ON DETERMINING THE DEFORMED SHELL MIDSURFACE FROM PRESCRIBED SURFACE STRAINS AND BENDINGS

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The intrinsic formulation of the geometrically non-linear theory of thin elastic shells, proposed in [1], allows one to find strains $\gamma_{\alpha\beta}$ and bendings $\kappa_{\alpha\beta}$ of the shell midsurface. Then the position vector $\mathbf{y}$ of the midsurface of the deformed shell can be found from known $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ by one of two methods proposed in [2].

In this report we develop an alternative novel method of determining the vector $\mathbf{y}$ from prescribed $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. The present approach uses the right polar decomposition of the midsurface deformation gradient $\mathbf{R} = \mathbf{RU}$, where $\mathbf{U}$ is the surface right stretch tensor and $\mathbf{R}$ is the 3D rotation tensor. Applying the method developed here the vector $\mathbf{y}$ is calculated in three consecutive steps described briefly below.

Let $\mathbf{x} = \mathbf{x}(\theta^\alpha)$, $\alpha = 1,2$, be the position vector of the shell midsurface M in the reference (undeformed) configuration. At each point $x \in M$ we define the natural base vectors $\mathbf{a}_\alpha = \partial \mathbf{x} / \partial \theta^\alpha \equiv \mathbf{x},_\alpha$, the unit normal vector $\mathbf{n} = \frac{1}{\sqrt{a}} \mathbf{a}_1 \times \mathbf{a}_2$, the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \bullet \mathbf{a}_\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta})$, and the covariant components $b_{\alpha\beta} = -\mathbf{a}_a \bullet \mathbf{n},_\beta$ of the curvature tensor $\mathbf{b}$. In the deformed configuration the shell midsurface $\overline{M}$ is parameterized by the convected coordinates θ^α so that its geometry is described by the same symbols with a bar above them: $\overline{\mathbf{a}}_\alpha, \overline{\mathbf{n}}, \overline{a}, \overline{a}_{\alpha\beta}, \overline{b}_{\alpha\beta}$, *etc.* Then the deformation state of the shell midsurface is described by the covariant components $\gamma_{\alpha\beta} = \frac{1}{2}(\overline{a}_{\alpha\beta} - a_{\alpha\beta})$ and $\kappa_{\alpha\beta} = -(\overline{b}_{\alpha\beta} - b_{\alpha\beta})$ of the surface strain $\boldsymbol{\gamma}$ and bending $\boldsymbol{\kappa}$ tensors, respectively.

Introducing the midsurface deformation gradient $\mathbf{F} = \mathbf{y},_\alpha \otimes \mathbf{a}^\alpha$, by the theorem of Tissot we can justify the right polar decomposition $\mathbf{F} = \mathbf{RU}$. Then the field $\mathbf{y} = \mathbf{y}(\theta^\alpha)$ can be found in the three steps described below.

a) From known $\gamma_{\alpha\beta}$ the stretch field $\mathbf{U} = \mathbf{U}(\theta^\alpha)$ is found by pure algebra through the explicit formula

$$\mathbf{U} = \frac{\left(1 + \sqrt{1 + 2\,\mathrm{tr}\,\boldsymbol{\gamma} + 4\det\boldsymbol{\gamma}}\right)\mathbf{a} + 2\boldsymbol{\gamma}}{\sqrt{2\left(1 + \mathrm{tr}\,\boldsymbol{\gamma} + \sqrt{1 + 2\,\mathrm{tr}\,\boldsymbol{\gamma} + 4\det\boldsymbol{\gamma}}\right)}} \,. \tag{1}$$

b) From known $\mathbf{U}$ and $\kappa_{\alpha\beta}$ the rotation field $\mathbf{R} = \mathbf{R}(\theta^\alpha)$ is calculated by solving the system of two linear PDE's

$$\mathbf{R},_{\alpha} = \mathbf{R}\times\mathbf{k}_{\alpha}\ ,\quad \mathbf{k}_{\alpha} = \varepsilon^{\lambda\kappa}\mu_{\lambda\alpha}\mathbf{a}_{\kappa} + k_{\alpha}\mathbf{n}\ ,$$

(2)

$$\mu_{\alpha\beta} = b_{\alpha\beta} - \sqrt{\frac{a}{\bar{a}}}\varepsilon^{\lambda\kappa}\varepsilon_{\alpha\rho}U^{\rho}_{\kappa}(b_{\lambda\beta} - \kappa_{\lambda\beta})\ ,\quad k_{\alpha} = -\sqrt{\frac{a}{\bar{a}}}\varepsilon^{\kappa\rho}U^{\lambda}_{\alpha}U_{\lambda\kappa|\rho}\ ,$$

where $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\lambda\kappa}$ are components of the surface permutation tensor $\boldsymbol{\varepsilon}$, and $(.)_{|\alpha}$ denotes the surface covariant derivative in the metric $a_{\alpha\beta}$. The integrability conditions $\varepsilon^{\alpha\beta}\mathbf{R},_{\alpha\beta} = \mathbf{0}$ of the system $(2)_1$ are proved to be equivalent to the compatibility conditions of the non-linear theory of thin shells.

Using the theorem of Frobenius – Dieudonné it has been shown that the solutions to the problem $(2)_1$ can be converted into an infinite set of systems of ODE's along curves $C \subset M$, parameterized by the length coordinate s and covering densely the entire domain of M :

(3) $$\frac{d\mathbf{R}}{ds} = \mathbf{RK},\quad \mathbf{K} = \mathbf{I}\times\mathbf{k}\ ,\quad \mathbf{k} = \mathbf{k}_{\alpha}\frac{d\theta^{\alpha}}{ds}\ ,$$

where $\mathbf{I}$ is the identity tensor of the 3D vector space.

Solution to the initial value problem $(3)_1$ may be obtained with any of the well known techniques, numerical techniques inclusive. In particular, applying the method of successive approximations the solution to $(3)_1$ can be given in the form

$$\mathbf{R} = \mathbf{R}_0\mathbf{R}_s\ ,\quad \mathbf{R}_s = \sum_{i=0}^{\infty}\mathbf{O}_i\ ,$$

(4)

$$\mathbf{O}_0(s) = \mathbf{I}\ ,\quad \mathbf{O}_i(s) = \int_{s_0}^{s}\mathbf{O}_{i-1}(t)\mathbf{K}(t)dt\ ,\quad i \geq 1\ ,$$

where $\mathbf{R}_0 = \mathbf{R}(s_0)$ is the rotation tensor at $s = s_0$.

c) With $\mathbf{R}$ and $\mathbf{U}$ already known the system $\mathbf{y},_{\alpha} = \mathbf{Fa}_{\alpha}$ can be integrated by quadrature

(5) $$\mathbf{y} = \mathbf{y}_0 + \int_{x_0}^{x}\mathbf{RUa}_{\alpha}d\theta^{\alpha}\ ,\quad \mathbf{y}_0 = \mathbf{y}(x_0)\ .$$

The equation $(3)_1$ is identical with the one describing spherical motion of a rigid body about a fixed point. Thus, one can point out a number of special cases when the equation has the solution in closed form. This indicates that the novel method presented here might in some cases be more efficient in applications than those proposed in [2].

Details of the method will be published in [3].

A similar approach has recently been successfully applied to analyse the classical problem of differential geometry: recovery of the surface from components of its two fundamental forms.

References

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