INERTIAL MOVING LOADS

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1. Introduction

The problem of bridge spans under a moving inertial load [1, 2] has existed since the beginning of the railways development. Together with increasing velocity of trains, the influence of the wave phenomenon is rising as well. Dynamic effects are generated by the load of train current collectors, travelling through the power supply cable of the overhead contact line. Solutions of inertial moving load applied to discrete systems unfortunately are practically not reported. Inertial force, which should be considered as a couple of a force and a mass is usually replaced by a spring-mass system. Finally the problem is solved as a problem with a massless force. We must also emphasize that the ad-hoc mass distribution between neighbouring nodes simply fails. In the case of the beam at low speed ranges and low ratio of the moving mass to the beam mass results exhibit errors. Unfortunately, such formulations exist in spite of a wrong formulation and analysis.

In this presentation the differential equations of the motion of a string and beams were derived from the Lagrange equation of the 2nd kind. Moreover, the direct solution of the differential equation was obtained as an alternative solution. Both results coincide. We also present the numerical approach to the moving inertial load problem. Classical finite element method with Newmark time integration scheme mentioned in fails. The space-time finite element method is the only method which enables us to describe the mass passing through the spatial finite element in a continuous way. We present the solution in the case of a string and a Bernoulli-Euler beam.

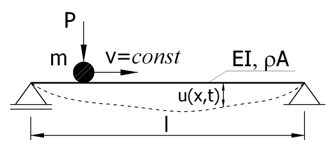

Figure 1. The point mass moving on the beam.

2. Formulation

The motion equation of the beam under a moving mass m coupled with a force P can be written as follows

$$EI\,\frac{\partial^4 u(x,t)}{\partial x^4} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,, \tag{1}$$

where EI is the beam stiffness, N is a tensile force and ρA is a linear mass density. Taking into account beam terms, we impose four boundary conditions

$$u(0,t) = 0\,,\quad u(l,t) = 0\,,\quad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=0} = 0\,,\quad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=l} = 0\,, \tag{2}$$

and two initial conditions $u(x,0) = 0$, $\partial u(x,t)/\partial t|_{t=0} = 0$. The equation can not be easily solved and we must integrate it in a numerical way. We use the matrix notation here

$$\mathbf{M}\begin{bmatrix}\ddot{\xi}_1(t)\\ \ddot{\xi}_2(t)\\ \vdots\\ \ddot{\xi}_n(t)\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{\xi}_1(t)\\ \dot{\xi}_2(t)\\ \vdots\\ \dot{\xi}_n(t)\end{bmatrix} + \mathbf{K}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_n(t)\end{bmatrix} = \mathbf{P}\,, \tag{3}$$

which results in a short form $\mathbf{M}\ddot{\boldsymbol{\xi}} + \mathbf{C}\dot{\boldsymbol{\xi}} + \mathbf{K}\boldsymbol{\xi} = \mathbf{P}$, where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are square matrices for $i = j = 1, 2, ..., n$.

When we calculate the value of general coordinates $\xi_i(t)$ for each i to n. Finally we can compute displacements of the string-beam $u(x,t)$

$$u(x,t) = \sum_{i=1}^{\infty} \xi_i(t) \sin \frac{i\pi x}{l} \, . \tag{4}$$

Displacements given in the example below are dimensionless. They were calculated in relation to the static deflection u_0 of the string-beam loaded in the mid point by the point force P: $u_0 = u_{0s}\, u_{0b}/(u_{0s}\, u_{0b})$. u_{0s} and u_{0b} are static deflections in the case of a string and a beam, respectively. The mass trajectory is depicted in Fig. 2.

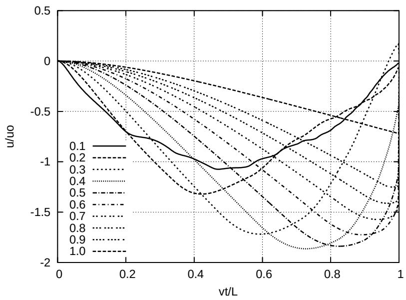

Figure 2. Mass trajectory for different speeds v (bending neglected).

3. Conclusions

We deal with the problem of the numerical treatment of the moving mass problem. The solution presented in the paper shows the way of mathematical analysis which results in a universal time stepping procedure. It enables us to solve the problem with the arbitrary speed. The solution in the case of the string exhibits discontinuous mass trajectory [3, 4] at the end support. This fact influences high gradients of the solution at the final stage of the motion. This phenomenon is the paradoxical property of the differential equation since considering boundary conditions we intuitively expect smooth curves. Numerical results of the string vibrations exhibit good accuracy, comparing with semi-analytical solution. In the case of the beam the coincidence of both curves is perfect.

[1] C.E. Inglis. *A Mathematical Treatise on Vibrations in Railway Bridges*. Cambridge University Press, 1934.

[2] L. Frỳba. *Vibrations of solids and structures under moving loads*. Academia, Prague, 1972.

[3] B. Dyniewicz i C.I. Bajer. Paradox of the particle's trajectory moving on a string. *Arch. Appl. Mech.*, 2008. DOI: 10.1007/s00419-008-0222-9.

[4] C.I. Bajer i B. Dyniewicz. Space-time approach to numerical analysis of a string with a moving mass. *Int. J. Numer. Meth. Engng.*, 2008. DOI: 10.1002/nme.2372.