ON THE IMPROVED MEMBRANE PART OF MIXED SHELL ELEMENTS

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1. Introduction

Mixed finite elements exhibit a higher accuracy of displacements and stresses and a better convergence rate in non-linear problems than elements based on other formulations. Since the pioneering paper by Pian of 1964, a lot of work has been done to improve mixed methods; the elements and their theoretical foundations.

Among the mixed elements, particularly well known is the Pian-Sumihara (PS) element [1], which is slightly more accurate than the EAS4 element for coarse distorted meshes. Afterwards, several elements were developed, which perform slightly better than the PS element, including the element by Yuan, Huang and Pian [5] and by Piltner and Taylor of [2].

The objective of the current paper is to present our recently developed mixed finite elements based on the incremental Hellinger-Reissner (HR) functional and the incremental Hu-Washizu (HW) functional, see [3] and [4]. They are directly applicable to 'solid-shell' elements and to the shell elements without the drilling rotation.

2. Characteristics of the approach

The key feature of our approach is the use of the skew coordinates associated with the natural basis at the element's center $\{\mathbf{g}_k^c\}$, and defined as follows

$$(1) \qquad x_S = \xi + \frac{(j_{,\eta})_c}{j_c}\,\xi\eta, \qquad y_S = \eta + \frac{(j_{,\xi})_c}{j_c}\,\xi\eta,$$

where j_c, $(j_{,\xi})_c$ and $(j_{,\eta})_c$ are the parts of the expansion of the Jacobian's determinant, $\det \mathbf{J} = j_c + (j_{,\xi})_c\,\xi + (j_{,\eta})_c\,\eta$. The motivation behind the use of these coordinates will be explained in the presentation. The representations of stress and strain are assumed in terms of the skew coordinates as follows.

The contra-variant components of stress are assumed in the basis $\{\mathbf{g}_k^c\}$, i.e. $\boldsymbol{\sigma} = \sigma^{kl}\,\mathbf{g}_k^c \otimes \mathbf{g}_l^c$, $k, l = 1, 2$. The representation of σ^{kl} is assumed (matrix $\mathbf{G}_1$) and transformed to the reference basis on use of

$$(2) \qquad \boldsymbol{\sigma}^{ref} = \mathbf{J}_c\,\mathbf{G}_1\,\mathbf{J}_c^T, \qquad \mathbf{G}_1 \doteq \begin{bmatrix} q_1 + q_2\,y_S & q_5 \\ \text{symm.} & q_3 + q_4\,x_S \end{bmatrix},$$

where $\mathbf{J}_c$ is the Jacobian matrix evaluated at the element center. The above 5-parameter representation in $\mathbf{G}$ is identical as in the PS element, but the skew coordinates are used.

The co-variant components of strain are assumed in the co-basis $\{\mathbf{g}_c^k\}$, i.e. $\boldsymbol{\varepsilon} = \varepsilon_{kl}\,\mathbf{g}_c^k \otimes \mathbf{g}_c^l$. The representation of $\varepsilon_{\alpha\beta}$ is assumed (matrix $\mathbf{G}_2$) and transformed to the reference basis on use of

$$(3) \qquad \boldsymbol{\varepsilon}^{ref} = \mathbf{J}_c^{-T}\,\mathbf{G}_2\,\mathbf{J}_c^{-1}, \qquad \mathbf{G}_2 \doteq \begin{bmatrix} q_6 + q_7\,y_S + q_8\,x_S & q_{12} + q_{13}\,x_S + q_{14}\,y_S \\ \text{symm.} & q_9 + q_{10}\,x_S + q_{11}\,y_S \end{bmatrix}.$$

The applied 9-parameter representation of strain is linear for all components.

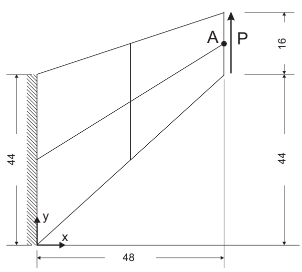
A beneficial consequence of using the above representations in skew coordinates is that, for a linear elastic case, the homogenous equilibrium equations and the compatibility condition are satisfied point-wise, regardless of the element's shape, which is an exceptional property. Note that for the representations in the natural coordinates, these equations are satisfied only for parallelograms.

3. Mixed and mixed/enhanced elements based on HR and HW functionals

Several mixed and mixed/enhanced 4-node elements are developed and tested in [3] and [4], using also the 7-parameter representation of stress, which requires an additional strain enhancement. We selected two elements, designated as HR5-S and HW14-S, as the best performers. The HR5-S is based on eq.(2), while the HW14-S on eq.(2) and eq.(3). These elements are mixed; although they use less parameters they still perform similarly as the mixed/enhanced elements.

The developed elements are based on the Green strain, and are applicable to large deformation problems and non-linear materials. They have a correct rank, and pass the patch test. They were subjected to a range of benchmark tests, to establish the coarse mesh accuracy and the sensitivity to mesh distortion. One of these tests is the Cook's membrane, see Fig.1, where $E = 1$, $\nu = 1/3$, $h = 1$, $P = 1$, which is very indicative, because the shear deformation dominates, and the elements are trapezoidal.

a)



b)

Element	Mesh 2×2	Mesh 32×32
HR5-S	21.353	23.940
HW14-S	21.353	23.940
EAS4	21.050	23.940
Q4	11.845	23.818
5β-A,B,C [5]	21.35	
QE2 [2]	21.35	
Ref.	23.81	

Figure 1. Cook's membrane. a) Initial geometry and load, b) Displacement u_y at point A for two meshes.

The numerical results confirm that our new HR5-S and HW14-S elements are more accurate and less sensitive to mesh distortion than the EAS4 element and the PS element. They use a smaller number of modes than the other top elements described in the literature, because the enhancement is not needed, yet they yield equally accurate results.

4. References

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