ON NATURAL STRAIN MEASURES OF THE NON-LINEAR MICROPOLAR CONTINUUM

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In the micropolar (or the Cosserat type) continuum each material particle can translate and independently rotate, that is it has six degrees of freedom of a rigid body, [1,2]. The micropolar continuum is used nowadays with success to model the behaviour, for example, of granular media, composites, polycrystalline solids, liquid crystals, magnetic fluids, nano-materials as well as thin bodies: rods, plated and shells.

Two strain measures of the micropolar continuum, called usually the stretch and wryness tensors, were originally proposed by Cosserats [1] in an awkward, now hardly understandable notation. In the contemporary literature the stretch and wryness tensors are defined in different ways using, for example: a) components in two different curvilinear coordinate systems associated with the undeformed (reference) and deformed (actual) placements of the body, b) components in the convected coordinate systems, c) Lagrangian or Eulerian descriptions, d) different representations of the rotation group SO(3), e) formally different definitions of gradient and divergence operators, f) different sign conventions, and f) requiring or not the measures to vanish in the undeformed placement of the body. As a result, definitions of the strain measures for the micropolar continuum used in different papers are in many cases not equivalent.

In this report we discuss three different methods of defining the strain measures of the non-linear micropolar continuum: 1) by a direct geometric approach, 2) introducing the strain measures as the fields work-conjugate to the respective internal stress and couple-stress fields, and 3) applying the principle of material frame-indifference to the polar-elastic strain energy density. All the three methods lead to the same definitions of the stretch and wryness tensors. Our strain measures expressed in the coordinate-free notation are of the relative type, for they are required to vanish in the undeformed placement of the body.

1. Geometric approach

In the undeformed placement the material particle of the micropolar body is given through the position vector $\mathbf{x} \in E$ and three orthonormal directors $\mathbf{h}_a \in E$, $a = 1,2,3$, where E is the 3D vector space. In the actual placement the same material particle becomes described by the position vector $\mathbf{y} \in E$ and three orthonormal directors $\mathbf{d}_a \in E$. Thus, the finite displacement of the body is described by

$$(1) \qquad \mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}), \quad \mathbf{d}_a = \varphi_a(\mathbf{h}_b) = \mathbf{Q}(\mathbf{x})\mathbf{h}_a ,$$

where $\mathbf{Q} = \mathbf{d}_a \otimes \mathbf{h}_a \in SO(3)$ is the microrotation tensor.

Analysing differences between position and orientation differentials $\mathrm{d}\mathbf{x}$, $\mathrm{d}\mathbf{y}$, $\mathrm{d}\mathbf{h}_a$, and $\mathrm{d}\mathbf{d}_a$ we can define the Euclidean norms

$$(2) \qquad \begin{aligned} \| \mathrm{d}\mathbf{y} - \mathbf{Q}\mathrm{d}\mathbf{x} \|^2 &= \mathrm{d}\mathbf{x} \bullet \mathbf{E}^T \mathbf{E} \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \bullet \mathbf{G}^T \mathbf{G} \mathrm{d}\mathbf{y} , \\ \| \mathbf{C}\mathrm{d}\mathbf{y} - \mathbf{Q}\mathbf{B}\mathrm{d}\mathbf{x} \|^2 &= \mathrm{d}\mathbf{x} \bullet \mathbf{\Gamma}^T \mathbf{\Gamma} \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \bullet \mathbf{\Delta}^T \mathbf{\Delta} \mathrm{d}\mathbf{y} . \end{aligned}$$

In (2), $\mathbf{B} = \frac{1}{2}\mathbf{h}_a \times Grad\,\mathbf{h}_a$ and $\mathbf{C} = \frac{1}{2}\mathbf{d}_a \times grad\,\mathbf{d}_a$ are the microstructure curvature tensors in the undeformed and deformed placements, respectively, with $Grad$ and $grad$ being the corresponding gradient operators and

$$
\begin{aligned}
&\mathbf{E} = \mathbf{Q}^T\mathbf{F} - \mathbf{I}\,, \quad \mathbf{G} = \mathbf{Q}\mathbf{E}\mathbf{F}^{-1} = \mathbf{I} - \mathbf{Q}\mathbf{F}^{-1}\,, \\
&\boldsymbol{\Gamma} = \mathbf{Q}^T\mathbf{C}\mathbf{F} - \mathbf{B}\,, \quad \boldsymbol{\Delta} = \mathbf{Q}\boldsymbol{\Gamma}\mathbf{F}^{-1} = \mathbf{C} - \mathbf{Q}\mathbf{B}\mathbf{F}^{-1}\,,
\end{aligned}
\tag{3}
$$

where $\mathbf{F} = Grad\,\mathbf{y}$. The measures $\mathbf{E}, \mathbf{G}$ are the natural stretch tensors while $\boldsymbol{\Gamma}, \boldsymbol{\Delta}$ are the natural wryness tensors of the micropolar continuum in the Lagrangian and Eulerian descriptions, respectively.

2. Work-conjugate strain and stress measures

The local equilibrium equations of the micropolar continuum in the Lagrangian description are

$$Div\,\mathbf{T} + \mathbf{f} = \mathbf{0}\,, \quad Div\,\mathbf{M} + ax(\mathbf{T}\mathbf{F}^T - \mathbf{F}\mathbf{T}^T) + \mathbf{m} = \mathbf{0}\,, \tag{4}$$

where $\mathbf{T}$ and $\mathbf{M}$ are the stress and couple stress tensors of the 1st Piola-Kirchhoff type, and $ax(\mathbf{A})$ denotes the axial vector of the skew tensor $(\mathbf{A})$.

Multiplying the vector equations (4) by the kinematically admissible virtual translation $\delta\mathbf{u}$ and virtual rotation $ax(\delta\mathbf{Q}\mathbf{Q}^T)$ fields, respectively, after appropriate transformations we can formulate for the micropolar continuum the principle of virtual work in which the internal virtual power density becomes expressed as

$$\Sigma = \mathbf{S} \bullet \delta\mathbf{E} + \mathbf{P} \bullet \delta\boldsymbol{\Gamma}\,. \tag{5}$$

In (5), the virtual measures $\delta\mathbf{E}, \delta\boldsymbol{\Gamma}$ - the virtual changes of $\mathbf{E}, \boldsymbol{\Gamma}$ defined by (3) - are work-conjugate to the corresponding stress measures $\mathbf{S} = \mathbf{Q}^T\mathbf{T}$, $\mathbf{P} = \mathbf{Q}^T\mathbf{M}$ of the 2nd Piola-Kirchhoff type.

3. Principle of material frame-indifference

The elastic micropolar body is usually defined by assuming the existence of the strain energy density $W = W(\mathbf{y}, \mathbf{F}, \mathbf{Q}, Grad\,\mathbf{Q}; \mathbf{x})$. The function W should be invariant under transformations following from a rigid-body motion of the reference frame $\mathbf{y} \to \mathbf{O}\mathbf{y} + \mathbf{a}$, $\mathbf{Q} \to \mathbf{O}\mathbf{Q}$ for arbitrary $\mathbf{O} \in SO(3)$ and $\mathbf{a} \in E$. Then W can be reduced to $\tilde{W}(\mathbf{E}, \mathbf{Q}^T\boldsymbol{\Gamma}; \mathbf{x})$, that is $\tilde{W}$ still depends on $\mathbf{Q}$. We bypass this inconvenience by postulating the strain energy density in the equivalent form $W = \bar{W}(\mathbf{y}, \mathbf{F}^T, \mathbf{Q}^T, Grad\,\mathbf{Q}^T; \mathbf{x})$ which under the transformations given above can be reduced to $\hat{W}(\mathbf{E}, \boldsymbol{\Gamma}; \mathbf{x})$. As a result, the density $\hat{W}$ depending only on $\mathbf{E}, \boldsymbol{\Gamma}$ at each $\mathbf{x}$ is the one which assures the principle of material frame-indifference to be identically satisfied.

We also present a review of alternative definitions of the strain measures for the micropolar continuum proposed in the literature.

4. References

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[2] A.C. Eringen and C.B. Kafadar (1976). Polar field theories. In: A.C. Eringen (Ed.), *Continuum Physics*, **4**, 1-75; Academic Press, New York.