EIGENSPECTRA AND ORDERS OF STRESS SINGULARITY AT A MODE I CRACK IN A POWER-LAW MEDIUM

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1. Introduction

Solutions for crack-tip fields are very important in understanding the mechanisms of crack initiation and propagation in elastic-plastic and creeping materials. The stress field in the vicinity of the crack tip in power-law materials (power-law hardening materials, power-law creeping materials) is widely discussed in literature. The stress singularity for a crack in a homogeneous power-hardening material with hardening exponent n was first studied by Hutchinson [1], Rice and Rosengren [2]. In [1] the problem of plastic stress singularity is reduced to a nonlinear eigenvalue problem and the shooting method is used to solve the homogeneous differential equation obtained in the analysis. It should be noted that for some time multi-term asymptotic solutions with the well-known HRR-field as the leading order term of the asymptotic expansion aroused considerable interest of many researchers. Nowadays the whole eigenspectrum and orders of stress singularity at the crack tip for a power-law medium are of prevailing interest. The present study offers a technique developed in the perturbation theory for analysis of nonlinear eigenvalue problems arising from fracture mechanics.

2. Mode I crack. Basic equations

Let us consider eigenspectra and orders of singularity of the stress field near a mode I crack tip in a power-law material. The power-law constitutive relations $\varepsilon_{ij} = (3/2)B\sigma_e^{n-1}s_{ij}$, where ε_{ij} is the strain, s_{ij} is the stress deviator, σ_e is the Mises equivalent stress, B, n are material constants, for the plane strain condition are described by $\varepsilon_{rr} = -\varepsilon_{\theta\theta} = 3B\sigma_e^{n-1}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)/4,\ \varepsilon_{r\theta} = 3B\sigma_e^{n-1}\sigma_{r\theta}/2$, where the equivalent stress is expressed by $\sigma_e^2 = 3\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2/4 + 3\sigma_{r\theta}^2$.

In analyzing the asymptotic behaviour of the stress field near the crack tip the Airy stress potential can be presented in the following form $F(r,\theta) = r^{\lambda+1}f(\theta)$. Using the constitutive equations and the compatibility equation one finds

$$
\begin{aligned}
& f_e^2 f^{IV}\left\{(n-1)\left[(1-\lambda^2)f + f''\right]^2 + f_e^2\right\} + f_e^4(1-\lambda^2)f'' + (n-1)(n-3)\times \\
& \times\left\{\left[(1-\lambda^2)f + f''\right]\left[(1-\lambda^2)f' + f'''\right] + 4\lambda^2 f'f''\right\}^2\left[(1-\lambda^2)f + f''\right] + \\
& +(n-1)f_e^2\left\{\left[(1-\lambda^2)f' + f'''\right]^2 + \left[(1-\lambda^2)f + f''\right](1-\lambda^2)f'' + 4\lambda^2\left(f''^2 + f'f'''\right)\right\}\times \\
& \times\left[(1-\lambda^2)f + f''\right] + 2(n-1)f_e^2\left\{\left[(1-\lambda^2)f + f''\right]\left[(1-\lambda^2)f' + f'''\right] + 4\lambda^2 f'f''\right\}\times \\
& \times\left[(1-\lambda^2)f' + f'''\right] + C_1(n-1)f_e^2\left\{\left[(1-\lambda^2)f + f''\right]\left[(1-\lambda^2)f' + f'''\right] + 4\lambda^2 f'f''\right\}f' + \\
& +C_1 f_e^4 f'' - C_2 f_e^4\left[(1-\lambda^2)f + f''\right] = 0,
\end{aligned}
\tag{1}
$$

where $f_e^2 = \left[(1-\lambda^2)f + f''\right]^2 + 4\lambda^2 f'^2,\ C_1 = 4\lambda\left[(\lambda-1)n + 1\right],\ C_2 = (\lambda-1)n\left[(\lambda-1)n + 2\right].$

The fourth order nonlinear ordinary differential equation (1) with the boundary conditions $f(\theta = \pm\pi) = 0,\ f'(\theta = \pm\pi) = 0$ defines a nonlinear eigenvalue problem in which the constant λ is the eigenvalue and $f(\theta)$ is the corresponding eigenfunction. The direct integration of the differential equation (1) is generally realized by the Runge-Kutta method in conjunction with the shooting method. Obviously, the eigenvalue λ and the initial value $f''(\theta = -\pi)$ are coupled with each other in general, and they have to be searched simultaneously. Only in some special cases one can assign a certain λ a prior through additional physical presumptions. Now the whole eigenspectrum and orders of stress singularity at the crack tip are of interest. The whole eigenspectrum stipulates the possible stress distributions in the neighbourhood of the crack tip. The purpose of this study is to obtain the whole eigenspectrum for the stress field near a mode I crack in a power-law material.

3. The perturbation theory approach

The underlying idea of the method is to consider the expansion representing the eigenvalue λ of the nonlinear eigenvalue problem formulated for an arbitrary exponent n to be a sum of the eigenvalue λ_0 corresponding to the "undisturbed" linear problem $(n = 1)$ and a small parameter ε which quantitatively describes the nearness of the eigenvalues: $\lambda = \lambda_0 + \varepsilon$. The exponent n and the stress function $f(\theta)$ can be presented as formal series with respect to $\varepsilon : n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \dots$, $f(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \dots$, where $f_0(\theta)$ denotes the solution of the linear problem $(n = 1)$. Introducing the asymptotic expansions for λ, n and $f(\theta)$ into (1) and collecting terms of equal power in ε, the set of linear differential equations is obtained. Thus, the boundary value problems for the nonhomogeneous fourth order linear differential equations with respect to $f_i(\theta)$ are formulated. It is known that if the boundary value problem for the homogeneous differential equation has a nontrivial solution then there can exist no solution of the corresponding nonhomogeneous differential equation unless the solvability condition is realized.

Analysis of the solvability condition for the boundary value problems obtained results in the three-term asymptotic expansions of the exponent $n : n = 1 - 2\varepsilon/(\lambda_0 - 1) + \varepsilon^2 n_2 + O(\varepsilon^3)$, where for $\lambda_0 \leqslant -\frac{3}{2}$ and for $\lambda_0 \geqslant \frac{3}{2}$ $\quad n_2 = -\frac{\lambda_0^5 - 2\lambda_0^4 - 7\lambda_0^3 + 11\lambda_0^2 + 4\lambda_0 - 5 - (\lambda_0^2 - 1)sgn(\lambda_0)}{(\lambda_0 + 1)(\lambda_0 - 1)^4}$.

For $\lambda_0 = 1/2$ corresponding to the classical HRR-problem the following closed form solution

$$n_k = -\frac{(-1)^k}{(\lambda_0 - 1)^{k+1}}, \quad n = 1 - \frac{1}{\lambda_0 - 1} \sum_{k=1}^{\infty} \left(-\frac{\varepsilon}{\lambda_0 - 1} \right)^k = -\frac{\lambda}{\lambda - 1}, \quad \lambda = \frac{n}{n+1} \tag{2}$$

is found. Hence, the well-known formula (2) connecting the hardening exponent n and the eigenvalue λ for the HRR-problem is derived.

4. Conclusions

Using the perturbation method the whole set of eigenvalues for a mode I crack tip in a power-law material is determined. The three-term asymptotic expansion for the exponent n allowing to find the eigenvalue via $\lambda = \lambda_0 + \varepsilon$ for the nonlinear eigenvalue problem is obtained.

The relative error of the three-term asymptotic expansion for a crack in the power-law material with $n = 2$ to the exact HRR-solution is 2%. The results obtained for $\lambda_0 = -1/2$ were compared with those found for the same problem by the Runge-Kutta method in conjunction with the shooting method. The comparison of the eigenvalues for $n = 2$ calculated by the three-term asymptotic expansion and by the numerical scheme $\lambda = -0.9801$ and $\lambda = -1.000$ shows the good agreement. The eigenvalues for $n = 3$ given by the four-term asymptotic expansion for $\lambda_0 = -1/2$ and by the Runge-Kutta method are $\lambda = -0.7716$ and $\lambda = -0.7755$. Consequently, a quite satisfactory solution is obtained by taking the asymptotic expansion achieved.

5. References

[1] J. W. Hutchinson (1968). Singular behavior at the end of tensile crack in a hardening material, *J. Mech. Phys. Solids* **16**, 13-31.

[2] J. R. Rice and G. F. Rosengren (1968). Plane strain deformation near a crack tip in a power-law hardening material, *J. Mech. Phys. Solids.,* **16**, 1-12.