\partial x_{i,N}}\right) \dot{x}_{i,N} + \sum_{\delta=1}^{s} \left(\bar{t}_{iNP_1 \ldots P_\delta} - \rho \frac{\partial \psi}{\partial x_{i,NP_1 \ldots P_\delta}}\right) \dot{x}_{i,NP_1 \ldots P_\delta} \tag{2.28}$$

$$(2.28)_{[\text{cont.}]} + \sum_{\lambda} \left(\overline{f}^{\#}_{\lambda N} - \rho \frac{\partial \psi}{\partial w_{\lambda N}} \right) \dot{w}_{\lambda N} + \sum_{\lambda} \sum_{\nu=1}^{T} \left(\overline{f}_{\lambda N Q_1 \ldots Q_\nu} - \rho \frac{\partial \psi}{\partial w_{\lambda N, Q_1 \ldots Q_\nu}} \right) \dot{w}_{\lambda N, Q_1 \ldots Q_\nu}$$

$$+ t^{d}_{iN} \dot{x}_{i,N} + \sum_{\lambda} f^{d}_{\lambda N} \dot{w}_{\lambda N} - \left(\rho \frac{\partial \psi}{\partial T} + \rho s \right) \dot{T} - \frac{q_j T_{,j}}{T} \geq 0 .$$

It is assumed here that ψ does not depend on $\dot{P}$ according to Eq. (2.19). This inequality should be satisfied in particular by the equilibrium state. Thus, if $t^{d}_{iN} \to 0$ and $f^{d}_{\lambda N} \to 0$, then

$$(2.29) \qquad \begin{aligned} \overline{t}^{\#}_{iN} &= \rho \frac{\partial \psi}{\partial x_{i,N}}, \quad \overline{t}_{iNP_1 \ldots P_\delta} = \rho \frac{\partial \psi}{\partial x_{i,NP_1 \ldots P_\delta}}, \\ \overline{f}^{\#}_{\lambda N} &= \rho \frac{\partial \psi}{\partial w_{\lambda N}}, \quad \overline{f}^{\#}_{\lambda N Q_1 \ldots Q_\nu} = \rho \frac{\partial \psi}{\partial w_{\lambda N, Q_1 \ldots Q_\nu}}, \quad s = -\frac{\partial \psi}{\partial T} . \end{aligned}$$

Taking into considerations (2.28) and (2.29) we can assume that the constraints imposed on the constitutive equations by the Clausius–Duhem inequality are given by

$$(2.30) \qquad t^{d}_{MN} \dot{x}_{i,N} + \sum_{\lambda} f^{d}_{\lambda N} \dot{w}_{\lambda N} - \frac{q_j T_{,j}}{T} \geq 0 .$$

Following Eqs. (2.26) and (2.27), the total stress t_{iM} and force $f_{\lambda N}$ is assumed as $t_{iM} = t^{\#}_{iM} + t^{d}_{iM}$ and $f_{\lambda N} = f^{\#}_{\lambda N} + f^{d}_{\lambda N}$. Equilibrium stresses and forces defined in (2.29) are connected with the tensor $t^{\#}_{iM}$ and forces $f^{\#}_{\lambda N}$. This connection is given by equations $t^{\#}_{iM} = \rho A_{iM}$, $f^{\#}_{\lambda N} = \rho B_{\lambda N}$, where A_{iM} and $B_{\lambda N}$ are defined by Eq. (2.4).

The phenomenon of the micro-nonhomogeneous deformation suggests a model of dissipation. The micro-nonhomogeneous deformation induced by micro-homogeneous deformation causes relative motion of atoms. Motion of this kind is evidently a source of heat. Therefore, we can postulate that the amount of heat emitted during the deformation is proportional to the relative velocities of crystal sublattices. Bearing in mind that the Clausius-Duhem inequality must be satisfied, we assume the positive definite quadratic form

$$(2.31) \qquad L = \sum_{\lambda} C_{\lambda M N} \dot{w}_{\lambda M} \dot{w}_{\lambda N} + \sum_{(\lambda,\nu) \in I} C_{\lambda \nu M N} (\dot{w}_{\lambda M} - \dot{w}_{\nu M})(\dot{w}_{\lambda N} - \dot{w}_{\nu N}) \geq 0$$

as the starting point, where $I = \{(\lambda, \nu) : \lambda < \nu, \lambda, \nu = 1, \ldots, N_\lambda - 1\}$.

The L determines the amount of heat emitted per unit time in a material point during the body motion. We can transform Eq. (2.31) to the form

$$(2.33) \qquad L = \sum_{\lambda=1}^{N_\lambda - 1} f^{d}_{\lambda N} \dot{w}_{\lambda N} = \sum_{\lambda=1}^{N_\lambda - 1} \left(C_{\lambda M N} \dot{w}_{\lambda M} + \sum_{\nu=\lambda+1}^{N_\lambda - 1} C_{\lambda \nu M N} (\dot{w}_{\lambda M} - \dot{w}_{\nu M}) \right.$$

$$\left. - \sum_{\alpha=1}^{\lambda-1} C_{\alpha \lambda M N} (\dot{w}_{\alpha M} - \dot{w}_{\lambda M}) \right) \dot{w}_{\lambda N} ,$$

where for $\lambda = N_\lambda - 1$ the second summand on the right-hand side vanishes, and for $\lambda = 1$ the third summand vanishes. The formula (2.32) defines the dissipative force $f^{d}_{\lambda N}$.

3. Thermodynamics of materials which undergo martensitic phase transformations with shuffles

3.1. The free energy

The free energy connected with martensitic phase transformation have been considered in [10]. In this description one takes into account the micro-nonhomogeneous deformation, internal rotations towards habit planes and flat character of the propagating interfaces.

The free energy suggested in [10] is expressed in the form

$$\psi = F_t + M + A + G_H \,. \tag{3.1}$$

The first term F_t describes mainly the deformation given by the Green strain tensor and the temperature. The second, M, in connected with the micro-nonhomogeneous deformation. This deformation is introduced with the help of the relative displacement vectors. The component G_H takes into account the flat character of propagating interfaces. It contains higher gradients of deformation. The term A describes internal rotations towards habit planes. These rotations are described with the aid of a variable **a**. This variable has a different nature than those hitherto used. Therefore **a** will be considered in more detail.

It should be pointed out that all terms M, A and G_H may depend on the variables appearing in F_t.

In order to define **a**, four special configurations have been distinguished. The first one is the reference configuration which coincides with the natural state of the austenite phase (parent phase). The considered martensitic transformation can take place in six shear systems [22, 23, 30], as it was mentioned previously. Let us consider one of them. Let $e_{12}^{*} < e_{12}^{**}$ stand for the fixed shear strains in this system. After exceeding the strain e_{12}^{*}, the phase transformation starts and after exceeding the strain e_{12}^{**} we pass to the elastic martensite. Thus, e_{12}^{*} and e_{12}^{**} can be considered as a boundary of the spinodal region.

The second configuration $x_\alpha = x_\alpha(X_N, t)$ is connected with the deformation for which the strain e_{12}^{*} is attained. The third configuration $x_\mu = x_\mu(x_\alpha, t)$ appears when the strain e_{12}^{*} is attained. The fourth configuration x_i is the actual configuration. The deformation gradients which are assigned to these configurations relative to the previous one are $\mathbf{F}^{ea} = [x_{\alpha,N}]$, $\mathbf{F}^{s} = [x_{\nu,\alpha}]$, $\mathbf{F}^{em} = [x_{i,\nu}]$, and therefore the total deformation gradient is given by $\mathbf{F} = \mathbf{F}^{em}\mathbf{F}^{s}\mathbf{F}^{ea}$.

If the configuration x_μ is not exceeded, then $\mathbf{F}^{s} = [x_{i,\alpha}]$ and $\mathbf{F} = \mathbf{F}^{s}\mathbf{F}^{ea}$. Let $\mathbf{F}^{s} = \mathbf{R}^{s}\mathbf{U}^{s}$ and $\mathbf{F}^{ea} = \mathbf{R}\mathbf{U}$ be the polar decompositions of the deformation gradients. The rotation **R** considered here should not be mistaken for the rotation from Eq. (2.2). In the formula (2.2) the total rotation tensor is introduced. With the aid of these decompositions one defines the variables

$$\mathbf{a}_K = \mathbf{R}^{-1}(\mathbf{R}^{s}\mathbf{U}^{s}\mathbf{R}\mathbf{U} - \mathbf{R})\mathbf{p}_K, \quad K = 1, 2. \tag{3.2}$$

The vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are material directors ([24]) connected with two habit planes which are assigned to the forming martensite variant. We assume that the internal rotation $\overline{\mathbf{R}}$ is defined just as $\overline{\mathbf{R}} = \mathbf{R}^{s}$. In this case it is assumed that the total rigid rotation of the whole body is eliminated from our considerations. This is necessary, in turn, in order to

escape the imposition of constitutive equations determined by part A of the free energy (3.1) on the rigid rotation mentioned above.

The variable $\mathbf{a}_K$ is a measure of deviation of the director $\mathbf{p}_K$ from its initial position under stretch and internal rotation $\overline{\mathbf{R}} = \mathbf{R}^s$. The variable $\mathbf{a}$ means this variable $\mathbf{a}_K$, which is connected with forming the martensite variant.

The part A of the free energy depends on $\mathbf{a}_K$ and describes the bifurcation process connected with internal rotation $\overline{\mathbf{R}}$ [10]. Thus this rotation is a variable in constitutive relations. It implies that a moment of force acting in the body is related to the material point in the spinodal region. As a consequence, the Cauchy stress tensor should be nonsymmetric in the spinodal region. Indeed, if we assume that $t_{iM} = \varrho \partial \psi / \partial x_{iM}$ and ψ depends on $\mathbf{a}_K$, then we obtain a nonsymmetric Cauchy stress tensor σ_{ij} from t_{iM}. It suggests that the balance moment of momentum equation should be investigated in more detail.

The total moment of force which acts on an element of the body $\mathcal{P}$ is assumed as

$$M_i = \int\limits_{\partial\chi(\mathcal{P})} (\varepsilon_{ijk} x_j p_k) da + \int\limits_{\chi(\mathcal{P})} \varepsilon_{ijk} x_j b_k dv \,, \tag{3.3}$$

where p_k is a surface force. The moment of momentum is given by

$$H_i = \int\limits_{\chi(\mathcal{P})} (\rho \varepsilon_{ijk} x_j v_k + i\dot{\alpha}_i) dv \,, \tag{3.4}$$

where $\alpha_i = \alpha_i(\overline{\mathbf{R}})$ is the internal rotation vector determined by $\overline{\mathbf{R}}$ and i is a moment of inertia assigned to the material point and connected with this rotation. Then, the balance moment of momentum equation can be expressed as

$$\int\limits_{\chi(\mathcal{P})} \varepsilon_{ijk} x_j (\sigma_{mk,m} + b_k - \rho \ddot{u}_k) + (\varepsilon_{ijk} \sigma_{jk} - i\ddot{\alpha}_i) dv = 0 \,. \tag{3.5}$$

In view of the motion equation (2.11), Eq. (3.5) leads to the condition

$$\varepsilon_{ijk} \sigma_{jk} = i\ddot{\alpha}_i \,. \tag{3.6}$$

This equation justifies the introduction of the second term in Eq. (3.4). It is connected with nonsymmetry of the Cauchy stress tensor in the spinodal region. Omission of this term in Eq. (3.4) would be in contradiction with our previous considerations.

However, introduction of $i\dot{\alpha}_i$ into Eq. (3.4) has greater consequences. Namely, the term $i\ddot{\alpha}_i \dot{\alpha}_i$ connected with kinetic energy should be taken into consideration in the balance energy equation (2.3). This leads to a new equation following from Eq. (2.3). Thus, Eqs. (2.11)–(2.13) should be suplemented by the equation

$$m_i + i\ddot{\alpha}_i = 0 \,, \tag{3.7}$$

where

$$m_i = \rho \frac{\partial \psi}{\partial a_m} \frac{\partial a_m}{\partial \overline{R}_{pq}} \frac{\partial \overline{R}_{pq}}{\partial \alpha_i} \,. \tag{3.8}$$

Then the constitutive equation for t_{iM} takes the same form as in (2.20), (2.29) with the relation

$$\bar{t}^{\#}_{iN} = \rho \frac{\partial \psi}{\partial e_{KL}} \frac{\partial e_{KL}}{\partial x_{i,N}} \,, \tag{3.9}$$

where $\psi = \psi(\mathbf{e}, \mathbf{e}_{,M_1}, \ldots, \mathbf{e}_{,M_1 \ldots M_L}, \mathbf{a}, T)$ is given by Eq. (3.1).

3.2. The material structure

The material structure for a body which undergoes martensitic phase transformations and micro-nonhomogeneous deformation is introduced in this section. Two new aspects should be taken into consideration in comparison with the material structure for a body with micro-nonhomogeneous deformation only. The first one is the variable **a** which should complete the actual deformation-temperature configuration. The second one is connected with dissipation. In the Sec. 2.2 a model of dissipation associated with the micro-nonhomogeneous deformation has been suggested. In this case the material structure of differential type of the first order was introduced. It is well known that materials connected with martensitic phase transformations have good damping properties [25]. The material damping connected with these phase transformations has a different mechanism than that for the micro-nonhomogeneous deformation. Heat emitted in this transformation will be mainly related to the acceleration connected with breaking passing through the energetic barrier. Thus the second aspect is that the material structure suggested here should be of differential type of a higher order.

In view of the above considerations one postulates the actual deformation temperature configuration in the form

$$g(t) = (\mathbf{e}, \mathbf{e}^G, \mathbf{w}_\lambda, \mathbf{w}_\lambda^G, \mathbf{a}, T, T_{,j})(t). \tag{3.10}$$

The actual reaction of particle X should be assumed in the form

$$s(t) = (\psi, t_{iM}, m_j, f_{\lambda N}, s, q_k)(t). \tag{3.11}$$

The balance equations for this material structure are Eqs. (2.11), (2.12), (2.13) and (3.7). In Eq. (2.13) the term $-m_i\dot{\alpha}_i$ appears additionally. Introduction of Eq. (2.13) into Eq. (2.28) leads to Eq. (3.8). This does not change the inequality (2.30) in the case of neglecting the dissipation related to α_k.

In the case of martensitic phase transformations the heat flux should be also separately considered. In general the body considered is a multiphase medium. Heat coefficients are different for the austenite phase and for the martensite one. Furthermore, the martensite can undergo internal rotations. It has an effect on the heat coefficients expressed in the basis **b** which are connected with the Bain axes. On the other hand, these coefficients are unchanged in the basis **d** which is connected with the shear system and can undergo internal rotation. It is because the structure rotates with this basis. Thus, with the help of basis **d** it will be convenient to introduce the dependence of heat coefficients on the internal rotations.

The evolution of the relative displacement vectors during the phase transformation leads to the formation of different martensite variants with different structures. These structures are connected with different heat coefficients which will depend on $\mathbf{w}_\lambda$.

Thus we assume the Fourier equation expressed in the basis **d** as a starting point. Let $\overline{q}_l$ be the heat flux, $\overline{\lambda}_{lk}$ be the heat coefficients and $T_{,k}$ be the gradient of temperature. These quantities are expressed in the basis **d**. The Fourier equation takes the form $\overline{q}_l = \overline{\lambda}_{lk}T_{,k}$ in this basis. The heat coefficient $\overline{\lambda}_{lk}$ do not depend on internal rotation $\overline{\mathbf{R}}$ because the basis **d** rotates in agreement with internal rotations. Thus, as a consequence of the structure transformation during deformation, $\overline{\lambda}_{lk}$ should depend on shear deformation measure e_{12} and relative displacement vectors $\mathbf{w}_\lambda$ only. Therefore, we assume $\overline{\lambda}_{lk} = \overline{\lambda}_{lk}(e_{12}, \mathbf{w}_\lambda)$. The relation between the gradient of temperature $T_{,k}$ given in basis $\mathbf{d}_k$ and the gradient

of temperature $T_{,n}$ expressed in the basis $\mathbf{b}_n$ has the form $T_{,k} = (\mathbf{d}_k \cdot \mathbf{b}_n)T_{,n}$, where $(\cdot)$ stands for the symbol of the scalar product. Connections between heat flux q_m in the basis $\mathbf{b}_m$ and $\overline{q}_l$ given in the basis $\mathbf{d}_l$ can be written as $q_m = (\mathbf{b}_m \cdot \mathbf{d}_l)\overline{q}_l$. Internal rotations will be taken into account by assuming that $\mathbf{d}_k = \mathbf{d}_k(\overline{\mathbf{R}})$. Bearing in mind the above considerations we can propose the constitutive equations for the heat flux in the basis $\mathbf{b}_m$ in the form

$$q_m = (\mathbf{b}_m \cdot \mathbf{d}_l(\overline{\mathbf{R}}))\overline{\lambda}_{lk}(e_{12}, \mathbf{w}_\lambda)(\mathbf{d}_k(\overline{\mathbf{R}}) \cdot \mathbf{b}_n)T_{,n}\,. \tag{3.12}$$

Now let us define the sets

$$\mathcal{B} = \left\{ \frac{d}{d\tau}P_B(\tau)|_{\tau=t},\ \frac{d^2}{d\tau^2}P_B(\tau)|_{\tau=t}, \ldots, \frac{d^n}{d\tau^n}P_B(\tau)|_{\tau=t} : P_B \in \Phi_B,\ t \in \operatorname{Dom} P_B \right\} \tag{3.13}$$

$$\mathcal{B}_g = \left\{ \frac{d}{d\tau}P_B(\tau)|_{\tau=t}\frac{d^2}{d\tau^2}P_B(\tau)|_{\tau=t}, \ldots, \frac{d^n}{d\tau^n}P_B(\tau)|_{\tau=t} : P_B \in \Phi_B,\ t \in \operatorname{Dom} P_B, P_B(t) = Bg \in \mathcal{G}_B \right\}. \tag{3.14}$$

The method of preparation space is assumed as $\mathcal{K} = \mathcal{B}$, and the intrinsic state space as $\Sigma = \{(g,k) : g \in \mathcal{G},\ k \in \mathcal{B}_g\}$. Furthermore, we assume that the detailed conditions suitable for the differential-type structure [14, 16] are satisfied. We can postulate the constitutive mapping (2.18) in the similar way as in Sec. 2.2 for processes with $\operatorname{Dom} P = \{0\}$ in the form (2.19)–(2.21). We assume that entropy takes the form $s = s^{\#} + s^d$, similarly to the stress. Furthermore, the equation for heat conductivity should take the form

$$q_i = q_i(g) \tag{3.15}$$

based on Eq. (3.12). The free energy is determined by Eq. (3.1). Furthermore one assumes that the free energy does not depend on the time derivatives of g.

In the frame of this differential structure one can suggest a simple model of dissipation connected with the phase transformation. The heat emited during this transformation will be mainly connected with acceleration which appears as a consequence of breaking after passing through the energetic barrier. It suggests that the dissipative entropy, in the simplest case, should be proportional to the acceleration of the deformation

$$s^d = c_{PQ}\ddot{e}_{PQ}\,. \tag{3.16}$$

In this case the formula (2.30) should be completed by the summand $-\rho s^d T$.

3.3. The material structure with internal state variables

It is convenient for calculations to introduce the material structure with internal state variables. The question is whether such a material structure is suitable for description of dissipation in the materials considered. Previously it has been suggested that dissipation depends on time derivatives of the deformation. Thus the internal state variables should include this situation.

Let us consider an evolution equation in the form

$$\dot{\mu} = A(P, \mu), \mu(t_0) = \mu_0\,. \tag{3.17}$$

Following LUBLINER [35], KOSIŃSKI, WOJNO [36] we can consider the solution of Eq. (3.14) in the form

$$\mu(t) = \Phi_t(P_r(\tau), \mu_0)\,, \tag{3.18}$$

where P_r is the past history of the process P, and Φ_t is a functional which assigns the value $\mu(t)$ to the function $P_r(\tau)$. We can expand this function into the Taylor series in the right-hand side of point t. Then, the functional (3.18) can be approximated by a function of the form $\overline{\Phi}_t(P(t), \dot{P}(t), \ddot{P}(t), \ldots, \mu_0)$. Therefore, Eq. (3.17) can describe the previously considered models connected with the differential material structures. However, description with the help of internal state variables is more general.

We introduce two kinds of internal variables. Variables δ describe the viscosity and variables β describe the dissipation connected with the phase transformation. Let us consider a simplified model. Then, we assume that only the stress $\mathbf{t} = \mathbf{t}(g, \delta)$ and forces $\mathbf{f}_\lambda = \mathbf{f}_\lambda(g, \delta)$ depend on δ. Thus the viscosity influences mainly the stress and the forces. The variables β are related to the dissipation which appear during breaking after passing through the energetic barrier. This breaking causes emission of heat. We assume that transformations have no influences on the shape of the potential energy of the deformation. Therefore we assume that only entropy $s = s(g, \beta)$ depends on β.

This example is based on simplified considerations which are carried out in order to characterize the role of these internal state variables. However, in general, it should allow for the dependence of all constitutive relations on δ and β. It is expected that the properties of the simplified model will be a distinctive feature of these generalized relations. In this case the following form of the Clausius–Duhem inequality should be considered:

$$\left(t_{iN} - \frac{\partial \psi}{\partial x_{i,N}}\right)\dot{x}_{i,N} + \sum_\lambda \left(f_{\lambda N} - \rho \frac{\partial \psi}{\partial w_{\lambda N}}\right)\dot{w}_{\lambda N} - \left(\rho \frac{\partial \psi}{\partial T} + \rho s\right)\dot{T} + \rho \frac{\partial \psi}{\partial \delta_i}\dot{\delta}_i + \rho \frac{\partial \psi}{\partial \beta_i}\dot{\beta}_i - \frac{q_j T_{,j}}{T} \geq 0\,. \tag{3.19}$$

In the above considerations the „size" of the material point is smaller than the thickness of the interface. Thus the suggested model could be a starting point to the description of a body with martensitic transformation in which the material point contains averaged properties of a composition of martensite variants. In such a case internal state variable can describe more complex effects connected with the phase transformation. In this description the deformation measure is the Green strain tensor $\mathbf{e}$ which is a function $\mathbf{e} = e(z_k, \mathbf{e}_k)$, where z_k is the mass contribution of the k-th martensite variant, and $\mathbf{e}_k$ is the strain connected with the k-th variant. Then the free energy has the form $F = F(\mathbf{e}, T)$.

Let us consider a part of the material with the volume V which will be treated in the limit as a material point. Let us assume that there exist different interfaces marked by m. Then the m-th interface is at a distance of $|x_m|$ from the origin of coordinates. We assume that the interface can undergo parallel translations only. Let us introduce the internal state variables defined as

$$\mu_m = \frac{1}{V}\int |x_m| dS_m\,, \tag{3.20}$$

where the integration is carried out on the area S_m of the m-th interface in the vol-

ume V. Let $\mu = (\mu_1, \ldots, \mu_D)$. This variable can describe dissipation connected with the motion of interfaces. The free energy should be plotted in these variables and takes the form $F = F(\mathbf{e}, T, \mu)$.

For determination of the free energy $F = F(\mathbf{e}, T, \mu)$, the function $\mathbf{e} = \mathbf{e}(z_k, \mathbf{e}_k)$ and the evolution function for the variable μ, the model studied in the paper could be useful.

4. Final remarks

In the present paper and [10] a three-dimensional model of the martensitic phase transformations with shuffles is suggested. This description is introduced in two stages. First, a model of the free energy connected with these phase transformations was studied in [10]. In this model the author made efforts in order to take into considerations the complicated phenomena which appear in the materials. In the second stage, in this paper, a model of thermodynamics which takes into account properties of the previous description [10] has been introduced.

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Extension of a plane crack due to plane SH-waves in a prestressed infinite elastic medium

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IN AN INFINITE isotropic elastic medium, initially in a state of uniform anti-plane shear, the problem has been analyzed of one-way extension of an infinitesimal flaw into a plane shear crack due to two identical linearly varying plane SH-waves with non-parallel wave fronts. Fracture is assumed to initiate at a point a finite time after the waves intersect there, and one of the crack edges is assumed to extend along the trace of the wave intersection. Following CHEREPANOV [7], CHEREPANOV and AFANAS'EV [8], the general solution of the problem has been derived in terms of an analytic function of complex variable. The results include the expression for the stress intensity factors at the crack tips.

1. Introduction

SINCE BROBERG'S [1] investigation of the solution of a crack expanding symmetrically with constant velocity, under conditions of plane stress or strain, in a homogeneous isotropic elastic medium in a field of spatially and time-invariant tensile stress, a number of papers have appeared analyzing different geometrical situations. CRAGGS [2] later solved the same problem as that done by BROBERG, but he used the method of homogeneous functions to obtain the solution. ACHENBACH and BROCK [3] considered the wave motion generated by a uniformly extending shear crack in a state of uniform antiplane shear. All the problems mentioned above are, however, self-similar ones with index $(0, 0)$, and they are concerned with symmetric expansion of cracks.

Problems involving non-symmetric extension of cracks under uniform loading along the crack surface are not easily found, perhaps due to severe mathematical complexity encountered in solving such problems. Following the method of homogeneous functions developed by CRAGGS [2], non-symmetric extension of a small flaw into a plane crack under polynomial form of loading was solved by BROCK [4]. Following the same procedure, BROCK [5] also solved the problem of non-symmetric extension of a crack due to incidence of plane dilatational waves. Recently, the elastodynamic problem of non-uniform extension of a crack under homogeneous form of loading solved by GEORGIADIS [6].

To the best knowledge of the author, the problem of one-way extension of a crack due to the action of two identical non-parallel plane SH-waves has not been investigated so far. In this paper, the problem of extension of an infinitesimal flaw into a plane shear crack has been solved, under the assumption of a constant extension rate due to the action of two identical non-parallel plane SH-waves propagating towards each other in an infinite isotropic elastic medium which is initially in a state of uniform anti-plane shear. In a finite time after the crossing of the plane wave fronts, a fracture is assumed to initiate along the line where the wave fronts crossed, and one of the crack edges is then assumed to travel with constant speed. Superposition considerations allow the original problem to be separated into three self-similar problems with $(0, 0)$, $(0, 1)$ and $(1, 0)$ as the indices of self-similarity. Following CHEREPANOV [7], CHEREPANOV and AFANAS'EV

[8], the mentioned self-similar problems have all been formulated as some problems of Riemann and Hilbert for a half-plane, which can be easily solved. Analytical expressions for the dynamic stress intensity factors at the crack tips have been derived.

2. Formulation of the problem

Let two identical plane waves defined by the formulae

$$\sigma_{yz} = A_0 W_{\pm} H(W_{\pm}), \qquad \sigma_{xz} = \pm A_0 \operatorname{ctg}\theta_0 W_{\pm} H(W_{\pm}) \tag{2.1}$$

referred to coordinate system (x, y, z), where

$$W_{\pm} = c_2 t \pm y \sin\theta_0 + x \cos\theta_0, \qquad 0 \leq \theta_0 \leq \pi/2,$$

and $H(\ )$ is Heaviside's unit function, propagate through the infinite solid which is pre-stressed so that

$$\sigma^0_{yz} = \sigma, \qquad \sigma^0_{xz} = 0. \tag{2.1'}$$

Let us assume that at $t = 0$ the non-parallel plane waves intersect at $x = y = 0$. A microcrack is assumed to appear at $t = t_0$ at $x = y = 0$ and one of its edges starts to extend along the trace of the wave intersection with uniform velocity v. The expanding crack, the circular wave front associated with its motion and the plane wave front are shown in Fig. 1a.

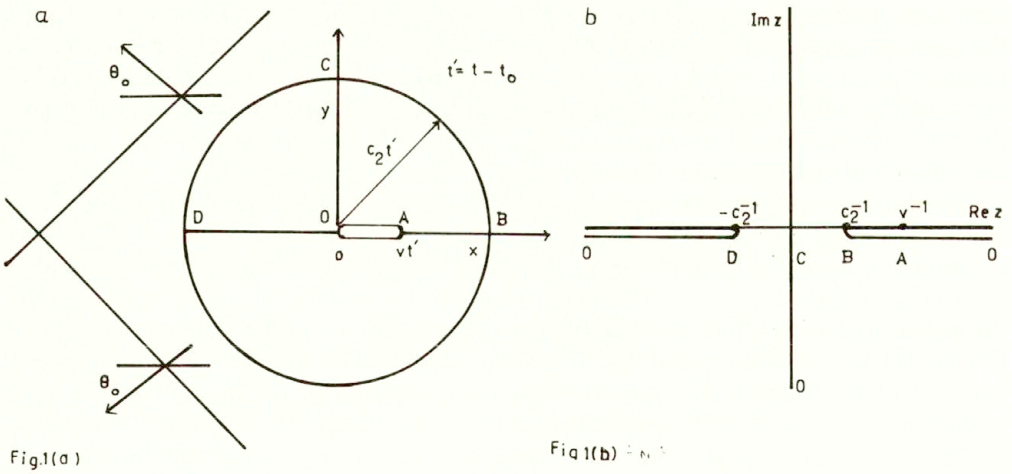

Fig.1(a) Fig.1(b)

FIG. 1.

In effect crack extension occurs by removing the stresses which would be generated in the crack plane by the combined applied static and dynamic fields if no cracks were present.

Accordingly, both the crack faces are subjected to shear tractions equal to $-\sigma - 2A_0(c_2 t + x\cos\theta_0)$.

The anti-symmetry of this loading about the crack plane implies that it is sufficient to consider the half-plane $y > 0$ with bounding surface $y = 0$. The boundary conditions for

this half-plane are then given by

$$
\begin{aligned}
&y = 0, \quad 0 < x < vt' : \quad \sigma_{yz} = -\sigma - 2A_0c_2t_0 - 2A_0(c_2t' + x\cos\theta_0)\,, \\
&y = 0, \quad x > vt', \quad x < 0 : W = 0\,,
\end{aligned} \tag{2.2}
$$

where $t' = t - t_0$.

Equation $(2.2)_1$ shows that, by invoking the superposition principle, the proposed problem can be divided into three separate problems of a constant shear traction, a shear stress linearly varying with time and a shear linearly varying with distance along the crack plane.

3. Constant shear traction on the crack faces

The wave motion generated by constant shear tractions on the faces of the crack defined by $y = 0$, $0 < x < vt$ has been considered in this section and, for simplicity, t instead of t' has been used. The boundary conditions are

$$
\begin{aligned}
&y = 0, \quad 0 < x < vt : \quad \sigma_{yz} = -p_0\,, \\
&y = 0, \quad x > vt, \quad x < 0, \quad W = 0\,,
\end{aligned} \tag{3.1}
$$

where $p_0 = \sigma + 2A_0c_2t_0$.

The displacement W which satisfies the wave equation

$$
\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \tag{3.2}
$$

is to be determined subject to the boundary conditions given by (3.1). From the boundary conditions we observe that $\partial W/\partial t$ shows dynamic similarity and is a homogeneous function of degree zero in x/t and y/t. Therefore, by the functionally invariant method of SMIRNOFF and SOBOLEV [9], we can write

$$
\frac{\partial W}{\partial t} = \operatorname{Re}\phi_0(z)\,, \tag{3.3}
$$

where

$$
t - xz + y\sqrt{c_2^{-2} - z^2} = 0\,. \tag{3.4}
$$

The sign of the radical is to be fixed by the condition that as $z \to \infty$,

$$
\sqrt{c_2^{-2} - z^2} = iz + O(z^{-1})\,. \tag{3.5}
$$

Equation (3.4) maps the semi-circular region of the cylindrical waves defined by OABCD onto the lower half of the complex z-plane with cuts, given by

$$
z = \frac{xt - iy\sqrt{t^2 - c_2^{-2}(x^2 + y^2)}}{x^2 + y^2} \tag{3.6}
$$

as shown in Fig. 1b.

In view of Eqs. (3.3) and (3.4) we find

$$
\frac{\partial \sigma_{yz}}{\partial t} = \mu \operatorname{Re}\left[\phi_0'(z)\frac{\partial z}{\partial y}\right],
$$

so that

$$\frac{\partial \sigma_{yz}(x,0,t)}{\partial t} = \frac{1}{t}\,\mathrm{Re}\left[-\mu z\phi_0'(z)\sqrt{c_2^{-2}-z^2}\right]. \tag{3.7}$$

Therefore the boundary conditions (3.1) are transformed to the following conditions in the z-plane

$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z < v^{-1}, \quad \mathrm{Re}\,\phi_0(z) = 0, \tag{3.8}$$

$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z > v^{-1}, \quad \mathrm{Im}\,\phi_0'(z) = 0. \tag{3.9}$$

In order to determine the analytic function $\phi_0(z)$ subject to the condition (3.8) and (3.9) it is necessary to know the behaviour of the function $\phi_0(z)$ when $z \to v^{-1}, 0$. The zero point of the z-plane corresponds to the point $x = 0, y = c_2 t$ in the physical plane, where the displacement derivative $\partial W/\partial t$ is zero. Hence, taking the representation (3.3) into account, we obtain

$$\mathrm{Re}\,\phi_0(0) = 0. \tag{3.10}$$

Further, the condition (3.9), after integration with respect to z, may be put in the form

$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z > v^{-1}, \quad \mathrm{Im}\,\phi_0(z) = 0. \tag{3.11}$$

Moreover, the displacement derivative $\partial W/\partial t$ near the moving crack tip $x = vt$, $y = 0$ should show square root singularity, so that at $z \to v^{-1}$

$$\phi_0(z) = O\left(1/\sqrt{z - v^{-1}}\right). \tag{3.12}$$

The above boundary conditions given by (3.8) and (3.11), together with the consideration (3.10) and (3.12), suggest that

$$\phi_0(z) = \frac{A}{\sqrt{z - v^{-1}}}, \tag{3.13}$$

where A is a real unknown constant to be determined.

By integrating Eq. (3.7) with respect to t it can be easily shown that for $x > 0$

$$\begin{aligned}
\sigma_{yz}(x,0,t) &= -\mu\,\mathrm{Re}\left\{\left[\phi_0(z)\sqrt{c_2^{-2}-z^2}\right]_{c_2^{-1}}^{t/x} + \int_{c_2^{-1}}^{t/x} \frac{z\phi_0(z)dz}{\sqrt{c_2^{-2}-z^2}}\right\}, \\
\sigma_{yz}(-x,0,t) &= -\mu\,\mathrm{Re}\left\{\left[\phi_0(z)\sqrt{c_2^{-2}-z^2}\right]_{-c_2^{-1}}^{-t/x} \int_{-c_2^{-1}}^{-t/x} \frac{z\phi_0(z)dz}{\sqrt{c_2^{-2}-z^2}}\right\}.
\end{aligned} \tag{3.14}$$

Next, using the boundary condition $(3.1)_1$ in Eq. $(3.14)_1$ we obtain

$$A = p_0/\mu I,$$

where

$$I = \frac{2}{\sqrt{v}}\left[\frac{\sqrt{c_2(c_2+v)}}{v}E(p) - \frac{v}{\sqrt{c_2(c_2+v)}}F(p)\right], \quad p = \sqrt{\frac{c_2 - v}{c_2 + v}}, \tag{3.15}$$

and $F(\,)$, $E(\,)$ are complete elliptic integrals of the first and second kind, respectively.

The stress intensity factors at the crack tips $x = vt$, $y = 0$ and $x = 0$, $y = 0$, defined by

$$N_{01} = lt_{x\to vt}\sqrt{x - vt}\sigma_{yz}(x, 0, t),$$
$$N_{02} = lt_{x\to 0}\sqrt{x}\sigma_{yz}(-x, 0, t),$$

respectively, are obtained with the help of Eqs. (3.13) and (3.14) as

$$N_{01} = \frac{\mu A}{c_2}\sqrt{(c_2^2 - v^2)t}, \tag{3.16}$$
$$N_{02} = \mu A\sqrt{t}.$$

4. Problem of shear traction linearly increasing with time on the crack faces

For the case of shear tractions on the faces of the crack increasing linearly with time, the boundary conditions are

$$y = 0, \quad 0 < x < vt, \quad \sigma_{yz} = -p_1 t, \tag{4.1}$$
$$y = 0, \quad x > vt, \quad x < 0, \quad W = 0, \tag{4.2}$$

where $p_1 = 2A_0c_2$.

The second order derivative $\partial^2 W/\partial t^2$ now shows dynamic similarity which can be taken as the real part of the analytic function $\phi_1(z)$, so that

$$\frac{\partial^2 W}{\partial t^2} = \operatorname{Re}\phi_1(z) \tag{4.3}$$

which implies

$$\frac{\partial^2 \sigma_{yz}(x, 0, t)}{\partial t^2} = \frac{1}{t}\operatorname{Re}[-\mu z\phi_1'(z)\sqrt{c_2^{-2} - z^2}], \tag{4.4}$$

where z is given by Eq. (3.6) and $\phi_1(z)$ satisfies the conditions

$$\operatorname{Im} z = 0, \quad \operatorname{Re} z < v^{-1}, \quad \operatorname{Re}\phi_1(z) = 0, \tag{4.5}$$
$$\operatorname{Im} z = 0, \quad \operatorname{Re} z > v^{-1}, \quad \operatorname{Im}\phi_1'(z) = 0. \tag{4.6}$$

Integrating Eq. (4.3), we obtain

$$\frac{d}{dz}\left[\frac{1}{x}\frac{\partial W}{\partial t}\right] = \operatorname{Re}\phi_1(z). \tag{4.7}$$

Taking into consideration the facts that near the moving crack tip $x = vt, y = 0$ the displacement derivative $\partial W/\partial t$ varies inversely to the factor $\sqrt{v_1 t - x}$, and that

$$\operatorname{Re}\phi_1(0) = 0,$$

we obtain (in view of the conditions (4.5), (4.6) and Eq. (4.7)) the result that

$$\phi_1(z) = \frac{d}{dz}\left[\frac{Bz}{\sqrt{z - v^{-1}}}\right]; \tag{4.8}$$

the real constant B is to be determined from the condition that on the crack surface stress $\sigma_{yz} = -p_1 t$.

From Eq. (4.4) after integration, we derive for $x > 0$

$$\sigma_{yz}(x,0,t) = -\mu x \,\mathrm{Re} \int_{c_2^{-1}}^{t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} + \frac{\tau(t/x - \tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau) d\tau \,,$$

(4.9)

$$\sigma_{yz}(-x,0,t) = \mu x \,\mathrm{Re} \int_{-c_2^{-1}}^{-t/x} \left\{ \sqrt{c_2^{-2} - \tau^2} - \frac{\tau(t/x + \tau)}{\sqrt{c_2^{-2} - \tau^2}} \right\} \phi_1(\tau) d\tau \,.$$

Therefore, using the boundary condition (4.1) in $(4.9)_1$, we obtain after simplifications

$$B = p_1/\mu J \,,$$

where

$$J = \frac{2}{3\sqrt{vc_2}} \left[\frac{\sqrt{c_2 + v}}{v} (3c_2 + v) E(p) + \frac{v}{2c_2\sqrt{(c_2 + v)}} (v - 8c_2) F(p) \right] ;$$

$F(\,)$, $E(\,)$ and p have been defined earlier.

The stress intensity factors at the crack tips defined by

$$N_{11} = lt_{x \to vt} \sqrt{x - vt}\, \sigma_{yz}(x,0,t) \,,$$
$$N_{12} = lt_{x \to 0} \sqrt{x}\, \sigma_{yz}(-x,0,t) \,,$$

are found to be

$$N_{11} = \frac{\mu t}{c_2} \sqrt{(c_2^2 - v^2)t}\, B \,,$$
$$N_{12} = \mu B t^{3/2} \,.$$

5. Problem of shear traction linearly varying with distance along the crack plane

Consider the initially undisturbed half-space $y \geq 0$ subjected to the shear traction $-p_2 x$ over $y = 0$, $0 < x < vt$. The boundary conditions are

$$y = 0, \quad 0 < x < vt : \quad \sigma_{yz} = -p_2 x \,,$$
$$y = 0, \quad x > vt, \quad x < 0, \quad W = 0 \,,$$

(5.1)

where $p_2 = 2A_0 \cos\theta_0$.

In this case, $\dfrac{\partial^2 W}{\partial x \partial t}$ shows dynamic similarity. So we take, keeping Eq. (3.6) in mind

$$\frac{\partial^2 W}{\partial x \partial t} = \mathrm{Re}\, \phi_2(z)$$

with

$$\frac{\partial^2 \sigma_{yz}}{\partial x \partial t} = -\frac{\mu}{t} \mathrm{Re} \left[z\phi_2'(z) \sqrt{c_2^{-2} - z^2} \right] ,$$

(5.2)

where $\phi_2(z)$ satisfies the conditions

$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z < v^{-1}, \quad \mathrm{Re}\, \phi_2(z) = 0 \,,$$
$$\mathrm{Im}\, z = 0, \quad \mathrm{Re}\, z > v^{-1}, \quad \mathrm{Im}\, \phi_2'(z) = 0 \,.$$

(5.3)

From Eq. $(5.2)_1$ after integration it is found that

$$W = -x^2 \operatorname{Re} \int_{v_1^{-1}}^{z} \tau^{-1}(z-\tau)\phi_2(\tau)d\tau ,$$

so that

$$-z^2 \frac{d}{dz}\left\{\frac{1}{t}\frac{\partial W}{\partial t}\right\} = \operatorname{Re}\phi_2(z) .$$

Since $\partial W/\partial t$ near the moving crack tip should exhibit square root singularity and also since $\operatorname{Re}\phi_2(0) = 0$, we have, in view of the conditions (5.3), the result

$$\phi_2(z) = z^2 \frac{d}{dz}\left[\frac{C}{z\sqrt{z-v^{-1}}}\right], \tag{5.4}$$

where the real constant C is to be determined from Eq. $(5.1)_1$.

Equation $(5.2)_2$ can be integrated to obtain for $x > 0$

$$\begin{aligned}
\sigma_{yz}(x,0,t) &= \mu x \operatorname{Re} \int_{c_2^{-1}}^{t/x} \left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2}-\tau^2} + \frac{t-\tau x}{x\sqrt{c_2^{-1}-\tau}}\right\}\phi_2(\tau)d\tau , \\
\sigma_{yz}(-x,0,t) &= \mu x \operatorname{Re} \int_{-c_2^{-1}}^{-t/x} \left\{\frac{t}{x\tau^2}\sqrt{c_2^{-2}-\tau^2} + \frac{t+\tau x}{x\sqrt{c_2^{-2}-\tau^2}}\right\}\phi_2(\tau)d\tau .
\end{aligned} \tag{5.5}$$

So, using the boundary condition $(5.1)_1$ and Eqs. (5.4), $(5.5)_1$ it is found that

$$C = -p_2/\mu K ,$$

where

$$K = \frac{3}{\sqrt{v}}\left[\frac{\sqrt{c_2(c_2+v)}}{v}E(p) - \frac{v}{\sqrt{c_2(c_2+v)}}F(p)\right];$$

$F(\ )$, $E(\ )$ and p have been defined earlier. In this case, the stress intensity factors are obtained as

$$\begin{aligned}
N_{21} &= lt_{x\to vt}\sqrt{x-vt}\,\sigma_{yz}(x,0,t) = -\frac{\mu t}{c_2}v\sqrt{(c_2^2-v^2)t}C , \\
N_{22} &= lt_{x\to 0}\sqrt{x}\,\sigma_{yz}(-x,0,t) = 0 .
\end{aligned} \tag{5.6}$$

6. Discussions

The solution of the orginal crack problem is obtained by taking $p_0 = \sigma + 2A_0c_2t_0$, $p_1 = 2A_0c_2$ and $p_2 = 2A_0\cos\theta_0$ and superposing the results obtained in Secs. 3–5 over the stress fields given by Eq. (2.1). Using the results obtained in the Secs. 3–5 it is possible to write the stress intensity factors at the edges as

(6.1)

$$\begin{aligned}
S_1 &= \frac{N_{01}+N_{11}+N_{21}}{\sigma\sqrt{c_2t_0}} = \frac{\mu}{\sqrt{v}}\left[\frac{1+\Delta}{p_0}A + \frac{\Delta\tau}{p_1}B - \frac{\Delta\tau m\cos\theta_0}{p_2}C\right]\sqrt{m\tau(1-m^2)} , \\
S_2 &= \frac{N_{02}+N_{12}+N_{22}}{\sigma\sqrt{c_2t_0}} = \frac{\mu}{\sqrt{v}}\left[\frac{1+\Delta}{p_0}A + \frac{\Delta\tau}{p_1}B\right]\sqrt{m\tau} ;
\end{aligned}$$

where the parameter $\tau = (t/t_0) - 1$ is the non-dimensionalized time after crack initiation, and $\Delta = (2A_0c_2t_0)/\sigma$ is the ratio at $x = y = 0$ at initiation of the crack plane stress due to the plane waves and the prestress. Also m is the dimensionless crack tip velocity given by $m = v/c_2$. The expressions for the stress intensity factors and the constants A, B and C given in Secs. 3–5 are very simple. One may easily obtain information about the stress in the crack line and their intensities for each problem, as well as for the original problem considered in this paper.

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Algebra of the transfer matrix for layered elastic material

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THE COMPLEX-VALUED 4×4 transfer matrix M for periodic elastic layers possesses the following symmetries

$$M_{21} = \overline{M_{12}}, \quad M_{22} = \overline{M_{11}}, \quad M_{23} = -\overline{M_{14}}, \quad M_{24} = -\overline{M_{13}},$$

$$M_{41} = -\overline{M_{32}}, \quad M_{42} = -\overline{M_{31}}, \quad M_{43} = \overline{M_{34}}, \quad M_{44} = \overline{M_{33}},$$

$$\det M = 1, \quad M_{14}M_{23}M_{31}M_{42} = M_{13}M_{24}M_{32}M_{41}.$$

In the paper it has been proved that, due to the above symmetries, two real or complex parameters φ, ψ and a set pp of further 18 scalar parameters may be defined and calculated. The transfer matrix may then be represented as the function $M = M(\varphi, \psi, pp)$. This new function for each integer n satisfies the identity

$$[M(\varphi, \psi, pp)]^n = M(n\varphi, n\psi, pp).$$

The derived identity drastically simplifies the calculation of displacements and stresses in the periodically layered medium.

Introduction

THE TRANSFER 4×4 matrix governing the reflection and transmission of the elastic wave at the boundary between two homogeneous materials possesses a very special, non-Hermitian symmetry. It is shown that due to this symmetry it is possible to represent the matrix M in a special form, which leads to drastic simplification of the calculation of the powers M^n.

In the earlier paper [3], in connection with propagation of transverse wave (plane problem with displacement orthogonal to the plane) in periodic system of layers, was considered the 2×2 matrix H with the following symmetry:

$$H = \begin{bmatrix} z_1 & z_2 \\ \overline{z}_1 & \overline{z}_1 \end{bmatrix}, \quad \det H = 1,$$

where the bar over the complex number denotes the complex conjugate. Due to this special symmetry the parameter φ may be introduced leading to the very useful in computations and elegant formula $H(\varphi)^n = H(n\varphi)$. The trivial generalisation of the above 2×2 matrix is the 4×4 matrix

$$H = \begin{bmatrix} z_1 & z_2 & 0 & 0 \\ \overline{z}_2 & \overline{z}_1 & 0 & 0 \\ 0 & 0 & z_3 & z_4 \\ 0 & 0 & \overline{z}_4 & \overline{z}_3 \end{bmatrix},$$

provided the determinants of the upper-left 2×2 part H_1, and the lower-right 2×2 part H_2 are equal 1. There exist two different parameters φ and ψ connected with the 2×2 matrices H_1, H_2. The 4×4 matrix H satisfies the identity $H(\varphi, \psi)^n = H(n\varphi, n\psi)$ because each of its 2×2 constituents satisfies the above mentioned formula, the first with parameter φ, the other with parameter ψ. The purpose of the present paper is to show

that the more general 4×4 matrix M governing the wave propagation of longitudinal and transverse waves satisfies the same identity.

1. Transfer matrix

Consider two elastic materials separated by the plane $x = x_2$. The layer situated between x_1 and x_2, its displacement, strain and stress shall be identified by the label 1, whereas the layer between x_2 and x_3 will be identified by the label 2. The waves propagate in some directions inclined to the x-direction, Fig. 1. The longitudinal wave propagating in medium 1, reflected by a plane produces the reflected, longitudinal and transverse waves propagating in medium 1 and transmitted, longitudinal and transverse waves propagating in the medium 2. These waves are multiply reflected leading to the wave pattern shown at Fig. 1. Similar pattern is obtained for the incident transverse wave propagating in the medium 1.

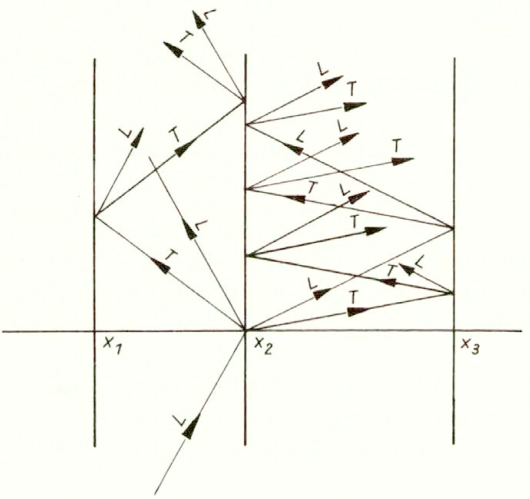

FIG. 1.

In each region the displacements are represented by the displacement potentials φ_K and ψ_K, where K is the layer label, $K = 1, 2$. The potential φ_K represents the sum of all longitudinal waves propagating in the layer K, and the potential ψ_K—the sum of all transverse waves propagating in the layer K. In [1] was considered the antiplane wave with the displacement in the z direction. Here we consider only the displacements in the plane x, y. The results of [1] and of the present paper, taken together correspond to the general reflection and refraction at the plane boundary between two elastic materials.

The following potentials in the regions K are expected:

$$
\begin{aligned}
\varphi_K &= A_K \exp i[-\omega t + p_K(x - x_K) + sy] \\
&\quad + B_K \exp i[-\omega t - p_K(x - x_K) + sy], \\
\psi_K &= C_K \exp i[-\omega t + q_K(x - x_K) + sy] \\
&\quad + D_K \exp i[-\omega t - q_K(x - x_K) + sy].
\end{aligned} \tag{1.1}
$$

The terms proportional to A_K, B_K represent two harmonic longitudinal waves of amplitudes A_K, B_K. The terms proportional to C_K, D_K represent two transverse waves of amplitudes C_K, D_K. The total displacement in each region is equivalent to two longitudinal and two transverse waves. The parameter s defines the incidence angle. The parameters p_K, q_K may be derived from the equations of motion. All waves possess the same frequency ω. If any wave does not exist, then the corresponding amplitude equals zero.

The displacement components u_x, u_y, u_z in each layer may be calculated from the formulae

$$u_{Kx} = \partial\varphi_K/\partial x - \partial\psi_K/\partial y, \quad u_{Ky} = \partial\varphi_K/\partial y + \partial\psi_K/\partial x, \quad u_z = 0. \tag{1.2}$$

The above potentials must satisfy the equations of motion

$$\partial^2\varphi_k/\partial x^2 + \partial^2\varphi_K/\partial y^2 = \frac{1}{c_{LK}^2}\partial^2\varphi_K/\partial t^2, \tag{1.3}$$

$$\partial^2\psi_k/\partial x^2 + \partial^2\psi_K/\partial y^2 = \frac{1}{c_{TK}^2}\psi_K/\partial t^2, \tag{1.4}$$

where c_{LK} and c_{TK} are the longitudinal and transverse wave speeds, respectively. It follows that the five parameters s, p_K, q_K in Eq. (1.3) are not arbitrary, but must satisfy the relations

$$\begin{aligned} p_K^2 + s^2 &= \omega^2/c_{LK}^2, \quad c_{LK}^2 = (\lambda_K + 2\mu_K)\rho_K, \\ q_K^2 + s^2 &= \omega^2/c_{TK}^2, \quad c_{TK}^2 = \mu_K/\rho_K, \end{aligned} \tag{1.5}$$

where ρ_K is the density of the material. If, in paricular, s is given in advance, then the other parameters are defined by Eqs. (1.5). Only one wave direction is arbitrary, the other wave directions must match the first one (Snellius rule).

The expressions for the potentials allow us to calculate the displacements and stresses. We omit here the very simple calculations resulting in the formulae

$$\begin{bmatrix} -iu_x \\ -iu_y \\ \tau_{xx} \\ \tau_{xy} \end{bmatrix}_1 = \begin{bmatrix} p_1 & -p_1 & -s & -s \\ s & s & q_1 & -q_1 \\ -z_1 & -z_1 & 2\mu q_1 s & 2\mu q_1 s \\ -2\mu_1 p_1 s & 2\mu_1 p_1 s & w_1 & w_1 \end{bmatrix} \tag{1.6}$$

$$\times \begin{bmatrix} \exp ip_1 l_1 & 0 & 0 & 0 \\ 0 & \exp(-ip_1 l_1) & 0 & 0 \\ 0 & 0 & \exp iq_1 l_1 & 0 \\ 0 & 0 & 0 & \exp(-iq_1 l_1) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix},$$

$$\begin{bmatrix} -iu_x \\ -iu_y \\ \tau_{xx} \\ \tau_{xy} \end{bmatrix}_2 = \begin{bmatrix} p_2 & -p_2 & -s & -s \\ s & s & q_2 & -q_2 \\ -z_2 & -z_2 & 2\mu q_2 s & 2\mu q_2 s \\ -2\mu_2 p_2 s & 2\mu_2 p_2 s & w_2 & w_2 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix}, \tag{1.7}$$

where

$$w_K = \mu_K(s^2 - q_K^2), \quad l_1 = x_2 - x_1, \tag{1.8}$$

$$z_K = \lambda_K(p_K^2 + s^2) + 2\mu_K p_K^2. \tag{1.9}$$

At both sides of the boundary between the regions the stress vector and the displacement vector have the same values. In order to express the amplitudes A_2, B_2, C_2, D_2 by the amplitudes A_1, B_1, C_1, D_1 the inverse matrix of that in (1.7) must be calculated. As the final result the relation

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} = M_1 \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} \tag{1.10}$$

is obtained. The transfer matrix M_1 allows us to calculate the amplitudes of the waves propagating in the region 2 provided the amplitudes of the waves propagating in the region 1 are known. Its components are given by the relations

$$\begin{aligned}
M_{11} &= [a_2(2\mu_1 p_1 s^2 - p_1 w_2) + b_2(2\mu_2 q_2 s^2 + q_2 z_1)] \exp ip_1 l_1 , \\
M_{12} &= [a_2(-2\mu_1 p_1 s^2 + p_1 w_2) + b_2(2\mu_2 q_2 s^2 + q_2 z_1)] \exp(-ip_1 l_1) , \\
M_{13} &= [a_2 s(w_2 - w_1) + 2b_2 q_1 q_2 s(\mu_2 - \mu_1)] \exp iq_1 l_1 , \\
M_{14} &= [a_2 s(w_2 - w_1) - 2b_2 q_1 q_2 s(\mu_2 - \mu_1)] \exp(-iq_1 l_1) , \\
M_{31} &= [-2a_2 p_1 p_2 s(\mu_2 - \mu_1) + b_2 s(z_2 - z_1)] \exp ip_1 l_1 , \\
M_{32} &= [2a_2 p_1 p_2 s(\mu_2 - \mu_1) + b_2 s(z_2 - z_1)] \exp(-ip_1 l_1) , \\
M_{33} &= [a_2(2\mu_2 p_2 s^2 - p_2 w_1) + b_2(2\mu_1 q_1 s^2 + q_1 z_2)] \exp iq_1 l_1 , \\
M_{34} &= [a_2(2\mu_2 p_2 s^2 - p_2 w_1) - b_2(2\mu_1 q_1 s^2 + q_1 z_2)] \exp(-iq_1 l_1) ,
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
&M_{21} = \overline{M_{12}}, \quad M_{22} = \overline{M_{11}}, \quad M_{23} = -\overline{M_{14}}, \quad M_{24} = -\overline{M_{13}} , \\
&M_{41} = -\overline{M_{32}}, \quad M_{42} = -\overline{M_{31}}, \quad M_{43} = \overline{M_{34}}, \quad M_{44} = \overline{M_{33}} ,
\end{aligned} \tag{1.12}$$

where

$$a_2 = \frac{1}{2}\frac{1}{2\mu_2 p_2 s^2 - p_2 w_2}, \qquad b_2 = \frac{1}{2}\frac{1}{2\mu_2 q_2 s^2 + q_2 z_2} . \tag{1.13}$$

The matrix of symmetry (1.12) will be called w-symmetric. The product of two w-symmetric matrices is w-symmetric. The above transition matrix was derived, with a slightly different notation, in [2].

The following identity may be obtained from (1.10)

$$\begin{aligned}
p_1(A_1\overline{A_1} - B_1\overline{B_1}) + q_1(C_1\overline{C_1} - D_1\overline{D_1}) \\
= p_2(A_2\overline{A_2} - B_2\overline{B_2}) + q_2(C_2\overline{C_2} - D_2\overline{D_2}) .
\end{aligned} \tag{1.14}$$

It expresses the fact that the energy fluxes in both regions are equal.

Assume that the two above layers of material 1 of thickness l_1 and material 2 of thickness l_2 are repeated in space, Fig. 2. Consider the dynamics of this periodic structure. The elementary cell consists of two elastic layers. The transfer matrix for the transition $1 \Rightarrow 2, 3 \Rightarrow 4$ etc. is the matrix M_1 (here we added the suffix 1) given by (1.11) and (1.12). The transfer matrix for the transition $2 \Rightarrow 3, 4 \Rightarrow 5$ etc. will be denoted by M_2. It may be obtained from formulae (1.11), (1.12) by interchanging the indices 1 and 2. The transfer matrix M for one cell is the product of M_2 and M_1

$$M = M_2 M_1 . \tag{1.15}$$

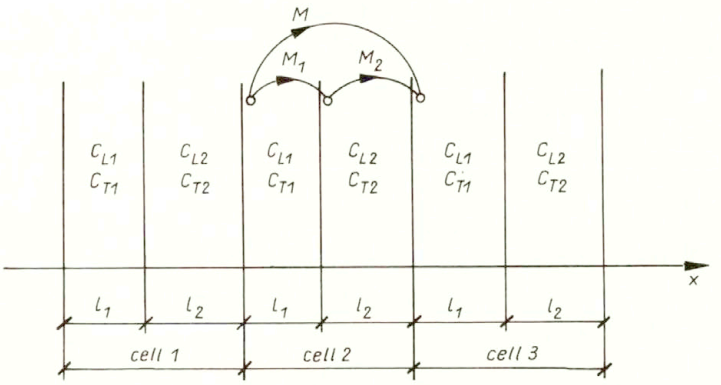

FIG. 2.

In order to save space we do not give the full, long expressions for the components of M. Note only that this transfer matrix for one cell is determined by nine real quantities

$$s,\ \lambda_1,\ \mu_1,\ \rho_1,\ l_1,\ \lambda_2,\ \mu_2,\ \rho_2,\ l_2\,. \tag{1.16}$$

The frequency ω does not influence M. The matrix M is the product of two w-symmetric matrices, and therefore is w-symmetric. The displacement field in the subsequent cells may be expressed by the amplitudes in the first cell and the powers of the transfer matrix $M^1, M^2, M^3, \ldots$.

2. Preliminary numerical analysis

The complex-valued 4×4 transition matrix derived above has the special, non-Hermitean symmetry. In order to make the symmetries better visible we write M in the shorthand notation

$$M = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ \overline{z}_2 & \overline{z}_1 & -\overline{z}_4 & -\overline{z}_3 \\ z_5 & z_6 & z_7 & z_8 \\ -\overline{z}_6 & -\overline{z}_5 & \overline{z}_8 & \overline{z}_7 \end{bmatrix}.$$

Note that the eight complex numbers $z_1, z_2, \ldots, z_8$ are determined by the nine real physical parameters mentioned above. Therefore there exist additional relations between the components of M.

Consider now the powers M^n of the matrix M for arbitrary integer n. The components $(M^n)_{ij}$ are the functions of the discrete variable n. Guided by the behaviour of the spatially periodic mechanical systems we expect that M^n is a periodic function of n. More precisely we expect to define the periodic function of the continuous variable n, which for integer n coincides with M^n.

In order to verify this proposal, about 100 succesive powers of M for numerical components given in advance were calculated from (1.11). The results fully supported the expectation.

Assume that, similarly to [1], some scalar parameter φ may be introduced leading to the representation of M in the form $M = M(\varphi)$ and M^n in the form $M(n\varphi)$. Then each component of M must be of the form $(a_{ij} \cos\varphi + b_{ij} \sin\varphi)$ and the corresponding

component of the n-th power M^n should be of the form $(a_{ij} \cos n\varphi + b_{ij} \sin n\varphi)$. Easy numerical and analytical computations show that there is not enough freedom to meet the data for M. On the other hand, the matrix H defined by (2.3) requires two parameters φ, ψ. Because H is a special case of M as given by (2.1), the matrix M requires the knowledge of at least two such parameters.

The simplest choice is therefore the assumption that there exist exactly two parameters φ and ψ and for each integer n there is

$$(2.1) \quad (M_{ij})^n = a_{ij} \sin n\varphi + b_{ij} \sin n\psi + c_{ij} \cos n\varphi + d_{ij} \cos n\psi + i(e_{ij} \sin n\varphi + f_{ij} \sin n\psi + g_{ij} \cos n\varphi + h_{ij} \cos n\psi),$$

where

$$(2.2) \quad a_{ij},\ b_{ij},\ c_{ij},\ d_{ij},\ e_{ij},\ f_{ij},\ g_{ij},\ h_{ij},$$

are constants. We shall show that the formula (2.1) follows from the symmetry (1.12). In order to check this hypothesis it is necessary to calculate $a_{ij}, b_{ij}, \ldots, g_{ij}, h_{ij}$ from the w-symmetric matrix M given in advance, and then to check if the formula (2.2) holds for each n.

In order to calculate the ten constants φ, ψ, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, f_{ij}, g_{ij}, h_{ij} (for fixed i, j) we must have the (i,j) component of five powers of M, namely of M, M^2, M^3, M^4, M^5. In order to check the formula (2.2) we must calculate aditionally at least M^6. Each component of the matrix M is the complex number z_1. Each component of the matrix M^2 is a sum of 2 complex products $z_1\ z_2$. Each component of the matrix M^3 is a sum of 2^2 complex products $z_1\ z_2\ z_3$, and so on. Finally, each component of the matrix M^6 is the sum of 2^5 complex products $z_1\ z_2\ z_3\ z_4\ z_5\ z_6$. The last product is equivalent to $2^6 = 64$ products of real numbers. Therefore to calculate the matrix M^6 alone it is necessary to determine $16 \times 32 \times 64 = 32768$ real numbers. Note that already the components of M, as given by (1.14), have a very complex structure. The number of necessary transformations at this stage entirely excludes the purely analytical approach.

In order to prove the formula (2.1) it was decided to base on the mixed numerical and analytical calculations. However, the final proof is purely analytic. The numerical calculations are not essential, they give only some suggestions concerning the structure of the results.

For the elastic constants given in advance and dimensions in both regions 1 and 2, and for the prescribed frequency ω, the six powers of M were calculated. The calculations were repeated many times for other elastic constants and other dimensions. These six powers of M allow us to calculate 64 coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij} and φ, ψ from the following system of equations:

$$(2.3) \quad a_{ij} \sin\varphi + b_{ij} \sin\varphi + c_{ij} \cos\varphi + d_{ij} \cos\psi = \mathrm{Re}(M)_{ij},$$

$$(2.4) \quad \begin{aligned} &a_{ij} \sin 2\varphi + b_{ij} \sin 2\psi + c_{ij} \cos 2\varphi + d_{ij} \cos 2\psi = \mathrm{Re}(M^2)_{ij}, \\ &a_{ij} \sin 2\varphi + b_{ij} \sin 3\psi + c_{ij} \cos 3\varphi + d_{ij} \cos 3\psi = \mathrm{Re}(M^3)_{ij}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{ij} \sin 6\varphi + b_{ij} \sin 6\psi + c_{ij} \cos 6\varphi + d_{ij} \cos 6\psi = \mathrm{Re}(M^6)_{ij}, \end{aligned}$$

(2.5)
$$e_{ij}\sin\varphi + f_{ij}\sin\psi + g_{ij}\cos\varphi + h_{ij}\cos\psi = \mathrm{Im}(M)_{ij}\,,$$
$$e_{ij}\sin 2\varphi + f_{ij}\sin 2\psi + g_{ij}\cos 2\varphi + h_{ij}\cos 2\psi = \mathrm{Im}(M^2)_{ij}\,,$$
(2.6)
$$e_{ij}\sin 3\varphi + f_{ij}\sin 3\psi = g_{ij}\cos 3\varphi + h_{ij}\cos 3\psi = \mathrm{Im}(M^3)_{ij}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e_{ij}\sin 6\varphi + f_{ij}\sin 6\psi + g_{ij}\cos 6\varphi + h_{ij}\cos 6\psi = \mathrm{Im}(M^6)_{ij}\,.$$

Theoretically ten equations are sufficient, but because of numerical reasons we prefer the overdetermined system (2.3)–(2.6). In this case the first six equations with six unknowns may be solved independently of the further six equations. To check the correctness of the approach and correctness of the program, the same coefficients after obvious change of the above system of equations were calculated basing on other powers of M, e.g. M^3, M^5, M^7, M^9, M^{11} and M^{13}.

As the result, a set of identities was obtained. The first four identities are

(2.7)
$$c_{11} = p, \quad d_{11} = 1 - p, \quad c_{33} = 1 - p, \quad d_{33} = p\,,$$

where p is a new constant. We do not repeat the formulae following from the w-symmetry. The further identities are

(2.8)
$$\begin{aligned} &c_{12} = d_{12} = g_{12} = h_{12} = 0\,,\\ &c_{34} = d_{34} = g_{34} = h_{34} = 0\,,\\ &g_{11} = h_{11} = g_{33} = h_{33} = 0\,. \end{aligned}$$

Note that the matrix M is determined by nine quantities: four elastic moduli λ_K, μ_K, thicknesses l_1, l_2, two densities ρ_K and the propagation direction m of one wave in one material (the other directions are then automatically determined). After eliminating the coefficients $a_{ij}, \ldots, f_{ij}$ it follows that between the coefficients some additional relations must exist.

The numerical analysis of the non-zero coefficients leads to the relations

(2.9)
$$d_{13} = -c_{13}, \quad d_{14} = -c_{14}\,,$$

(2.10)
$$\begin{aligned} &e_{12} = a_{12}/m_1, \quad f_{12} = b_{12}/m_1\,,\\ &e_{34} = a_{34}/m_2, \quad f_{34} = b_{34}/m_2\,, \end{aligned}$$

(2.11)
$$\begin{aligned} &e_{13} = -a_{13}/m_3, \quad f_{13} = -b_{13}/m_3, \quad g_{13} = c_{13}m_3, \quad h_{13} = -c_{13}m_3\,,\\ &e_{14} = a_{14}/m_4, \quad f_{14} = b_{14}/m_4, \quad g_{14} = -c_{14}m_4, \quad h_{14} = c_{14}m_4\,, \end{aligned}$$

(2.12)
$$\begin{aligned} &a_{31} = -ha_{13}, \quad b_{31} = -hb_{13}, \quad c_{31} = hc_{13}, \quad d_{31} = -hc_{13}\,,\\ &e_{31} = -ha_{13}/m_3, \quad f_{31} = -hb_{13}/m_3, \quad g_{31} = -hm_3c_{13}, \quad h_{31} = hm_3c_{13}\,,\\ &a_{32} = -ha_{14}, \quad b_{32} = -hb_{14}, \quad c_{32} = hc_{14}, \quad d_{32} = -hc_{14}\,,\\ &e_{32} = -ha_{14}/m_4, \quad f_{32} = -hb_{14}/m_4, \quad g_{32} = -hm_4c_{14}, \quad h_{32} = hm_4c_{14}\,, \end{aligned}$$

where h, m_1, m_2, m_3, m_4 are constants. Note that the calculations were performed for the physical situation defined by the nine real quantities: dimensions and elastic constants (1.16). For such data real h, m_1, m_2, m_3, m_4 were obtained. In the general case complex values must be allowed.

It is necessary to remember that these relations were obtained numerically, where the numbers of absolute values smaller than 10^{-12} were considered as equal zero. Further we give the analytical verification, that the identities (2.7)–(2.12) are not approximate, but exact.

Now there remain only 20 unknowns

$$\begin{aligned} &p,\ h,\ m_1,\ m_2,\ m_3,\ m_4,\ e_{11},\ f_{11},\ e_{33},\ f_{33},\\ &a_{12},\ b_{12},\ a_{34},\ b_{34},\ a_{13},\ b_{13},\ c_{13},\ a_{14},\ b_{14},\ c_{14}. \end{aligned} \tag{2.13}$$

Their number is still large, but the numerical analysis has reduced the problem to a rather decent situation. Instead of the previously expected 128 coefficients $a_{ij}, \ldots, h_{ij}$ (or 64 if the w-symmetry is taken into account) we face now only 20 unknowns.

3. Analytic approach

The redundant parameters may now be eliminated. We prefer to eliminate them in two stages. At the first stage only the zero parameters are eliminated and the relations (2.12) with parameter h are taken into account. This does not destroy the symmetry of computations. The transition matrix M after the first stage elimination has the following components:

$$\begin{aligned} M_{11} &= p\cos\varphi + (1-p)\cos\psi + i[e_{11}\sin\varphi + f_{11}\sin\psi],\\ M_{33} &= (1-p)\cos\varphi + p\cos\psi + i[e_{33}\sin\varphi + f_{33}\sin\psi],\\ M_{12} &= a_{12}\sin\varphi + b_{12}\sin\psi + i[e_{12}\sin\varphi + f_{12}\sin\psi],\\ M_{34} &= a_{34}\sin\varphi + b_{34}\sin\psi + i[e_{34}\sin\varphi + f_{34}\sin\psi],\\ M_{13} &= a_{13}\sin\varphi + b_{13}\sin\psi + c_{13}(\cos\varphi - \cos\psi)]\\ &\quad + i[e_{13}\sin\varphi + f_{13}\sin\psi + g_{13}(\cos\varphi - \cos\psi)],\\ M_{14} &= a_{14}\sin\varphi + b_{14}\sin\psi + c_{14}(\cos\varphi - \cos\psi)]\\ &\quad + i[e_{14}\sin\varphi + f_{14}\sin\psi + g_{14}(\cos\varphi - \cos\psi)],\\ M_{31} &= h[-a_{13}\sin\varphi\sin\varphi - b_{13}\sin\psi + c_{13}(\cos\varphi - \cos\psi)]\\ &\quad + ih[e_{13}\sin\varphi + f_{13}\sin\psi - g_{13}(\cos\varphi - \cos\psi)],\\ M_{32} &= h[-a_{14}\sin\varphi - b_{14}\sin\psi + c_{14}(\cos\varphi - \cos\psi)]\\ &\quad + ih[-e_{14}\sin\varphi - f_{14}\sin\psi + g_{14}(\cos\varphi - \cos\psi)]. \end{aligned} \tag{3.1}$$

The remaining components of the matrix M are determined by the w-symmetry. In the further calculations of this chapter we shall use (3.1), possessing large symmetry. Later we use the relations obtained at the second stage, when additionally the remaining relations (2.11), (2.12) are taken into account. The relations given below are obtained.

$$\begin{aligned} M_{11} &= p\cos\varphi + (1-p)\cos\psi + i[e_{11}\sin\varphi + f_{11}\sin\psi],\\ M_{33} &= (1-p)\cos\varphi + p\cos\psi + i[e_{33}\sin\varphi + f_{33}\sin\psi],\\ M_{12} &= a_{12}\sin\varphi + b_{12}\sin\psi + \frac{i}{m_1}[a_{12}\sin\varphi + b_{12}\sin\psi],\\ M_{34} &= a_{34}\sin\varphi + b_{34}\sin\psi + \frac{i}{m_2}[a_{34}\sin\varphi + b_{34}\sin\psi],\\ M_{13} &= a_{13}\sin\varphi + b_{13}\sin\psi + c_{13}(\cos\varphi - \cos\psi) \end{aligned} \tag{3.2}$$

(3.2) [cont.)
$$+i\left[-\frac{1}{m_3}(a_{13}\sin\varphi + b_{13}\sin\psi) + m_3c_{13}(\cos\varphi - \cos\psi)\right],$$

$$M_{14} = a_{14}\sin\varphi + b_{14}\sin\psi + c_{14}(\cos\varphi - \cos\psi)$$
$$+ i\left[\frac{1}{m_4}(a_{14}\sin\varphi + b_{14}\sin\psi) - m_4c_{14}(\cos\varphi - \cos\psi)\right],$$

$$M_{31} = h[-(a_{13}\sin\varphi + b_{13}\sin\psi) + c_{13}(\cos\varphi - \cos\psi)]$$
$$+ ih\left[-\frac{1}{m_3}(a_{13}\sin\varphi + b_{13}\sin\psi) - m_3c_{13}(\cos\varphi - \cos\psi)\right],$$

$$M_{32} = h[-(a_{14}\sin\varphi + b_{14}\sin\psi) + c_{14}(\cos\varphi - \cos\psi)]$$
$$+ ih\left[-\frac{1}{m_4}(a_{14}\sin\varphi + b_{14}\sin\psi) - m_4c_{14}(\cos\varphi - \cos\psi)\right].$$

In the next chapter we return to the calculation of the parameters $\varphi, \psi, a_{11}, b_{11}, \ldots$ $\ldots, f_{44}$, provided the components of the matrix M are given in advance.

Here we assume that the 20 parameters $p, h, m_1, \ldots, c_{14}$ at the right-hand side of (3.1) and (3.2) are given, and prove that the matrix defined by this relations for each n satisfies the expected identity (2.2). Anticipating the results we note that not all the parameters may be given in advance. There exist some relations between them, which will be discussed later.

Calculate first the matrix M^2. In accord with the above relations we obtain the real and imaginary parts of the first component in the following form:

(3.3)
$$\mathrm{Re}(M^2)_{11} = [p^2 + h(c_{13}^2 + g_{13}^2 - c_{14}^2 - g_{14}^2)]\cos^2\varphi$$
$$+[(1-p)^2 + h(c_{13}^2 + g_{13}^2 - c_{14}^2 - g_{14}^2)]\cos^2\psi$$
$$+[-e_{11}^2 + a_{12}^2 + e_{12}^2 - h(a_{13}^2 + e_{13}^2 - a_{14}^2 - e_{14}^2)]\sin^2\varphi$$
$$+[-f_{11}^2 + b_{12}^2 + f_{12}^2 - h(b_{13}^2 + f_{13}^2 - b_{14}^2 - f_{14}^2)]\sin^2\psi$$
$$+2[p(1-p) - h(c_{13}^2 + g_{13}^2 - c_{14}^2 - g_{14}^2)]\cos\varphi\cos\psi$$
$$+2[-e_{11}f_{11} + a_{12}b_{12} + e_{12}f_{12}$$
$$+h(-a_{13}b_{13} - e_{13}f_{13} + a_{14}b_{14} + e_{14}f_{14})]\sin\varphi\sin\psi,$$

(3.4)
$$\mathrm{Im}(M^2)_{11} = [pe_{11} + h(-a_{13}g_{13} + c_{13}e_{13} + a_{14}g_{14} - c_{14}e_{14})]\sin 2\varphi$$
$$+[(1-p)f_{11} + h(b_{13}g_{13} - c_{13}f_{13} - b_{14}g_{14} + c_{14}f_{14})]\sin 2\psi$$
$$+[pf_{11} + h(-b_{13}g_{13} + c_{13}f_{13} + b_{14}g_{14} - c_{14}f_{14})]2\cos\varphi\sin\psi$$
$$+[(1-p)e_{11} + h(a_{13}g_{13} - c_{13}e_{13} - a_{14}g_{14} + c_{14}e_{14})]2\sin\varphi\cos\psi.$$

Relations of similar structure are obtained for the remaining 15 components of M^2. Quoting them here would take too much place. We prefer to give here only the full system of relations between the parameters.

We expect that, independently of other parameters, φ and ψ may be arbitrary. We are guided here by the results for the 2×2 w-symmetric matrix discussed in [1]. We prove that this does not lead to a contradiction. If we do not make this assumption the analysis is more difficult. Moreover, we face then many trivial situations we prefer to avoid. Because $\mathrm{Re}(M^2)_{11}$ must be identically equal (for each φ and ψ) to $[p\cos 2\varphi + (1-p)\cos 2\psi]$, the coefficients of $\cos\varphi\cos\psi$ and of $\sin\varphi\sin\psi$ must vanish, the coefficient of $\cos^2\varphi$ must be equal p, the coefficient of $\sin^2\varphi$ must be equal $-p$, the coefficient of $\cos\varphi\cos\psi$ must be

zero, and so on. The resulting set of equations may be divided into four groups. When writing the equations we use the relations (2.9)–(2.11), provided this is not destroying the symmetry of the notation. If the symmetry was destroyed, we would prefer not to take them into account.

Exceptional position in the calculations has the equation

$$p(1-p) + g[-c_{13}^2(1+m_3^2) + c_{14}^2(1+m_4^2)] = 0, \tag{3.5}$$

since it is essential not for one component of M only, but for all the diagonal elements.

The first group consists of 12 equations

$$\begin{aligned}
&(1-p)e_{11} + he^{\#}{}_{11} = 0, \quad pf_{11} + hf^{\#}{}_{11} = 0,\\
&(1-p)f_{33} + hf^{\#}{}_{33} = 0, \quad pe_{33} + he^{\#}{}_{33} = 0,\\
&(1-p)a_{12} + ha^{\#}{}_{12} = 0, \quad pb_{12} + hb^{\#}{}_{12} = 0,\\
&(1-p)b_{34} + hb^{\#}{}_{34} = 0, \quad pa_{34} + ha^{\#}{}_{34} = 0,\\
&(1-p)e_{12} + he^{\#}{}_{12} = 0, \quad pf_{12} + hf^{\#}{}_{12} = 0,\\
&(1-p)f_{34} + hf^{\#}{}_{34} = 0, \quad pe_{34} + he^{\#}{}_{34} = 0,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
e^{\#}{}_{11} &= a_{13}c_{13}(m_3 + 1/m_3) + a_{14}c_{14}(m_4 + 1/m_4),\\
f^{\#}{}_{11} &= -b_{13}c_{13}(m_3 + 1/m_3) - b_{14}c_{14}(m_4 + 1/m_4),\\
e^{\#}{}_{33} &= a_{13}c_{13}(m_3 + 1/m_3) - a_{14}c_{14}(m_4 + 1/m_4),\\
f^{\#}{}_{33} &= -b_{13}c_{13}(m_3 + 1/m_3) + b_{14}c_{14}(m_4 + 1/m_4),\\
a^{\#}{}_{12} &= -a_{13}c_{14}(1 - m_4/m_3) + a_{14}c_{13}(1 - m_3/m_4),\\
b^{\#}{}_{12} &= b_{13}c_{14}(1 - m_4/m_3) - b_{14}c_{13}(1 - m_3/m_4),\\
a^{\#}{}_{34} &= a_{13}c_{14}(1 + m_4/m_3) - a_{14}c_{13}(1 + m_3/m_4),\\
b^{\#}{}_{34} &= -b_{13}c_{14}(1 + m_4/m_3) + b_{14}c_{13}(1 + m_3/m_4),\\
e^{\#}{}_{12} &= a_{13}c_{14}(m_4 + 1/m_3) + a_{14}c_{13}(m_3 + 1/m_4),\\
f^{\#}{}_{12} &= -b_{13}c_{14}(m_4 + 1/m_3) - b_{14}c_{13}(m_3 + 1/m_4),\\
e^{\#}{}_{34} &= -a_{13}c_{14}(m_4 - 1/m_3) + a_{14}c_{13}(m_3 - 1/m_4),\\
f^{\#}{}_{34} &= b_{13}c_{14}(m_4 - 1/m_3) - b_{14}c_{13}(m_3 - 1/m_4).
\end{aligned} \tag{3.7}$$

These relations close the first group of equations. Note that the internal (compare (2.3)) coefficients 11, 12, 33, 34 occupy other positions than the external coefficients 13, 14. The equations of the first group express the internal coefficients by the external ones.

The second group expresses the external coefficients of sine by the external coefficients of cosine and the internal coefficients

$$\begin{aligned}
&a_{13} + g_{13}(e_{11} + e_{33}) + c_{14}(a_{12} - a_{34}) + g_{14}(e_{12} - e_{34}) = 0,\\
&b_{13} + g_{13}(-f_{11} + f_{33}) + c_{14}(-b_{12} + b_{34}) + g_{14}(-f_{12} + f_{34}) = 0,\\
&e_{13} + c_{13}(-e_{11} - e_{33}) + g_{14}(-a_{12} - a_{34}) + c_{14}(e_{12} + e_{34}) = 0,\\
&f_{13} + c_{13}(-f_{11} - f_{33}) + g_{14}(-b_{12} - b_{34}) + c_{14}(e_{12} + e_{34}) = 0,\\
&a_{14} + g_{14}(e_{11} - e_{33}) + c_{13}(a_{12} - a_{34}) + g_{13}(e_{12} + e_{34}) = 0,
\end{aligned} \tag{3.8}$$

(3.8) [cont.]
$$b_{14} + g_{14}(-f_{11} + f_{33}) + c_{13}(-b_{12} + b_{34}) + f_{13}(-f_{12} - f_{34}) = 0\,,$$
$$e_{14} + c_{14}(-e_{11} + e_{33}) + g_{13}(-a_{12} - a_{34}) + c_{13}(e_{12} - e_{34}) = 0\,,$$
$$f_{14} = c_{14}(f_{11} - f_{33}) + g_{13}(b_{12} + b_{34}) + c_{13}(-f_{12} + f_{34}) = 0\,.$$

The third group consists of 4 inhomogeneous and 2 homogeneous equations

(3.9)
$$-e_{11}^2 + a_{12}^2 + e_{12}^2 + hP_1 = -p\,,$$
$$-f_{11}^2 + b_{12}^2 + f_{12}^2 + hP_2 = -(1-p)\,,$$
$$-e_{33}^2 + a_{34}^2 + e_{34}^2 + hP_3 = -(1-p)\,,$$
$$-f_{33}^2 + b_{34}^2 + f_{34}^2 + hP_4 = -p\,,$$

(3.10)
$$-e_{11}f_{11} + a_{12}b_{12} + e_{12}f_{12} + hP_5 = 0\,,$$
$$-e_{33}f_{33} + a_{34}b_{34} + e_{34}f_{34} + hP_6 = 0\,,$$

where

(3.11)
$$P_1 = P_3 = -a_{13}^2 - e_{13}^2 + a_{14}^2 + e_{14}^2\,,$$
$$P_2 = P_4 = -b_{13}^2 - f_{13}^2 + b_{14}^2 + f_{14}^2\,,$$
$$P_5 = P_6 = -a_{13}b_{13} - e_{13}f_{13} + a_{14}b_{14} + e_{14}f_{14}\,.$$

Finally we have the fourth group of relations

(3.12)
$$a_{13}(f_{11} + f_{33}) + e_{14}(b_{12} + b_{34}) + a_{14}(-f_{12} - f_{34}) + b_{13}(e_{11} + e_{33}) + f_{14}(a_{12} + a_{34}) + b_{14}(-e_{12} - e_{34}) = 0\,,$$
$$a_{14}(f_{11} - f_{33}) + e_{13}(b_{12} + b_{34}) + a_{13}(-f_{12} + f_{34}) + b_{14}(e_{11} - e_{33}) + f_{13}(a_{12} + a_{34}) + b_{13}(-e_{12} + e_{34}) = 0\,,$$
$$e_{13}(-f_{11} - f_{33}) + a_{14}(-b_{12} + b_{34}) + e_{14}(-f_{12} - f_{34}) + f_{13}(-e_{11} - e_{33}) + b_{14}(-a_{12} + a_{34}) + f_{13}(-e_{12} + e_{34}) = 0\,,$$
$$e_{14}(-f_{11} + f_{33}) + a_{13}(-b_{12} + b_{34}) + e_{13}(-f_{12} - f_{34}) + f_{14}(-e_{11} + e_{33}) + b_{13}(-a_{12} + a_{34}) + f_{13}(-e_{12} - e_{34}) = 0\,.$$

Further we show that the last four relations may be omitted in the calculations. Note that the total number of equations is 31, and the total number of unknown parameters is only 19. Therefore if the nontrivial solution exists, some equations are mutually dependent. As it was mentioned above, not all the coefficients may be assumed arbitrarily. There exists a set of free coefficients that may be given in advance, the remaining coefficients have to be calculated from the above equations. Because the equations are very complex and mutually dependent, even the number of free coefficients may not be calculated from the number of unknowns and the number of equations. Again the task seems to be extremely difficult. The physics suggests that the number of independent parameters equals nine, since one cell is defined by nine numbers (1.16). The only possibility is to try one choice, and it if fails try another. Obviously most of the proposed sets of independent parameters will lead to contradiction, and this anticipation forces us to base again on the numerical approach.

The previous equation and the next equation are separated by hundreds of hours spent at the computer. We do not list the approaches that failed, discuss only the one leading to M and satisfying the expected identity $M^2(\varphi, \psi) = M(2\varphi, 2\psi)$.

Assume that the following nine parameters:

$$\varphi,\ \psi,\ p,\ m_3,\ k_{14},\ a_{13},\ a_{14},\ b_{13},\ b_{14}, \tag{3.13}$$

are given, and define

$$c_{13} = \nu k_{13}, \quad c_{14} = \nu k_{14}, \quad k_{13} = 1. \tag{3.14}$$

The parameter p is assumed to be different from 0 and 1. For symmetry of the formulae assume $k_{13} = 1$. The degenerate cases $p = 0$ and $p = 1$ will be considered in the last chapter. The parameter ν will be determined later. Note that m_4 is not given and must be calculated from the above system of equations, together with other parameters. Note that the physical problem is governed by nine parameters, too (cf. (1.16)).

Define the new sixteen functions

$$\begin{aligned}
K_1 &= g_{13}e^{\#}{}_{11} + c_{14}a^{\#}{}_{12} + g_{14}e^{\#}{}_{12}, & N_1 &= g_{13}d^{\#}{}_{33} - c_{14}a^{\#}{}_{34} - g_{14}e^{\#}{}_{34},\\
K_2 &= -g_{13}e^{\#}{}_{33} + c_{14}b^{\#}{}_{34} + g_{14}e^{\#}{}_{34}, & N_2 &= -g_{13}f^{\#}{}_{11} - c_{14}b^{\#}{}_{12} - g_{14}f^{\#}{}_{34},\\
K_3 &= -c_{13}e^{\#}{}_{11} - f_{14}a^{\#}{}_{12} + c_{14}e^{\#}{}_{12}, & N_3 &= -c_{13}d^{\#}{}_{33} - f_{14}a^{\#}{}_{34} + c_{14}e^{\#}{}_{34},\\
K_4 &= -c_{13}e^{\#}{}_{33} - f_{14}b^{\#}{}_{34} + c_{14}e^{\#}{}_{34}, & N_4 &= -c_{13}f^{\#}{}_{11} - f_{14}b^{\#}{}_{12} + c_{14}f^{\#}{}_{34},\\
K_5 &= g_{14}e^{\#}{}_{11} + c_{13}a^{\#}{}_{12} + g_{13}e^{\#}{}_{12}, & N_5 &= -g_{14}d^{\#}{}_{33} - c_{13}a^{\#}{}_{34} + g_{13}e^{\#}{}_{34},\\
K_6 &= g_{14}e^{\#}{}_{33} + c_{13}b^{\#}{}_{34} - g_{13}e^{\#}{}_{34}, & N_6 &= -g_{14}f^{\#}{}_{11} - c_{13}b^{\#}{}_{12} - g_{13}f^{\#}{}_{34},\\
K_7 &= -c_{14}e^{\#}{}_{11} - f_{13}a^{\#}{}_{12} + c_{13}e^{\#}{}_{12}, & N_7 &= c_{14}d^{\#}{}_{33} - f_{13}a^{\#}{}_{34} - c_{13}e^{\#}{}_{34},\\
K_8 &= -c_{14}e^{\#}{}_{33} + f_{13}b^{\#}{}_{34} + c_{13}e^{\#}{}_{34}, & N_8 &= c_{14}f^{\#}{}_{11} + f_{13}b^{\#}{}_{12} - c_{13}f^{\#}{}_{34}.
\end{aligned} \tag{3.17}$$

where quantities marked by # are defined by (3.7).

It is easy to verify that for each p, h, m_4 and ν the identities are valid

$$K_i = N_i, \quad i = 1, 2, \ldots, 8. \tag{3.18}$$

Take into account (3.6) to obtain $g_{13}e_{33} + c_{14}b_{34} + g_{14}e_{34} = -hK_1/p$, etc. The system of equations (3.8) reduces now to the system

$$\begin{aligned}
&a_{13} - \frac{h}{1-p}K_1 - \frac{h}{p}N_1 = 0,\\
&b_{13} - \frac{h}{1-p}K_2 - \frac{h}{p}N_2 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&f_{14} - \frac{h}{1-p}K_8 - \frac{h}{p}N_8 = 0.
\end{aligned} \tag{3.19}$$

Due to Eqs. (3.18) the above relations simplify to the eight relations

$$\begin{aligned}
a_{13} &= \frac{h}{p(1-p)}K_1, \quad b_{13} = \frac{h}{p(1-p)}K_2,\\
e_{13} &= \frac{h}{p(1-p)}K_3, \quad f_{13} = \frac{h}{p(1-p)}K_4,
\end{aligned} \tag{3.20}$$

(3.20) [cont.)

$$a_{14} = \frac{h}{p(1-p)} K_5, \quad b_{14} = \frac{h}{p(1-p)} K_6,$$
$$e_{14} = \frac{h}{p(1-p)} K_7, \quad f_{14} = \frac{h}{p(1-p)} K_8.$$

Calculate now the parameter h from (3.5) to obtain

$$h = -\frac{p(1-p)}{\nu^2[-k_{13}^2(1+m_3^2) + k_{14}^2(1+m_4^2)]}. \tag{3.21}$$

Substitute this relation into each of the relations (3.19). Then they are identically satisfied. The set of relations (3.8) therefore follows from the set (3.18) and the relation (3.5).

Substitute now (3.13) and (3.15) into (3.7) and then calculate the coefficients e_{11}, f_{33}, f_{12}, e_{34} from (3.6). Substitution of the results shows that (3.12) is identically satisfied for each h and ν. This set of equations follows therefore from the set (3.6). From now on we disregard (3.12).

Note that up to this point the parameters v, m_4 have been arbitrary. Now we may define them to match the third group of equations (3.9). Before performing the calculations introduce six new functions

$$L_1 = -e^{\#}{}_{11}{}^2 + a^{\#}{}_{12}{}^2 + e^{\#}{}_{12}{}^2, \quad L_2 = -f^{\#}{}_{11}{}^2 + b^{\#}{}_{12}{}^2 + f^{\#}{}_{12}{}^2,$$
$$L_3 = -e^{\#}{}_{33}{}^2 + a^{\#}{}_{34}{}^2 + e^{\#}{}_{34}{}^2, \quad L_4 = -f^{\#}{}_{33}{}^2 + b^{\#}{}_{34}{}^2 + f^{\#}{}_{34}{}^2,$$
$$L_5 = -e^{\#}{}_{11} f^{\#}{}_{11} + a^{\#}{}_{12} b^{\#}{}_{12} + e^{\#}{}_{12} f^{\#}{}_{12}, \tag{3.22}$$
$$L_6 = -e^{\#}{}_{33} f^{\#}{}_{33} + a^{\#}{}_{34} b^{\#}{}_{34} + e^{\#}{}_{34} f^{\#}{}_{34}.$$

where the quantities at the right-hand side are defined by (3.7). Note that the third set of equations (3.9) reduce now to the following six equations:

$$-\frac{h^2}{(1-p)^2} L_1 + hP_1 = -p,$$
$$-\frac{h^2}{p^2} L_2 + hP_2 = -(1-p),$$
$$-\frac{h^2}{p^2} L_3 + hP_3 = -(1-p),$$
$$-\frac{h^2}{(1-p)^2} L_4 + hP_4 = -p \tag{3.23}$$
$$-\frac{h^2}{p(1-p)} L_5 + hP_5 = 0,$$
$$-\frac{h^2}{p(1-p)} L_6 + hP_6 = 0.$$

Substitution of (3.7) into the expressions (3.22) proves that for each ν and m_4 there is

$$L_1 = L_3, \quad L_2 = L_4, \quad L_5 = L_6. \tag{3.24}$$

Moreover, there is

$$\frac{L_1}{P_1} = \frac{L_2}{P_2} = \frac{L_3}{P_3} = \frac{L_4}{P_4} = -\frac{L_5}{P_5} = -\frac{L_6}{P_6} = -\frac{p(1-p)}{h}. \tag{3.25}$$

Therefore the functions $L_1, L_2, \ldots, L_6$ may be expressed as the product of $p(1-p)/h$ and $P_1, P_2, \ldots, P_6$. Substitute the expressions into (3.23). The last two relations are now satisfied, and in the first four relations two pairs are identical. There remain two very simple relations

$$hP_1 = -p(1-p), \quad hP_2 = -p(1-p). \tag{3.26}$$

It follows that the remaining freedom of defining ν and m_4 must be used to obtain $P_1 = P_2$, not contradicting (3.21). From (3.11) and (3.21) follow the necessary relations

$$\begin{aligned} a_{13}^2(1+m_3^{-2}) - a_{14}^2(1+m_4^{-2}) &= p(1-p)/h, \\ b_{13}^2(1+m_3^{-2}) - b_{14}^2(1+m_4^{-2}) &= p(1-p)/h, \\ \nu^2[k_{13}^2(1+m_3^2) + k_{14}^2(1+m_4^2)] &= p(1-p)/h. \end{aligned} \tag{3.27}$$

The first two equations after subtracting give the equation

$$(1+m_3^{-2})(a_{13}^2 - b_{13}^2) = (1+m_4^{-2})(a_{14}^2 - b_{14}^2),$$

leading to the following expression for the parameter m_4

$$m_4 = \pm\sqrt{\frac{(1+m_3^{-2})(a_{13}^2 - b_{13}^2)}{a_{14}^2 - b_{14}^2} - 1}. \tag{3.28}$$

The two first equations of (3.27) are now identidal. Together with the third one each of them gives the equation

$$a_{13}^2(1+m_3^{-2}) - a_{14}^2(1+m_4^{-2}) = \nu^2[k_{14}^2(1+m_3^2) + k_{14}^2(1+m_4^2)],$$

which allows to calculate the last unknown parameter ν

$$\nu = \pm\sqrt{\frac{a_{13}^2(1+m_3^{-2}) - a_{14}^2(1+m_4^{-2})}{k_{13}^2(1+m_3^2) + k_{14}^2(1+m_4^2)}}. \tag{3.29}$$

Now the equations (3.9), and the equivalent equations (3.23) are satisfied. Note that for some data (3.13) the parameters m_4 and ν are real, and for other data they are imaginary or complex.

Up to now we have proved that for arbitrary φ and ψ there is

$$[M(\varphi, \psi, pp)]^2 = M(2\varphi, 2\psi, pp), \tag{3.30}$$

where pp stands for the whole set of constant parameters, $pp = \{h, m_1, m_2, m_3, m_4, e_{11}, f_{11}, a_{12}, b_{12}, c_{12}, \ldots, e_{44}, f_{44}\}$.

There remains to prove the identity

$$[M(\varphi, \psi, pp)]^n = M(n\varphi, n\psi, pp). \tag{3.31}$$

for arbitrary integer n. The proof by mathematical induction is straightforward. Assuming that (3.31) holds for n we multiply M^n, as given by (3.30), by M. Since the equations must be satisfied for each φ and ψ, the coefficients of the trigonometric functions at both sides of the resulting equation must be the same. The resulting equations are exactly the equations (3.6), (3.8), (3.12). Since the calculated parameters satisfy them, the identity (3.31) induces its validity for $n+1$. This completes the proof of (3.31).

Calculate now the complex-valued matrix M, as given by (2.4). All parameters are known and each component of M may be expressed by the nine parameters

$$\varphi,\ \psi,\ h,\ m_3,\ a_{13},\ a_{14},\ b_{13},\ b_{14},\ c_{13}/c_{14}\,. \tag{3.32}$$

Tedious analytic calculations lead to the identity

$$\det M = 1\,. \tag{3.33}$$

Summarize now the results of the present chapter. The nine parameters (3.32) may be given in advance. The formulae (3.28), (3.29) allow to calculate m_4 and ν and the formula (3.14) allows to calculate c_{13}, c_{14}. The value of h follows from (3.5). There remains to calculate $e_{11}, f_{11}, \ldots, e_{34}, f_{34}$ from (3.6) and $a_{31}, b_{31}, \ldots, g_{32}, h_{32}$ from (2.12). Examination of (3.2) proves that all components of the transfer matrix M may now be expressed by the nine parameters (3.13). The determinant of the resulting matrix equals 1.

Note that the transfer matrix M may be expressed either by the nine physical quantities (1.16) or the nine parameters (3.13). Either of the two possibilities may be used, when constructing the transfer matrix. The second choice gives more insight into the mathematical structure of M. The above relations did not allow to calculate the parameters (2.13) and the coefficients of (2.4), provided M was known. The formulae allowing such calculations are obtained in the next chapter.

We do not intend to discuss here the limiting and degenerate cases corresponding to $p \to 0, p \to 1, h \to 0, m_3 \to 0, m_4 \to 0, m_3 \to m_4, a_{14} \to \pm b_{14}$. This would consume too much space. We shall give the analysis elsewhere. Here we mention only that if $p \to 0$ and h remains finite, then the only possibility corresponds to (2.3). If $p \to 0$ and $h \to 0$, then the nontrivial matrix M is obtained. If $m_3 \to m_4$ then $h \to 0$ and $\nu, c_{13}, c_{14} \to \infty$. If $a_{14} \to b_{14}$ then $\nu, c_{13}, c_{14} \to \infty$.

4. Calculation of the parameters

In the previous chapter from the nine parameters given in advance the set of parameters $\varphi, \psi, p, h, m_1, a_{12}, \ldots, h_{ij}$ has been calculated. This set could be used to calculate effectively the complex-valued matrix M satisfying, for each integer n, the two identities $[M(\varphi, \psi)]^n = M(n\varphi, n\psi), \det M = 1$. We shall show that M satisfies one additional relation that was overlooked in the previous calculations.

In the present chapter we assume that all components of the matrix M are given in advance, e.g. they are calculated from the physical data for the elementary cell. For this M we will effectively calculate the parameters $p, h, m_1, m_2, m_3, m_4, d_{11}, e_{11}, \ldots, b_{34}$. The formulae are therefore essential for consideration of periodic systems of layers. The given components of M possess the w-symmetry and satisfy the relation $\det M = 1$. We shall discover one additional relation that was not essential for the previous calculations. It was shown in Secs. 2 and 3 how to calculate these parameters from the first five powers of M. Now we know much more about the structure of M and shall deduce the parameters from the first two powers only.

In order to simplify the formulae let us separate each component of M into the real and imaginary parts,

$$M_{ij} = R_{ij} + iI_{ij}\,. \tag{4.1}$$

First we summarize some necessary results of Secs. 2 and 3. It has been proved that the transfer matrix M may be represented in the following form:

$$
\begin{aligned}
R_{11} &= p\cos\varphi + (1-p)\cos\psi, \qquad I_{11} = d_{11}\sin\varphi + e_{11}\sin\psi,\\
R_{12} &= a_{12}\sin\varphi + b_{12}\sin\psi, \qquad I_{12} = \frac{1}{m_1}[a_{12}\sin\varphi + b_{12}\sin\psi],\\
R_{33} &= (1-p)\cos\varphi + p\cos\psi, \qquad I_{33} = d_{33}\sin\varphi + e_{33}\sin\psi,\\
R_{34} &= a_{34}\sin\varphi + b_{34}\sin\psi, \qquad I_{34} = \frac{1}{m_2}[a_{34}\sin\varphi + b_{34}\sin\psi],\\
R_{13} &= a_{13}\sin\varphi + b_{13}\sin\psi + c_{13}(\cos\varphi - \cos\psi),\\
I_{13} &= -\frac{1}{m_3}[a_{13}\sin\varphi + b_{13}\sin\psi] + m_3 c_{13}(\cos\varphi - \cos\psi),\\
R_{14} &= a_{14}\sin\varphi + b_{14}\sin\psi + c_{14}(\cos\varphi - \cos\psi),\\
I_{14} &= \frac{1}{m_4}[a_{14}\sin\varphi + b_{14}\sin\psi] - m_4 c_{14}(\cos\varphi - \cos\psi),\\
R_{31} &= -h(a_{13}\sin\varphi + b_{13}\sin\psi) + h c_{13}(\cos\varphi - \cos\psi),\\
I_{31} &= -\frac{h}{m_3}(a_{13}\sin\varphi + b_{13}\sin\psi) - h m_3 c_{13}(\cos\varphi - \cos\psi),\\
R_{32} &= -h(a_{14}\sin\varphi + b_{14}\sin\psi) + h c_{14}(\cos\varphi - \cos\psi),\\
I_{32} &= -\frac{h}{m_4}(a_{14}\sin\varphi + b_{14}\sin\psi) - h m_4 c_{14}(\cos\varphi - \cos\psi).
\end{aligned}
\tag{4.2}
$$

The remaining components of M are determined by the w-symmetry.

From the above set of relations immediately follow the expressions for m_1, m_2,

$$
m_1 = \frac{R_{12}}{I_{12}}, \qquad m_2 = \frac{R_{34}}{I_{34}}. \tag{4.3}
$$

We pass to the calculation of the further three parameters h, m_3, m_4.

From (4.2) follow the relations

$$
\begin{aligned}
R_{13} + m_3 I_{13} &= (1 + m_3^2)c_{13}(\cos\varphi - \cos\psi),\\
R_{31} - m_3 I_{31} &= h(1 + m_3^2)c_{13}(\cos\varphi - \cos\psi),
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
m_3 R_{13} - I_{13} &= (m_3 + 1/m_3)(a_{13}\sin\varphi + b_{13}\sin\psi),\\
m_3 R_{31} + I_{31} &= -h(m_3 + 1/m_3)(a_{13}\sin\varphi + b_{13}\sin\psi),
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
R_{14} - m_4 I_{14} &= (1 + m_4^2)c_{14}(\cos\varphi - \cos\psi),\\
R_{32} - m_4 I_{32} &= h(1 + m_4^2)c_{14}(\cos\varphi - \cos\psi),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
m_4 R_{14} + I_{14} &= (m_4 + 1/m_4)(a_{14}\sin\varphi + b_{14}\sin\psi),\\
m_4 R_{32} + I_{32} &= -h(m_4 + 1/m_4)(a_{14}\sin\varphi + b_{14}\sin\psi).
\end{aligned}
\tag{4.7}
$$

The equations (4.4) and (4.5) lead to two relations

$$h = \frac{R_{31} - m_3 I_{31}}{R_{13} + m_3 I_{13}} = \frac{m_3 R_{31} + I_{31}}{m_3 R_{13} - I_{13}}. \tag{4.8}$$

Calculate now m_3 from (4.8) to obtain

$$m_3 = \frac{R_{31} - hR_{13}}{I_{31} + hI_{13}} = \frac{-I_{31} + hI_{13}}{R_{31} + hR_{13}}. \tag{4.9}$$

The last relation leads to the following expression for h^2:

$$h^2 = \frac{R_{31}^2 + I_{31}^2}{R_{13}^2 + I_{13}^2}. \tag{4.10}$$

On the other hand, it is possible to perform parallel calculations basing on the components M_{14} and M_{41}. From the equations (4.6), (4.7) we obtain two equations

$$h = \frac{R_{32} - m_4 I_{32}}{R_{14} - m_4 I_{14}} = \frac{-m_4 R_{32} - I_{32}}{m_4 R_{14} + I_{14}}, \tag{4.11}$$

and to further relations parallel to (4.10)

$$m_4 = \frac{-R_{32} + hR_{14}}{-I_{32} + hI_{14}} = \frac{-I_{32} - hI_{14}}{R_{32} + hR_{14}}. \tag{4.12}$$

These relations lead to the expression for h^2 other than that in (4.10)

$$h^2 = \frac{R_{32}^2 + I_{32}^2}{R_{14}^2 + I_{14}^2}. \tag{4.13}$$

Therefore there must exist one additional relation between the components of M, namely

$$\frac{R_{31}^2 + I_{31}^2}{R_{13}^2 + I_{13}^2} = \frac{R_{32}^2 + I_{32}^2}{R_{14}^2 + I_{14}^2}. \tag{4.14}$$

Taking into account the symmetries of M, the above relation may be expressed by the complex-valued components of M, namely

$$M_{14} M_{23} M_{31} M_{42} = M_{13} M_{24} M_{32} M_{41}. \tag{4.15}$$

It may be verified that the transition matrix M calculated from (1.11) satisfies in fact the above relation.

Let us now pass to the calculation of p. Subtraction of the appropriate expressions of Eq. (4.2) leads to the relations

$$\begin{aligned} R_{11} - R_{33} &= (2p-1)(\cos\varphi - \cos\psi), \\ R_{13} + m_3 I_{13} &= (1 + m_3^2) c_{13} (\cos\varphi - \cos\psi), \\ R_{14} - m_4 I_{14} &= (1 + m_4^2) c_{14} (\cos\varphi - \cos\psi). \end{aligned} \tag{4.16}$$

After eliminating the trigonometric functions we obtain

$$\begin{aligned} c_{13} &= (2p-1) \frac{R_{13} + m_3 I_{13}}{(1 + m_2^2)(R_{11} - R_{33})}, \\ c_{14} &= (2p-1) \frac{R_{14} - m_4 I_{14}}{(1 + m_4^2)(R_{11} - R_{33})}. \end{aligned} \tag{4.17}$$

Our purpose is to calculate p. After determining the parameter p, these relations will be used as formulae for c_{13}, c_{14}. Now Eq. (3.5) derived in the previous chapter can be used. We write here this equation in a slightly different form

$$p(1-p) - h(1+m_3^2)c_{13}^2 + h(1+m_4^2)c_{14}^2 = 0\,. \tag{4.18}$$

Substitute into this equation the coeffients c_{13}, c_{14} as given by Eq. (4.17). The following quadratic equation for p is obtained

$$(4U-1)p^2 - (4U-1)p + U = 0\,, \tag{4.19}$$

where the function U is given by the relation

$$U = \frac{h}{(R_{11}-R_{33})^2}\left[-\frac{(R_{13}+m_3I_{13})^2}{1+m_3^2} + \frac{(R_{14}-m_4I_{14})^2}{1+m_4^2}\right]. \tag{4.20}$$

Other equivalent expression for the function U may be obtained after substitution of h as given by Eq. $(4.9)_1$ in the first product, and has given by Eq. $(4.10)_1$ in the second product of Eq. (4.18),

$$U = \frac{(R_{14}-m_4I_{14})(R_{32}-m_4I_{32})}{(1+m_4^2)(R_{11}-R_{33})} - \frac{(R_{13}+m_3I_{13})(R_{31}-m_3I_{31})}{(1+m_3^2)(R_{11}-R_{33})}\,.$$

The solution of the quadratic equation (4.19) is

$$p = \frac{1}{2} \pm \frac{1}{\sqrt{1-4U}}\,. \tag{4.21}$$

There remains to calculate the parameters φ and ψ. From Eq. (4.2) follow the two equations

$$\begin{aligned} pR_{11} - (1-p)R_{33} &= (2p-1)\cos\varphi\,,\\ pR_{33} - (1-p)R_{11} &= (2p-1)\cos\psi\,. \end{aligned} \tag{4.22}$$

Because p is already known, there is

$$\begin{aligned} \varphi &= \arccos\frac{pR_{11}-(1-p)R_{33}}{2p-1}\,,\\ \psi &= \arccos\frac{pR_{33}-(1-p)R_{11}}{2p-1}\,. \end{aligned} \tag{4.23}$$

It may be checked that the change of the sign in Eq. (4.21) corresponds to the exchange between φ and ψ. Because this is not essential for the representation of M, we decide to use in Eq. (2.21) the upper + sign only. Note that for some M given in advance the parameters φ, ψ may be complex.

The above calculations show that a part of the parameters may be calculated without squaring the matrix M. Now we must calculate the remaining parameters. Note that because the values of h, p, m_3, m_4 are already known, Eqs. (4.17) are in fact the formulae for c_{13}, c_{14}. The above remark applies therefore to c_{13}, c_{14}, too. The exceptional position of Eq. (3.5) has been again stressed by the fact that its validity may be established at the level of M, and not at the level of M^2.

There remain the coefficients of the sine function. Note that the coefficients of sine appear in all expressions (4.2) as the sum of two terms $(a_{ij}\sin\varphi + b_{ij}\sin\psi)$. Therefore a_{ij}, b_{ij} can not be calculated without calculating the second power of M. If we calculate

M^2, then e.g. the system of equations for e_{11}, f_{11} reads

$$\begin{aligned} e_{11}\sin\varphi + f_{11}\sin\psi &= T_1, \quad T_1 = \operatorname{Im} M_{11}, \\ e_{11}\sin 2\varphi + f_{11}\sin 2\psi &= T_2, \quad T_2 = \operatorname{Im} M_{11}^2, \end{aligned} \tag{4.24}$$

and has the solution

$$\begin{aligned} e_{11} &= \frac{1}{\xi}[T_1\sin 2\psi - T_2\sin\psi], \\ f_{11} &= \frac{1}{\xi}[-T_1\sin 2\varphi + T_2\sin\varphi], \end{aligned} \tag{4.25}$$

where

$$\xi = \sin\varphi\sin 2\psi - \sin 2\varphi\sin\psi. \tag{4.26}$$

Exactly the same structure possess the remaining equations for a_{ij}, b_{ij}, e_{ij}, f_{ij}. In order to save space we give only the resulting formulae

$$\begin{aligned} e_{33} &= \frac{1}{\xi}[T_3\sin 2\psi - T_4\sin\psi], \quad T_3 = \operatorname{Im} M_{33}, \quad T_4 = \operatorname{Im} M_{33}^2, \\ f_{33} &= \frac{1}{\xi}[-T_3\sin 2\varphi + T_4\sin\varphi], \end{aligned} \tag{4.27}$$

$$\begin{aligned} a_{12} &= \frac{1}{\xi}[T_5\sin 2\psi - T_6\sin\psi], \quad T_5 = \operatorname{Re} M_{12}, \quad T_6 = \operatorname{Re} M_{12}^2, \\ b_{12} &= \frac{1}{\xi}[-T_5 sim 2\varphi + T_6\sin\varphi], \end{aligned} \tag{4.28}$$

$$\begin{aligned} a_{34} &= \frac{1}{\xi}[T_7\sin 2\psi - T_8\sin\psi], \quad T_7 = \operatorname{Re} M_{34}, \quad T_8 = \operatorname{Re} M_{34}^2, \\ b_{34} &= \frac{1}{\xi}[-T_7\sin 2\varphi + T_8\sin\varphi], \end{aligned} \tag{4.29}$$

$$\begin{aligned} a_{13} &= \frac{1}{\xi}[T_9\sin 2\psi - T_{10}\sin\psi], \\ b_{13} &= \frac{1}{\xi}[-T_9\sin 2\varphi + T_{10}\sin\varphi], \end{aligned} \tag{4.30}$$

$$T_9 = \frac{\operatorname{Re} M_{13} - \operatorname{Im} M_{13}/m_3}{1 + m_3^{-2}}, \quad T_{10} = \frac{\operatorname{Re} M_{13}^2 - \operatorname{Im} M_{13}^2/m_3}{1 + m_3^{-2}},$$

$$\begin{aligned} a_{14} &= \frac{1}{\xi}[T_{11}\sin 2\psi - T_{12}\sin\psi], \\ b_{14} &= \frac{1}{\xi}[-T_{11}\sin 2\varphi + T_{12}\sin\varphi], \end{aligned} \tag{4.31}$$

$$T_{11} = \frac{\operatorname{Re} M_{14} + \operatorname{Im} M_{14}/m_4}{1 + m_4^{-1}}, \quad T_{12} = \frac{\operatorname{Re} M_{14}^2 + \operatorname{Im} M_{14}^2/m_4}{1 + m_4^{-2}}.$$

The above relations close the formulae necessary to calculate the parameters $\varphi, \psi, a_{ij}, \ldots$ $\ldots, h_{ij}$ appearing in Eq. (2.1) and express them in terms of M. In practice, from the physical parameters (1.16) we calculate M, and then evaluate the mathematical parameters (2.2).

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Plane problem for the steady heat conduction of material with circular inclusions

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THE METHOD of functional equations for boundary R-value problem is proposed, the latter being considered as a plane problem for the steady heat conduction of composite material. The simplest version of the method has been suggested in [1].

1. Introduction

LET US CONSIDER the mutually disjoint circles $D_k = \{z \in \mathbb{C}, |z - a_k| < r_k\}$ $(k = 0, 1, ..., n)$ on the complex plane $\mathbb{C}$. Let $D = \mathbb{C} \setminus U_{k=0}^{n} \overline{D}_k$, the point $z = 0$ belongs to D. We shall consider steady heat conduction in the complex plane $\mathbb{C}$, when $D, D_0, ..., D_n$ are occupied by materials with the thermal conductivities $\lambda, \lambda_0, ..., \lambda_n$ (> 0), respectively, contact between materials being perfect. Let us assume that we have sources of heat in the domain D only. Let $f(z)$ be a complex potential of the sources [2]. It means that the known function $f(z)$ is analytic in all D_k. As an example, if we have sources at $z = 0$, $z = \infty$, then $f(z) = \ln z$, where the branch of logarithm is fixed in such a way that the cut belongs to D.

In order to determine the temperature function we find the harmonic functions $u(z)$, $u_0(z), ..., u_n(z)$ in $D, D_0, ..., D_n$, respectively, according to the boundary conditions

$$u(t) = u_k(t), \qquad \lambda \frac{\partial u}{\partial n} = \lambda_k \frac{\partial u_k}{\partial n}, \qquad |t - a_k| = r_k, k = 0, 1, ..., n,$$

where $\partial/\partial n$ is a normal derivative. The function $u(z)$ has the same singularities as $f(z)$. This problem is equivalent to the following boundary $\mathbb{R}$-value problem [2]:

$$\phi(t) = \phi_k(t) + \varrho_k \overline{\phi_k(t)} - f(t), \qquad |t - a_k| = r_k, \tag{1.1}$$

where $\varrho_k := (\lambda - \lambda_k)(\lambda + \lambda_k)^{-1}$, the unknown functions $\phi(z), \phi_0(z), \ldots, \phi_n(z)$ are analytic in $D, D_0, \ldots, D_n$, respectively, and continuously differentiable in $\overline{D}, \overline{D}_0, \ldots, \overline{D}_n$. The harmonic and analytic functions are related by the identities $u(z) = \operatorname{Re} \phi(z)$, $u_k(z) = 2\lambda(\lambda + \lambda_k)^{-1} \operatorname{Re} \phi_k(z)$. The problem (1.1) is of the so-called elliptic type [3, 4] because $|\varrho_k| < 1$. That notation will help us essentially to solve the problem (1.1).

In [1, 2, 5] the problem (1.1) has been reduced to the following system of functional equations:

$$\phi_k(z) = \sum_{\substack{m=0 \\ m \neq k}}^{n} \varrho_m [\overline{\phi_m(z_m^*)} - \overline{\phi_m(0_m^*)}] + f(z) + \phi(0) - \varrho_k \overline{\phi_k(0_k^*)}, \tag{1.2}$$

$$|z - a_k| < r_k, \qquad k = 0, 1, \ldots, n,$$

where $\phi_k(z)$ are unknown functions. Here $z_k^* := r_k^2/(\overline{z - a_k}) + a_k$ — is inversion with respect to the circumference $|z - a_k| = r_k$. For a example, the point $0_k^* := r_k^2 \overline{(-a_k)} + a_k$.

Let us denote the sequence of inversions with respect to the circumferences with numbers $k_1, k_2, \ldots, k_m$ by

$$z^*_{k_m k_{m-1} \ldots k_1} := (z^*_{k_{m-1} \ldots k_1})^*_{k_m}.$$

There are no equal neighbour numbers in the sequence $k_1, k_2, \ldots, k_m$. Also, we have

$$\phi(z) = \sum_{\substack{m=0 \\ m \neq k}}^{n} \frac{\lambda - \lambda_m}{2\lambda} [\overline{\phi_m(z^*_m)} - \overline{\phi_m(0^*_m)}] + \phi(0), z \in \overline{D}. \tag{1.3}$$

G. M. GOLUZIN [1] has solved the system (1.2) by the method of successive approximations under special assumptions. His result has been developed in [2, 5–8]. In this paper we prove convergence of the method of successive approximations in general situation and get a solution of Eq. (1.1) in analytic form.

2. Solution of the system of functional equations

Let us suppose that the given functions $g_k(z)$ are analytic in $|z - a_k| < r_k$ and continuously differentiable in $|z - a_k| \leq r_k (k = 0, 1, \ldots, n)$.

THEOREM 1. *The system of functional equations*

$$\psi_k(z) = \sum_{\substack{m=0 \\ m \neq k}}^{n} \varrho_m \overline{\psi_m(z^*_m)} + g_k(z), \quad |z - a_k| \leq r_k, \quad k = 0, 1, \ldots, n, \tag{2.1}$$

has the solution

$$\begin{aligned} \psi_k(z) = {} & [g_k(z) - g_k(O^*_k)] + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \varrho_{k_1} [\overline{g_{k_1}(z^*_{k_1})} - \overline{g_{k_1}(O^*_{k_1})}] \\ & + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \varrho_{k_1} \varrho_{k_2} [g_{k_2}(z^*_{k_1 k_2}) - g_{k_2}(O^*_{k_1 k_2})] + \ldots + c_k \end{aligned} \tag{2.2}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \cdots \sum_{\substack{k_m=0 \\ k_m \neq k_{m-1}}}^{n} \varrho_{k_1} \varrho_{k_2} \cdots \varrho_{k_m} \mathfrak{S}^m [g_{k_m}(z^*_{k_m} \ldots k_1) - g_{k_m}(O^*_{k_m \ldots k_1 k})] + c_k,$$

where c_k are constants, $\mathfrak{S}$ is the operator of complex conjugation. The series (2.2) *converges absolutely and uniformly in $|z - a_k| \leq r_k$.*

Proof. Let us consider the Banach space C which consists of functions continuous on $U_{k=0}^{n} \partial \overline{D}_k$. The norm $\|f\| = \max_k \max_{\partial D_k} |f(t)|$. Here $\partial D_k := \{t \in \mathbb{C}, |t - a_k| = r_k\}$. We denote by C^+ the subspace of functions analytic in $|z - a_k| < r_k$, i.e. $f \in C^+$ denotes that $f_k(z) = f(z)$ are analytic in $|z - a_k| < r_k$ and continuous on $|z - a_k| \leq r_k (k = 0, 1, \ldots, n)$. The functions $\overline{\psi_m(z^*_m)}$ are analytic in $|z - a_k| \leq r_k$, where k is fixed, $m = 0, 1, \ldots, n$, $m \neq k$. The function $g_k(z)$ is continuously differentiable. Hence $\psi_k(z)$ is continuously differentiable too. Set $\Psi_k(z) = \psi'_k(z)$; then

$$\Psi_k(z) = \sum_{\substack{m=0 \\ m \neq k}}^{n} \varrho_m (\overline{z^*_m})' \overline{\Psi_m(z^*_m)} + g'_k(z), \quad |z - a_k| \leq r_k. \tag{2.3}$$

Let us rewrite Eq. (2.3) as the equation in C^+

$$\Psi = A\Psi + g'.$$

The operator A is compact in C^+, because $A : C^+ \to C^+$ and A is compact in C as a sum of compositions of the bounded operators

$$f(t) \to \overline{f(t)}, \qquad f(t) \to \varrho_m(\overline{t_m^*})' f(t)$$

and the compact operator

$$f(t) \to f(t_m^*) = \frac{1}{2\pi} \int_{\partial D_k} \frac{t(\tau)}{\tau - t_m^*} d\tau, \qquad |t - a_k| = r_k,$$

when $k, m = 0, 1, \ldots, n$, $m \neq k$. We shall prove that the spectral radius of A is less than 1. This will mean the convergence of the method of successive approximations in C^+ [9]. We have to prove that if $\Psi = \mu A\Psi$, where the constant μ satisfies the inequality $|\mu| \leq 1$, then $\Psi = 0$. Integrating $\Psi = \mu A\Psi$ we obtain

$$\psi_k(z) = \mu \sum_{\substack{m=0 \\ m \neq k}}^{n} \varrho_m \overline{\psi_m(z_m^*)} + \gamma_k, |z - a_k| \leq r_k, \tag{2.4}$$

where γ_k are constants. We introduce the function

$$\omega(z) := \mu \sum_{\substack{m=0 \\ m \neq k}}^{n} \varrho_m \overline{\psi_m(z_m^*)},$$

which is analytic in $\overline{D}$. From Eq. (2.4) we obtain the boundary $\mathbb{R}$-value problem

$$\omega(t) = \psi_k(t) + \mu\varrho_k \overline{\psi_k(t)} - \gamma_k, |t - a_k| = r_k.$$

Using the inequality $1 > |\mu\varrho_k|$ and the general theory [3, 4] we conclude that the last problem, where γ_k are fixed, has the solution depending on arbitrary additive constant. In our case $\omega(z) \equiv \text{const}$. Then $\Psi_k(z) = \psi_k'(z) \equiv 0$. Therefore, convergence in C^+ has been proved. We can write the solution of the system (2.3)

$$\Psi_k(z) = g_k'(z) + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \varrho_{k_1} (\overline{g_{k_1}(z_{k_1}^*)})' + \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k}}^{n} \varrho_{k_1}\varrho_{k_2} (g_{k_2}(z_{k_2 k_1}^*))' + \ldots$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{k_1=0 \\ k_2 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \cdots \sum_{\substack{k_m=0 \\ k_m \neq k_{m-1}}}^{n} \varrho_{k_1}\varrho_{k_2} \cdots \varrho_{k_m} (\mathfrak{G}^m g_{k_m}(z_{k_m \ldots k_1}^*))', \qquad |z - a_k| \leq r_k.$$

That series converges uniformly in C^+. Hence it may be integrated term by term

$$\psi_k(z) - \psi_k(0_k^*) = \int_{0_k^*}^{z} \Psi(z) dz \tag{2.5}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \cdots \sum_{\substack{k_m=0 \\ k_m \neq k_{m-1}}}^{n} \varrho_{k_1}\varrho_{k_2} \cdots \varrho_{k_m} \mathfrak{G}^m [g_{k_m}(z_{k_m \ldots k_1}^*) - g_{k_m}(0_{k_m \ldots k_1 k}^*)].$$

Assume now that $\lambda_k < \lambda$, i.e. $0 < \varrho_k < 1$ for all k. Let us pick out μ such that $0 < \mu < 1$, and the constants

$$\varepsilon_k := \frac{1}{\lambda}\frac{(1-\mu)(\lambda^2 - \lambda_k^2)}{\mu(\lambda + \lambda_k) + \lambda - \lambda_k}$$

which satisfy the inequalities $\lambda_k - \lambda\varepsilon_k > 0$ $(k = 0, 1, \ldots, n)$. Then $\varrho_k = \mu\varrho_k'$ where $0 < \varrho_k' = \dfrac{\lambda - (\lambda_k - \lambda\varepsilon_k)}{\lambda + (\lambda + \lambda\varepsilon_k)} < 1$. If we substitute $\mu\varrho_k'$ instead of ϱ_k in Eq. (2.5), then we obtain the series

$$\sum_{m=0}^{\infty} \mu^m \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \varrho_{k_1}' \varrho_{k_2}' \ldots \varrho_{k_m}' \mathfrak{G}^m [g_{k_m}(z^*_{k_m \ldots k_1}) - g_{k_m}(0^*_{k_m \ldots k_1 k})]$$

converging absolutely as a geometric series. This proves the theorem.

3. Solution of the boundary $\mathbb{R}$-value problem (1.1)

We return to the system (1.2). The functions $\phi_k(z)$ have the form

$$\phi_k(z) = \sum_{m=0}^{\infty} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{n} \varrho_{k_1} \varrho_{k_2} \ldots \varrho_{k_m} \mathfrak{G}^m [f(z^*_{k_m \ldots k_1}) - f(0^*_{k_m \ldots k_1 k})] + c_k, \tag{3.1}$$

$$|z - a_k| \leq r_k,$$

where $c_k = \phi_k(0_k^*)$. From Eq. (1.3) we have

$$\phi(z) = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{\substack{k_1=0 \\ k_1 \neq k}}^{n} \sum_{\substack{k_2=0 \\ k_2 \neq k}}^{n} \cdots \sum_{\substack{k_m=0 \\ k_m \neq k_{m-1}}}^{n} \varrho_{k_1} \varrho_{k_2} \ldots \varrho_{k_m} \varrho_k \mathfrak{G}^{m+1} [f(z^*_{k_m \ldots k_1}) \tag{3.2}$$

$$- f(0^*_{k_m \ldots k_1 k})] + c, \qquad z \in \overline{D},$$

where $c = \phi(0)$. The problem (1.1) has the solution depending on an arbitrary additive constant [3]. We assume c to be that arbitrary constant. Substituting $z = 0_m^*$ in Eq. (3.1) we obtain the values

$$R_{mk} := [\overline{\phi_m(0^*_{mk})} - \overline{\phi_m(0^*_m)}], \qquad k, m = 0, 1, \ldots, n, k \neq m$$

(here we write m instead of k and inversely). Then, by substituting $z = 0_k^*$ in Eq. (1.2) we have the $\mathbb{R}$-linear algebraic system

$$c_k + \varrho_k \bar{c}_k = c + R_k, \qquad k = 0, 1, \ldots, n,$$

where $R_k = \sum_{m=0}^{n} \varrho_m R_{mk}$. After solving that system we obtain

$$c_k = \frac{1}{1 + \varrho_k} \operatorname{Re} R_k + \frac{i}{1 - \varrho_k} \operatorname{Im} R_k + \frac{\operatorname{Re} c}{1 + \varrho_k} + \frac{i \operatorname{Im} c}{1 - \varrho_k}, \qquad k = 0, 1, \ldots, n. \tag{3.3}$$

So we have proved the following

THEOREM 2. *The problem* (1.1) *has the solution* (3.1), (3.2), (3.3) *depending on the arbitrary constant* c.

4. Concluding remarks

The series

$$(4.1)\qquad \sum_{m=0}^{\infty}\sum_{\substack{k_1=0\\k_1\neq k}}^{n}\sum_{\substack{k_2=0\\k_2\neq k_1}}^{n}\cdots\sum_{\substack{k_m=0\\k_m\neq k_{m-1}}}^{n}\varrho_{k_1}\varrho_{k_2}\cdots\varrho_{k_m}\mathfrak{G}^m f(z^*_{k_m\ldots k_1})$$

can be divergent. For example, if $f(z)\equiv 1$, $\varrho_m=\dfrac{1}{n}(m=0,1,\ldots,n)$, we obtain the divergent series $\sum_{m=0}^{\infty}1$. It can be easily seen that if $|\varrho_m|<\dfrac{1}{n}$ then (4.1) converges absolutely.

Let us consider the example when we have sources at $z=0$ and $z=\infty$. Then $f(z)=\ln z$. Let $\varrho_m=\varrho(m=0,1,\ldots,n)$. Then the following relation is valid:

$$\operatorname{Re}\phi_k(z)=\ln\prod_{m=0}^{\infty}\prod_{\substack{k_1=0\\k_1\neq k}}^{n}\cdots\prod_{\substack{k_m=0\\k_m\neq k_{m-1}}}^{n}\left|\frac{z^*_{k_m\ldots k_1}}{0^*_{k_m\ldots k_1 k}}\right|^{\varrho^m}+\text{const},\qquad |z-a_k|\leq r_k\,.$$

The function $\phi(z)$ has an analogous form.

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Optimization problems in nonlinear systems

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AN OPZIMIZATION problem has been considered in the one-dimensional formulation on continua and compact subsets of Banach spaces. This formulation has been found to be useful from the point of view of synthesis of analog processor systems for optimal control. Stochastic modeling of physical systems, as well as mechanical automata and synthesis of electronic control systems are the fields of possible applications of the results obtained.

1. Introduction

LET $\mathcal{C}$ BE A NONEMPTY PEANO CONTINUUM [6], where a nonempty, connected and locally-connected compact subset of a Banach space is an example. Let $\mathcal{J}$ denote the interval $[0, 1]$ of the real line $\mathbb{R}$. By the Hahn–Mazurkiewicz theorem [6], there exists a continuous map $\zeta_{\mathcal{C}}(\cdot) : \mathcal{J} \ni t \to \zeta_{\mathcal{C}}(t) \in \mathcal{C}$, which defines continuous surjection of the interval $\mathcal{J}$ onto $\mathcal{C}$.

Let $g(\cdot)$ be a function defined on a Peano continuum $\mathcal{C}$ and with values in $\mathbb{R}$. In a number of considerations, one can analyse the given function $g(\cdot)$ by taking into consideration the composite function $g \circ \zeta_{\mathcal{C}}(\cdot)$ of one variable t in $\mathcal{J}$, where $\zeta_{\mathcal{C}}(\cdot)$ is a continuous surjection of $\mathcal{J}$ onto $\mathcal{C}$, as in the Hahn–Mazurkiewicz theorem.

The problem of estimating the global minimum value and the global maximum value of the function, and the problem of selecting the points of global minimum and global maximum in the domain of the function are the subject of these considerations. By the Hahn–Mazurkiewicz theorem, the optimization problem considered admits the one-dimensional formulation on continua.

Let $g(\cdot)$ be a continuous function defined on a Peano continuum $\mathcal{C}$ and having values in $\mathbb{R}$. For example, $g(\cdot) : \mathbb{B} \to \mathbb{R}$ is a continuous function defined on a Banach space $\mathbb{B}$, and $\mathcal{C}$ is a nonempty, connected and locally-connected compact subset of $\mathbb{B}$. Let $\zeta_{\mathcal{C}}(\cdot)$ be a continuous surjection of $\mathcal{J}$ onto $\mathcal{C}$.

The value

$$g_{\underline{\min}|e} \triangleq \min g(x) \quad \text{for } x \in \mathcal{C},$$

of the global minimum of the function $g(\cdot)$ on $\mathcal{C}$, and the value

$$g_{\overline{\max}|e} \triangleq \max g(x) \quad \text{for } x \in \mathcal{C},$$

of the global maximum of the function $g(\cdot)$ on $\mathcal{C}$, as well as the points in $\mathcal{C}$, where the function $g(\cdot)$ has the value of the global minimum or the value of the global maximum, respectively (the Weierstrass theorem [6]), are now available as a result of the analysis of the continuous function $g \circ \zeta_{\mathcal{C}}(\cdot)$ defined on the interval $\mathcal{J}$.

The following formulation is also considered.

Let $g(\cdot) : \mathbb{B} \to \mathbb{R}$ be a C^1-function defined on a Banach space $\mathbb{B}$, and let W be a nonempty and compact subset of $\mathbb{B}$, where it has been assumed for the set W that there exists a continuous trajectory $\tilde{x}(t), t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty)$, which is enclosed and dense in W,

cl Im $\tilde{x}(\cdot) = W$. Additionally, the trajectory assumed has to be continuously differentiable for all t in $\mathbb{R}^+$, except for a subset of isolated points [2]. The values $g_{\underline{\min}}|_W$ and $g_{\overline{\max}}|_W$, as well as the points in W where the function $g(\cdot)$ assumes the value of global minimum or the value of global maximum on W, are now to be found as a result of the analysis of the function $\tilde{g}(\cdot) \triangleq g \circ \tilde{x}(\cdot)$ defined on $\mathbb{R}^+$.

The integral equation formula has been proposed, such that the function obtained as the unique solution of the appropriate equation reaches the limit value equal to $g_{\underline{\min}}|_W$ (or the limit value equal to $g_{\overline{\max}}|_W$, respectively), as $t \to \infty$. The equivalent differential equation formula has been presented. The estimation theorem has been enclosed, where the trajectories defined for $t \in \mathcal{J}$ are taken into consideration.

The optimization problem formulation which is the subject of this paper is important for a number of fields of applications, e.g. in synthesis of systems of analog processors for optimal control [3, 15], numerical continuous-time processing systems [15, 16], mechanical automata [1], [10, 11], and stochastic modeling [9, 12]. Both deterministic [2, 5, 7, 8, 13, 14] and stochastic trajectories [4, 12, 16] have been taken into account.

2. A dynamical system for points of global minimum and global maximum of a function identification

Let $g(\cdot) : \mathbb{B} \to \mathbb{R}$ be a $\mathcal{C}^1$-function defined on a Banach space $\mathbb{B}$. The problem of estimating the global minimum value $g_{\underline{\min}}|_W$ and the global maximum value $g_{\overline{\max}}|_W$ of the function $g(\cdot)$ on a compact and nonempty subset W of the space $\mathbb{B}$ is considered. The problem of identifying the points from the following two sets is also considered:

$$\mathcal{A}_{\underline{\min} g}|_W = \{x \in W : g(x) = g_{\underline{\min}}|_W\},$$

and

$$\mathcal{A}_{\overline{\max} g}|_W = \{x \in W : g(x) = g_{\overline{\max}}|_W\}.$$

It is assumed for the subset W that there exists a continuous map $\tilde{x}(\cdot) : \mathbb{R}^+ \to \mathbb{B}$, $\mathbb{R}^+ = [0, \infty)$, which is continuously differentiable for all t in $\mathbb{R}^+$, except for a subset of isolated points, and which defines the trajectory $\tilde{x}(t)$, $t \in \mathbb{R}^+$, which is dense in W [2].

For the given $\mathcal{C}^1$-function $g(\cdot)$ and the trajectory $\tilde{x}(t), t \in \mathbb{R}^+$, $\dot{\tilde{g}}(t)$ denotes the right-hand derivative of the function $\tilde{g}(\cdot) = g \circ \tilde{x}(\cdot)$, at $t \in \mathbb{R}^+$.

Define the functions

$$\underline{\alpha}(\cdot) : \mathbb{R} \ni y \to \underline{\alpha}(y) = \begin{cases} 1 & \text{for } y \leq 0, \\ 0 & \text{for } y > 0, \end{cases}$$

and

$$\overline{\alpha}(\cdot) : \mathbb{R} \ni y \to \overline{\alpha}(y) = \begin{cases} 0 & \text{for } y < 0, \\ 1 & \text{for } y \geq 0. \end{cases}$$

The following theorem is valid.

THEOREM 1. *Let $g(\cdot) : \mathbb{B} \to \mathbb{R}$ be a $\mathcal{C}^1$-function defined on a Banach space $\mathbb{B}$, and let W be a nonempty and compact subset of $\mathbb{B}$, where it has been assumed for W that there exists a continuous trajectory $\tilde{x}(t)$, $t \in \mathbb{R}^+$, which is enclosed and dense in W and which is continuously differentiable for all t in $\mathbb{R}^+$ [2], except for a subset of isolated points.*

The following is satisfied, where $\tilde{g}(\cdot) = g \circ \tilde{x}(\cdot)$:

1. The function $\phi(\cdot)$ defined as the unique solution of the integral equation

$$\phi(t) = \int_0^t \underline{\alpha}(\tilde{g}(\tau) - \phi(\tau)) \cdot \underline{\alpha}(\dot{\tilde{g}}(\tau)) \cdot \dot{\tilde{g}}(\tau) d\tau + \tilde{g}(0) \tag{2.1}$$

reaches the limit value equal to $g_{\underline{\min}|W}$, as $t \to \infty$.

2. The function $\psi(\cdot)$ defined as the unique solution of the integral equation

$$\psi(t) = \int_0^t \overline{\alpha}(\tilde{g}(\tau) - \psi(\tau)) \cdot \overline{\alpha}(\dot{\tilde{g}}(\tau)) \cdot \dot{\tilde{g}}(\tau) d\tau + \tilde{g}(0) \tag{2.2}$$

reaches the limit value equal to $g_{\overline{\max}|W}$, as $t \to \infty$. (The derivative $\dot{\tilde{g}}(t)$ is defined as the right-hand derivative of the function $\tilde{g}(\cdot)$, at $t \in \mathbb{R}^+$).□

The proof of the theorem has been omitted.

It is to be noted that the integral equation (2.1) is equivalent to the following nonautonomous and discontinuous differential equation [5]:

$$\frac{d\phi}{dt} = \underline{\alpha}(\tilde{g}(t) - \phi) \cdot \underline{\alpha}(\dot{\tilde{g}}(t)) \cdot \dot{\tilde{g}}(t), \qquad t \in \mathbb{R}^+, \tag{2.3}$$

with the initial condition given by

$$\phi(0) = \tilde{g}(0).$$

The integral equation (2.2) is equivalent to the following nonautonomous and discontinuous differential equation [5]:

$$\frac{d\psi}{dt} = \overline{\alpha}(\tilde{g}(t) - \psi) \cdot \overline{\alpha}(\dot{\tilde{g}}(t)) \cdot \dot{\tilde{g}}(t), \qquad t \in \mathbb{R}^+, \tag{2.4}$$

with the initial condition given by

$$\psi(0) = \tilde{g}(0).$$

Assume now that the function $g(\cdot)$, a compact set W and the trajectory $\tilde{x}(t), t \in \mathbb{R}^+$ (such as it has been assumed in the Theorem 1) are given. The function $\phi(\cdot)$ is defined by the integral expression (2.1) ($\phi(\cdot)$ is the unique solution of the integral equation (2.1)).

Let $\underline{\alpha}_0(\cdot)$ be the function

$$\underline{\alpha}_0(\cdot) : \mathbb{R} \ni y \to \underline{\alpha}_0(y) = \begin{cases} 1 & \text{for } y < 0, \\ 0 & \text{for } y \geq 0. \end{cases}$$

Define the function

$$\tilde{\underline{y}}(\cdot) : \mathbb{R}^+ \to \{0, 1\} : \underline{\tilde{y}}(t) = \underline{\alpha}(\tilde{g}(t) - \phi(t)) \cdot \underline{\alpha}_0(\dot{\tilde{g}}(t)),$$

and let $\underline{\mathcal{T}}_\Delta$ be the subset of $\mathbb{R}^+$ given by

$$\underline{\mathcal{T}}_\Delta = \{t \in \mathbb{R}^+ : \underline{\tilde{y}}(t) = 1\}.$$

The following situations are possible:

i. $\underline{\mathcal{T}}_\Delta$ is an empty set, which takes place iff (if and only if) $\tilde{x}(0) \in \mathcal{A}_{\underline{\min} g|W}$.

ii. $\underline{\mathcal{T}}_\Delta = \mathbb{R}^+$, which takes place iff $\dot{\tilde{g}}(t) < 0$ for each $t \in \mathbb{R}^+$. In that case, $\phi(t) = \tilde{g}(t)$ for all $t \in \mathbb{R}^+$. The limit set $\Omega(\tilde{x}(\cdot))$ of the map $\tilde{x}(t), t \in \mathbb{R}^+$, [7, 8], is a subset of $\mathcal{A}_{\underline{\min} g|W}$,

$$\Omega(\tilde{x}(\cdot)) \subseteq \mathcal{A}_{\underline{\min} g|W}.$$

iii. $\underline{\mathcal{T}}_{\Delta}$ is a nonempty and proper subset of $\mathbb{R}^{+}$. The set $\underline{\mathcal{T}}_{\Delta}$ is the set union of some number of maximal intervals in $\mathbb{R}^{+}$, where t'_{υ} and $t''_{\upsilon} \in [0, \infty]$, respectively, are the values of t which define, respectively, the begin and the end of the subsequent interval considered, υ being the ordinal number assigned to the subsequent (maximal) interval enclosed in $\underline{\mathcal{T}}_{\Delta}$ [6]. Each of the considered (maximal) intervals in $\underline{\mathcal{T}}_{\Delta}$ is open at the end, and it is open or closed at the begin.

Consider the case where the set of intervals becomes well-ordered for the well-order that satisfies $\upsilon' \leq \upsilon''$ iff $\underline{t}'_{\upsilon'} \leq \underline{t}'_{\upsilon''}$.

Let $\underline{\kappa}_{\Delta}$ be the ordinal number of the (well-ordered) set of maximal intervals enclosed in $\underline{\mathcal{T}}_{\Delta}$ [6].

If the ordinal number $\underline{\kappa}_{\Delta}$ of the set of maximal intervals in $\underline{\mathcal{T}}_{\Delta}$ is finite, $\underline{\kappa}_{\Delta} \geq 1$, and $\underline{t}''_{\underline{\kappa}_{\Delta}} < \infty$, then $\tilde{x}(\underline{t}''_{\underline{\kappa}_{\Delta}}) \in \mathcal{A}_{\underline{\min} g|W}$.

In an another case, if $\underline{\kappa}_{\Delta}$ is equal to the ordinal number ω of the set of natural numbers [6], and if there is exactly one point $\underline{x} \in W$ where the function $g(\cdot)$ attains the global minimum value in the compact set W, then $\lim_{k \to \infty} \tilde{x}(\underline{t}''_{k}) = \underline{x}$.

Let $\overline{\alpha}_{0}(\cdot)$ be the function

$$\overline{\alpha}_{0}(\cdot) : \mathbb{R} \ni y \to \overline{\alpha}_{0}(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ 1 & \text{for } y > 0. \end{cases}$$

Define the function

$$\overline{y}^{\sim}(\cdot) : \mathbb{R}^{+} \to \{0, 1\} : \overline{y}^{\sim}(t) = \overline{\alpha}(\tilde{g}(t) - \psi(t)) \cdot \overline{\alpha}_{0}(\dot{\tilde{g}}(t)),$$

and let $\overline{\mathcal{T}}_{\Delta}$ be the subset of $\mathbb{R}^{+}$ given by

$$\overline{\mathcal{T}}_{\Delta} = \{t \in \mathbb{R}^{+} : \overline{y}^{\sim}(t) = 1\}.$$

The following situations are possible:

j. $\overline{\mathcal{T}}_{\Delta}$ is an empty set, which takes place iff $\tilde{x}(0) \in \mathcal{A}_{\overline{\max} g|W}$.

jj. $\overline{\mathcal{T}}_{\Delta} = \mathbb{R}^{+}$, which takes place iff $\dot{\tilde{g}}(t) > 0$ for each $t \in \mathbb{R}^{+}$. In that case, $\psi(t) = \tilde{g}(t)$ for all $t \in \mathbb{R}^{+}$. The limit set $\Omega(\tilde{x}(\cdot))$ of the map $\tilde{x}(t), t \in \mathbb{R}^{+}$, [7, 8], is a subset of $\mathcal{A}_{\overline{\max} g|W}$

$$\Omega(\tilde{x}(\cdot)) \subseteq \mathcal{A}_{\overline{\max} g|W}.$$

jjj. $\overline{\mathcal{T}}_{\Delta}$ is a nonempty and proper subset of $\mathbb{R}^{+}$. The set $\overline{\mathcal{T}}_{\Delta}$ is the union of some number of maximal intervals in $\mathbb{R}^{+}$, where $\overline{t}'_{\upsilon}$ and $\overline{t}''_{\upsilon} \in [0, \infty]$, respectively, are the values of t which define, respectively, the begin and the end of subsequent interval considered, υ being the ordinal number assigned to the subsequent (maximal) interval enclosed in $\overline{\mathcal{T}}_{\Delta}$ [6]. Each of the considered (maximal) intervals in $\overline{\mathcal{T}}_{\Delta}$ is open at the end, and it is open or closed at the begin.

Consider the case where the set of intervals becomes well-ordered for the well-order that satisfies $\upsilon' \leq \upsilon''$ iff $\overline{t}'_{\upsilon'} \leq \overline{t}'_{\upsilon''}$.

Let $\overline{\kappa}_{\Delta}$ be the ordinal number of the (well-ordered) set of maximal intervals enclosed in $\overline{\mathcal{T}}_{\Delta}$ [6].

If the ordinal number $\overline{\kappa}_{\Delta}$ of the set of maximal intervals in $\mathcal{T}_{\Delta}$ is finite, $\overline{\kappa}_{\Delta} \geq 1$, and $\overline{t}''_{\overline{\kappa}_{\Delta}} < \infty$, then $\tilde{x}(\overline{t}''_{\overline{\kappa}_{\Delta}}) \in \mathcal{A}_{\overline{\max} g|W}$.

In an another case, if $\overline{\kappa}_{\Delta}$ is equal to the ordinal number ω of the set of natural numbers [6], and if there is exactly one point $\overline{x} \in W$ where the function $g(\cdot)$ attains the global maximum value in the compact set W, then $\lim_{k\to\infty} \tilde{x}(\overline{t}''_k) = \overline{x}$.

The following theorem is proposed, which concerns the approximate estimation (in finite time) of the values $g_{\underline{\min}|W}$ and $g_{\overline{\max}|W}$.

THEOREM 2. *Let $g(\cdot) : \mathbb{B} \to \mathbb{R}$ be a $\mathcal{C}^1$-function defined on a Banach space $\mathbb{B}$ and let W be a nonempty and compact subset of $\mathbb{B}$, where it has been assumed for W that for each number $\varrho > 0$ there exists a continuous trajectory $\tilde{\gamma}_{\varrho}(t), t \in \mathcal{J}$, which is enclosed in W and satisfies the condition*

$$\sup_{x'} \inf_{x''} \{||x' - x''|| : x' \in W, x'' \in \operatorname{Im} \tilde{\gamma}_{\varrho}(\cdot)\} < \varrho. \tag{2.5}$$

Additionally, the trajectory assumed has to be continuously differentiable for all t in $\mathcal{J}$, except for a subset of isolated points in $\mathcal{J}$.

The following theorem is valid.

1. The function $\phi_{\varrho}(\cdot)$ defined as the unique solution of the ϱ-parametrized integral equation

$$\phi_{\varrho}(t) = \int_0^t \underline{\alpha}(\tilde{g}_{\varrho}(\tau) - \phi_{\varrho}(\tau)) \cdot \underline{\alpha}(\dot{\tilde{g}}_{\varrho}(\tau)) \cdot \dot{\tilde{g}}_{\varrho}(\tau)\, d\tau + \tilde{g}_{\varrho}(0), \qquad t \in \mathcal{J}, \tag{2.6}$$

reaches the limit value equal to $g_{\underline{\min}|W}$, for $t = 1$ and $\varrho \to 0$.

2. The function $\psi_{\varrho}(\cdot)$ defined as the unique solution of the ϱ-parametrized integral equation

$$\psi_{\varrho}(t) = \int_0^t \overline{\alpha}(\tilde{g}_{\varrho}(\tau) - \psi_{\varrho}(\tau)) \cdot \overline{\alpha}(\dot{\tilde{g}}_{\varrho}(\tau)) \cdot \dot{\tilde{g}}_{\varrho}(\tau) d\tau + \tilde{g}_{\varrho}(0), \qquad t \in \mathcal{J}, \tag{2.7}$$

reaches the limit value equal to $g_{\overline{\max}|W}$, for $t = 1$ and $\varrho \to 0$.

Both in the expression (2.6) and in (2.7), $\tilde{\gamma}_{\varrho}(t)(t \in \mathcal{J})$ is an arbitrarily chosen trajectory, which is enclosed in W and satisfies (2.5). The trajectories are continuous on $\mathcal{J}$ and they are continuously differentiable for all $t \in \mathcal{J}$, except for a subset of isolated points in $\mathcal{J}$. $\tilde{g}_{\varrho}(\cdot)$ is defined as the composition $g \circ \tilde{\gamma}_{\varrho}(\cdot)$.

The value of the right-hand derivative of the function $\tilde{g}_{\varrho}(\cdot)$ is taken for $\dot{\tilde{g}}_{\varrho}(t)$ at $t \in [0, 1)$, and the value of the left-hand derivative of $\tilde{g}_{\varrho}(\cdot)$ is taken for $\dot{\tilde{g}}_{\varrho}(t)$ at $t = 1$. □

The proof of the theorem has been omitted.

It is to be noted that the integral equation (2.6) is equivalent to the following nonautonomous and discontinuous differential equation [5]

$$\frac{d\phi_{\varrho}}{dt} = \underline{\alpha}(\tilde{g}_{\varrho}(t) - \phi_{\varrho}) \cdot \underline{\alpha}(\dot{\tilde{g}}_{\varrho}(t)) \cdot \dot{\tilde{g}}_{\varrho}(t), \qquad t \in \mathcal{J}, \tag{2.8}$$

with the initial condition given by

$$\phi_{\varrho}(0) = \tilde{g}_{\varrho}(0).$$

The integral equation (2.7) is equivalent to the following nonautonomous and discontinuous differential equation [5]

$$\frac{d\psi_{\varrho}}{dt} = \overline{\alpha}(\tilde{g}_{\varrho}(t) - \psi_{\varrho}) \cdot \overline{\alpha}(\dot{\tilde{g}}_{\varrho}(t)) \cdot \dot{\tilde{g}}_{\varrho}(t), \qquad t \in \mathcal{J}, \tag{2.9}$$

with the initial condition given by

$$\psi_\varrho(0) = \tilde{g}_\varrho(0).$$

In the reference [15], the block diagram of an analog processor system for estimating the global minimum value and the global maximum value of a function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$ on a compact subset of $\mathbb{R}^n$ has been designed. The formulas presented in the Theorems 1 and 2 have been applied.

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Homogenization of heterogeneous magnetoelastic medium

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THE PROCEDURE of homogenization is used to obtain the homogenized system of field equations and macroscopic constitutive laws of magnetoelasticity. The effective material properties are derived as functions of microperiodic structure. The one-dimensional example of layered medium is calculated.

1. Introduction

DURING THE LAST two decades an increasing amount of research has been conducted to develop methods and procedures for improving the description of macroproperties for given micro-inhomogeneous structure of solid media. In the case of interaction between different physical fields in solids, the problem of macrodescription is especially interesting both from the theoretical and experimental point of view. The spectrum of physical phenomena in coupled fields is discussed e.g. in [6, 7, 8]. The method used in this paper called homogenization [1] consists on replacing the model of heterogeneous medium by an equivalent homogeneous model. Equivalence is understood in the sense that the solution of any initial boundary value problem for a heterogeneous body is "close" to the solution of a related initial boundary value problem for the homogeneous body, the effective coefficients of which do not depend of microstructure. Homogenization was applied previously by many authors to calculate macrobehaviour of thermoelastic field, piezoelectricity in solids [3, 4], perfectly conducting solid [2], and many others. The example of practical meaning of the method is shown in [5], where superconducting multifilamentary composites in presence of a weak electromagnetic field is studied. The macroscopic transverse conductivity describing the loss of energy dissipated in a matrix (fibres are superconductors) is in agreement with experimental data. The more practical interest of homogenization technique are eddy-current non-destructive tests for electromagnetoelastic materials. In [10] the theoretical modelling of composite structure is based on the model similar to the self-consistent model of matrix-inclusion composites, and even in such a case interesting results are obtained. The method of homogenization is more promising than the self-consistent scheme if the 2-dimensional examples are calculated.

In the present paper the heterogeneous, electrically conducting solid placed in an initial strong magnetic field is considered. The body has a periodic structure, what means that all the material coefficients: elastic constants, magnetic permeability, electric conductivity and dielectric coefficients are periodic functions, with the same cell of periodicity. Our goal is to obtain the macroscopic behaviour of such a microheterogeneous magnetoelastic solid. By applying the theory of homogenization, the method of two-scale asymptotic expansion is exploited. As a result, the homogenized system of field equations and constitutive laws are obtained. All formulae include the solutions of so-called "problems on the cell". The semigroup theory is used to derive the effective electric conductivity and effective dielectric constants. In the formulae for the coefficients mentioned above the integro-differential operator appears. Similar effect was obtained in viscoelasticity [9]. The formulae obtained

are formal and complicated. In order to illustrate the solutions, the two-layer structure is assumed and calculations are done explicitly for the arbitrary anisotropic layers.

2. Basic equations of magnetoelasticity

We consider the heterogeneous, elastic solid with finite electric conductivity. Within the limits of the phenomenological theory for "slowly" moving elastic bodies, i.e. with relativistic effects neglected, such a physical problem is governed by Maxwell equations, elasticity equations and constitutive relations with appropriate couplings between the fields.

Under the assumption of strong initial magnetic induction and small deformations, there will be only a slight change in the magnetic field vector with respect to the primary field. In this connection we may write:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{b},$$

where $\mathbf{h}$ and $\mathbf{b}$ are small magnitudes.

Using this relation and disregarding all combinations of small magnitudes higher than linear, we obtain the following set of equations [7]:

Maxwell equations

$$\begin{aligned} &\operatorname{rot}\mathbf{h} = \mathbf{j} + \dot{\mathbf{D}}, \\ &\operatorname{rot}\mathbf{E} = -\dot{\mathbf{b}}, \\ &\operatorname{div}\mathbf{b} = 0, \\ &\operatorname{div}\mathbf{D} = \varrho e, \end{aligned} \tag{2.1}$$

elasticity equations

$$\varrho\ddot{\mathbf{u}} = \operatorname{div}\mathbf{c}\operatorname{grad}\mathbf{u} + \mathbf{j} \times \mathbf{B}_0 + \mathbf{P}, \tag{2.2}$$

constitutive laws

$$\begin{aligned} &\mathbf{j} = \eta\mathbf{E} + \eta(\dot{\mathbf{u}} \times \mathbf{B}_0), \\ &\mathbf{b} = \mu\mathbf{h}, \\ &\mathbf{D} = \epsilon[\mathbf{E} + (\dot{\mathbf{u}} \times \mathbf{B}_0)] - \frac{1}{c^2}(\dot{\mathbf{u}} \times \mathbf{H}_0), \end{aligned} \tag{2.3}$$

where

$\mathbf{E}$ electric field,
$\mathbf{D}$ electric induction,
$\mathbf{h}$ magnetic field,
$\mathbf{b}$ magnetic induction,
$\mathbf{j}$ electric current,
$\mathbf{u}$ elastic displacement,
$\mathbf{c}$ elastic Hooke's tensor,
μ magnetic permeability,
η electric conductivity,
ϵ electric permeability,
$\mathbf{H}_0(x)$ initial magnetic field,
$\mathbf{B}_0$ initial magnetic induction,
ϱ mass density,
$\mathbf{P}$ body forces.

The above equations are valid in domain $\mathcal{B}$ of the space E^3. The fields $\mathbf{u}$, $\mathbf{h}$, $\mathbf{E}$ are assumed to be known on the boundary of the domain occupied by the medium under consideration. Besides, the homogeneous initial conditions for the time equal 0 are assumed. It is the case when the motion of the material is caused by forces $\mathbf{P}$ acting inside.

The heterogeneous structure of the medium is given by the Y-periodic tensorial functions of variable $y = x/\varepsilon$ satisfying the following conditions:

$$\begin{aligned} &\mu_{ij} = \mu_{ji}, \quad \mu_{ij}\xi_i\xi_j \geq \gamma_1|\xi|^2, \\ &\eta_{ij} = \eta_{ji}, \quad \eta_{ij}\xi_i\xi_j \geq \gamma_2|\xi|^2, \\ &\epsilon_{ij} = \epsilon_{ji}, \quad \epsilon_{ij}\xi_i\xi_j \geq \gamma_3|\xi|^2 \end{aligned} \tag{2.4}$$

for all $\xi \in \mathbb{R}^3$,

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad c_{ijkl}\zeta_{ij}\zeta_{kl} \geq \gamma_4|\zeta|^2$$

for all $\zeta \in \mathbb{R}^3 \times \mathbb{R}^3$.

where γ_1, γ_2, γ_3 and γ_4 are constans.

We look for the solutions of the form

$$\begin{aligned} \mathbf{h}^\varepsilon(\mathbf{x}, t) &= \mathbf{h}^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon\mathbf{h}^1(\mathbf{x}, \mathbf{y}, t) + \dots, \\ \mathbf{u}^\varepsilon(\mathbf{x}, t) &= \mathbf{u}^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon\mathbf{u}^1(\mathbf{x}, \mathbf{y}, \mathbf{t}) + \dots, \\ \mathbf{E}^\varepsilon(\mathbf{x}, t) &= \mathbf{E}^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon\mathbf{E}^1(\mathbf{x}, \mathbf{y}, t) + \dots, \end{aligned} \tag{2.5}$$

where $\mathbf{h}^i$, $\mathbf{u}^i$, $\mathbf{E}^i$ are $\mathbf{y}$-periodic functions.

Substituting Eqs. (2.5) to Eqs. (2.1), (2.2) and assuming

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i},$$

we obtain the hierarchy of equations:

at the order ε^{-2}

$$\operatorname{div}_y[\mathbf{c}(\mathbf{y})\operatorname{grad}_y \mathbf{u}^0(\mathbf{x}, \mathbf{y})] = 0. \tag{2.6}$$

It implies that $\mathbf{u}^0$ does not depend on y and we have $\mathbf{u}^0(\mathbf{x})$.

At the order ε^{-1} we have

$$\begin{aligned} \operatorname{rot}_y \mathbf{h}^0(\mathbf{x}, \mathbf{y}) &= 0, \\ \operatorname{rot}_y \mathbf{E}^0(\mathbf{x}, \mathbf{y}) &= 0, \\ \operatorname{div}_y \mathbf{c}(\mathbf{y})\operatorname{grad}_x \mathbf{u}^0(\mathbf{x}) + \operatorname{div}_y \mathbf{c}(y)\operatorname{grad}_y \mathbf{u}^1(\mathbf{x}, \mathbf{y}) &= 0. \end{aligned} \tag{2.7}$$

Equation $(2.7)_3$ implies that

$$\mathbf{u}^1(\mathbf{x}, \mathbf{y}) = -\chi(\mathbf{y})\operatorname{grad}_x \mathbf{u}^0(x),$$

where $\chi(\mathbf{y})$ is a solution of the problem on the cell

$$\operatorname{div}_y[\mathbf{c}(\mathbf{y})\operatorname{grad}_y \chi(\mathbf{y})] = -\operatorname{div}_y \mathbf{c}(\mathbf{y}). \tag{2.8}$$

At the order ε^0 we have

$$\begin{aligned} \operatorname{rot}_x \mathbf{h}^0(\mathbf{x}, \mathbf{y}) + \operatorname{rot}_y \mathbf{h}^1(\mathbf{x}, \mathbf{y}) &= \epsilon\dot{\mathbf{E}}^0(\mathbf{x}, \mathbf{y}) - \frac{1}{c^2}[\ddot{\mathbf{u}}^0(x) \times \mathbf{H}_0(\mathbf{y})] + \varepsilon(\ddot{\mathbf{u}}^0 \times \mathbf{B}_0) \\ &\quad + \eta[\mathbf{E}^0(\mathbf{x}, \mathbf{y}) + \dot{\mathbf{u}}^0(\mathbf{x}) \times \mathbf{B}_0], \\ \operatorname{rot}_x \mathbf{E}^0(\mathbf{x}, \mathbf{y}) + \operatorname{rot}_y \mathbf{E}^1(\mathbf{x}, \mathbf{y}) &= -\mu\dot{\mathbf{h}}^0(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{2.9}$$

$$\varrho\ddot{\mathbf{u}}^0 = \operatorname{div}_x(\mathbf{c}\operatorname{grad}_x \mathbf{u}^0) + \operatorname{div}_y(\mathbf{c}\operatorname{grad}_x \mathbf{u}^1) + \operatorname{div}_x(\mathbf{c}\operatorname{grad}_y \mathbf{u}^1) + \operatorname{div}_y(\mathbf{c}\operatorname{grad}_y \mathbf{u}^2) + \eta\mathbf{E}^0 \times \mathbf{B}_0 + [\eta(\dot{\mathbf{u}}^0 \times \mathbf{B}_0)] \times \mathbf{B}_0 + \mathbf{P}. \tag{2.9 [cont.]}$$

Taking the mean value $\langle\cdot\rangle$ on Y in Eqs. $(2.9)_{1,2}$ we obtain

$$\operatorname{rot}_x\langle\mathbf{h}^0\rangle = \langle\epsilon\dot{\mathbf{E}}^0\rangle + \langle\epsilon\rangle(\ddot{\mathbf{u}}^0 \times \mathbf{B}_0) - \frac{1}{c^2}\langle\ddot{\mathbf{u}}^0 \times \mathbf{H}_0\rangle + \langle\eta\mathbf{E}^0\rangle + \langle\eta\rangle(\dot{\mathbf{u}}^0 \times \mathbf{B}_0), \tag{2.10}$$

$$\operatorname{rot}_x\langle\mathbf{E}^0\rangle = -\langle\mu\dot{\mathbf{h}}^0\rangle,$$

where

$$\langle\cdot\rangle = \int_Y \cdot\, dy.$$

The terms $\operatorname{rot}_y \mathbf{h}^1$, $\operatorname{rot}_y \mathbf{E}^1$ of Eqs. (2.9) have a zero mean value what we can see by integrating by parts and taking into account the Y-periodicity. From the condition of uniqueness of $\mathbf{u}^2$ as a function of y we obtain the homogenized equation of elasticity from Eq. $(2.9)_3$

$$\langle\varrho\rangle\ddot{\mathbf{u}}^0 = \operatorname{div}_x \mathbf{c}^h \operatorname{grad}_x \mathbf{u}^0(\mathbf{x}) + \langle\mathbf{j}^0\rangle \times \mathbf{B}_0 + \mathbf{P}, \tag{2.11}$$

where

$$\mathbf{c}^h = \langle\mathbf{c} - \mathbf{c}\operatorname{grad}_y \chi\rangle. \tag{2.12}$$

Now we proceed to obtain the homogenized constitutive laws. Let us consider the divergence of Eqs. $(2.1)_{1,2}$

$$\operatorname{div}\dot{\mathbf{D}} = -\operatorname{div}\mathbf{j}, \qquad \operatorname{div}\dot{\mathbf{b}} = 0.$$

Using the asymptotic expansion (2.5) we obtain at the order ε^{-1}

$$\operatorname{div}_y\left\{\epsilon[\dot{\mathbf{E}}^0 + (\ddot{\mathbf{u}}^0 \times \mathbf{B}_0)] - \frac{1}{c^2}[\ddot{\mathbf{u}}^0 \times \mathbf{H}_0(y)]\right\} = -\operatorname{div}_y[\eta(\mathbf{E}^0 + \dot{\mathbf{u}}^0 \times \mathbf{B}_0)], \tag{2.13}$$

$$\operatorname{div}_y[\mu\dot{\mathbf{h}}^0] = 0.$$

From Eq. $(2.13)_2$ we have

$$\operatorname{div}_y \mathbf{b}^0 = 0.$$

Equations $(2.7)_{1,2}$ imply that $\mathbf{E}^0$ and $\mathbf{h}^0$ have the form

$$\begin{aligned} \mathbf{E}^0 - \langle\mathbf{E}^0\rangle &= \operatorname{grad}_y \Phi, \\ \mathbf{h}^0 - \langle\mathbf{h}^0\rangle &= \operatorname{grad}_y \Psi. \end{aligned} \tag{2.14}$$

Functions Φ and Ψ are Y-periodic. Substituting Eqs. (2.14) into Eq. (2.13) we have

$$\operatorname{div}_y\left[\left(\epsilon\frac{\partial}{\partial t} + \eta\right)(\langle\mathbf{E}^0\rangle + \operatorname{grad}_y \Phi)\right] + \operatorname{div}_y[\epsilon(\ddot{\mathbf{u}}^0 \times \mathbf{B}_0)] - \frac{1}{c^2}\operatorname{div}_y[\ddot{\mathbf{u}}^0 \times \mathbf{H}_0] + \operatorname{div}_y[\eta(\dot{\mathbf{u}}^0 \times \mathbf{B}_0)] = 0, \tag{2.15}$$

$$\operatorname{div}_y[\mu(\operatorname{grad}_y \Psi + \langle\mathbf{h}^0\rangle)] = 0.$$

Equations $(2.15)_2$ contain x and t as parameters. We have

$$\operatorname{div}_y(\mu \operatorname{grad}_y \Psi) = -\operatorname{div}_y \mu\langle\mathbf{h}^0\rangle$$

which implies

$$\Psi = \chi^1(\mathbf{y})[\langle \mathbf{h}^0 \rangle], \tag{2.16}$$

where $\chi^1(\mathbf{y})$ is a solution of the equation on the cell

$$\operatorname{div}_y[\mu(\mathbf{y}) \operatorname{grad}_y \chi^1(\mathbf{y})] = -\operatorname{div}_y \mu(y). \tag{2.17}$$

Then the mean values of $\mathbf{h}^0$ and $\mathbf{b}^0$ are related by

$$\langle \mathbf{b}^0 \rangle = \mu^h \langle \mathbf{h}^0 \rangle, \tag{2.18}$$

where

$$\mu^h = \langle \mu + \mu \operatorname{grad}_y \chi_1(\mathbf{y}) \rangle. \tag{2.19}$$

Equation $(2.15)_1$ contains time derivative of $\mathbf{E}^0$ and it will be solved by a different method.

We introduce the space

$$\tilde{V}_Y = \{\theta; \theta \in H^1_{\text{loc}}(R^3), \quad Y - \text{per}, \ \langle \theta \rangle = 0\}, \tag{2.20}$$

equipped with the scalar product

$$(\Phi, \theta)_{\tilde{V}_Y} = \int_Y \epsilon_{ij}(\mathbf{y}) \frac{\partial \Phi}{\partial y_i} \frac{\partial \theta}{\partial y_j} d\mathbf{y}; \tag{2.21}$$

now Eq. $(2.15)_1$ becomes

$$\begin{aligned} \frac{\partial}{\partial t}\Big\{ \int_Y \epsilon(\langle \mathbf{E}^0 \rangle + \operatorname{grad}_y \Phi) \operatorname{grad}_y \theta d\mathbf{y} \\ + \int_Y \epsilon(\mathbf{v}^0 \times \mathbf{B}_0) \operatorname{grad}_y \theta dy - \frac{1}{c^2} \int_Y (\mathbf{v}^0 \times \mathbf{H}_0) \operatorname{grad}_y \theta dy \Big\} \\ + \int_Y \eta(\mathbf{v}^0 \times \mathbf{B}_0) \operatorname{grad}_y \theta dy + \int_Y \eta[\langle \mathbf{E}^0 \rangle + \operatorname{grad}_y \Phi] \operatorname{grad}_y \theta dy = 0, \end{aligned} \tag{2.22}$$

where $\mathbf{v}^0 = \dot{\mathbf{u}}^0$.

We then define the operator A (which is bounded and symmetric from $\tilde{V}_Y$ into itself) and the elements f^1_j, f^2_j, f^4_j, $j = 1, 2, 3$

$$\begin{aligned} (A\Phi, \theta)_{\tilde{V}_Y} &= \int_Y \eta_{ij} \frac{\partial \Phi}{\partial y_i} \frac{\partial \theta}{\partial y_j} dy, \\ (f^1_j, \theta)_{\tilde{V}_Y} &= \int_Y \epsilon_{ij} \frac{\partial \theta}{\partial y_i} dy, \\ (f^2_j, \theta)_{\tilde{V}_Y} &= \int_Y \eta_{ij} \frac{\partial \theta}{\partial y_i} dy, \\ (f^4_j, \theta)_{\tilde{V}_Y} &= \int_Y \mu^{-1}_{ij} \frac{\partial \theta}{\partial y_i} dy. \end{aligned} \tag{2.23}$$

Using the above definitions, Eq. (2.22) is equivalent to

$$\begin{aligned} \Big(\frac{\partial}{\partial t}[f^1_j \langle E^0_j \rangle + \Phi + f^1_j (\mathbf{v}^0 \times \mathbf{B}_0)_j - \frac{1}{c^2} \varepsilon_{jkl} f^4_j v^0_k B_{0l}] \\ + f^2_j (\mathbf{v}^0 \times \mathbf{B}_0)_j + f^2_j \langle E^0_j \rangle + A\Phi, \theta \Big)_{\tilde{V}_Y} = 0, \qquad \forall \theta \in \tilde{V}_Y. \end{aligned} \tag{2.24}$$

The unknown ϕ must of course satisfy the initial condition $\phi(0) = 0$. According to the semigroup theory, the solution is:

$$\phi = -f_j^1\langle E_j^0\rangle - f_j^1(\mathbf{v}^0 \times \mathbf{B}_0)_j + \frac{1}{c^2}\varepsilon_{jkl}f_j^4 v_k^0 B_{ol} + \int_0^t e^{-A(t-s)}f_j^3[\langle E_j^0\rangle + (\mathbf{v}^0 \times \mathbf{B}_0)_j]ds - \frac{1}{c^2}\int_0^t e^{-A(t-s)}\varepsilon_{jkl}Af_j^4 v_k^0 B_{ol}ds, \tag{2.25}$$

where

$$f_j^3 = Af_j^1 - f_j^2.$$

If $\phi(t)$ is known, $\mathbf{E}^0$ is calculated from Eq. $(2.14)_1$ and we have

$$\begin{aligned} D_i^0 &= \epsilon_{ij}E_j^0 + \epsilon_{ij}(\mathbf{v}^0 \times \mathbf{B}_0)_j - \frac{1}{c^2}\varepsilon_{ikl}\mu_{lm}^{-1}v_k^0 B_{om}, \\ j_i^0 &= \eta_{ij}E_j^0 + \eta_{ij}(\mathbf{v}^0 \times \mathbf{B}_0)_j. \end{aligned} \tag{2.26}$$

Now we calculate the mean values of D_i^0 and j_i^0.

Finally we obtain the macroscopic constitutive laws in the form

$$\begin{aligned} \langle D_i^0\rangle &= b_{ij}^\epsilon\langle E_j^0\rangle + \int_0^t d_{ij}^\epsilon\langle E_j^0\rangle ds + b_{ij}^\epsilon(\mathbf{v}^0 \times \mathbf{B}_0)_j + \int_0^t d_{ij}^\epsilon(\mathbf{v}^0 \times \mathbf{B}_0)_j ds \\ &\quad - \frac{1}{c^2}\langle\mu_{lm}^{-1}\rangle\varepsilon_{ikl}v_k^0 B_{om} + p_{ij}^\epsilon\varepsilon_{jnl}v_n^0 B_{ol} - \frac{1}{c^2}\int_0^t q_{ij}^\epsilon\varepsilon_{jnl}v_n^0 B_{ol}ds, \\ \langle j_i^0\rangle &= b_{ij}^\eta\langle E_j^0\rangle + \int_0^t d_{ij}^\eta\langle E_j^0\rangle ds + b_{ij}^\eta(\mathbf{v}^0 \times \mathbf{B}_0)_j + \int_0^t d_{ij}^\eta(\mathbf{v}^0 \times \mathbf{B}_0)_j ds \\ &\quad - \frac{1}{c^2}p_{ij}^\eta\varepsilon_{jnl}v_n^0 B_{ol} - \frac{1}{c^2}\int_0^t q_{ij}^\eta\varepsilon_{jnl}v_n^0 B_{ol}ds, \end{aligned} \tag{2.27}$$

where the homogenized coefficients have the following representation:

$$\begin{aligned} b_{ij}^\epsilon &= \langle\epsilon_{ij} - \epsilon_{ik}\partial_k f_j^1\rangle, & b_{ij}^\eta &= \langle\eta_{ij} - \eta_{ik}\partial_k f_j^1\rangle, \\ d_{ij}^\epsilon &= \langle\epsilon_{ik}\partial_k f_j^3 e^{-A\xi}\rangle, & d_{ij} &= \langle\eta_{ik}\partial_k f_j^3 e^{-A\xi}\rangle, \\ p_{ij}^\epsilon &= \langle\epsilon_{ik}\partial_k f_j^4\rangle, & p_{ij}^\eta &= \langle\eta_{ik}\partial_k f_j^4\rangle, \\ q_{ij}^\epsilon &= \langle\epsilon_{ik}\partial_k Af_j^4 e^{-A\xi}\rangle, & q_{ij}^\eta &= \langle\eta_{ik}\partial_k Af_j^4 e^{-A\xi}\rangle, \qquad \xi = t - s. \end{aligned} \tag{2.28}$$

3. One-dimensional example of a layered medium

We assume that the medium possesses a layered structure. The periodic "cell" consists of two different homogeneous but arbitrary anisotropic layers. It means that all material coefficients have the form

$$\mathbf{c}(\mathbf{y}) = \kappa(y)[\![\mathbf{c}]\!] + \mathbf{c}^{(2)},$$

$$\eta(\mathbf{y}) = \kappa(y)[\![\eta]\!] + \eta^{(2)},$$

$$\epsilon(\mathbf{y}) = \kappa(y)[\![\epsilon]\!] + \epsilon^{(2)},$$

$$\mu(\mathbf{y}) = \kappa(y)[\![\mu]\!] + \mu^{(2)},$$

$$\kappa(y) = \begin{cases} 1 & y \in \text{material with properties (1)}, \\ 0 & y \in \text{material with properties (2)}, \end{cases}$$

$$[\![\,\cdot\,]\!] = (\cdot)^{(1)} - (\cdot)^{(2)},$$

$$\langle \kappa(y) \rangle = \xi \quad \text{volume fraction of material (1)},$$

$$(1 - \xi) \quad \text{volume fraction of material (2)},$$

$$\mathbf{y} = (y_1, y_2, y_3), \qquad y_2 \equiv y.$$

In order to obtain the $\mathbf{c}^h$ we use Eq. (2.12) and under the assumptions made above we get

$$c^h_{ijkl} = \langle c_{ijkl} - c_{ij2n} \frac{\partial}{\partial y} \chi^{kl}_n \rangle, \tag{3.1}$$

where χ^{kl}_n fulfil the equations

$$\frac{\partial}{\partial y} c_{i22n} \frac{\partial}{\partial y} \chi^{kl}_n = \frac{\partial}{\partial y} c_{i2kl}.$$

Integrating both sides of the above equations we have

$$c_{i22n} \frac{\partial}{\partial y} \chi^{kl}_n = c_{i2kl} + s_{ikl},$$

$s_{ikl} \equiv$ const and it is determined from the condition of uniqueness of solution χ^{kl}_n in the class of periodic functions:

$$s_{ikl} = -\langle c^{-1}_{i22n} \rangle^{-1} \langle c^{-1}_{j22n} c_{j2kl} \rangle.$$

Then

$$\frac{\partial}{\partial y} \chi^{kl}_n [c_{p22n}]^{-1} [c_{p2kl}] - [c_{p22n}]^{-1} \langle c^{-1}_{p22s} \rangle^{-1} \langle c^{-1}_{r22s} c_{rskl} \rangle.$$

Thus Eq. (3.1) reads as

$$c^h_{ijkl} = \langle c_{ijkl} \rangle - \langle c_{ij2n} (c_{p22n})^{-1} c_{p2kl} \rangle + \langle c_{ij2n} (c_{p22n})^{-1} \rangle \langle c^{-1}_{p22s} \rangle^{-1} \langle c^{-1}_{r22s} c_{rskl} \rangle.$$

Finally, after calculations we get

$$c^h_{ijkl} = \langle c_{ijkl} \rangle - \xi^2 (1 - \xi) [\![c_{i2js}]\!] c^{(2)}_{s22p} [\![c_{p2kl}]\!] + \xi (1 - \xi)^2 [\![c_{i2js}]\!] c^{(1)}_{s22p} [\![c_{p2kl}]\!]. \tag{3.2}$$

Analogically, we calculate

$$\mu^h_{ij} = \langle \mu_{ij} \rangle - \langle \mu_{i2} (\mu_{22})^{-1} \mu_{2j} \rangle + \langle \mu_{22})^{-1} \mu_{2j} \rangle \langle \mu^{-1}_{22} \rangle^{-1} \langle \mu_{i2} (\mu_{22})^{-1} \rangle \tag{3.3}$$

and the final form is

$$\mu^h_{ij} = \langle \mu_{ij} \rangle - \xi^2 (1 - \xi) [\![\mu_{i2}]\!] \mu^{(2)}_{22} [\![\mu_{2j}]\!] + \xi (1 - \xi)^2 [\![\mu_{i2}]\!] \mu^{(1)}_{22} [\![\mu_{2j}]\!]. \tag{3.4}$$

In order to obtain the effective coefficients in homogenized constitutive laws (2.27) we solve $(2.15)_1$, which in the one-dimensional case has the form

$$\frac{\partial}{\partial y}\left(\epsilon_{2i}\frac{\partial}{\partial t}+\eta_{2i}\right)\left[\langle E_i^0\rangle+\delta_{i2}\frac{\partial \Phi}{\partial y}+(\mathbf{v}^0\times\mathbf{B}_0)_i\right]-\frac{1}{c^2}\frac{\partial}{\partial y}\frac{\partial}{\partial t}(\mathbf{v}^0\times\mathbf{H}_0)_2=0.$$

Denoting

$$\Omega_i=\langle E_i^0\rangle+\delta_{i2}\frac{\partial \Phi}{\partial y}+(\mathbf{v}^0\times\mathbf{B}_0)_i$$

we have

$$\frac{\partial}{\partial y}\left(\epsilon_{2i}\frac{\partial}{\partial t}+\eta_{2i}\right)\Omega_i-\frac{1}{c^2}\frac{\partial}{\partial y}\frac{\partial}{\partial t}(v^0\times H_0)_2=0.$$

After integration with respect to y we get

$$\left(\frac{\epsilon_{21}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{21}}{\epsilon_{22}}\right)\Omega_1+\left(\frac{\partial}{\partial t}+\frac{\eta_{22}}{\epsilon_{22}}\right)\Omega_2+\left(\frac{\epsilon_{23}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{23}}{\epsilon_{22}}\right)\Omega_3$$
$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{1}{\epsilon_{22}}(\mathbf{v}^0\times\mathbf{H}_0)_2\right]=\frac{a(t)}{\epsilon_{22}}.$$

We use the following form of the above equation

$$\left(\frac{\partial}{\partial t}+\frac{\eta_{22}}{\epsilon_{22}}\right)\Omega_2=B(t),$$

where

$$B(t)=-\left(\frac{\epsilon_{21}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{21}}{\epsilon_{22}}\right)\Omega_1-\left(\frac{\epsilon_{23}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{23}}{\epsilon_{22}}\right)\Omega_3+\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{1}{\epsilon_{22}}(\mathbf{v}^0\times\mathbf{H}_0)_2\right]+\frac{a(t)}{\epsilon_{22}}$$

to get the solution

$$\Omega_2=\langle E_2^0\rangle+\frac{\partial \Phi}{\partial y}+(\mathbf{v}^0\times\mathbf{B}_0)_2=\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}B(s)ds.$$

Taking the mean value over the cell

$$\langle E_2^0\rangle+\langle\mathbf{v}^0\times\mathbf{B}_0\rangle_2=\int_0^t\langle e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}B(s)ds$$

and differentiating with respect to the time we have

$$\frac{\partial}{\partial t}\langle E_2^0\rangle+\frac{\partial}{\partial t}(\mathbf{v}^0\times\mathbf{B}_0)_2=\langle B(t)\rangle$$

and

$$a(t)=\left[\left\langle\frac{1}{\epsilon_{22}}\right\rangle\right]^{-1}\left\{\frac{\partial}{\partial t}\langle E_2^0\rangle+\frac{\partial}{\partial t}(\mathbf{v}^0\times\mathbf{B}_0)_2\right.$$
$$\left.+\left\langle\left(\frac{\epsilon_{21}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{21}}{\epsilon_{22}}\right)\Omega_1\right\rangle+\left\langle\left(\frac{\epsilon_{23}}{\epsilon_{22}}\frac{\partial}{\partial t}+\frac{\eta_{23}}{\epsilon_{22}}\right)\Omega_3\right\rangle-\frac{1}{c^2}\frac{\partial}{\partial t}\left\langle\frac{1}{\epsilon_{22}}(\mathbf{v}^0\times\mathbf{H}_0)_2\right\rangle\right\}.$$

Thus

$$\frac{\partial \Phi}{\partial y}=-\langle E_2^0\rangle-(\mathbf{v}^0\times\mathbf{B}_0)_2-\frac{\epsilon_{21}}{\epsilon_{22}}[\langle E_1^0\rangle+(\mathbf{v}^0\times B_0)_1$$
$$+\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_1^0\rangle+(\mathbf{v}^0\times\mathbf{B}_0)_1)ds]-\frac{\eta_{21}}{\epsilon_{22}}\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}[\langle E_1^0\rangle+(\mathbf{v}^0\times\mathbf{B}_0)_1]ds \tag{3.5}$$

(3.5) [cont.]

$$-\frac{\epsilon_{23}}{\epsilon_{22}}\Big[\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3 + \int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3)ds\Big]$$
$$-\frac{\eta_{23}}{\epsilon_{22}}\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3)ds + \frac{1}{c^2}\frac{1}{\epsilon_{22}}\Big[(\mathbf{v}^0\times\mathbf{H}_0)_2$$
$$+\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\mathbf{v}^0\times\mathbf{H}_0)_2\Big] + \left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\frac{1}{\epsilon_{22}}\Big\{\langle E_2^0\rangle$$
$$+\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\langle E_2^0\rangle ds\Big\} + (\mathbf{v}^0\times B_0)_2 + \int_0^t e^{\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\mathbf{v}^0\times\mathbf{B}_0)_2 ds$$
$$+\left\langle\frac{\epsilon_{21}}{\epsilon_{22}}\right\rangle\Big[\langle E_1^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_1 + \int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_1^0\rangle + \mathbf{v}^0\times\mathbf{B}_0)_1 ds\Big]$$
$$+\left\langle\frac{\eta_{21}}{\epsilon_{22}}\right\rangle\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_1^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_1)ds + \left\langle\frac{\epsilon_{23}}{\epsilon_{22}}\right\rangle\Big[\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3$$
$$+\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}(\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3)ds\Big] + \left\langle\frac{\eta_{23}}{\epsilon_{22}}\right\rangle\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}[\langle E_3^0\rangle + (\mathbf{v}^0\times\mathbf{B}_0)_3]ds$$
$$-\frac{1}{c^2}\left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\frac{1}{\epsilon_{22}}\left[\left\langle\frac{1}{\epsilon_{22}}(\mathbf{v}^0\times\mathbf{H}_0)_2\right\rangle\right] + \frac{1}{c^2}\int_0^t e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\left\langle\frac{1}{\epsilon_{22}}(\mathbf{v}^0\times\mathbf{H}_0)_2\right\rangle ds.$$

The constitutive laws have the form

$$\langle D_i^0\rangle = \langle\epsilon_{ij}\rangle\langle E_j^0\rangle + \left\langle\epsilon_{i2}\frac{\partial\Phi}{\partial y}\right\rangle + \langle\epsilon_{ij}\rangle(\mathbf{v}^0\times\mathbf{B}_0)_j - \frac{1}{c^2}\langle(\mathbf{v}^0\times\mathbf{H}_0)_i\rangle,$$

$$\langle j_i^0\rangle = \langle\eta_{ij}\rangle\langle E_j^0\rangle + \left\langle\eta_{i2}\frac{\partial\Phi}{\partial y}\right\rangle + \langle\eta_{ij}\rangle(\mathbf{v}^0\times\mathbf{B}_0)_j.$$

Inserting Eq. (3.5) to the above equations and collecting the corresponding terms we get the homogenized coefficients

$$b_{ij}^\epsilon = \langle\epsilon_{ij}\rangle - \left\langle\frac{\epsilon_{i2}\epsilon_{2j}}{\epsilon_{22}}\right\rangle + \left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\left\langle\frac{\epsilon_{2i}}{\epsilon_{22}}\right\rangle\left\langle\frac{\epsilon_{2j}}{\epsilon_{22}}\right\rangle,$$
$$d_{ij}^\epsilon = [d_{i1}^\epsilon, d_{i2}^\epsilon, d_{i3}^\epsilon],$$
$$d_{i1}^\epsilon(t-s) = -\left\langle\frac{\epsilon_{i2}\epsilon_{21}}{\epsilon_{22}}e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\right\rangle - \left\langle\frac{\epsilon_{21}\eta_{21}}{\epsilon_{22}}e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\right\rangle$$
$$+\left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\left\langle\frac{\epsilon_{21}}{\epsilon_{22}}\right\rangle\left\langle\frac{\epsilon_{2i}}{\epsilon_{22}}e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\right\rangle + \left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\left\langle\frac{\eta_{21}}{\epsilon_{22}}\right\rangle\left\langle\frac{\epsilon_{2i}}{\epsilon_{22}}e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\right\rangle,$$
$$d_{i2}^\epsilon(t-s) = \left\langle\frac{1}{\epsilon_{22}}\right\rangle^{-1}\left\langle\frac{\epsilon_{2i}}{\epsilon_{22}}e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)}\right\rangle,$$

$$d^{\varepsilon}_{i3}(t-s) = +\left\langle \frac{\epsilon_{i2}\epsilon_{23}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle - \left\langle \frac{\epsilon_{21}\eta_{23}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle$$
$$+\left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\epsilon_{23}}{\epsilon_{22}} \right\rangle \left\langle \frac{\epsilon_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle + \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{23}}{\epsilon_{22}} \right\rangle \left\langle \frac{\epsilon_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle.$$

$$p^{\varepsilon}_{ij} = \delta_{i2}\left[\left\langle \frac{\epsilon_{2k}}{\epsilon_{22}} \mu^{-1}_{kj} \right\rangle - \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\epsilon_{2k}}{\epsilon_{22}} \right\rangle \left\langle \frac{1}{\epsilon_{22}} \mu^{-1}_{kj} \right\rangle \right],$$

$$q^{\varepsilon}_{ij}(t-s) = \delta_{i2}\left[\left\langle \frac{\epsilon_{2k}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \mu^{-1}_{kj} \right\rangle - \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\epsilon_{2k}}{\epsilon_{22}} e^{-\frac{\eta_{2k}}{\epsilon_{22}}(t-s)} \right\rangle \left\langle \frac{\mu^{-1}_{kj}}{\epsilon_{22}} \right\rangle \right],$$

$$b^{\eta}_{ij} = \langle \eta_{ij} \rangle - \left\langle \frac{\eta_{i2}\epsilon_{2j}}{\epsilon_{22}} \right\rangle + \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{2i}}{\epsilon_{22}} \right\rangle \left\langle \frac{\epsilon_{2j}}{\epsilon_{22}} \right\rangle,$$

$$d^{\eta}_{ij} = [d^{\eta}_{i1}, d^{\eta}_{i2}, d^{\eta}_{i3}],$$

$$d^{\eta}_{i1}(t-s) = -\left\langle \frac{\eta_{i2}\epsilon_{21}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle - \left\langle \frac{\eta_{2i}\eta_{21}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle$$
$$+\left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\epsilon_{21}}{\epsilon_{22}} \right\rangle \left\langle \frac{\eta_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle + \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{21}}{\epsilon_{22}} \right\rangle \left\langle \frac{\eta_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle,$$

$$d^{\eta}_{i2}(t-s) = \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle,$$

$$d^{\eta}_{i3}(t-s) = -\left\langle \frac{\eta_{i2}\epsilon_{23}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle - \left\langle \frac{\eta_{2i}\eta_{23}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle$$
$$+\left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\epsilon_{23}}{\epsilon_{22}} \right\rangle \left\langle \frac{\eta_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle + \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{23}}{\epsilon_{22}} \right\rangle \left\langle \frac{\eta_{2i}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle,$$

$$p^{\eta}_{ij} = \delta_{i2}\left[\left\langle \frac{\eta_{2k}\mu^{-1}_{kj}}{\epsilon_{22}} \right\rangle - \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{2k}}{\epsilon_{22}} \right\rangle \left\langle \frac{\mu^{-1}_{kj}}{\epsilon_{22}} \right\rangle \right],$$

$$q^{\eta}_{ij}(t-s) = \delta_{i2}\left[\left\langle \frac{\eta_{2k}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \mu^{-1}_{kj} \right\rangle - \left\langle \frac{1}{\epsilon_{22}} \right\rangle^{-1} \left\langle \frac{\eta_{2k}}{\epsilon_{22}} e^{-\frac{\eta_{22}}{\epsilon_{22}}(t-s)} \right\rangle \left\langle \frac{1}{\epsilon_{22}} \mu^{-1}_{kj} \right\rangle \right].$$

4. Conclusions

The homogenized set of equations combines the macrofields $\mathbf{u}^0$, $\langle \mathbf{E}^0 \rangle$ and $\langle \mathbf{h}^0 \rangle$. Let us stress that the zero-order terms in ε-expansion series of the electric and magnetic fields $\mathbf{E}^0$ and $\mathbf{h}^0$ are rapidly fluctuating fields, i.e. they depend on microvariable $\mathbf{y}$, contrary to the pure dependence of displacement $\mathbf{u}$ on macrovariable $\mathbf{x}$. In the effective relations for mean values of the current $\langle \mathbf{j}^0 \rangle$ and electric induction $\langle \mathbf{D}^0 \rangle$ we obtain the integro-differential terms (corresponding to the memory effects) which describe nonlocality in time of the effective laws. This result is due to the dynamical character of couplings in Eqs. (2.1)–(2.3). The operator A in the Eqs. (2.24) is positive defined; then the kernels in integral terms decay exponentially as t tends to infinity and, consequently, the "memory" vanishes exponentially.

The initial static field $\mathbf{H}_0$ generated in such a way that the constant static magnetic induction $\mathbf{B}_0$ inside the body is produced, is responsible for the restrictions imposed on the type of heterogeneities which can be treated by the applied method. Namely, taking into account that rot $\mathbf{H}_0 = 0$ and $\mathbf{H}_0 = \mu^{-1}(x)\mathbf{B}_0$, the following conditions on the tensorial field $\mu^{-1}(x)$ must be fulfiled:

$$\varepsilon_{ijk}\partial_j \mu^{-1}_{kl}(x) B_{0l} = 0.$$

One should take into account the above conditions when performing the numerical calculations.

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Calculation of coefficients of a power series approximation of a center manifold for nonlinear integro-differential equations

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A NUMERICAL ALGORITHM for center manifold reduction of integro-differential equations with kernel of convolution type is presented. Approximation up to the desired order can be obtained provided that derivatives of the transfer functions of the system of equations are known.

1. Introduction

THE PROBLEM dealt with in this paper is an integro-differential equation with the convolution-type kernel

$$\left\{\frac{dx}{dt}\right\} = [D]\{x(t)\} + \int_{-\infty}^{0} [G(-\Theta;U)]\{x(t+\Theta)\}d\Theta + \{f(x)\}, \tag{1.1}$$

where $\{x(t)\}$ is an n-dimensional vector function of time, $[D]$ is a $n \times n$ matrix of real numbers, $[G(-\Theta;U)]$ is a square matrix of order n composed of given functions of time which also depend on the parameter U. The only nonlinear term is a vector function $\{f(x)\}$.

Such an equation occurs frequently in many different fields of applied sciences, for instance in the flutter analysis of flexible aircraft. In this case $\{x(t)\}$ represents generalized coordinates of the structure, all equal to zero at steady motion. The integral in Eq. (1.1) represents unsteady aerodynamic forces and also reflects the history of motion. Elements of the matrix $[G(-\Theta;U)]$ are response functions corresponding to the impulsive changes of generalized coordinates. The vector function $\{f(x)\}$ describes nonlinear properties of an aircraft structure. The parameter U denotes the flight velocity.

The aircraft flutter phenomenon occurs when the flight velocity exceeds certain critical value changing the structural response to the shedding of vortices behind the aircraft body. The structural response itself induces additional aerodynamic self-excited forces causing loss of stability of the motion. It is known that there is a qualitative difference between the behavior of the linear and nonlinear aircraft structure above the critical flutter speed. In the linearized case loss of stability leads to oscillations the amplitude of which grows very fast and an aircraft is usually destroyed in a very short period of time. If the structure or aerodynamics is nonlinear then the finite amplitude limit-cycle oscillations take place.

The linearized second-order flutter equation exhibits loss of stability of a stationary solution if a root of the characteristic equation crosses the imaginary axis. At this point the matrix of the linear part of the corresponding set of first-order integro-differential equation (1.1) has a complex-conjugate, pure imaginary pair of eigenvalues. Therefore, the Hopf bifurcation of time-periodic solutions occurs in the nonlinear flutter system [1].

The aim is to investigate the asymptotic stability of equilibrium solution $\{x(t)\} = 0$ of Eq. (1.1) in the neighbourhood of a critical bifurcation point. It is known that all bifurcating solutions tend asymptotically to a finite-dimensional attracting subspace — center manifold [1]. The dimension of the center manifold is usually small and, therefore, it is beneficial to reduce the initial dynamical system to that describing solutions lying in this manifold.

The center manifold reduction and bifurcations for integro-differential equations have been studied much less extensively than those problems for ordinary differential equations. The difference between dealing with ordinary differential equations and integro-differential equations is crucial. In the first case all calculations are placed in the Euclidean space while the second case involves two functional subspaces of initial conditions and solutions. There is also a significant difference from the numerical point of view. Ordinary differential equations allow for evaluation of a single iterative numerical scheme for both the center manifold and the normal form reductions [7]. For integro-differential equations these two steps must be separated.

HASSARD, KAZARINOFF and WAN [2], using previous results of CHAFEE [3] and LIMA [4], gave an example of center manifold reduction and the Hopf bifurcation formulae for a certain two-dimensional integro-differential equation of the type (1.1). They took into account only the first term in a formal power series expansion for nonlinear function $\{f(x)\}$ and put a lot of effort going through very tedious algebraic calculations. For that reason, their procedure cannot be straightforwardly extended to multi-dimensional systems, neither they can account for higher nonlinear terms. However, the method is general and can be used to build up an algorithm for numerical calculations instead of forcing final algebraic formulae. Such an algorithm of center manifold reduction forms the fundamental part of the asymptotic stability analysis and will be presented in the paper after a brief outline of the method.

Equation (1.1) can be written in the form

$$\left\{\frac{dx}{dt}\right\} = L_0\{x\} + \{f(x)\}, \tag{1.2}$$

where the linear operator L_0 is given by

$$L_0\{x\} = [D]\{x(t)\} + \int_{-\infty}^{0} [G(-\Theta; U)]\{x(t+\Theta)\}d\Theta. \tag{1.3}$$

There exists one basic difficulty in a formal extension of the center manifold theorem, and the Hopf bifurcation theory as well, worked out for ordinary differential equations, to integro-differential equations. In order to solve Eq. (1.2) and initial vector-function $\{x(\Theta)\}$, describing the history of motion, must be given, continuous for every $\Theta \in (-\infty, 0]$. Consequently, Eq. (1.2) defines a map from the space $\mathbb{C}^-$ of continuous vector-functions over the interval $(-\infty, 0]$ onto the Euclidean space $\mathbb{R}^n$, parametrized by time t. Thus, operator (1.3) describes a transformation between two distinct spaces and, therefore, an eigenvalue problem for L_0 cannot be formulated (the point of bifurcation should be defined by pure imaginary eigenvalues of the operator L_0 [5]). The space $\mathbb{C}^-$ of initial functions must satisfy some additional requirements, forming essentially conditions of existence of improper integral in Eq. (1.3). These requirements are always satisfied for any physical system, as well as the continuity condition imposed on initial functions. Neither unbounded velocities nor infinite accelerations, generated by discontinuities, are allowed in physical systems.

2. Extended integro-differential equation

The idea of solving the above-mentioned difficulties consists of such an extension of the definition of the operator L_0 to have it acting in a single space $\mathbb{C}^-$. The extended definition is given by [2]:

(2.1)
$$L\{x(\Theta)\} = \begin{cases} \left\{\dfrac{dx(\Theta)}{d\Theta}\right\}, & \text{for} \quad -\infty < \Theta < 0, \\ [D]\{x(0)\} + \int\limits_{-\infty}^{0} [G(-\tau;U)]\{x(\tau)\}\, d\tau, & \text{for} \quad \Theta = 0, \end{cases}$$

where the operand is a vector function $\{x(\Theta)\} \in \mathbb{C}^-$. Now the integro-differential equation can be written as a mapping of the space $\mathbb{C}^-$ onto itself, with time t as a parameter:

$$\frac{dx_t(\Theta)}{dt} = Lx_t(\Theta) + Rx_t(\Theta), \tag{2.2}$$

where the following notation has been applied:

$$x_t(\Theta) = \{x(t+\Theta)\},$$
$$Rx_t(\Theta) = \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ \{f(x(t))\}, & \text{for} \quad \Theta = 0. \end{cases}$$

For $-\infty < \Theta < 0$ equation (2.2) gives the straightforward relation

$$\frac{d\{x(t+\Theta)\}}{dt} = \frac{d\{x(t+\Theta)\}}{d\Theta}.$$

Equation (1.2) is obtained from Eq. (2.2) for any given time t by putting $\Theta = 0$. It means that the space $\mathbb{C}^-$ of initial vector functions is always "the past" (history of motion). The adjoint to the space $\mathbb{C}^-$ is the space $\mathbb{C}^+$, composed of continuous vector functions $\{y(\Theta)\}$ defined over the interval $[0, +\infty)$ [3]. Any two functions from $\mathbb{C}^-$ and $\mathbb{C}^+$ are involved in a bilinear form $\langle y, x\rangle$:

$$\langle y, x\rangle = \{\bar{y}(0)\}^T\{x(0)\} - \int\limits_{-\infty}^{0}\int\limits_{0}^{\eta} \{\bar{y}(\xi-\eta)\}^T[G(-\eta;U)]\{x(\xi)\}\, d\xi d\eta, \tag{2.3}$$
$$\{x(\Theta)\} \in \mathbb{C}^-, \qquad \{y(\Theta)\} \in \mathbb{C}^+,$$

where $\{\ldots\}^T$ denotes the transposed vector. This form will be called "outer product". The outer product allows for a formal definition of an adjoint operator L^* by using the relation $\langle y, Lx\rangle = \langle L^*y, x\rangle$:

$$L^*\{y(\Theta)\} = \begin{cases} -\left\{\dfrac{dy(\Theta)}{d\Theta}\right\}, & \text{for} \quad 0 < \Theta < +\infty, \\ [D]^T\{y(0)\} + \int\limits_{-\infty}^{0} [G(-\tau;U)]^T\{y(-\tau)\}\, d\tau, & \text{for} \quad \Theta = 0. \end{cases}$$

Now an eigenvalue problem for the operator L can be formulated:

$$L\{\varphi(\Theta)\} = \lambda\{\varphi(\Theta)\}. \tag{2.4}$$

The form of an eigenfunction $\{\varphi(\Theta)\}$ follows immediately from Eqs. (2.1) and (2.4) by taking $-\infty < \Theta < 0$:

$$\{\varphi(\Theta)\} = \{\varphi_0\}e^{\lambda\Theta}. \tag{2.5}$$

The unknown vector $\{\varphi_0\}$ belongs to Euclidean space $\mathbb{R}^n$ and is given as an eigenvector of the eigenvalue problem posed for $\Theta = 0$:

$$\lambda\{\varphi_0\} = [D]\{\varphi_0\} + \left(\int_{-\infty}^{0} [G(-\tau;U)]e^{\lambda\tau}d\tau\right)\{\varphi_0\}.$$

In the same way an eigenvalue problem for the adjoint operator L^* can be formulated,

$$L^*\{\psi(\Theta)\} = \lambda^*\{\psi(\Theta)\}. \tag{2.6}$$

Eigenvalues and eigenvectors of both problems satisfy the following relations:

$$\lambda^* = \overline{\lambda}, \qquad \langle\psi_k, \varphi_l\rangle = \delta_{kl}, \tag{2.7}$$

where δ_{kl} is Kronecker's symbol. It can be easily found, that if $k \neq l$, then $\langle\psi_k, \varphi_l\rangle = 0$, no matter what the value of the scalar product $\{\overline{\psi}_k(0)\}^T\{\psi_l(0)\}$ in $\mathbb{R}^n$ is, and that for $k = l$ the following formula holds:

$$\{\overline{\psi}(0)\}^T\left([I] - \frac{d}{ds}[A(s;U)]\right)_{s=\lambda_k}\{\varphi(0)\} = 1,$$

where $[A(s;U)]$ is the matrix transfer function

$$[A(s;U)] = \int_{-\infty}^{0} [G(-\Theta;U)]e^{s\Theta}\,d\Theta. \tag{2.8}$$

In general case the space $\mathbb{C}^-$ of initial functions is of infinite dimensions. If, however, the elements of matrix $[G(-\Theta;U)]$ in Eq. (1.1), being functions of Θ, are solutions to certain linear ordinary differential equations with constant coefficients of order not exceeding p, then the space $\mathbb{C}^-$ is p-dimensional and the integro-differential equation (1.1) can be reduced to a system of p ordinary differential equations.

3. Center manifold

If, for a certain critical value U_c of a parameter U, there exists a set of m eigenvalues λ_k $(m \leq n)$, such that

$$\mathrm{Re}(\lambda_k) = 0, \qquad \text{for} \quad k = 1, 2, \ldots, m, \tag{3.1}$$

then there exists an m-dimensional subspace $\mathbf{M}$ of an infinite-dimensional space of all solutions to Eq. (2.2), called center manifold, containing all asymptotic solutions for $t \to \infty$. The center manifold is locally attracting for solutions remaining sufficiently close to the equilibrium solution $\{x\} = 0$. This is the most important property of the center manifold, despite the fact that the area of attractivity is in general unknown. The aim is to derive the equations for dynamical system restricted to the center manifold. These equations will be obtained corresponding to Eq. (1.1), but the method works also for the suspended system with the parameter U considered as an additional variable, satisfying

the equation:

$$\frac{dU}{dt} = 0 .$$

The suspended system allows for getting results valid in the neighbourhood of the critical value U_c.

An algorithm for center manifold reduction essentially outlines two properties of a center manifold. First, the center manifold is invariant. This means that if the initial function $\{x(t)\} \in \mathbf{M}$, for $t \leq 0$, then also for $t > 0$ $\{x(t)\} \in \mathbf{M}$. Second, the center manifold is tangent to the m-dimensional linear subspace spanned by the eigenvectors corresponding to the critical eigenvalues (3.1). The last property allows for a formal splitting of the unknown vector function $\{x_t(\Theta)\}$ on the center manifold into two parts:

$$\{x_t(\Theta)\} = \sum_{j=1}^{m} z_j(t)\{\varphi_j(\Theta)\} + \{w(\Theta, t)\} , \tag{3.2}$$

where $\{\varphi_j(\Theta)\}$ are eigenvectors of Eq. (2.4) corresponding to the critical eigenvalues λ_j, $\{w(\Theta, t)\}$ is a still unknown vector function, and scalar functions $z_j(t)$ describe new coordinates. At the bifurcation point $U = U_c$ all coordinates $z_j(t)$ are equal to zero. The function (3.2) must be invariant which means, that

$$\{w(\Theta, t)\} = \{w(z_1(t), \ldots, z_m(t), \Theta)\} . \tag{3.3}$$

Moreover, function $\{w\}$ must satisfy the following conditions at the bifurcation point $U = U_c$:

$$\{w(0, \ldots, 0, \Theta)\} = 0, \quad \left\{\frac{\partial w(0, \ldots, 0, \Theta)}{\partial z_j}\right\} = 0, \quad j = 1, 2, \ldots, m . \tag{3.4}$$

For convenience, new coordinates $z_j(t)$ are chosen such that

$$z_j(t) = \langle \psi_j(\Theta), x_t(\Theta) \rangle, \quad j = 1, 2, \ldots, m , \tag{3.5}$$

where $\{\psi_j(\Theta)\}$ are eigenvectors of Eq. (2.6). Then, orthogonality conditions follow immediately:

$$\langle \psi_j(\Theta), w(z_1, \ldots, z_m, \Theta) \rangle = 0, \quad j = 1, 2, \ldots, m . \tag{3.6}$$

It is worth noting that these orthogonality conditions involve the outer product (2.3) and hold in the space $\mathbb{C}^- \times \mathbb{C}^+$ (not in $\mathbb{R}^n$) and, therefore, cannot be used to reduce the number of components of the vector $\{w\}$.

Equation (2.2) on the center manifold takes the form

$$\langle \psi_j(\Theta), dx_t(\Theta)/dt \rangle = \langle \psi_j(\Theta), Lx_t(\Theta) + Rx_t(\Theta) \rangle, \quad j = 1, 2, \ldots, m ,$$

and, after applying Eqs. (3.2) and (3.6) and the following equalities:

$$\langle \psi_j(\Theta), dx_t(\Theta)/dt \rangle = dz_j(t)/dt , \langle \psi_j(\Theta), Lx_t(\Theta) \rangle = \lambda_j z_j(t) ,$$
$$\langle \psi_j(\Theta), Rx_t(\Theta) \rangle = \{\overline{\psi}_j(0)\}^T \{f_0\} ,$$

where

$$\{f_0\} = \{f(x_t(0))\} ,$$

gives the system of m ordinary differential equations for new coordinates $z_j(t)$:

$$dz_j(t)/dt = \lambda_j z_j(t) + \{\overline{\psi}_j(0)\}^T \{f_0\}, \quad j = 1, 2, \ldots, m . \tag{3.7}$$

These are no more integro-differential equations, however, they contain an unknown vector function $\{w\}$ as a part of the argument of $\{f_0\}$:

$$\{f_0\} = f\Big(\sum_{j=1}^{m} z_j(t)\{\varphi_j(0)\} + \{w(z_1, \ldots, z_m, 0)\}\Big).$$

The function $\{w\}$ plays a crucial role — it gives a condition to be satisfied by coordinates $z_1, \ldots, z_m$, in order that solution $\{x_t(\Theta)\}$ would remain in the center manifold all the time. Essentially, the function $\{w\}$ defines the center manifold.

The function $\{w\}$ must satisfy certain differential equation which can be obtained after substitution of $\{x_t(\Theta)\}$ of the form Eq. (3.2) into Eq. (2.2):

$$\text{(3.8)} \quad \left\{\frac{dw}{dt}\right\} - L\{w\} = \begin{cases} -\sum_{j=1}^{m} (\{\overline{\psi}_j(0)\}^T\{f_0\})\{\varphi_j(\Theta)\}, & \text{for} \quad -\infty < \Theta < 0, \\ -\sum_{j=1}^{m} (\{\overline{\psi}_j(0)\}^T\{f_0\})\{\varphi_j(0)\} + \{f_0\}, & \text{for} \quad \Theta = 0. \end{cases}$$

Equations (3.7) and (3.8) constitute a full set of $n + m$ scalar equations which can be solved for n components of the vector $\{w\}$ and for m functions $z_j(t)$. It is easy to check that the right-hand side of Eq. (3.8) is always orthogonal to the set of functions $\{\psi_j(\Theta)\}$, according to the outer product (2.3).

4. Center manifold reduction

The problem of practical solving of the set of Eqs. (3.7) and (3.8) remains still open. Up to now, the only general method deals with formal power series expansions for vector functions $\{w\}$ and $\{f_0\}$, with respect to variables z_j.

It follows from the conditions (3.4) that the formal multi-variable Taylor series expansion for the function $\{w\}$ has to be of the form:

$$\text{(4.1)} \qquad \{w(z_1, \ldots, z_m, \Theta)\} = \sum_{\mu \geq 2} \frac{1}{\mu!}[w_\mu(\Theta)]\{z^\mu\},$$

where each component of the vector $\{z^\mu\}$ is a product:

$$\{z^\mu\} = \{z_1^{\mu_1} \cdot z_2^{\mu_2} \ldots z_m^{\mu_m}\},$$

with restrictions: $\sum_{j=1}^{m} \mu_j = \mu, \qquad \mu_j \geq 0.$

The number of components of $\{z^\mu\}$ grows as μ increases and is equal to the number $c(\mu, m)$ of compositions of μ into m parts

$$\text{(4.2)} \qquad c(\mu, m) = \binom{m + \mu - 1}{\mu - 1}.$$

The coefficients of the series (4.1) are rectangular functional matrices $[w_\mu(\Theta)]$ composed of n-component vectors $\{w_{\mu k}(\Theta)\}$:

$$\text{(4.3)} \qquad [w_\mu(\Theta)] = [w_{\mu 1}(\Theta) \vdots \ldots \vdots w_{\mu k}(\Theta) \vdots \ldots \vdots w_{\mu m}(\Theta)].$$

The number of columns of each matrix (4.3) is equal to $c(\mu, m)$. Substitution of series (4.1) into (3.2) gives a consistent power series expansion for the function $\{x_t(\Theta)\}$ on the center manifold:

$$\{x_t(\Theta)\} = \sum_{\mu \geq 1} \frac{1}{\mu!} [w_\mu(\Theta)]\{z^\mu\}, \tag{4.4}$$

where the first coefficient for $\mu = 1$ is a square matrix of order n built of eigenvectors $\{\varphi_j(\Theta)\}$:

$$[w_1(\Theta)] = [\varphi_1(\Theta) \vdots \ldots \vdots \varphi_k(\Theta) \vdots \ldots \vdots \varphi_m(\Theta)]. \tag{4.5}$$

It is assumed that the power series expansion for the nonlinear function $\{f(x)\}$ at the right-hand side of Eq. (1.1) is known:

$$\{f(x)\} = \sum_{\nu \geq 2} \frac{1}{\nu!} [f_\nu]\{x^\nu\}, \tag{4.6}$$

where

$$\{x^\nu\} = \{x_1^{\nu_1} \cdot x_2^{\nu_2} \ldots x_n^{\nu_n}\}, \qquad \sum_{j=1}^{n} \nu_j = \nu, \qquad \nu_j \geq 0.$$

By setting $\Theta = 0$ and substituting Eq. (4.4) into Eq. (4.6) a corresponding power series expansion can be obtained, with respect to variables $z_j(t)$

$$\{f_0\} = \sum_{\mu \geq 2} \frac{1}{\mu!} [f_{0\mu}]\{z^\mu\}. \tag{4.7}$$

Substitution of Eqs. (4.1) and (4.7) into Eq. (3.8) leads to equations for coefficients $[w_\mu(\Theta)]$. Differentiation of Eq. (3.3) yields

$$\left\{\frac{dw}{dt}\right\} = \sum_{j=1}^{m} \left\{\frac{\partial w}{\partial z_j}\right\} \frac{dz_j}{dt},$$

and, after substituting the derivatives of variables z_j Eq. (3.7) on the center manifold, results in

$$\left\{\frac{dw}{dt}\right\} = \sum_{j=1}^{m} \lambda_j \left\{\frac{\partial w}{\partial z_j}\right\} z_j + \sum_{j=1}^{m} (\{\overline{\psi}_j(0)\}^T \{f_0\}) \left\{\frac{\partial w}{\partial z_j}\right\}.$$

Differentiation of the series (4.1) term by term with respect to the variable z_j yields

$$\left\{\frac{\partial w}{\partial z_j}\right\} = \sum_{\mu \geq 2} \frac{\mu_j}{\mu!} [w_\mu(\Theta)]\{z^{\mu - \delta_j}\},$$

where

$$\{z^{\mu - \delta_j}\} = \{z_1^{\mu_1} \cdots z_{j-1}^{\mu_{j-1}} \cdot z_j^{\mu_j - 1} \cdot z_{j+1}^{\mu_{j+1}} \cdots z_m^{\mu_m}\}.$$

Now an expression for the derivative of $\{w\}$ in the form of a series can be obtained:

$$\left\{\frac{dw}{dt}\right\} = \sum_{\mu \geq 2} \frac{1}{\mu!} [w_\mu(\Theta)][\Lambda_\mu]\{z^\mu\} + \sum_{j=1}^{m} (\{\overline{\psi}_j(0)\}^T \{f_0\}) \frac{\partial}{\partial z_j} \sum_{\mu \geq 2} \frac{1}{\mu!} [w_\mu(\Theta)]\{z^\mu\},$$

where matrix $[\Lambda_\mu]$ is a diagonal matrix of order $c(\mu, m)$, given by Eq. (4.2), with elements

$$(\Lambda_\mu)_{kk} = \sum_{j=1}^{m} \lambda_j \mu_j, \qquad k = 1, 2, \ldots, c(\mu, m),$$

where numbers μ_j correspond to the k-th composition of μ (the k-th component of the vector $\{z^\mu\}$). After substituting the above expression into Eq. (3.8) the final differential equation for $[w_\mu(\Theta)]$ is obtained:

$$\sum_{\mu \geq 2} \frac{1}{\mu!}([w_\mu(\Theta)][\Lambda_\mu] - L[w_\mu(\Theta)])\{z^\mu\} = \sum_{\nu \geq 2} \frac{1}{\nu!}[r_\nu(\Theta)]\{z^\nu\} + \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ \sum_{\nu \geq 2} \frac{1}{\nu!}[f_{0\nu}]\{z^\nu\}, & \text{for} \quad \Theta = 0, \end{cases} \tag{4.8}$$

where $L[w_\mu(\Theta)]$ denotes a matrix with columns given by the operator L acting on columns of the matrix $[w_\mu(\Theta)]$. The right-hand side series with coefficients $[r_\nu(\Theta)]$ results from differentiating and multiplying two series:

$$\sum_{\nu \geq 2} \frac{1}{\nu!}[r_\nu(\Theta)]\{z^\nu\} = -\sum_{j=1}^{m} (\{\overline{\psi}_j(0)\}^T \{f_0\}) \frac{\partial}{\partial z_j} \sum_{\mu \geq 2} \frac{1}{\mu!}[w_\mu(\Theta)]\{z^\mu\}. \tag{4.9}$$

This is a relatively simple task to create a numerical scheme based on Eq. (4.9) for calculating the elements of matrices $[r_\nu(\Theta)]$, but it is very hard to find analytic formulae for this purpose. The reason is that the analytical formula relating the given μ-composition to its position in the vector $\{z^\mu\}$ does not exist and, moreover, such a relation is not unique [6].

It follows from comparison in Eq. (4.7) of the left and right-hand side terms of the same order, that the value of μ corresponding to the matrix $[w_\mu(\Theta)]$ on the right-hand side is always less than that on the left-hand side. Therefore, Eq. (4.7) can be solved recursively for $\mu = 1, 2, \ldots$, and $\mu = \nu$. The unknown matrix $[w_2(\Theta)]$ in the first step ($\mu = 2$) depends on the known matrix $[w_1(\Theta)]$ composed of eigenvectors of the operator L Eq. (4.5). The first coefficient $[f_{02}]$ in the series (4.7) can be also calculated by using only the matrix $[w_1(0)]$ (i.e. for $\Theta = 0$).

In each consecutive step of the algorithm the following matrix equation has to be solved for the functional matrix $[w_\mu(\Theta)]$:

$$[w_\mu(\Theta)][\Lambda_\mu] - L[w_\mu(\Theta)] = [r_\mu(\Theta)] + \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ [f_{0\mu}], & \text{for} \quad \Theta = 0, \end{cases} \tag{4.10}$$

for $\mu = 2, 3, \ldots,$.

Equation (4.10) can be rewritten for each column of $[w_\mu(\Theta)]$ separately,

$$\sigma_k\{w_{\mu k}(\Theta)\} - L\{w_{\mu k}(\Theta)\} = \{r_{\mu k}(\Theta)\} + \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ [f_{0\mu k}], & \text{for} \quad \Theta = 0, \end{cases} \tag{4.11}$$

where consecutive compositions of μ are numbered by an index $k = 1, 2, \ldots, c(\mu, m)$ (i.e. each Eq. (4.11) for a given k corresponds to a single composition of μ). Hence, the total of $c(\mu, m)$ equations of the type Eq. (4.11) have to be solved in order to obtain one

matrix $[w_\mu(\Theta)]$. The number σ_k (corresponding to the k-th composition) is given by

$$\sigma_k = \sum_{j=1}^{m} \lambda_j \mu_j .$$

Both operators $B\{\varphi\} = \sigma_k\{\varphi\} - L\{\varphi\}$ and L share the same eigenvectors. The eigenvalue problem for the operator B is of the form

$$B\{\varphi\} = (\sigma_k - \lambda)\{\varphi\} ,$$

what means that the eigenvalue of B is equal to $\sigma_k - \lambda$. Consequently, if for a certain k-th composition the following equality holds:

$$\sum_{j=1}^{m} \lambda_j \mu_j - \lambda_l = 0, \quad \text{for} \quad 1 \leq l \leq m , \tag{4.12}$$

then the vector $\{w_{\mu k}(\Theta)\}$ is not unique and the full solution to Eq. (4.11) contains as a part the eigenvector $\{\varphi_l(\Theta)\}$ corresponding to λ_l. The orthogonality conditions allow for uniqueness of the full solution in this case.

Equation (4.11) must be solved in two stages. First, an ordinary differential equation is solved

$$\sigma_k\{w_{\mu k}(\Theta)\} - \frac{d}{d\Theta}\{w_{\mu k}(\Theta)\} = \{r_{\mu k}(\Theta)\} , \tag{4.13}$$

derived from Eqs. (4.11) and (2.1) in the interval $-\infty < \Theta < 0$. It follows from Eqs. (4.9) and (4.5) that for $\mu = 2$ the right-hand side of Eq. (4.13) is a linear combination of eigenvectors of the form Eq. (2.5). Moreover, the solution to Eq. (4.13) corresponding to a given μ affects the right-hand side of Eq. (4.13) for $\mu + 1$. Consequently, the right-hand side of Eq. (4.13) is always of the form of a linear combination of elementary terms:

$$\{r(\Theta)\} = \{r_0\}\Theta^p e^{\beta\Theta} , \tag{4.14}$$

where $p \geq 0$, and β is a complex number equal either to σ_k or to an eigenvalue of the operator L. Therefore, it is sufficient to look only into a simple form of Eq. (4.13) with the right-hand side given by Eq. (4.14):

$$\sigma_k\{w^{(e)}(\Theta)\} - \left\{\frac{dw^{(e)}(\Theta)}{d\Theta}\right\} = \{r(\Theta)\}, \tag{4.15}$$

where $\{w^{(e)}(\Theta)\}$ denotes an elementary component of the solution.

The elementary solution is of the form

$$\{w^{(e)}(\Theta)\} = \begin{cases} \{r_0\}e^{\beta\Theta} \sum_{j=0}^{p} \dfrac{p!}{j!(\sigma - \beta)^{p-j+1}}\Theta^j, & \text{for} \quad \sigma_k \neq \beta, \\ -\{r_0\}e^{\beta\Theta}\dfrac{1}{p+1}\Theta^{p+1}, & \text{for} \quad \sigma_k = \beta. \end{cases} \tag{4.16}$$

The general solution to Eq. (4.15) is a sum of Eq. (4.16) and the solution $\{w_h(\Theta)\}$ to be homogeneous equation corresponding to $\{r(\Theta)\} \equiv 0$

$$\{w_h(\Theta)\} = \{w_{h0}\}e^{\sigma_k\Theta} .$$

The second stage of the algorithm consists in finding an unknown vector $\{w_{h0}\}$. The general solution to Eq. (4.13) is a sum of two vectors

$$\{w_{\mu k}(\Theta)\} = \{w_{\mu k}^{(r)}(\Theta)\} + \{w_{h0}\}e^{\sigma_k \Theta}, \tag{4.17}$$

where $\{w_{\mu k}^{(r)}(\Theta)\}$ is the full particular solution forming a linear combination of elementary terms of the form Eq. (4.14) — this follows immediately from Eq. (4.16). Substitution of the sum into Eq. (4.11) for $\Theta = 0$ yields the system of linear algebraic equations which can be solved for the unknown vector $\{w_{h0}\}$:

$$\left(\sigma_k[I] - [D] - \int_{-\infty}^{0} [G(-\tau; U)]e^{\sigma_k \tau} d\tau\right)\{w_{h0}\} = \{r_{\mu k}(0)\} + \{f_{0\mu k}\} - \sigma_k\{w_{\mu k}^{(r)}(0)\} + L\{w_{\mu k}^{(r)}(0)\}. \tag{4.18}$$

It can be seen, after recalling the implicit form of the operator L in Eq. (2.1) that the last term on the right-hand side of Eq. (4.18) contains integrals of the form

$$J(p, \beta) = \int_{-\infty}^{0} [G(-\tau; U)]\tau^p e^{\beta \tau} d\tau .$$

These integrals can be rewritten as derivatives of the matrix transfer function $[A]$ Eq. (2.8)

$$J(p, \beta) = \frac{d^p}{d\beta^p} \int_{-\infty}^{0} [G(-\tau; U)]e^{\beta \tau} d\tau = \frac{d^p}{d\beta^p}[A(\beta)] .$$

Equation (4.18) has a unique solution for every $\sigma_k \neq \lambda_l$, where λ_l can be any eigenvalue of the operator L. However, if $\sigma_k = \lambda_l$, then the matrix of the system of equations is singular and solution is not unique. The term causing the lack of uniqueness is an eigenvector of L corresponding to λ_l:

$$\{\varphi_l(\Theta)\} = \{\varphi_{0l}\}e^{\lambda_l \Theta} .$$

In this case the right-hand side of Eq. (4.18) must be orthogonal (according to the scalar product in $\mathbb{R}^n$) to the vector $\{\psi_{0l}\} = \{\psi_l(0)\}$, where

$$\{\psi_l(\Theta)\} = \{\psi_{0l}\}e^{\bar{\lambda}_l \Theta}$$

is the corresponding eigenvector of the adjoint operator L^* (the property (2.7) has been taken into account). It will be shown here that this condition is always satisfied.

Let $\{v(\Theta)\}$ stand for the right-hand side of Eq. (4.11):

$$\{v(\Theta)\} = \{r_{\mu k}(\Theta)\} + \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ [f_{0\mu k}], & \text{for} \quad \Theta = 0. \end{cases}$$

It follows from Eqs. (3.6) and (3.8) that the following orthogonality condition holds:

$$\langle \psi_j(\Theta), v(\Theta) \rangle = 0, \quad \text{for} \quad j = 1, 2, \ldots, m .$$

Hence, by using the definition formula (2.3) of the outer product, the scalar product (ψ_{0j}, v_0) in $\mathbb{R}^n$ can be written as

$$(\psi_{0j}, v_0) = \int_{-\infty}^{0} \int_{0}^{\eta} \{\overline{\psi}_j(\xi - \eta)\}^T [G(-\eta; U)]\{v(\xi)\} d\xi d\eta .$$

The function $\{v(\xi)\}$ in the integrand can be replaced by

$$\{v(\xi)\} = \sigma_k\{w_{\mu k}^{(r)}(\xi)\} - L\{w_{\mu k}^{(r)}(\xi)\},$$

for the reason that $\{w_{\mu k}^{(r)}(\xi)\}$ is a particular solution to Eq. (4.11) for $\Theta = 0$, and the scalar product is given by

$$(\psi_{0j}, v_0) = \int_{-\infty}^{0}\int_{0}^{\eta} \{\overline{\psi}_j(\xi - \eta)\}^T[G(-\eta; U)](\sigma_k\{w_{\mu k}^{(r)}(\xi)\} - L\{w_{\mu k}^{(r)}(\xi)\})\, d\xi\, d\eta .$$

The above formula holds for all eigenfunctions $\{\psi_j\}$ $(j = 1, 2, \ldots, m)$, in particular for $\{\psi_l\}$ corresponding to $\sigma_k = \lambda_l$. Now the scalar product (ψ_{0l}, b_0) can be calculated:

$$\begin{aligned}(\psi_{0l}, b_0) &= (\psi_{0l}, v_0) - (\psi_{0l}, \lambda_l w_{\mu k}^{(r)}(0) - L w_{\mu k}^{(r)}(0)) \\ &= \int_{-\infty}^{0}\int_{0}^{\eta} \{\overline{\psi}_l(\xi - \eta)\}^T[G(-\eta; U)](\lambda_l\{w_{\mu k}^{(r)}(\xi)\} - L\{w_{\mu k}^{(r)}(\xi)\})\, d\xi\, d\eta \\ &\quad -(\psi_{0l}, \lambda_l w_{\mu k}^{(r)}(0) - L w_{\mu k}^{(r)}(0)) \\ &= -\langle \psi_l(\Theta), \lambda_l w_{\mu k}^{(r)}(\Theta) - L w_{\mu k}^{(r)}(\Theta)\rangle = 0,\end{aligned}$$

where

$$\{b_0\} = \{r_{\mu k}(0)\} + \{f_{0\mu k}\} - \lambda_l\{w_{\mu k}^{(r)}(0)\} + L\{w_{\mu k}^{(r)}(0)\}$$

denotes the right-hand side of Eq. (4.18). The final zero value of the scalar product comes from the orthogonality of the vector

$$\lambda_l\{w_{\mu k}^{(r)}(\Theta)\} - L\{w_{\mu k}^{(r)}(\Theta)\}$$

to the eigenvector $\{\psi_l(\Theta)\}$ and proves that Eq. (4.18) is always solvable. It means that the condition (4.12), when fulfilled, does not cause the lack of uniqueness in the center manifold reduction, contrary to the normal form reduction for systems of ordinary differential equations [7]. Nevertheless, the problem of assuring uniqueness of the solution for $\sigma_k = \lambda_l$ during numerical calculations has to be solved. Let $\{w_\sigma(\Theta)\}$ be any of the infinite number of solutions $\{w_{\mu k}(\Theta)\}$ Eq. (4.17) corresponding to $\sigma_k = \lambda_l$. The general solution $\{w_{\mu k}(\Theta)\}$ to Eq. (4.11) must be orthogonal, according to the outer product (2.3), to all eigenvectors $\{\psi_j(\Theta)\}$ corresponding to bifurcation eigenvalues (3.1). Hence, the general solution must be of the form

$$\begin{aligned}\{w_{\mu k}(\Theta)\} = \{w_\sigma(\Theta)\} &+ \sum_{j=1}^{m} \alpha_j\{\varphi_{0j}\}e^{\lambda_l\Theta} \\ &+ \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ -\sum\limits_{\substack{j=1 \\ j\neq l}} \alpha_j\{\varphi_{0j}\}, & \text{for} \quad \Theta = 0. \end{cases}\end{aligned}$$

From the orthogonality condition $\langle \psi_i(\Theta), w_{\mu k}(\Theta)\rangle = 0$ for $i = 1, 2, \ldots, m$, a system of m linear algebraic equations can be obtained and solved for unknowns α_j,

$$\sum_{j=1}^{m} \alpha_j\langle \psi_i(\Theta), \varphi_{0j}e^{\lambda_l\Theta}\rangle - \sum_{\substack{j=1 \\ j\neq l}}^{m} \alpha_j(\psi_{0i}, \varphi_{0j}) = -\langle \psi_i(\Theta), w_\sigma(\Theta)\rangle, \qquad \text{for } i = 1, 2, \ldots, m.$$

In order to calculate the coefficients of the system of equations the values of outer products $\langle \psi(\Theta), u^{(k)}(\Theta) \rangle$ have to be calculated, where $\{\psi(\Theta)\}$ is any eigenvector of adjoint eigenproblem (2.6), and $\{u^{(k)}(\Theta)\}$ is a vector of the form:

$$\{u^{(k)}(\Theta)\} = \{u_0\}\Theta^k e^{\beta\Theta},$$

for $k \geq 0$, $\beta \neq \lambda^*$. For $k = 0$, after setting $\lambda^* = \overline{\lambda}$, formula (2.3) gives

$$\langle \psi(\Theta), u^{(0)}(\Theta) \rangle = (\psi_0, u_0) - \{\overline{\psi}_0\}^T \frac{[A(\beta)] - [A(\lambda)]}{\beta - \lambda} \{u_0\},$$

where $[A(\beta)]$ and $[A(\lambda)]$ are matrix transfer functions (2.8) with arguments β and λ, respectively. In particular case, for $\beta = \lambda$,

$$\lim_{\beta \to \lambda} \langle \psi(\Theta), u^{(0)}(\Theta) \rangle = (\psi_0, u_0) - \{\overline{\psi}_0\}^T \left[\frac{dA(\beta)}{d\beta}\right]_{\beta=\lambda} \{u_0\}.$$

For $k \neq 0$ formula (2.3) yields the relation:

$$\langle \psi(\Theta), u^{(k)}(\Theta) \rangle = -\frac{d^k}{d\beta^k}\left(\{\overline{\psi}_0\}^T \frac{[A(\beta)] - [A(\lambda)]}{\beta - \lambda} \{u_0\}\right),$$

that allows for deriving the recurrence formula for $k > 0$,

$$\langle \psi(\Theta), u^{(k)}(\Theta) \rangle = -\frac{1}{\beta - \lambda}\left(\{\overline{\psi}_0\}^T \left[\frac{dA^k(\beta)}{d\beta^k}\right] \{u_0\} + k\langle \psi(\Theta), u^{(k-1)}(\Theta) \rangle\right).$$

In the particular case, for $\beta = \lambda$, the above formula reduces to

$$\lim_{\beta \to \lambda} \langle \psi(\Theta), u^{(k)}(\Theta) \rangle = -\frac{1}{k+1} \{\overline{\psi}_0\}^T \left[\frac{dA^{k+1}(\beta)}{d\beta^{k+1}}\right]_{\beta=\lambda} \{u_0\}.$$

At this stage all formulae needed for center manifold reduction have been obtained. The problem that still remains unsolved is the estimation of accuracy and validity range induced by the method of power series expansions. This problem sometimes may be crucial because power series expansions do not always converge. In numerical practice all calculations refer to the finite number of terms of any series involved and, therefore, it is better to consider them as a polynomial approximation rather than the Taylor series expansion.

5. Numerical example

Up to now, there are no numerical results available for center manifold reduction of the aircraft flutter equation. Therefore, in order to verify the algorithm a simple set of two integro-differential equations investigated previously by HASSARD *et al.* [2] has been chosen. The system is the following:

$$\left\{\frac{dx}{dt}\right\} = [D]\{x(t)\} + \int_{-\infty}^{0} [G(-\Theta; U)]\{x(t+\Theta)\} d\Theta + \{f(x)\} + \{h(x)\}, \tag{5.1}$$

where matrices $[D]$ and $[G(-\Theta; U)]$ are given by

$$[D] = \begin{bmatrix} 0 & -1/\beta \\ U \cdot (\beta - 1) & 0 \end{bmatrix},$$

$$[G(-\Theta;U)] = \begin{bmatrix} \dfrac{U\cdot\Theta}{\beta\cdot T^2}e^{\Theta/T} & 0 \\ 0 & 0 \end{bmatrix},$$

with β and T being real numbers. The Taylor series expansion for the first nonlinear vector $\{f(x)\}$, Eq. (4.6), reduces to the single term corresponding to $\nu = 2$:

$$\{f(x)\} = \frac{1}{2!}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 2\beta & 0 \end{bmatrix}\begin{Bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{Bmatrix}.$$

The second nonlinear vector $\{h(x)\}$ is of the form:

$$\{h(x)\} = \beta x_1(t)\int_{-\infty}^{0}[G(-\Theta;U)]\{x(t+\Theta)\}d\Theta,$$

and has not been included in Eq. (1.1). However, such nonlinear terms can be added without any qualitative changes made in the algorithm. They affect only calculation of the right-hand side of Eq. (4.10) in each consecutive step of the algorithm. After substituting the series expansion (4.14) for $\{x(t+\Theta)\}$ on the center manifold, the following series for $\{h(x)\}$ is obtained:

$$\{h(x)\} = \beta x_1(t)\sum_{\mu\geq 1}\frac{1}{\mu!}\int_{-\infty}^{0}[G(-\Theta;U)][w_\mu(\Theta)]d\Theta\{z^\mu\}.$$

Each column $\{w_{\mu k}(\Theta)\}$ of the matrix $[w_\mu(\Theta)]$, where $k = 1,2,\ldots,c(\mu,m)$, can be only of the elementary form

$$\{w_{\mu k}(\Theta)\} = \{w_{\mu k}\}\Theta^p e^{s\Theta},$$

with an integer number $p \geq 0$ and s being a complex imaginary number. Consequently, all integrals can be carried out as follows:

$$\int_{-\infty}^{0}[G(-\Theta;U)]\Theta^p e^{s\Theta}d\Theta\{w_{\mu k}\} = \left(\frac{d^p}{ds^p}[A(s)]\right)\{w_{\mu k}\},$$

where matrix $[A(s)]$ is given by Eq. (2.8).

The above example is especially suitable for testing because the kernel $[G(-\Theta;U)]$ is finite-dimensional; it follows that Eq. (5.1) can be reduced to the system of ordinary differential equations and, therefore, the results of the algorithm can be compared with results of other methods worked out for this kind of equations.

The system of ordinary differential equations corresponding to the integro-differential equation (5.1) is following:

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_2(t)/\beta - x_3(t)\cdot U/\beta - x_1(t)\cdot x_2(t) - x_1(t)\cdot x_3(t)\cdot U, \\
\frac{dx_2}{dt} &= x_1(t)\cdot U\cdot(\beta-1) + x_1(t)\cdot x_2(t)\cdot\beta, \\
\frac{dx_3}{dt} &= x_4(t), \\
\frac{dx_4}{dt} &= x_1(t)/T^2 - x_3(t)/T^2 - x_4(t)\cdot 2/T,
\end{aligned} \tag{5.2}$$

and has been obtained after introducing two new variables x_3 and x_4:

$$x_3(t) = -(1/T^2) \int_{-\infty}^{0} \Theta e^{\Theta/T} x_1(t+\Theta) d\Theta, \qquad x_4(t) = \frac{dx_3}{dt}.$$

There exists one critical value U_c of the parameter U,

$$U_c = \frac{2\beta}{T + 2T^2(\beta - 1)},$$

corresponding to two complex conjugate pure imaginary eigenvalues (3.1) of the linear operator L Eq. (2.1)

$$\lambda_1 = i/T, \qquad \lambda_2 = -i/T.$$

In this case the center manifold is two-dimensional ($m = 2$) and contains an asymptotic system of reduced equations of the form Eq. (3.7), which can be written as a power series

$$\left\{\frac{dz}{dt}\right\} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \{z\} + \sum_{\mu \geq 2} [d_\mu] \{z^\mu\}.$$

The presented algorithm of center manifold reduction, corresponding to

$$\beta = 1.001, \qquad T = 0.125, \qquad U_c = 16.011997,$$

gives the following values for the elements of the first seven matrices $[d_\mu]$:

$$[d_1]\{z^1\} = \begin{bmatrix} 0.00000 & 8.00000i \\ 0 & \end{bmatrix} \quad \begin{matrix} 0 \\ 0.00000 \quad -8.00000i \end{matrix} \Bigg] \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix},$$

$$[d_2]\{z^2\} = \begin{bmatrix} 1.60146 & 3.20347i & -1.60146 & 3.20347i \\ & 0 & & 0 \\ -1.60146 & -3.20347i & 1.60146 & -3.20347i \end{bmatrix}^T \begin{Bmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{Bmatrix},$$

$$[d_3]\{z^3\} = \begin{bmatrix} -0.56110 & 0.08007i & -0.40079 & -0.40081i \\ -0.40069 & -0.40072i & 0.08007 & -0.56102i \\ 0.08007 & 0.56102i & -0.40069 & 0.40072i \\ -0.40079 & 0.40081i & -0.56110 & -0.08007i \end{bmatrix}^T \begin{Bmatrix} z_1^3 \\ z_1^2 z_2^1 \\ z_1^1 z_2^2 \\ z_2^3 \end{Bmatrix},$$

$$[d_4]\{z^4\} = \begin{bmatrix} 0.12349 & -0.10488i & 0.15802 & 0.03585i \\ 0.47890 & 0.01985i & 0.27152 & 0.39503i \\ 0.12846 & -0.06426i & 0.12846 & 0.06426i \\ 0.27152 & -0.39503i & 0.47890 & -0.01985i \\ 0.15802 & -0.03585i & 0.12349 & 0.10488i \end{bmatrix}^T \begin{Bmatrix} z_1^4 \\ z_1^3 z_2^1 \\ z_1^2 z_2^2 \\ z_1^1 z_2^3 \\ z_2^4 \end{Bmatrix},$$

$$[d_5]\{z^5\} = \begin{bmatrix} -0.04440 & 0.04163i & -0.05995 & -0.01053i \\ -0.29168 & 0.14012i & -0.28713 & -0.14928i \\ -0.26271 & 0.17039i & -0.34411 & -0.00760i \\ -0.34411 & 0.00760i & -0.26271 & -0.17039i \\ -0.28713 & 0.14928i & -0.29168 & -0.14012i \\ -0.05995 & 0.01053i & -0.04440 & -0.04163i \end{bmatrix}^T \begin{Bmatrix} z_1^5 \\ z_1^4 z_2^1 \\ z_1^3 z_2^2 \\ z_1^2 z_2^3 \\ z_1^1 z_2^4 \\ z_2^5 \end{Bmatrix},$$

$$[d_6]\{z^6\} = \begin{bmatrix} 0.02936 & -0.02501i & 0.03763 & 0.00848i \\ 0.16004 & -0.18080i & 0.26747 & -0.00054i \\ 0.22524 & -0.18292i & 0.31328 & -0.04130i \\ 0.48793 & -0.14352i & 0.48793 & 0.14352i \\ 0.31328 & 0.04130i & 0.22524 & 0.18292i \\ 0.26747 & 0.00054i & 0.16004 & 0.18080i \\ 0.03763 & -0.00848i & 0.02936 & 0.02501i \end{bmatrix}^T \begin{Bmatrix} z_1^6 \\ z_1^5 z_2^1 \\ z_1^4 z_2^2 \\ z_1^3 z_2^3 \\ z_1^2 z_2^4 \\ z_1^1 z_2^5 \\ z_2^6 \end{Bmatrix},$$

$$[d_7]\{z^7\} = \begin{bmatrix} -0.01959 & 0.02461i & -0.03390 & -0.00046i \\ -0.06837 & 0.16610i & -0.19204 & 0.07969i \\ -0.21957 & 0.24556i & -0.37570 & 0.08582i \\ 120.24909 & 241.68046i & -201.70191 & 402.36405i \\ -201.70191 & -402.36405i & 120.24909 & -241.68046i \\ -0.37570 & -0.08582i & -0.21957 & -0.24556i \\ -0.19204 & -0.07969i & -0.06837 & -0.16610i \\ -0.03390 & 0.00046i & -0.01959 & -0.02461i \end{bmatrix}^T \begin{Bmatrix} z_1^7 \\ z_1^6 z_2^1 \\ z_1^5 z_2^2 \\ z_1^4 z_2^3 \\ z_1^3 z_2^4 \\ z_1^2 z_2^5 \\ z_1^1 z_2^6 \\ z_2^7 \end{Bmatrix}.$$

The corresponding normal form obtained by iterative method of HSU and FAVRETTO [8] based on nonlinear transformation of variables $\{z\} \to \{Z\}$ is given by

$$\begin{aligned} \frac{dZ_1}{dt} &= (-0.00000 + 8.00000i) \cdot Z_1 + (-0.40069 - 1.46963i) \cdot Z_1^2 Z_2^1 \\ &\quad + (0.08277 - 0.50758i) \cdot Z_1^3 Z_2^2 + (120.69879 + 241.09039i) \cdot Z_1^4 Z_2^3, \\ \frac{dZ_2}{dt} &= (-0.00000 - 8.00000i) \cdot Z_2 + (-0.40069 + 1.46963i) \cdot Z_1^1 Z_2^2 \\ &\quad + (0.08277 + 0.50758i) \cdot Z_1^2 Z_2^3 + (120.69879 - 241.09039i) \cdot Z_1^3 Z_2^4. \end{aligned}$$

The same results are obtained after applying the iterative method [8] directly to the system of ordinary differential equations (5.2).

The matrix transfer function (2.8) in the case of aircraft flutter is the aerodynamic matrix used in linear flutter analysis and can be easily calculated by using well-known numerical methods. However, the kernel $[G(-\Theta; U)]$ is, in general, of infinite dimensions and the integro-differential flutter equation cannot be replaced by a system of ordinary differential equations. The only exception occurs when the elements of aerodynamic matrix have been approximated by rational functions.

6. Concluding remarks

From the numerical point of view the most important is the fact that the method of center manifold reduction does not require the calculation of response matrix $[G(-\tau; U)]$. In explicit form only the matrix transfer function (2.8) appears, which is much easier to calculate than the response matrix, especially for the pure imaginary values of the argument. In certain problems also the derivatives of the matrix transfer function can be calculated without employing numerical differentiation schemes. Such schemes, however, may be necessary in order to obtain the power series expansion for the nonlinear part $\{f(x)\}$ of Eq. (1.1).

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Eigenfrequencies of a system of elastic layers

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A HARMONIC WAVE propagates across the chain of elastic layers. The displacements of the ends of the chain are equal zero. The characteristic determinant has been calculated and expressed as a continued fraction. The eigenfrequencies may be calculated as the frequencies for which this determinant equals zero. The probability density of the ratio of amplitudes for a random distribution of layers has been derived. Numerical example has been given and discussed.

1. Homogeneous layers

THE SYSTEM of layers were dealt with in many papers, e.g. in [1]–[8]. In the present paper we consider the random system of N different homogeneous elastic layers, Fig. 1. The situation is close to that considered in [9] and [10]. The layer situated between x_k and x_{k+1} is identified by the natural number k, $k = 1, 2, 3, \ldots, N$. The propagation speed, density and thickness of the layer k are denoted by c_k, ϱ_k and h_k, respectively. In the layer k, two sinusoidal waves of frequency ω propagate, one of amplitude A_k in the $+x$-direction, and the other of amplitude B_k in the $-x$-direction. The total displacement in the layer k is (Fig. 1)

$$u_k = A_k \exp\{i\omega[t - (x - x_k)/c_k]\} + B_k \exp\{i\omega[t + (x - x_k)/c_k]\}, \tag{1.1}$$

where t is time, $x_k \leq x \leq x_{k+1}$. The displacement u_k satisfies the equation of motion

$$c_k^2 u_{k,xx} = u_{k,tt}. \tag{1.2}$$

Physical displacement is the real part of the complex-valued function $u_k(x, t)$.

At both sides of the boundary between layers k and $k + 1$ both the displacement and the stress vector have the same values. This continuity leads to the following relation between the wave amplitudes

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = M_k \begin{bmatrix} A_k \\ B_k \end{bmatrix}, \tag{1.3}$$

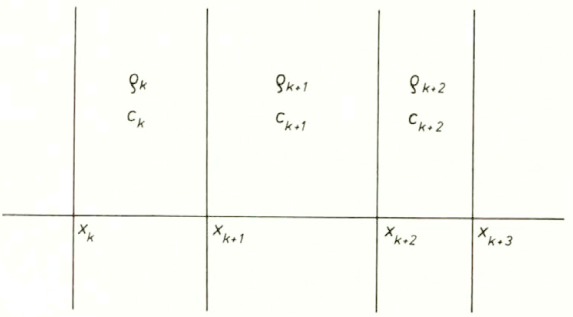

FIG. 1.

where

$$M_k = \begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix}, \tag{1.4}$$

$$P_k = \frac{1}{2}(1 + \kappa_k)\exp(-i\alpha_k), \qquad S_k = \overline{P_k}, \tag{1.5}$$

$$Q_k = \frac{1}{2}(1 - \kappa_k)\exp(i\alpha_k), \qquad R_k = \overline{Q_k},$$

$$\alpha_k = \omega h_k / c_k, \tag{1.6}$$

$$\kappa_k = \varrho_k c_k / \varrho_{k+1} c_{k+1}, \tag{1.7}$$

$$h_k = x_{k+1} - x_k. \tag{1.8}$$

The transfer matrix M_k allows us to express the amplitudes A_{k+1}, B_{k+1} by the amplitudes A_k, B_k. The above results may be chained to express the amplitudes in the N-th layer by the amplitudes in the first layer. The frequency ω influences the transfer matrix M_k via the functions α_k. The determinant of M_k does not depend on the frequency ω

$$\det M_k = \kappa_k. \tag{1.9}$$

If there exists an access to the ends of the set of layers, then the displacement at the beginning of the set may be given in advance as a function of time. The displacement of the other end may then be calculated, taking into account total number of the layers, their dimensions and elastic properties. In particular, the displacement of the beginning of the system may be a harmonic function of arbitrary frequency ω.

An entirely other situation arises if there is no access to the layers, e.g. if the system of N layers is situated between two rigid walls. If the first and the last layers adhere to the walls, no relative displacement between the walls and the ends of the system are allowed, and the displacements of the ends of the system are zero. Only a discrete spectrum of frequencies corresponds to a non-zero displacement field inside the layers. In this case frequency ω is not arbitrary, but must be calculated from the equations of motion. Such frequencies are the eigenfrequencies of the system.

In this paper we consider the last situation only, when there exists no access to the ends of the chain. The purpose of the analysis is calculation of the frequencies, for which there exists a non-zero, time-dependent displacement field.

2. Chain of elastic layers

The considered system of N layers will be further called a chain. In accord with notation (1.1), the boundary conditions $u = 0$ at the beginning and the end of the chain may be expressed by the wave amplitudes

$$A_1 + B_1 = 0, \qquad A_{N+1} + B_{N+1} = 0. \tag{2.1}$$

Displacement of the beginning of the first layer is $A_1 + B_1$. The layer $N + 1$ is not present in the system. The expression $A_{N+1} + B_{N+1}$ is used, because it equals the displace-

ment of the end of the layer N and simplifies the notation. Write now the system (1.4) and (2.1) as the matrix equation for the wave amplitudes $A_1, B_1, \ldots, A_N, B_N, A_{N+1}, B_{N+1}$

$$
(2.2)\qquad
\begin{bmatrix}
1 & 1 & & & & & & & & & \\
P_1 & Q_1 & -1 & & & & & & & & \\
R_1 & S_1 & 0 & -1 & & & & & & & \\
 & & P_2 & Q_2 & -1 & & & & & & \\
 & & R_2 & S_2 & 0 & -1 & & & & & \\
 & & & & P_3 & Q_3 & -1 & \vdots & & & \\
 & & & & & \cdots & \cdots & \cdots & & & \\
 & & & & & & \vdots & P_N & Q_N & -1 & \\
 & & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & & 1 & 1
\end{bmatrix}
\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ \\ A_N \\ B_N \\ A_{N+1} \\ B_{N+1} \end{bmatrix} = 0.
$$

The non-trivial set of amplitudes $A_1, B_1, \ldots, A_{N+1}, B_{N+1}$ exists, if the determinant Δ of the matrix in (2.2) equals zero. This determinant is a function of frequency ω, $\Delta = \Delta(\omega)$. In order to calculate $\Delta(\omega)$, define for $k = 1, 2, \ldots, N$ the following five auxiliary determinants D_k, E_k, F_k, G_k, H_k

$$
(2.3)\qquad D_k = \begin{vmatrix}
Q_k & -1 & & & & & & & & \\
S_k & 0 & -1 & & & & & & & \\
 & P_{k+1} & Q_{k+1} & -1 & & & & & & \\
 & R_{k+1} & S_{k+1} & 0 & -1 & & & & & \\
 & & & P_{k+2} & Q_{k+2} & -1 & \vdots & & & \\
 & & & & \cdots & \cdots & \cdots & & & \\
 & & & & & \vdots & P_N & Q_N & -1 & \\
 & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & 1 & 1
\end{vmatrix},
$$

$$
(2.4)\qquad E_k = \begin{vmatrix}
P_k & -1 & & & & & & & & \\
R_k & 0 & -1 & & & & & & & \\
 & P_{k+1} & Q_{k+1} & -1 & & & & & & \\
 & R_{k+1} & S_{k+1} & 0 & -1 & & & & & \\
 & & & P_{k+2} & Q_{k+2} & -1 & \vdots & & & \\
 & & & & \cdots & \cdots & \cdots & & & \\
 & & & & & \vdots & P_N & Q_N & -1 & \\
 & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & 1 & 1
\end{vmatrix},
$$

$$
(2.5)\qquad F_K = \begin{vmatrix}
0 & -1 & & & & & & & \\
P_k & Q_k & -1 & & & & & & \\
R_k & S_k & 0 & -1 & & & & & \\
 & & P_{k+1} & Q_{k+1} & -1 & & & & \\
 & & R_{k+1} & S_{k+1} & 0 & -1 & \vdots & & \\
 & & & & \cdots & \cdots & \cdots & & \\
 & & & & & \vdots & P_N & Q_N & -1 \\
 & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & 1 & 1
\end{vmatrix},
$$

$$
(2.6)\qquad G_k = \begin{vmatrix}
S_k & & & & & & & & \\
 & Q_{k+1} & -1 & & & & & & \\
 & S_{k+1} & 0 & -1 & & & & & \\
 & & P_{k+2} & Q_{k+2} & -1 & & & & \\
 & & R_{k+2} & S_{k+2} & 0 & -1 & \vdots & & \\
 & & & & \cdots & \cdots & \cdots & & \\
 & & & & & \vdots & P_N & Q_N & -1 \\
 & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & 1 & 1
\end{vmatrix},
$$

$$
(2.7)\qquad H_k = \begin{vmatrix}
R_k & & & & & & & & \\
 & Q_{k+1} & -1 & & & & & & \\
 & S_{k+1} & 0 & -1 & & & & & \\
 & & P_{k+2} & Q_{k+2} & -1 & & & & \\
 & & R_{k+2} & S_{k+2} & 0 & -1 & \vdots & & \\
 & & & & \cdots & \cdots & \cdots & & \\
 & & & & & \vdots & P_N & Q_N & -1 \\
 & & & & & & R_N & S_N & 0 & -1 \\
 & & & & & & & & 1 & 1
\end{vmatrix}.
$$

The elements not marked are equal zero. The determinants D_k, E_k have the rank $2(N-k)+3$, the determinant F_k has the rank $2(N-k)+4$, and the determinants G_k, H_k have the rank $2(N-k)+2$. They differ only in the structure of the upper left corner. Note that in the upper left corner of the determinant with suffix k, constants of the k-th layer are situated. The following relations hold

$$(2.8)\qquad D_k = Q_k F_{k+1} + G_k\,,$$

$$(2.9)\qquad E_k = P_k F_{k+1} + H_k\,,$$

$$(2.10)\qquad F_k = E_k\,,$$

$$(2.11)\qquad G_k = S_k D_{k+1}\,,$$

$$(2.12)\qquad H_k = R_k D_{k+1}\,.$$

The functions F_k, G_k, H_k may be eliminated from the above formulae. The resulting recursive relations are

$$D_k = \overline{P_k} D_{k+1} + Q_k E_{k+1}\,,$$
$$E_k = \overline{Q_k} D_{k+1} + P_k E_{k+1}\,. \tag{2.13}$$

They must be complemented by the values of D_k, E_k for some fixed k. Note that, in accord with (2.3), (2.4), there is

$$D_N = \overline{P_N} + Q_N, \qquad E_N = P_N + \overline{Q_N}\,. \tag{2.14}$$

Each D_k, E_k may now be calculated.

We prove that the values of E_k and D_k for each index k are complex conjugate numbers. Assume that for some k there is

$$E_k = \overline{D_k}\,. \tag{2.15}$$

Then from Eqs. (2.13) it follows that this equality holds for $k + 1$

$$E_{k+1} = \overline{D_{k+1}}\,. \tag{2.16}$$

The fact that, in accord with Eqs. (2.14), there is $E_N = \overline{D_N}$, completes the proof of Eq. (2.16).

Equations (2.13) may now be written as a single recursive formula

$$D_k = \overline{P_k} D_{k+1} + Q_k \overline{D_{k+1}}\,. \tag{2.17}$$

The above calculations were performed in order to evaluate the determinant Δ. From (2.2) follows the relation

$$\Delta = D_1 - E_1\,, \tag{2.18}$$

which, after taking into account (2.16), reduces to the relation

$$\Delta = D_1 - \overline{D_1} = 2i \operatorname{Im} D_1\,. \tag{2.19}$$

It follows, that $\Delta(\omega) = 0$, if $\operatorname{Im} D_1 = 0$. Obviously, D_1 is a function of ω. The recursive relations (2.13), or (2.17) allow us to calculate $D_1(\omega)$, and finally the roots ω for which $\Delta(\omega) = 0$. Note that the boundary condition for the N-th layer was taken into account already when calculating the determinants D_k, E_k. If $\Delta = 0$, then also the boundary condition for $k = 1$ is satisfied. If, for some frequency, Δ is not equal zero, then the boundary condition for the N-th layer is satisfied, but the boundary condition for the first layer is not. The only solution of (2.2) is then the trivial solution $A_k = B_k = 0$.

3. Eigenfrequencies of periodic structure

Especially simple relations would be obtained for the functions P_k, Q_k and the transfer matrix M_k independent of k. The situation, when all parameters of the layer are independent of k is trivial, since only one layer is present then. The other possibility corresponds to $\varrho_k c_k$ being proportional to $\exp(k)$. This, however, is not acceptable for large N.

The only interesting system with P_k, Q_k, M_k independent of k is the periodic system of equal elementary cells. An elementary cell may consist of arbitrary number of different elastic layers. In this case the transfer matrix M for one cell equals the product of transfer

matrices for all layers of one cell. As the product of W-symmetric matrices, it possesses the W-symmetry. If k denotes the number of the cell (not the number of the layer, as in the previous chapter), then all the above relations may be used. In this case P, Q, R, S are the components of the transfer matrix for one cell.

In the periodic system of layers the elastic moduli after passing one cell return to the previous values. Due to this fact, one additional relation for the transfer matrix exists [19], namely

$$\det M = P\overline{P} - Q\overline{Q} = 1 . \tag{3.1}$$

Instead of Eq. (2.13) we face now the simple system

$$\begin{aligned} D_k &= \overline{P} D_{k+1} + Q E_{k+1} , \\ E_k &= \overline{Q} D_{k+1} + P E_{k+1} , \end{aligned} \tag{3.2}$$

where the coefficients are independent of k. The solution of the above system is expected in the following form

$$D_k = L_D \exp\{(N+1-k)\psi\}, \qquad E_k = L_E \exp\{(N+1-k)\psi\} , \tag{3.3}$$

where L_D, L_E and ψ are the complex parameters to be calculated. Substitution into Eqs. (3.2) leads to the system of equations

$$\begin{aligned} (\exp i\psi - \overline{P}) L_D - Q L_E &= 0 , \\ -\overline{Q} L_D + (\exp i\psi - P) L_E &= 0 . \end{aligned} \tag{3.4}$$

Since the determinant must be equal to zero, the following equation for ψ is obtained

$$\exp(2i\psi) - (P + \overline{P}) \exp(i\psi) + 1 = 0 . \tag{3.5}$$

Its solution is

$$\begin{aligned} \exp i\psi &= \operatorname{Re} P \pm i\sqrt{1 - (\operatorname{Re} P)^2} , \\ \cos\psi &= \operatorname{Re} P, \qquad \sin\psi = \pm\sqrt{1 - (\operatorname{Re} P)^2} . \end{aligned} \tag{3.6}$$

For $1 - (\operatorname{Re} P)^2 > 0$ the parameter ψ is real-valued. There are two values of ψ satisfying Eq. (3.5). We assume that $\sin\psi > 0$, $\cos\psi = \operatorname{Re} P$. The other value is then $-\psi$.

Note that from (2.3) follows $D_N = \overline{P_N} + Q_N$. In accord with Eqs. (3.2) (but not from (2.3)) we may define

$$D_{N+1} = E_{N+1} = 1 . \tag{3.7}$$

Since there are two roots, the expected formula (3.3) should be replaced by the formula

$$D_k = L_{D1} \exp\{i(N+1-k)\psi\} + L_{D2} \exp\{-i(N+1-k)\psi\} , \tag{3.8}$$

and an expression of analogous structure for E_k. Now the constants L_{D1}, L_{D2} must be calculated. In accord with (3.8) and the above mentioned values of D_N, D_{N+1}, there is

$$\begin{aligned} L_{D1} + L_{D2} &= 1 , \\ L_{D1} \exp(i\psi) + L_{D2} \exp(-1\psi) &= \overline{P} + Q . \end{aligned} \tag{3.9}$$

Elementary calculations lead to the following expressions for the complex-valued coefficients L_{D1}, L_{D2}

$$
\begin{aligned}
L_{D1} &= \frac{1}{2\sin\psi}[-\operatorname{Im} P + \operatorname{Im} Q + \sin\psi + i(-\operatorname{Re} P - \operatorname{Re} Q + \cos\psi)]\,, \\
L_{D2} &= \frac{1}{2\sin\psi}[\operatorname{Im} P - \operatorname{Im} Q + \sin\psi + i(\operatorname{Re} P + \operatorname{Re} Q - \cos\psi)]\,.
\end{aligned} \tag{3.10}
$$

The determinant Δ may now be calculated from Eqs. (3.8) and (2.19) to be given by the formula

$$
\Delta = \frac{2i}{\sin\psi}(-\operatorname{Im} P + \operatorname{Im} Q)\sin N\psi\,. \tag{3.11}
$$

4. Determinant expressed as continued fractions

The analysis presented in the second chapter allows us to calculate the determinant Δ as the function of frequency ω. Here the alternative approach to the calculation of $\Delta(\omega)$ will be given.

Define the $2N + 2$ complex-valued parameters

$$
\begin{aligned}
&s_1 = D_1, \quad s_2 = E_1, \quad s_3 = D_2, \quad s_4 = E_2, \ldots, \quad s_{k-1} = D_k, \\
&s_{2k} = E_k, \ldots, \quad s_{2N-1} = D_N, \quad s_{2N} = E_N, \quad s_{2N+1} = 1, \quad s_{2N+2} = 1\,.
\end{aligned} \tag{4.1}
$$

The last two relations were added in accordance with (3.7). Remember that we are interested only in the value of s_1 and more exactly — in the value $\operatorname{Im} s_1$, cf. (2.19). Obviously, in accordance with (2.14) there is $s_{2N-1} = \overline{P_N} + Q_N$, $s_{2N} = P_N + \overline{Q_N}$. Basing on the relations (2.13) we construct now the relations between the variables (4.1), trying to write them in a simple form. Substitution of Eqs. (4.1) into (2.13) gives

$$
\begin{aligned}
s_{2k-1} &= \overline{P_k} s_{2k+1} + Q_k s_{2k+2}\,, \\
s_{2k} &= \overline{Q_k} s_{2k+1} + P_k s_{2k+2}\,.
\end{aligned} \tag{4.2}
$$

The last equation connects the three successive s_m, s_{m+1}, s_{m+2} and possesses already the desired structure. The first equation must be transformed. Substituting s_{2k+2} calculated from the second equation into the first equation we obtain a relation between three successive s_m

$$
s_{2k-1} = \frac{Q_k}{P_k} s_{2k} + \frac{P_k\overline{P_k} - Q_k\overline{Q_k}}{P_k} s_{2k+1}\,. \tag{4.3}
$$

After defining the new material functions

$$
\begin{aligned}
&\alpha_1 = \frac{Q_1}{P_1}, \quad \alpha_2 = \overline{Q_1}, \quad \alpha_3 = \frac{Q_2}{P_2}, \ldots, \\
&\alpha_{2k-1} = \frac{Q_{2k}}{P_{2k}}, \quad \alpha_{2k} = \overline{Q_k}, \ldots, \quad \alpha_{2N} = \overline{Q_n}\,,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&\beta_1 = \frac{P_1\overline{P_1} - Q_1\overline{Q_1}}{P_1}, \quad \beta_2 = P_1, \ldots, \\
&\beta_{2k-1} = \frac{P_k\overline{P_k} - Q_k\overline{Q_k}}{P_k}, \quad \beta_{2k} = P_k, \ldots, \beta_{2N} = P_N\,,
\end{aligned} \tag{4.5}
$$

the relations between the variables s_m may be written in the simple form

$$
\begin{aligned}
s_1 &= \alpha_1 s_2 + \beta_1 s_3 \\
s_2 &= \alpha_2 s_3 + \beta_2 s_4 \\
s_3 &= \alpha_3 s_4 + \beta_3 s_5 \\
&\dots \\
s_k &= \alpha_k s_{k+1} + \beta_k s_{k+2} \\
&\dots \\
s_{2N} &= \alpha_{2N} s_{2N+1} + \beta_{2N} s_{2N+2} .
\end{aligned} \tag{4.6}
$$

Note that the last two relations of (4.1) give $s_{2N+1} = s_{2N+2} = 1$. Divide the first equation of (4.6) by s_2. There follows

$$
\frac{s_1}{s_2} = \alpha_1 + \beta_1 \frac{1}{\frac{s_2}{s_3}} . \tag{4.7}
$$

Calculate the ratio s_2/s_3 from the second equation of (4.6) and substitute into (4.7). Repeating this procedure we obtain the finite continued fraction

$$
\frac{s_1}{s_2} = \alpha_1 + \beta_1 \cfrac{1}{\alpha_2 + \beta_2 \cfrac{1}{\alpha_3 + \beta_3 \cfrac{1}{\alpha_4 + \ddots \cfrac{}{\alpha_{2N-1} + \beta_{2N-1} \cfrac{1}{\alpha_{2N} + \beta_{2N}}}}}} . \tag{4.8}
$$

For s_m/s_{m+1} we obtain the lower part of the above fraction starting with α_m. In general for $1 \leq m \leq 2N$ there is

$$
\begin{aligned}
&\frac{s_m}{s_{m+1}} = v_m , \\
&v_m = \alpha_m + \beta_m \cfrac{1}{\alpha_{m+1} + \beta_{m+1} \cfrac{1}{\alpha_{m+2} + \beta_{m+2} \ddots \cfrac{}{\beta_{2N-1} \cfrac{1}{\alpha_{2N} + \beta_{2N}}}}} .
\end{aligned} \tag{4.9}
$$

Since $\alpha_{2N} + \beta_{2N} = P_N + \overline{Q_N}$, the above formula allows to calculate v_m. In the previous chapter we demonstrated that it is essential for the analysis to calculate s_1. In accord with the above notation, s_1 equals the product

$$
s_1 = v_1 v_2 v_3 \dots v_{2N} . \tag{4.10}
$$

In the case of large N it is easier to handle the equivalent logarithmic form of (4.10)

$$
\log s_1 = \log v_1 + \log v_2 + \log v_3 + \dots + \log v_{2N} . \tag{4.11}
$$

5. Random distribution of layers

Above were discussed the chains with a distribution of cells or layers specified in advance. In this chapter we consider the chain consisting of two different types of cells. The distribution of cells is not deterministic, in particular it is not periodic. The chain consists of N cells. Therefore we face the random distribution of two different types of cells over N places. The investigations are based on the excellent paper by H. Smith, who considered a chain of interacting masses, [9]. Denote the two different kinds of cells by α, β. They are distributed over N places, e.g. $\{\alpha, \alpha, \beta, \alpha, \beta, \ldots, \beta, \beta, \alpha\}$. The total number of cells is fixed and equal N; the particular cells may be repeated several times at different places. The transfer matrices for the two kinds of cells are, respectively,

$$M_\alpha = \begin{bmatrix} P_\alpha & Q_\alpha \\ R_\alpha & S_\alpha \end{bmatrix}, \qquad M_\beta = \begin{bmatrix} P_\beta & Q_\beta \\ R_\beta & S_\beta \end{bmatrix}. \tag{5.1}$$

The subscripts in the above relations are not the numbers of cells, as in (1.4), but the types of cells. The probability that the place k is occupied by the cell of type α will be denoted by $p_{k\alpha}$, and the probability, that the place k is occupied by the cell of type β will be denoted by $p_{k\beta}$. Obviously, $p_{k\alpha} + p_{k\beta} = 1$. In the special case of homogeneous distribution, the probability is independent of k and the index k may be omitted, e.g. $p_{k\alpha} = p_\alpha$.

Consider a stastically large number of chains of N cells with random distribution of the above two types of cells. The number of cells of type α differs from chain to chain. The number of cells of type β equals N minus the number of cells of type α. Calculate for each chain the N complex-valued vectors

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix} \cdots \begin{bmatrix} P_3 & Q_3 \\ R_3 & S_3 \end{bmatrix} \begin{bmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{5.2}$$

$k = 1, 2, 3, \ldots, N$. Each of the transfer matrices at the right-hand side equals either M_α or M_β. For each chain the vector $[A_k, B_k]$ is, in general, a different function of frequency ω. Note that the vector $[A_1, B_1] = [p, -p]$, $p = \text{const}$ satisfies the boundary condition for one end of the chain, since then $A_1 + B_1 = 0$, and the displacement of that end equals zero. Without loosing generality we may take $p = 1$. This was the motivation to take $[A_1, B_1] = 1, -1$ at the right-hand side of (5.2). The resulting vector at the left-hand side satisfies the boundary condition for the other end, if the sum of its components equals zero, $A_{N+1} + B_{N+1} = 0$, since then the displacement of that end (end of the N-th cell or beginning of the $(N + 1)$-th cell) equals zero.

Define the complex-valued parameter

$$z_k = A_k / B_k . \tag{5.3}$$

The complex number z_k uniquely defines a point z_k on the $x + iy$ plane. In accord with (1.4), (1.5), the parameters z_k and z_{k+1} are connected by the relations

$$z_{k+1} = \frac{P_k z_k + Q_k}{R_k z_k + S_k}, \qquad z_k = \frac{S_k z_{k+1} - Q_k}{-R_k z_{k+1} + P_k}. \tag{5.4}$$

The second relation is the inverse of the first one.

Concentrate the attention on a fixed place k. In the situation considered in this chapter, the components P_k, Q_k, R_k, S_k of the transfer matrix M_k are equal either to P_α, Q_α, R_α, S_α if the k-th cell is a cell of type α, or to P_β, Q_β, R_β, S_β, if the k-cell

is a cell of type β. For each chain, in general, z_k has a different value, since the vector $[A_k B_k]$ is different for each chain. Define an arbitrary fixed surface dv_k on the $x + iy$ plane. For some chains the point z_k is situated inside dv_k, for other — outside dv_k. The area of dv_k will be denoted by dV_k. Define the probability distribution of the points z_k over the plane x, y

$$w_k(z_k)dV_k = \frac{\text{number of chains for which } z_k \in dv_k}{\text{total number of chains}}. \tag{5.5}$$

In order to derive the equations for $w_k(z_k)$ assume, for the time being, that in all chains the k-th place is occupied by a cell of the same type. Assume first that this place is occupied by the cell of the type α, $M_k = M_\alpha$. Then, in accord with Eqs. (5.4), z_{k+1} is a definite function of z_k

$$z_{k+1} = \frac{P_\alpha z_k + Q_\alpha}{R_\alpha z_k + S_\alpha}, \qquad z_k = \frac{S_\alpha z_{k+1} - Q_\alpha}{-R_\alpha z_{k+1} + P_\alpha}. \tag{5.6}$$

Denote by dv_{k+1} the surface into which the surface dv_k is transformed by the above function. The ratio of their areas is the modulus of the derivative dz_{k+1}/dz_k. It may be expressed either by variable z_k or by z_{k+1}

$$\begin{aligned} \frac{dV_{k+1}}{dV_k} &= \left| \frac{P_\alpha S_\alpha - Q_\alpha R_\alpha}{(R_\alpha z_k + S_\alpha)^2} \right|, \\ \frac{dV_k}{dV_{k+1}} &= \left| \frac{P_\alpha S_\alpha - Q_\alpha R_\alpha}{(-R_\alpha z_{k+1} + P_\alpha)^2} \right|. \end{aligned} \tag{5.7}$$

In the special case discussed now, when in each chain the k-th cell is of type α, the transformation is not random, but deterministic. If the point z_k is situated inside (outside) dv_k, then the point z_{k+1} is situated inside (outside) dv_{k+1}. Therefore the probability of finding the point z_{k+1} inside the surface dv_{k+1} equals the probability of finding the point z_k inside the surface dv_k. Since the total probability equals the product of probability density and the corresponding areas, the relation holds

$$w_{k+1}(z_{k+1})dV_{k+1} = w_k(z_k)dV_k.$$

Since z_k and z_{k+1} are related by Eqs. (5.6), and dV_k and dV_{k+1} by Eqs. (5.7), there is

$$w_{k+1}(z_{k+1}) = \left| \frac{P_\alpha S_\alpha - Q_\alpha R_\alpha}{(-R_\alpha z_{k+1} + P_\alpha)^2} \right| w_k\left(\frac{S_\alpha z_{k+1} - Q_\alpha}{-R_\alpha z_{k+1} + P_\alpha} \right). \tag{5.8}$$

Note that z_{k+1} is in fact an independent variable. Further we omit the subscript $k + 1$ and use z instead of z_{k+1}. The relation between the probability densities w_k and w_{k+1} has the form

$$w_{k+1}(z) = \left| \frac{P_\alpha S_\alpha - Q_\alpha R_\alpha}{(-R_\alpha z + P_\alpha)^2} \right| w_k\left(\frac{S_\alpha z - Q_\alpha}{-R_\alpha z + P_\alpha} \right).$$

The above calculations were performed under the assumption that at the k-th place a cell of type α was situated. If at the k-th place the cell of type β was situated, analogous formula would be obtained, namely

$$w_{k+1}(z) = \left| \frac{P_\beta S_\beta - Q_\beta R_\beta}{(-R_\beta z + P_\beta)^2} \right| w_k\left(\frac{S_\beta z - Q_\beta}{-R_\beta z + P_\beta} \right).$$

Actually the k-th place may be occupied either by M_α with probability $p_{k\alpha}$, or by M_β with probability $p_{k\beta}, p_{k\alpha} + p_{k\beta} = 1$. The actual relation between $w_{k+1}(z)$ and $w_k(z)$

reads therefore

$$w_{k+1}(z) = p_{k\alpha}\left|\frac{P_\alpha S_\alpha - Q_\alpha R_\alpha}{(-R_\alpha z + P_\alpha)^2}\right| w_k\left(\frac{S_\alpha z - Q_\alpha}{-R_\alpha z + P_\alpha}\right) + p_{k\beta}\left|\frac{P_\beta S_\beta - Q_\beta R_\beta}{(-R_\beta z + P_\beta)^2}\right| w_k\left(\frac{S_\beta z - Q_\beta}{-R_\beta z + P_\beta}\right). \tag{5.9}$$

Since in each realization there is $A_1/B_1 = -1$, cf. (5.2), therefore $w_1(z)$ is different from zero only if $z = -1$. Taking into account that the integral over the whole plane must be equal 1, we have

$$w_1(z) = \delta(\operatorname{Re} z + 1)\delta(\operatorname{Im} z), \tag{5.10}$$

where δ is Kronecker's delta. Formula (5.9), together with Eq. (5.10), completely determine the probability distribution $w_k(z)$ for each k. As it was stressed above, if $A_N/B_N = -1$, then the boundary condition at the end $k = N$ of the chain is satisfied. The probability distribution of $w_1(-1)$ is of particular interest, since this probability corresponds to the case of the boundary condition satisfied at the end $k = 1$ of the chain.

The argument z for which $w_1(z)$ is different from zero is the support p_1 of w_1. The support of w_1 is the point $p_1 = -1$. From (5.9) it follows that the support p_2 of $w_2(z)$ consists of the two points $p_{2(1)} = (S_\alpha p_1 - Q_\alpha)/(-R_\alpha p_1 + p_\alpha)$ and $p_{2(2)} = (S_\beta p_1 - Q_\beta)/(-R_\beta p_1 + p_\beta)$. The support of $w(z)$ is, in turn, the set of four points p_3 into which the two points $p_{2(1)}, p_{2(2)}$ are transformed by the functions (5.4).

Concentrate attention on one particular realisation. The transition matrix M_k with components p_k, Q_k, R_k, S_k is known. The above discussion proves that the supports for successive cells are connected by the relation

$$p_{k+1} = \frac{S_k p_k - Q_k}{-R_k p_k + p_k}. \tag{5.11}$$

Multiply both sides of the above relation by the conjugate expressions to obtain

$$p_{k+1}\overline{p_{k+1}} = \frac{P_k S_k p_k \overline{p}_k - R_k S_k p_k - p_k Q_k \overline{p}_k + Q_k R_k}{R_k Q_k p_k \overline{p}_k - R_k S_k p_k - P_k Q_k \overline{p}_k + P_k S_k}.$$

Take into account that $p_1 = (-1, 0)$, therefore the modulus of the complex number p_1 equals 1, $|p_1| = 1$. The numerator for $k = 1$ equals the denominator, and the conclusion follows that $p_2\overline{p_2} = 1$. Consequently $p_3\overline{p}_3 = p_4\overline{p}_4 = \ldots = p_{k+1}\overline{p_{k+1}} = 1$, and for each k there is

$$|p_k| = 1. \tag{5.12}$$

Therefore all points of the support are situated on the circle of a unit radius.

Now the chain consisting of two different kinds of randomly distributed layers will be considered. The layers of type α are characterized by the propagation speed c_α, mass density ϱ_α and thickness h_α. The layers of type β are characterized by the corresponding data with subscript β. It would be very inconvenient for the computations if the transfer matrix (1.4) for the layer k depended not only on the data for the layer k, but also upon the mass density and propagation speed of the neighbouring layer $k + 1$. In order to remove this inconveniency, simple additional calculations are necessary.

Let us add between every two layers, a virtual layer characterized by propagation speed c_v, density ϱ_v and zero thickness. Equations (1.3)–(1.8) prove that such layers do not influence the dynamics of the chain. Now the virtual layer and the layer of type α

constitute a cell of type α. The transfer matrix M_α for the cell of type α is a product of the transfer matrix for the added virtual layer and the transfer matrix for the layer of type α

$$M_\alpha = \frac{1}{4}\begin{bmatrix} \left(1+\frac{s_\alpha}{s_v}\right)e^{-i\omega\frac{h_\alpha}{c_\alpha}} & \left(1-\frac{s_\alpha}{s_v}\right)e^{i\omega\frac{h_\alpha}{c_\alpha}} \\ \left(1-\frac{s_\alpha}{s_v}\right)e^{-i\omega\frac{h_\alpha}{c_\alpha}} & \left(1+\frac{s_\alpha}{s_v}\right)e^{i\omega\frac{h_\alpha}{c_\alpha}} \end{bmatrix}\begin{bmatrix} 1+\frac{s_v}{s_\alpha} & 1-\frac{s_v}{s_\alpha} \\ 1-\frac{s_v}{s_\alpha} & 1+\frac{s_v}{s_\alpha} \end{bmatrix}, \tag{5.13}$$

where $s_\alpha = \varrho_\alpha c_\alpha$. After multiplication, the following components of the transfer matrix for the cell of type α are obtained:

$$\begin{aligned} (M_\alpha)_{11} &= \frac{1}{4}\left(2+\frac{s_\alpha}{s_v}+\frac{s_v}{s_\alpha}\right)e^{-i\omega\frac{h_\alpha}{c_\alpha}} + \frac{1}{4}\left(2-\frac{s_\alpha}{s_v}-\frac{s_v}{s_\alpha}\right)e^{i\omega\frac{h_\alpha}{c_\alpha}}, \\ (M_\alpha)_{12} &= \frac{1}{4}\left(\frac{s_\alpha}{s_v}-\frac{s_v}{s_\alpha}\right)e^{-i\omega\frac{h_\alpha}{c_\alpha}} + \frac{1}{4}\left(-\frac{s_\alpha}{s_v}+\frac{s_v}{s_\alpha}\right)e^{i\omega\frac{h_\alpha}{c_\alpha}}. \end{aligned} \tag{5.14}$$

The remaining two components of M_α are defined by the w-symmetry. In the notation used in Eq. (5.11) we have

$$P_k = (M_\alpha)_{11}, \quad Q_k = (M_\alpha)_{12}, \quad R_k = \overline{(M_\alpha)_{12}}, \quad S_k = \overline{(M_\alpha)_{11}},$$

if the k-th cell is of type α. Note that the transfer matrix for the cell of type α defined above does not depend on the propagation speed and density of the neighbouring cell which is of type α or type β. The components of the transfer matrix for the cell of type β may be obtained by replacing in Eq. (5.14) the subscript α by β. The transfer matrices M_α and M_β may now be shuffled to obtain the transfer matrix for the whole chain.

Define the set of real parameters φ_k by the relation

$$\exp(i\varphi_k) = p_k. \tag{5.15}$$

The discrete set of φ_k is monotonically increasing. Since φ_k is a discrete set and in the definition (5.15) each branch of the inverse trigonometric functions is allowed, special care must be taken at passing to another branch of the function. The proof of monotonicity is based on the introduction of the continuous variable h

$$h = \nu h_k, \quad 0 \le \nu \le 1,$$

and replacement in Eq. (1.5), (1.6) of the discrete h_k by h. Now the discrete M_k, M_{k+1} are replaced by continuous $M(\nu)$ with $M(0) = M_k, M(1) = M_{k+1}$. The parameter φ_k is replaced by the continuous parameter $\varphi(\nu)$, and the discrete set p_k replaced by continuous $p(\nu)$. Only the values for $h = h_k$ are essential for the dynamics of the chain. In accord with Eqs. (5.12) and (5.13) we write

$$\exp(i\varphi(\nu)) = \frac{S_k(\nu)p_k - Q_k(\nu)}{-R_k(\nu)p_k + p_k(\nu)}. \tag{5.16}$$

Values of these functions (corresponding either to α, or β) are here not essential. Note that $\varphi(0) = \varphi_k$, $\varphi(1) = \varphi_{k+1}$. It may be proved that the derivative $d\varphi/d\nu$ is positive and φ is a monotonically increasing function of ν. We do not quote here the proof since the expression for $d\varphi/d\nu$ is rather long, and the proof takes a lot of space. The full proof will be published in [11]. The important final result is that the set of real numbers φ_k is ordered

$$\varphi_1 < \varphi_2 < \varphi_3 < \ldots. \tag{5.17}$$

Let us now pass to the numerical example. Since we are here interested in qualitative results only, we prefer to waste no space for definition of the dimensionless variables. Assume in a certain fixed coordinate system

$$(5.18)\quad c_\alpha = 1, \quad c_\beta = 2, \quad c_v = 4, \quad \varrho_\alpha = \varrho_\beta = 1, \quad h_\alpha = h_\beta = 1, \quad \omega = 1.$$

The calculations will be performed for the chain consisting of $N = 200$ cells. The probabilities $p_{k\alpha}$ and $p_{k\beta}$ are assumed to be independent of k and of the type of the cell, therefore $p_{k\alpha} = p_{k\beta} = 1/2$. In accord with Eqs. (5.9) and (5.10), the function w_N is zero everywhere, except for the 2^{200} separate points, into which the point $z = -1$ is transformed by multiple application of Eqs. (5.4). It was proved in Eq. (5.12) that all these points are situated on the circle of radius 1, centered at the point $(0, 0)$.

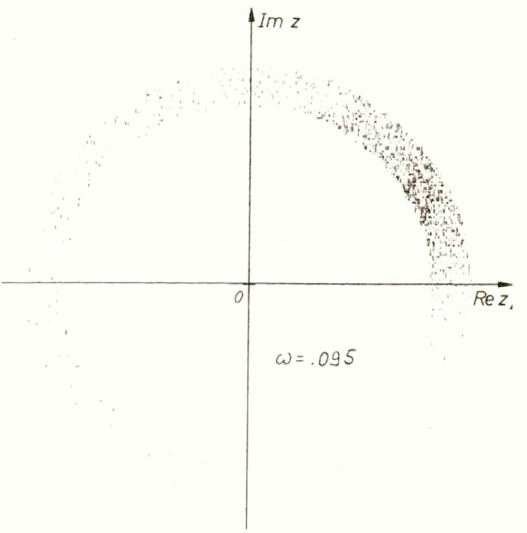

FIG. 2.

Obviously it is possible to plot only a small fraction of the 2^{200} points constituting the support. Figure 2 contains about 4000 of them for the frequency $\omega = .095$. In order to obtain a more clear picture, they were marked not on the circle of radius 1, but at a small random distance from the unit circle. It is seen that, for the above data, they are distributed non-homogeneously. Most of them are situated near the point (1,0), slightly above the real axis. Figure 3 gives the points for $\omega = .20$. Again, the regions of larger density of points are seen, but the distribution is more uniform. For very small frequencies the regions of large density of points are well defined. For larger frequencies the distribution is more uniform.

Let us now make an analysis of the above results. It is not intended to support it by rigorous proofs, but let us treat it just as an explanation of the results. Assume first $p_{k\alpha} = 1$, $p_{k\beta} = 0$. All cells are now of type α, and all subsequent z_k are deterministic. For $\omega = .095$ the first 30 points z_k are shown on the diagram in Fig. 4. The smaller is the frequency ω, the smaller will be the distance between the points. Denote the angle at which the point z_k is situated by $\varphi_{\alpha k}$, $\varphi_{\alpha k} \geq 0$. To the point $(-1, 0)$ corresponds $\varphi_{\alpha 0} = 0$. It has been proved above that angle $\varphi_{\alpha k}$ is a monotonically increasing function

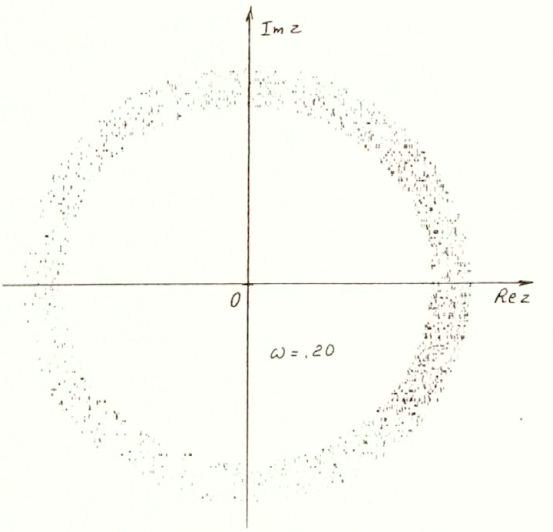

FIG. 3.

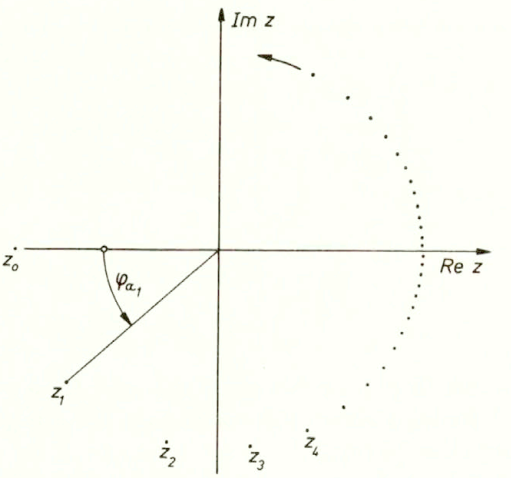

FIG. 4.

of k; a diagram similar to that shown at Fig. 4 is obtained, if all cells are of type β, Fig. 5. The corresponding angles are $\varphi_{\beta k}$. It is seen that for each k, we obtain $\varphi_{\alpha k} > \varphi_{\beta k}$. In this case the distances between the successive points are smaller than those at Fig. 4.

Produce now the diagram for one random distribution of cells of type α and type β. Denote the corresponding angles by φ_k. They are monotonically increasing functions of k, Fig. 6. Obviously, the difference $\varphi_k - \varphi_{k-1}$ depends only on the type of the k-th cell. If this is the cell of type α, then this difference is approximately equal to the difference of angles $\varphi_{\alpha k}$ at Fig. 4, for this particular value of φ_k (not for the number k, but for the value of φ_k). This differences are exactly equal, if accidentally for some m equality $\varphi_{\alpha k} = \varphi_m$ holds true. Analogous equality holds for the cell of type β. The inequalities

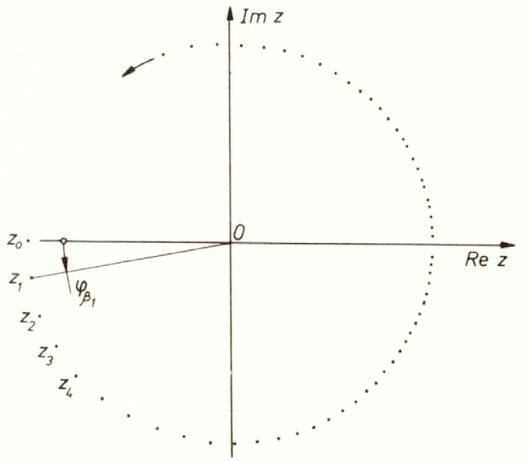

FIG. 5.

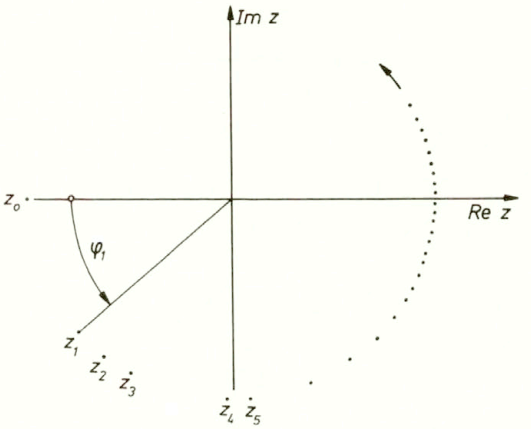

FIG. 6.

follow,

$$\varphi_{\alpha k} \geq \varphi_k \geq \varphi_{\beta k}. \tag{5.19}$$

Denote

$$\begin{aligned} \psi_{\alpha N} &= \varphi_{\alpha N} \bmod 2\pi, \qquad \psi_{\beta N} = \varphi_{\beta N} \bmod 2\pi, \\ 0 \leq \psi_{\alpha N} &< 2\pi, \qquad 0 \leq \psi_{\beta N} < 2\pi. \end{aligned} \tag{5.20}$$

If the frequency ω is small enough to satisfy the inequalities $\varphi_{\alpha N} < 2\pi$, $\varphi_{\beta N} < 2\pi$, then for all realisations the points z_N are situated in the sector $\psi_{\beta N} < \psi_N < \psi_{\alpha N}$. For such ω no point is situated in the sector $0 < \psi_N < \psi_{\beta N}$ or in the sector $\psi_{\alpha N} < \psi_N < 2\pi$. In this case the spectrum of points ψ_N is perfectly localized in one sector between ψ_β and ψ_α. If, however, $\varphi_{\alpha N} > \varphi_{\beta N} + 2\pi$, then the regions overlap and each ψ_N is possible. The points z_N may than be situated at any point of the circle.

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