THE ADAPTIVE NEM – DELAUNAY ELEMENTS

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1. Abstract

Meshless methods utilized in numerical solution of boundary problems have recently been widely investigated by many authors. The explicit connectivity between nodes does not exist for such methods. Therefore due to computational efforts in remeshing steps for FEM, the meshless methods seem to be an attractive alternative for adaptive process in computational mechanics. The Natural Element Method (NEM) proposed by Traversoni (1994), Brown and Sambridge (1995) [1] is treated as a meshless method. The shape functions for the NEM are constructed with help of the Voronoi diagram, which describes so called natural neighbours for each node P_i placed in the domain Ω. There are two main kinds of approximation for the NEM, the “non-sibsonian” with the Laplace coordinate [2] built on basis of the first order Voronoi diagram (1), and approximation with the Sibson functions [3] constructed with the help of locally created the second order Voronoi diagram (2).

(1) $$V_i = \left\{ \mathbf{x} \in R^n \;:\; d(\mathbf{x}, P_i) < d(\mathbf{x}, P_k) \;\forall\, i \neq k,\;\; \forall\, \mathbf{x} \in \Omega \right\}$$

(2) $$V_{ij} = \left\{ \mathbf{x} \in R^n \;:\; d(\mathbf{x}, P_i) < d(\mathbf{x}, P_j) < d(\mathbf{x}, P_k) \;\forall\, i \neq j \neq k,\;\; \forall\, \mathbf{x} \in \Omega \right\}$$

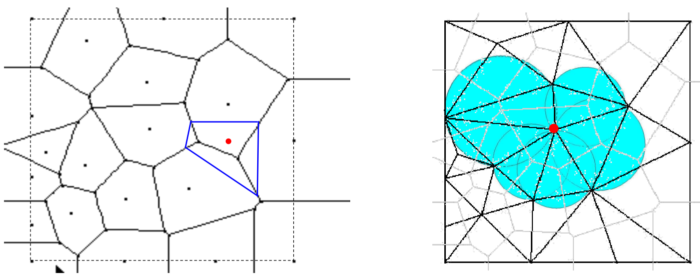

Fig 1) a) The Voronoi diagram (first order, and locally second order),
b) Delaunay triangles and support domain for selected node.

For both approximations the support domain of the shape function for the node P_i is the union of all the circumcircles about that node. In this work the Delaunay tessellation dual to the Voronoi diagram is utilized. The global stiffness matrix is obtained by summing over each Delaunay triangle instead the triangularized Voronoi region as in [4]. For each Delaunay triangle the proper stabilized numerical integration [5] is applied, i.e. 1 or 2-points Gauss quadrature along each edge of the triangle. For such an integration only values of the shape function are required, not the derivatives as usual. The error in energy norm (3) (or norm) of the solution is calculated by the local projection of the solution values over the Delaunay triangles.

(3) $$Err = \left\| \mathbf{u}^h - \bar{\mathbf{u}} \right\| \qquad \textit{where} \qquad \bar{\mathbf{u}} = \sum_k B_k\, \mathbf{u}^h(\mathbf{x}_k)$$

$\boldsymbol{u}^h$ is the NEM solution in Ω and B_k – bilinear shape functions for 3-node triangle.

In the adaptive procedure the new size of the Delaunay element is calculated from

$$h_i^{new} = h_i \left(\eta \frac{\left\| \mathbf{u}^h \right\|}{\left\| \mathbf{u}^h - \bar{\mathbf{u}} \right\|_i} \frac{\sqrt{A_i}}{\sqrt{A}} \right)^{1/p} \tag{4}$$

and new nodes are placed. Firstly at the edges of the Delaunay triangle and then if required in its interior.

For the proposed routine of the adaptive process with NEM the results (the energy norm) for the 2D linear elasticity problem (plane stress problem) (fig.2) and for the assumed permissible energy error level η=1% are presented in tab.1. The “real” energy ($\| \mathbf{u}^R \|$ = **3.48077**) was calculated for the uniformly divided domain to 400 elements (8-node finite elements) in AnSys system. The initial and the ultimate nodes location for adaptive process are shown in fig. (2).

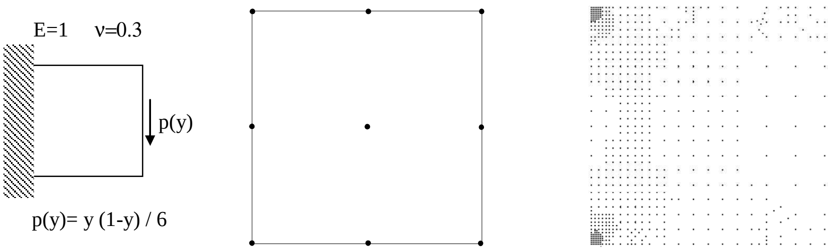

Fig 2) The test problem. The initial and the final set of nodes

Nodes	$\| \mathbf{u}^h \|$	*Err*	*Err* / $\| \mathbf{u}^h \|$ [%]	*Err* / $\| \mathbf{u}^R \|$ [%]
9	4.9287	2.8049 E 00	56.91	80.58
25	3.8187	6.7136 E-01	17.58	19.29
81	3.5844	2.0144 E-01	5.62	5.79
224	3.5184	6.9518 E-02	1,98	2.00
444	3.5045	3.7097 E-02	1.06	1.07
636	3.5001	2.7658 E-02	0.79	0.79

Table.1. Result for unity square test problem.

2. References

[1] Sukumar N., Moran B., Beltytschko T.(1998). The natural element method in solid mechanics. *Int. J. Numer. Meth. Engng.*, **43**, 839-887.

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