ON THE MECHANICS OF FUNCTIONALLY GRADED PLATES

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Metallic and polymeric foams are more and more used as a material for lightweight structures [1]. Such structures are applied in the automotive or airspace industries since they combine low weight, high strength and excellent possibilities to absorb energy. The foam itself can be modeled as a functionally graded material with mechanical properties changing over the thickness direction.

The aim of this contribution is a new theory based on the direct approach in the plate theory added by the effective properties concept.

1. Basic equations

Let us consider for the brevity the geometrically and physically linear theory. In addition, we assume plate-like structures. Here we use the so-called direct approach. In this case one states a two-dimensional deformable surface. On each part of this deformable surface forces and moments are acting – they are the primary variables. The next step is the introduction of the deformation measures. Finally, it is necessary to interlink the forces and the moments with the deformation variables (constitutive equations). Such a theory is formulated by a more natural way in comparison with the other approaches. But the identification of the stiffness and other parameters is a non-trivial problem and must be realized for each class of plates individually.

The motion equations and the kinematic equations are given by the relations [2–4]

$$\nabla \cdot \mathbf{T} + \mathbf{q} = \rho\ddot{\mathbf{u}} + \rho\mathbf{\Theta}_1\cdot\ddot{\boldsymbol{\varphi}}, \qquad \nabla \cdot \mathbf{M} + \mathbf{T}_\times + \mathbf{m} = \rho\mathbf{\Theta}_1^{\mathrm{T}}\cdot\ddot{\mathbf{u}} + \rho\mathbf{\Theta}_2\cdot\ddot{\boldsymbol{\varphi}}, \tag{1}$$

$$\boldsymbol{\mu} = \frac{1}{2}\left[\nabla\mathbf{u}\cdot\mathbf{a} + (\nabla\mathbf{u}\cdot\mathbf{a})^{\mathrm{T}}\right], \qquad \boldsymbol{\gamma} = \nabla\mathbf{u}\cdot\mathbf{n} + \mathbf{c}\cdot\boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla\boldsymbol{\varphi} \tag{2}$$

Here $\mathbf{T}$, $\mathbf{M}$ are the tensors of forces and moments, $\mathbf{q}$, $\mathbf{m}$ are the surface loads (forces and moments), $\mathbf{T}_\times$ is the vector invariant of the force tensor, ∇ is the nabla operator, $\mathbf{u}$, $\boldsymbol{\varphi}$ are the vectors of displacements and the rotations, $\mathbf{\Theta}_1, \mathbf{\Theta}_2$ are the first and the second tensor of inertia, ρ is the density, $(\ldots)^{\mathrm{T}}$ denotes transposed and $\dot{(\ldots)}$ is the time derivative. $\mathbf{a}$ is the first metric tensor, $\mathbf{n}$ is the unit normal vector, $\mathbf{c} = -\mathbf{a}\times\mathbf{n}$ is the discriminant tensor, $\boldsymbol{\mu}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are the tensor of in-plane strains, the vector of transverse shear strains and the tensor of the out-of-plane strains, respectively.

Limiting our discussion to the elastic behavior and small strains we assume the following constitutive equations of a plate

$$\begin{aligned} \mathbf{T}\cdot\mathbf{a} &= \mathbf{A}\cdot\cdot\boldsymbol{\mu} + \mathbf{B}\cdot\cdot\boldsymbol{\kappa} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma}_1, \quad \mathbf{T}\cdot\mathbf{n} = \mathbf{\Gamma}\cdot\boldsymbol{\gamma} + \mathbf{\Gamma}_1\cdot\cdot\boldsymbol{\mu} + \mathbf{\Gamma}_2\cdot\cdot\boldsymbol{\kappa}, \\ \mathbf{M}^{\mathrm{T}} &= \boldsymbol{\mu}\cdot\cdot\mathbf{B} + \mathbf{C}\cdot\cdot\boldsymbol{\kappa} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma}_2 \end{aligned} \tag{3}$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$ are 4$^{\mathrm{th}}$ rank tensors, $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$ are 3$^{\mathrm{rd}}$ rank tensors, $\mathbf{\Gamma}$ is a 2$^{\mathrm{nd}}$ rank tensor expressing the effective stiffness properties. They depend on the material properties and the cross-section geometry. In the general case the tensors contain 36 different values – a reduction is possible assuming some symmetries.

Let us consider an orthotropic material behavior and a plane mid-surface. In this case one gets

$$\begin{aligned} \mathbf{A} &= A_{11}\mathbf{a}_1\mathbf{a}_1 + A_{12}(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + A_{22}\mathbf{a}_2\mathbf{a}_2 + A_{44}\mathbf{a}_4\mathbf{a}_4, \\ \mathbf{B} &= B_{13}\mathbf{a}_1\mathbf{a}_3 + B_{14}\mathbf{a}_1\mathbf{a}_4 + B_{23}\mathbf{a}_2\mathbf{a}_3 + \mathbf{B}_{24}\mathbf{a}_2\mathbf{a}_4 + B_{42}\mathbf{a}_4\mathbf{a}_2, \\ \mathbf{C} &= C_{22}\mathbf{a}_2\mathbf{a}_2 + C_{33}\mathbf{a}_3\mathbf{a}_3 + C_{34}(\mathbf{a}_3\mathbf{a}_4 + \mathbf{a}_4\mathbf{a}_3) + C_{44}\mathbf{a}_4\mathbf{a}_4, \\ \mathbf{\Gamma} &= \Gamma_1\mathbf{a}_1 + \Gamma_2\mathbf{a}_2, \qquad \mathbf{\Gamma}_1 = \mathbf{0}, \qquad \mathbf{\Gamma}_2 = \mathbf{0} \end{aligned}$$

with $\mathbf{a}_1 = \mathbf{a} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2, \mathbf{a}_2 = \mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2, \mathbf{a}_3 = \mathbf{c} = \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1, \mathbf{a}_4 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1$, $\mathbf{e}_1, \mathbf{e}_2$ are unit basic vectors.

2. Stiffness tensors identification

The individuality of each class of plates in the framework of the direct approach is expressed by the effective properties (stiffness, density, inertia terms, etc.). Let us focus our attention on the stiffness expressions. The identification of the effective stresses should be performed on the base of the properties of the real material. Let us assume the generalized Hooke's law with material properties which depend on z. The identification of the effective properties can be performed with the help of static boundary value problems (two-dimensional, three-dimensional) and the comparison of the forces and moments (in the sense of averaged stresses or stress resultants).

Finally, we get the following expressions for the stiffness tensor components [2, 4]

$$
\begin{aligned}
(A_{11}; -B_{13}; C_{33}) &= \frac{1}{4}\left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}(1; z; z^2)\right\rangle, \\
(A_{22}; B_{24}; C_{44}) &= \frac{1}{4}\left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}(1; z; z^2)\right\rangle, \\
(A_{12}; -B_{23} = B_{14}; -C_{34}) &= \frac{1}{4}\left\langle \frac{E_1 - E_2}{1-\nu_{12}\nu_{21}} z(1; z; z^2)\right\rangle, \\
(A_{44}; -B_{42}; C_{22}) &= < G_{12}(1; z; z^2) >,
\end{aligned}
\tag{4}
$$

$$
\Gamma_1 = \frac{1}{2}(\lambda^2 + \eta^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}}, \quad \Gamma_2 = \frac{1}{2}(\eta^2 - \lambda^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}}, \tag{5}
$$

where $< \ldots >$ is the integral over the plate thickness h, while η^2 and λ^2 are the minimal eigen-values of the following Sturm-Liouville problems

$$
\frac{\mathrm{d}}{\mathrm{d}z}\left(G_{1n}\frac{\mathrm{d}Z}{\mathrm{d}z}\right) + \eta^2 G_{12} Z = 0, \; \frac{\mathrm{d}}{\mathrm{d}z}\left(G_{2n}\frac{\mathrm{d}Z}{\mathrm{d}z}\right) + \lambda^2 G_{12} Z = 0, \quad \left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{|z|=\frac{h}{2}} = 0
$$

The described above approach was applied to FGM plates made of metal or polymer foams with nonhomogeneous distribution of porosity [4, 5].

3. Conclusions

We presented the theory of FGM plates on the basis of the direct approach. The considered approach to model FGM plates within the framework of a 5-parametric theory of plates has an advantage with respect to theories of sandwich or laminated plates since many classical results can be improved without any difficulties.

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